Abstract

With the motivation of overcoming difficulties in studying systems of several one dimensional Tomonaga Luttinger liquid wires connected locally at a junction, we construct quadratic lattice field theories for the single non-chiral/chiral wire, the three wire fork and the chiral box junction. For a fork consisting of three identical non-chiral wires, we find that the permutation symmetries of the system, together with the requirement of charge and current continuity, determine the terms in the action governing the dynamics of the fields adjacent to the junction. These results are generalised to the case of $N \geq 3$ identical (as well as different) TLL wires. A study of the chiral box junction circuit model reveals that junctions of several chiral wires can be formed by relating the TLL interaction parameters of their constituents.
I. INTRODUCTION

The subject of continuum field theories of interacting fermions and bosons in one dimension is well studied. A major success of this pursuit has been the development of the Tomonaga-Luttinger liquid (TLL) as a paradigm for metallic behaviour in 1D. This has been facilitated by the fact that the dynamics of a system of interacting fermions in 1D (a many body problem involving the complications of electron electron interactions) can generically be described in terms of that of a system of free bosonic collective modes. This is captured in a powerful manner by the bosonisation formalism\textsuperscript{2,3}. While considerable attention has been devoted to understanding the various ordered phases that can arise from renormalisation group (RG) relevant bulk perturbations\textsuperscript{2,3,9}, some fascinating aspects of boundary critical phenomena in these systems have also been uncovered\textsuperscript{11}. Specifically, an interplay of electronic interactions and impurity scattering is known to lead to drastic consequences for the dynamics of a TLL: the strength of an infinitely weak local backscatterer turns out to be RG relevant, leading to the strong coupling physics being that of two TLLs effectively broken off from one another\textsuperscript{5}.

An understanding of geometries beyond strictly 1D is, however, not as well developed. A prototypical problem is that of a point junction connecting 3 or more TLLs: the TLLs are described by their velocity $v$ and interaction $K$ parameters (taken to be identical throughout our discussion) while the junction can be characterised by a matrix describing the transmission and reflection amplitudes connecting the various arms. Progress has, however, been made in understanding the phase diagram and transport properties of such a system in only certain special cases. These include (a) the case of arbitrarily strong interactions (i.e., arbitrary values of the $K$ parameters) while the junction matrix takes very special forms (e.g, completely disconnected wires in which all reflection amplitudes are nearly perfect and all transmission amplitudes perturbatively weak\textsuperscript{6,7,8,9,10} and (b) the case of perturbatively weak interactions in the various wires (i.e., all $K$ parameters just a little less than 1 for the case of spinless electrons) while the junction matrix can be arbitrary\textsuperscript{11,12}. However, it is important to first note that a study of the full problem, i.e., with arbitrary interactions and junction matrix, remains unsolved.

Progress in this context was made very recently in the case of transport in quantum Hall edge states with constrictions\textsuperscript{13}. In this work, a model was developed for a quantum Hall bar with a mesoscopic region of lowered filling fraction (in comparison to the bulk). An understanding of the edge state transport in this system needed the formulation of a quadratic field theory describing the dynamics of edge density waves. Most importantly, the junction of the chiral incoming and outgoing edges lying at the meeting points of the bulk and constriction Hall fluid regions needed a careful analysis in terms of matching conditions on the local edge fields as well as their spatial derivatives. In conjunction with the Hamiltonians (describing the energy cost for density fluctuations) and commutation relations for the edge fields, the matching conditions helped provide a complete description of a scenario of intermediate ballistic transmission in the system in terms of
a quadratic continuum field theory. An analysis of local interedge quasiparticle tunneling physics in this model then led to the generalisation of the Kane-Fisher quantum impurity model. With these findings as inspiration, the present work aims to take steps towards the formulation of quadratic continuum field theoretic descriptions of systems of several TLL (non-chiral as well as chiral) wires connected locally at junctions. In practice, however, this task has proved far easier to conduct by first placing the system on a lattice. We will, therefore, provide a lattice field theory analysis in this work, stopping to take note of the continuum equivalents for all important results obtained wherever necessary.

The are other motivations for constructing lattice variations of the familiar TLL continuum theory beyond gaining insights on the influence of geometry in such systems. The first relates to the possibility that, once the theory of a single junction connecting several TLL wires is well understood, an analysis of regular two and three dimensional arrays composed of such systems can be carried out. Such a study was, in fact, conducted by Kazymyrenko and Doucot in the continuum for a 2D regular array of 4 TLL-wire junctions, in the limit of very weak electronic interactions in the wires. Remarkably, the authors found that the RG physics of a single junction was also responsible for the RG phase diagram of the 2D array consisting of a system of a series of metal-insulator transitions (depending on the chemical potential of the array). However, this result is as yet to be confirmed for the case of arbitrarily strong interactions in the TLL wires. It is, therefore, clear that establishing the theory of a single junction represents a very important step towards the study of the dynamics of regular wire arrays. At the same time, it is worth noting that considerable success has been enjoyed by the numerical Monte Carlo methods employed in lattice gauge theory approaches to understanding, among other things, the strong coupling physics of continuum QCD. This presents the possibility that similar numerical approaches can be developed towards the study of lattice variants of continuum theories for connected TLL systems. Such a development holds the promise of a complementary approach towards the study of the dynamics such systems.

II. LATTICE MODELS FOR CONNECTED NON-CHIRAL 1D SYSTEMS

A. Lattice model for the 1D Tomonaga-Luttinger liquid

The goal here is to construct a lattice model for the prototypical correlated system of electrons in d=1, the Tomonaga-Luttinger liquid (TLL). As the continuum theory is known to be quadratic in terms of two bosonic fields, \( \phi \) and \( \theta \), we consider the following Hamiltonian for a Gaussian chain in one dimension

\[
H = -\frac{\nu}{2\pi a^2} \sum_{i=-\infty}^{\infty} \frac{\phi_i}{K}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + K\theta_i(\theta_{i+1} - 2\theta_i + \theta_{i-1})
\]  (1)
As shown below in Figure 1, the bosonic fields $\phi_i$ and $\theta_i$ are defined on the $i$th link between the lattice sites $i$ and $i+1$. $v$ is the equivalent of the wave velocity on the lattice while $K$ again denotes the interaction parameter. We define the following density operators on the $i$th lattice link while the densities $(\rho, \theta)$, which can be seen to be the differences of the equations of motion at sites $i$ and $i+1$, are defined on the lattice nodes.

We can then use the equation of continuity on the lattice $i\partial_x(\rho_{i+1} + \rho_i)/2 + (j_{i+1} - j_i)/a = 0$, to define the currents (in Matsubara frequencies $\bar{\omega}_n$) $j_{\phi,i} = \bar{\omega}_n(\phi_{i+1} + \phi_i)/2$ and $j_{\theta,i} = \bar{\omega}_n(\theta_{i+1} + \theta_i)/2$.

The commutation relations satisfied by these fields are

$$\left[ \frac{\phi_{i+1} + \phi_i}{2}, \frac{\theta_{j+1} - \theta_j}{\pi a} \right] = i\delta_{ij}$$
$$= \left[ \frac{\theta_{i+1} + \theta_i}{2}, \frac{\phi_{j+1} - \phi_j}{\pi a} \right] .$$

As these commutation relations give us that certain combinations of the adjacent $\phi$ and $\theta$ fields on every node define canonically conjugate variables, we can write down the action (in $\bar{\omega}_n$) via a Legendre transformation as

$$S = \sum \bar{\omega}_n \left\{ H + \sum_i \frac{\bar{\omega}_n}{\pi a} \left( \frac{\theta_{i+1} + \theta_i}{2} \right) \left( \frac{\phi_{i+1} - \phi_i}{\pi a} \right) \right\} .$$

We can now use the variational principle to derive the equation of motion for $\theta_i$ and $\phi_i$ as

$$\frac{\delta S}{\delta \theta_i} = -\frac{vK}{\pi a^2} (\theta_{i+1} - 2\theta_i + \theta_{i-1}) + \frac{\bar{\omega}_n}{2a} (\phi_{i+1} - \phi_{i-1}) = 0$$
$$\frac{\delta S}{\delta \phi_i} = -\frac{v}{\pi Ka^2} (\phi_{i+1} - 2\phi_i + \phi_{i-1}) - \frac{\bar{\omega}_n}{2a} (\phi_{i+1} - \phi_{i-1}) = 0 ,$$

which can be seen to be the differences of the equations of motion at sites $i+1$ and $i$

$$(\theta_{i+1} - \theta_i) = \frac{\bar{\omega}_n a}{2vK} (\phi_{i+1} + \phi_i) ,$$
$$(\phi_{i+1} - \phi_i) = \frac{\bar{\omega}_n aK}{2v} (\theta_{i+1} + \theta_i) .$$

Analogous relations for the (continuum) fields $\phi(x,t)$ and $\theta(x,t)$ were given earlier. As in the continuum theory, it is also possible to perform a (Gaussian) functional integral over either the field $\phi(x,t)$ or the field $\theta(x,t)$ obtain the lattice actions for simply the $\phi$ or $\theta$ fields. For instance, in the action given above, we can complete squares and integrate out the $\phi$ fields, leaving us with

$$S_{\theta} = \sum_{\bar{\omega}_n, i = -\infty}^{\infty} \left\{ \frac{\omega_n^2}{2\pi vK} \left( \frac{\theta_{i+1} + \theta_i}{2} \right)^2 - \frac{v}{2\pi Ka^2} (\theta_{i+1} - 2\theta_i + \theta_{i-1}) \right\} .$$
By starting from an action with the other Legendre transform term \( \sum \bar{\omega}_n (\phi_{i+1} + \phi_i)/2(\theta_{j+1} - \theta_j)/(\pi a) \), we can complete squares and integrate out the \( \theta \) fields, giving us
\[
S_\phi = \sum_{\bar{\omega}_n, i = -\infty}^{\infty} \left\{ \frac{\bar{\omega}_n^2 K}{2\pi v} (\phi_{i+1} + \phi_i)^2 - \frac{v K}{2\pi a^2} \phi_i (\phi_{i+1} - 2\phi_i + \phi_{i-1}) \right\}.
\]
(7)

From these, the equations of motion for \( \phi(x, t) \) and \( \theta(x, t) \) can be easily derived
\[
\bar{\omega}_n^2 \frac{\phi_{i+1} + \phi_i}{2} = \bar{\omega}_n J_{\phi, i} = v^2 (\rho_{\phi, i+1} - \rho_{\phi, i})
\]
\[
\bar{\omega}_n^2 \frac{\theta_{i+1} + \theta_i}{2} = \bar{\omega}_n J_{\theta, i} = v^2 (\rho_{\theta, i+1} - \rho_{\theta, i})
\]
(8)

These are the equivalent of the wave-equations of motion obtained for the fields \( \phi \) and \( \theta \) in the continuum. Using the translational symmetry in the system, we can try the following ansatz for the solution of the equation of motion (in real space and time)
\[
\phi_j(t) = \sum_{k, \omega} e^{i(k_j - \omega t)} \tilde{\phi}(k, \omega).
\]
(9)

Putting this into the equation of motion gives us the dispersion relation as \( \omega = \pm v \sin(k/2) \approx \pm v k \), in the limit of small wavevectors. Thus, we confirm that the lattice version of the TLL indeed has two (gapless) linear branches in the dispersion relation for long-wavelength excitations characterised by a wavevector \( k \ll k_F \) (where \( k_F \) is the one-dimensional Fermi wavevector). Thus, we see that we have been able to formulate a lattice field theoretic model for the TLL. Further, taking the continuum limit involves taking the limit of the lattice spacing \( a \to 0 \), while keeping the product of the lattice site index \( i = x/a \) and the discrete momentum \( k = a \tilde{k} \) constant (where \( \tilde{k} \) is the continuum wavevector). This process can be carried out in all relations given above, encountering no difficulties in rederiving the familiar continuum expression. The generalisation to the case of electrons with spin can also be carried out without any difficulties. Finally, all correlation functions on the lattice (including the two point propagator) can be computed from this action along the lines shown in Ref.(15).

**B. Lattice model for the 3 TLL fork**

We now turn to the formulation of a lattice model for the case of three identical TLL wires connected at a junction: the 3 wire fork. For the sake of convenience, we will number the nodes as increasing positively outwards from the zeroth junction node. This is shown below in Fig(2). We proceed by first writing down the lattice action for this system and then computing the equations of motion in the bulk and at the junction. Thereby, we highlight the role of symmetry in dictating the form of the terms in the action involving the fields surrounding the junction node. Thus, the complete action can be written as \( S = S_0 + S_1 \) where \( S_0 \) is the quadratic action for the fields on each of the three identical arms (from node index 2 and greater) which was given above in
FIG. 2: A schematic diagram for the lattice 3 TLL fork model. The fields, densities and currents are defined as shown in Fig.1.

equation (3). The term $S_1$ includes all terms of the action involving the fields surrounding the zeroth node

$$S_1 = \sum_{\omega_n, \alpha=1}^{3} \left[ -\frac{v}{2\pi a^2 K} \phi_1^\alpha (\phi_2^\alpha - \phi_1^\alpha) - \frac{vK}{2\pi a^2} \theta_1^\alpha (\theta_2^\alpha - \theta_1^\alpha) \right. \\
+ \frac{\omega_n}{2a} \theta_1^\alpha (\phi_2^\alpha + \phi_1^\alpha)] \\
+ \sum_{\omega_n} \frac{v}{2\pi a^2 K} (\tilde{\phi}_0)^2 + \frac{vK}{2\pi a^2} (\tilde{\theta}_0)^2 - \frac{\omega_n}{2a} \tilde{\theta}_0 \tilde{\phi}_0 ,$$  

(10)

where $\tilde{\phi}_0 = \sum_{\alpha=1}^{3} \phi_1^\alpha , \tilde{\theta}_0 = \sum_{\alpha=1}^{3} \theta_1^\alpha$. Computing the equations of motion for all fields with index $i \geq 2$, we find them to be exactly the same as that given above in equation (5). For the fields surrounding node zero, we get the two equations of motion

$$\frac{vK}{a} \tilde{\theta}_0 = \frac{\omega_n}{2} \tilde{\phi}_0 , \quad \frac{v}{aK} \tilde{\phi}_0 = \frac{\omega_n}{2} \tilde{\theta}_0 .$$  

(11)

In complete analogy with the equations of motion for all other fields away from the junction, these two relations are also easy to interpret in terms of currents and densities, $(j_{\theta,0} = (\omega_n/2)\tilde{\theta}_0, j_{\phi,0} = (\omega_n/2)\tilde{\phi}_0)$ and $(\rho_{\theta,0} = \tilde{\theta}_0/a, \rho_{\phi,0} = \tilde{\phi}_0/a)$ respectively. At this point, it’s important to note that the boundary terms (i.e., those that contain the fields surrounding the zeroth node) that appear in the action possess the interchange (or permutation) symmetries of $1 \leftrightarrow 2, 2 \leftrightarrow 3$ and $3 \leftrightarrow 1$. As the three wires are identical in every way, that such boundary terms should possess such permutation symmetries is to be expected. Further, the form of the current, density as well as equations of motion at the zeroth node are also seen to possess these symmetries as a consequence.

A direct consequence of the quadratic nature of the complete action is that the form of the equations of motion are simple and identical at all nodes. By going to a basis of incoming and outgoing (chiral) modes, it is simple to see that the outgoing currents at link 1, $I_{1\text{out}} = (I_{1\text{out}}^1, I_{1\text{out}}^2, I_{1\text{out}}^3)$, are simply related to the incoming currents at link 1, $I_{1\text{in}} = (I_{1\text{in}}^1, I_{1\text{in}}^2, I_{1\text{in}}^3)$,
by the following matrix relation $\mathbf{I}_1^{\text{out}} = \mathbf{T} \mathbf{I}_1^{\text{in}}$, where the matrix $\mathbf{T}$ is given by

$$
\mathbf{T} = \begin{pmatrix}
-1/3 & 2/3 & 2/3 \\
2/3 & -1/3 & 2/3 \\
2/3 & 2/3 & -1/3
\end{pmatrix}.
$$

The matrix $\mathbf{T}$ is real and orthogonal, and current conservation is clearly seen in the fact that the sum of all elements in every row (and column) of $\mathbf{T}$ add up to $1$. Further, as expected, the matrix $\mathbf{T}$ too possesses the three permutation symmetries mentioned earlier. The partitioning of current from any incoming arm impinging on the junction is symmetric between the the three outgoing arms. Further, this partitioning also has the maximum transmission possible, i.e., from any incoming arm into the other outgoing arms simultaneously. The presence of a finite backscattering into the same arm in this formulation is a new feature and non-perturbative in nature. Backscattering has traditionally been studied in the form of tunneling processes which are accounted for by terms depending exponentially on the various bosonic field degrees of freedom; such terms can be treated only perturbatively starting from the quadratic theory presented here. Thus, by following the dictates of the permutation symmetries and the need for the conservation of charge and current continuity everywhere, we are able to model a transport scenario of intermediate transmission without resorting to perturbation theory. Further, our lattice analysis makes clear the form of the boundary terms that must be included in a continuum analysis of the same system

$$
S_b = \int_0^\beta d\tau \int dx \delta(x) \left[ \frac{v}{2\pi K} \left( \sum_\alpha \phi^\alpha \right)^2 + \frac{vK}{2\pi} \left( \sum_\alpha \theta^\alpha \right)^2 + \frac{i}{2\pi} \partial_\tau \left( \sum_\alpha \theta^\alpha \right) \left( \sum_\alpha \phi^\alpha \right) \right].
$$

The treatment presented above can be simply extended to the case of $N$ identical TLL wires meeting at a single node. The results are identical to those given above, with the only difference that the densities and currents at the zeroth node are now sums over the $N$ $\phi$ and $\theta$ fields. Further, the case of a system of $N$ different TLLs meeting at a single node can also be treated in a similar fashion. In fact, by absorbing all factors of the different velocities $v_\alpha$ and interaction parameters $K_\alpha$ ($\alpha = 1, 2, 3$) into the definitions of the various $\phi$ and $\theta$ bosonic fields, we can write down the action and derive similar results for the junction equations of motion as those given above. Further, the elements of the partitioning matrix for the currents at the junction $\mathbf{T}$ are found to be

$$
T_{ij} = \frac{K_i - \sum_{j \neq i}^N K_j \delta_{ij}}{\sum_{i=1}^N K_i} = \frac{2K_j}{\sum_{i=1}^N K_i} \delta_{ij} \text{ for } j \neq i.
$$

For the case of 3 identical TLL wires $K_i = K$, it is easily seen that the form given above simplifies to that given earlier. Further, the case of 2 identical TLL wires meeting at a junction can also be reached from the formulation given above. This is done by first setting all the bosonic degrees of
freedom in any one arm, say 3, to zero, i.e., \( \phi^3_i = 0 \theta^3_i \) for all \( i = 1, \ldots, \infty \). Then, we can carry out the procedure of “unfolding” the two remaining wires into one. This is done by relabeling all the fields of one of the arms, say 1, \( \phi^1_i \to -\phi^1_i, \phi^1_i \to -\phi^1_i \) for all \( i = 1, \ldots, \infty \), drop the superscripts of 1 and 2 and set \( \bar{\omega}_n \to -\bar{\omega}_n \) in the Legendre transform terms involving \( \phi_{-1} \) and \( \theta_{-1} \). The final transformation is needed because while a universal sense of direction (right and left) can always be taken for the case of a single wire (i.e., two wires meeting at a junction), we have only the directions of in and out for the case of more than two wires meeting at a junction. This means that while in the latter case, the chirality between all incoming and outgoing modes must be opposite, in the former case, an incoming direction in one arm can have the same chirality as the outgoing in the other. Finally, for the case of two different TLL wires meeting at a junction, we find that our analysis coincides with that presented in Refs.\(^{16,17,18}\).

III. LATTICE MODELS FOR CONNECTED CHIRAL 1D SYSTEMS

A. A lattice model for the infinite chiral 1D system

We construct here a lattice model for the infinite chiral 1D system. Wen’s formulation\(^\text{19}\) describes the excitations of the quantum Hall edge system in terms of chiral bosonic hydrodynamic modes – the chiral Tomonaga-Luttinger liquid (TLL) — with the action being quadratic in the bosonic field \( \phi(x, \tau) \) (with edge velocity \( v \) and quantum Hall filling fraction \( \nu \)).

Thus, we consider the following model for the Gaussian chain in one dimension (we write the action as a sum over Matsubara frequencies \( \bar{\omega}_n = 2\pi n/\beta, n \in \mathbb{Z}, \beta = 1/k_BT \) at a finite temperature \( T \), and the lattice spacing is given by \( a \))

\[
S = \sum_{\bar{\omega}_n,i=-\infty}^{\infty} \left\{ -\frac{v\phi_i}{4\pi\nu a^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + \frac{\bar{\omega}_n\phi_i}{8\pi\nu a}(\phi_{i+1} - \phi_{i-1}) \right\} .
\]  

\( (14) \)

![FIG. 3: A schematic diagram for the discrete chiral lattice model. The fields, densities and currents are defined as shown in Fig. 1.](image)

As shown above in Figure 3, the bosonic field \( \phi_i \) is defined on the \( ith \) link between the lattice sites \( i \) and \( i+1 \). \( v \) is the equivalent of the edge velocity on the lattice, while \( \nu \) is the filling fraction. We define the density \( \rho_i \) and current \( j_i \) operators at the \( ith \) lattice point (or node) in terms of the bosonic fields \( \phi_i \) and \( \phi_{i-1} \) on the links on either side of it \( \rho_i = (\phi_{i+1} - \phi_i)/a \), \( j_i = \frac{\bar{\omega}_n}{2}(\phi_{i+1} + \phi_i) \), and which satisfy the following equation of continuity on the lattice \( i \partial_{\tau}(\rho_{i+1} + \rho_i) + (j_{i+1} - j_i)/a = 0 \).
We can now use the variational principle to derive the equation of motion for $\phi_i$ as
\[
\frac{\delta S}{\delta \phi_i} = \frac{v}{a} (2\phi_i - \phi_{i+1} - \phi_{i-1}) + \frac{\bar{\omega}_n}{2} (\phi_{i+1} - \phi_{i-1}) = 0 .
\] (15)

With some simple algebraic manipulations and by using the definitions for $\rho_i$ and $j_i$ given above, the expression obtained from the variation of the action can be rewritten as $j_i - j_{i-1} = v(\rho_i - \rho_{i-1})$, which can be easily seen to be the difference of the expressions for the equation of motion at sites $i$ and $i - 1$. $j_i = v\rho_i$. The equation of motion at the link $i$ is seen to have a very simple meaning: the current on the $ith$ node, $j_i$, is simply a propagating chiral density wave $\rho_i$ at that node (with an edge velocity $v$). Note that one gets the same equation of motion for Wen’s continuum theory as well. As we have translational symmetry in the model, we can try the following ansatz for the solution of the equation of motion (in real space and time)
\[
\phi_j(t) = \sum_{k,\omega} e^{i(kj - \omega t)} \tilde{\phi}(k, \omega)
\] (16)

Putting this into the equation of motion gives us the dispersion relation
\[
\omega = 2v \tan(k/2) \sim v k
\] (17)
in the continuum approximation of $k \to 0$. Note that by taking a one-sided lattice derivative of the $\phi$ fields in the expression for the density $\rho$, we have already broken the chirality symmetry explicitly. In analogy with the Nielsen-Ninomiya theorem for chiral fermions on the lattice, we find that there appears a second mode at the edges of the Brillouin zone ($k \sim \pm \pi/a$) with the dispersion $\omega = -2v \cot(k/2)$. However, this mode corresponds to a staggered order in the chiral $\phi$ field, i.e., this mode has an oscillatory $(-1)^j$ phase factor which alternates from one link to the next and, therefore, has no equivalent in a continuum formulation. Thus, at the expense of working with a set of solutions for the field $\phi$ which are not complete, we can ignore this solution. We thus conclude that our lattice action contains, upon taking the continuum limit, the same physical content as the Wen action for the FQH edge. Further, we can also see that the discrete Lagrangian from this action corresponds to the Hamiltonian defined on the lattice
\[
H = \frac{v}{4\pi \nu a^2} \sum_{\omega_n, n = -\infty}^\infty \phi_i (2\phi_i - \phi_{i+1} - \phi_{i-1})
\] (18)

which gives the energy cost of density-fluctuations at the nodes, together with the commutation relation
\[
[\frac{\phi_{k+1} + \phi_k}{2}, (\phi_{j+1} - \phi_j) \frac{a}{\bar{\omega}_n}] = i\pi \nu \delta_{jk}
\] (19)

where the Kronecker delta $\delta_{jk}$ ensures the correct commutation relations between fields which are on nearest-neighbour links. A few simple algebraic manipulations reveal that the second term in the lattice action (14) conveys, as expected, the same information as the commutation relation given above. Further, we note that upon taking the continuum limit, we rederive the familiar
Hamiltonian (upto a total spatial derivative term) and Kac-Moody commutation relation between the FQH edge field and the corresponding local density. Finally, all lattice correlation functions can again be computed from this action along the lines of Ref. (15).

B. The chiral box-junction

We now study a lattice model for the chiral box junction. This is shown in Fig. 4 below. A continuum version of this model has, in the context of edge state transport in constricted quantum Hall systems, been proposed recently in Ref. (13). The “constriction” region is characterised by a reduced filling-fraction \( \nu_2 \) (less than that of the bulk Hall fluid \( \nu_1 \)) as well as by the upper and lower edge states \( (\phi^u, \phi^d) \), which carry the currents transmitted across the constriction) and the right and left edge states \( (\phi^r, \phi^l) \), which carry the currents reflected at the boundaries of the bulk and constriction regions. As in the preceding subsections, we will proceed by writing down an action governing the dynamics of such a box-junction lattice model and compute the equations of motion for the various fields which describe the currents in terms of propagating chiral density disturbances. Further, we will see that the conservation of current at the corner nodes of the box give rise to a relation between the effective filling-fractions of the transmitted and reflected edge states.

\[
S_0 = \sum_{\omega_n,i=-\infty}^{2} \left( \sum_{\alpha=1in,1out} L_0[\phi^\alpha_i] + \sum_{\beta=2in,2out} L_1[\phi^\beta_i^1] \right), \tag{20}
\]

FIG. 4: A schematic diagram for the chiral 4-wire box-junction lattice model. The fields, densities and currents are defined as shown in Fig. 4. The parameters \( \nu_1 \) and \( \nu_2 \) govern the commutation relations of the bulk \( (\phi^{1in}, \ldots, \phi^{2out}) \) and \( (\phi^u, \phi^d) \) constriction transmission fields respectively.

We begin with the action for the box-junction model \( S = S_0 + S_1 + S_2 \) where the action for the outer incoming and outgoing arms is
where
\[ \mathcal{L}_0[\phi_{i}^{u}] = -v\phi_{i+1}^{u}(\phi_{i}^{u} - 2\phi_{i}^{u} + \phi_{i-1}^{u}) + \frac{\bar{\omega}_n}{2}\phi_{i}^{u}(\phi_{i+1}^{u} - \phi_{i-1}^{u}) \] (21)
and \( \mathcal{L}_0[\phi_{i}^{l}] \) has the same form as \( \mathcal{L}_0[\phi_{i}^{u}] \) but with \( \bar{\omega}_n \rightarrow -\bar{\omega}_n \). Note that we’ve set the lattice spacing \( a \) to unity here and in all that follows. Also, we have normalised the entire action with regards to the bulk filling-fraction \( \nu_1 \). Further, the action for the inner edges is

\[ S_1 = \sum_{\bar{\omega}_n} [f \{-v\phi_{0}^{u}(\phi_{1}^{u} - 2\phi_{0}^{u} + \phi_{-1}^{u}) + \frac{\bar{\omega}_n}{2}\phi_{0}^{u}(\phi_{1}^{u} - \phi_{-1}^{u})\} + f \{-v\phi_{0}^{d}(\phi_{1}^{d} - 2\phi_{0}^{d} + \phi_{-1}^{d}) + \frac{\bar{\omega}_n}{2}\phi_{0}^{d}(\phi_{1}^{d} - \phi_{-1}^{d})\} + g \{-v\phi_{0}^{u}(\phi_{1}^{l} - 2\phi_{0}^{l} + \phi_{-1}^{l}) + \frac{\bar{\omega}_n}{2}\phi_{0}^{l}(\phi_{1}^{l} - \phi_{-1}^{l})\} + g \{-v\phi_{0}^{d}(\phi_{1}^{r} - 2\phi_{0}^{r} + \phi_{-1}^{r}) + \frac{\bar{\omega}_n}{2}\phi_{0}^{r}(\phi_{1}^{r} - \phi_{-1}^{r})\}] \] (22)

where, by assuming that the properties of the upper and lower edge transmitted edge states of the constriction are determined by the effective filling-fraction inside the constriction \( \nu_2 \), the quantity \( f \) is simply given by \( f = \nu_1/\nu_2 \). The quantity \( g = \nu_1/\nu_{ref} \) (where \( \nu_{ref} \) is the effective filling-fraction for the reflected edge states on the left and right) will be determined from the analysis presently. It is worth noting that the Hamiltonians and commutation relations for the two incoming and outgoing edge fields are given by equations \[ \text{[15]} \] and \[ \text{[16]} \] respectively with \( \nu \equiv \nu_1 \), while those for the upper and lower edge fields are the same but with \( \nu \equiv \nu_2 \) and those for the left and right edge fields are the same but with \( \nu \equiv \nu_{ref} \). Finally, the action for the corner nodes is given by \( S_2 = S_2^{ul} + S_2^{lr} + S_2^{ru} + S_2^{lr} \), where each of the four terms represents the action for the fields at one of the four corners of the box. As the form of the four terms in \( S_2 \) are identical to one another, we present here only one of the terms, \( S_2^{ul} \), for the sake of brevity

\[ S_2^{ul} = \sum_{\bar{\omega}_n} [ -vf\phi_{-1}^{u}(\phi_{0}^{u} - 2\phi_{-1}^{u} + \frac{\phi_{1}^{in,1}}{f}) + f\bar{\omega}_n\phi_{0}^{u}(\phi_{0}^{u} - \frac{\phi_{1}^{in,1}}{f}) - vg\phi_{-1}^{l}(\phi_{0}^{l} - 2\phi_{-1}^{l} + \frac{\phi_{1}^{in,1}}{g}) + g\bar{\omega}_n\phi_{0}^{l}(\phi_{0}^{l} - \frac{\phi_{1}^{in,1}}{g}) - v\phi_{-1}^{in,1}(\phi_{-1}^{u} + \phi_{-1}^{l} - 2\phi_{-1}^{in,1} + \phi_{-1}^{in,2}) + \bar{\omega}_n\phi_{-1}^{in,1}(\phi_{-1}^{u} + \phi_{-1}^{l} - \phi_{-1}^{in,2}) ] \] (23)

As before, we can now compute the equations of motion for the various fields from the action. Beginning with the fields in the “bulk” (i.e., nodes/links numbered \( i \leq -2 \) and \( i \geq 2 \)), we obtain \( j_{i}^{1,\alpha} = v\rho_{i}^{1,\alpha} \), \( j_{i}^{2,\alpha} = -v\rho_{i}^{2,\alpha} \) where \( \alpha = (in, out) \) and the current and density at node \( i \) are given by \( j_{i}^{k,\alpha} = \bar{\omega}_n(\phi_{i+1}^{k,\alpha} + \phi_{i}^{k,\alpha})/2 \) and \( \rho_{i}^{k,\alpha} = (\phi_{i+1}^{k,\alpha} - \phi_{i}^{k,\alpha}) \). The difference of sign in the right-hand side of the two equations of motion arises from the opposite chirality of the two incoming and outgoing edge states. In the same way, we can compute the equations of motion of the edge states within the constriction region from the action \( S_1 \) as \( j_{i}^{u} = v\rho_{i}^{u} \), \( j_{i}^{d} = -v\rho_{i}^{d} \), \( j_{i}^{l} = v\rho_{i}^{l} \), \( j_{i}^{r} = v\rho_{i}^{r} \) where \( i = -1,0 \) and the currents and densities have been defined exactly as that in the bulk. While
the opposite chirality of the up(pper) and l(ower) transmitted edge states is apparent, the fact
that the r(ight) and l(eft) reflected edge states have opposite chirality to one another is learned
from the fact that the nodes/links on these two edges are numbered in opposite fashion. We now
proceed to compute the equations of motion at the corners of the box. At the top-left corner, we
find
\[ j_{1,\text{in}}^{-1} = \frac{\alpha_n}{2} \sum_{\alpha=u,l,1,\text{in}} \phi^0_{-1} = v(\phi^u_{-1} + \phi^l_{-1} - \phi^{1,\text{in}}_{-1}) \equiv v\rho_{-1}^{1,\text{in}} \]
\[ j_{u,1,\text{in}}^{-1} = \frac{\alpha_n}{2} (\phi^u_{-1} + \phi^{1,\text{in}}_{-1}) = v(\phi^u_{-1} - \phi^{1,\text{in}}_{-1}) \equiv v\rho_{-1}^{u,1,\text{in}} \]
\[ j_{l,1,\text{in}}^{-1} = \frac{\alpha_n}{2} (\phi^l_{-1} + \phi^{1,\text{in}}_{-1}) = v(\phi^l_{-1} - \phi^{1,\text{in}}_{-1}) \equiv v\rho_{-1}^{l,1,\text{in}} . \] (24)

In the above relations, the currents \( j_{u,1,\text{in}}^{-1}, j_{l,1,\text{in}}^{-1} \) and corresponding densities \( \rho_{u,1,\text{in}}^{1,\text{in}}, \rho_{l,1,\text{in}}^{1,\text{in}} \) are
those propagated from the incoming arm 1 in into the u(pper) and l(eft) edge states respectively.
Now, applying Kirchoff’s law for the conservation of current to these relations, we obtain
\[ \frac{1}{f} + \frac{1}{g} = 1, \]
which for \( f = \nu_1/\nu_2 \) gives \( g = \nu_1/(\nu_1 - \nu_2) \). This, then, gives us the effective filling-fraction of
the reflected edge states as \( \nu_{\text{ref}} = \nu_1 - \nu_2 \). Another way in which this result can be derived is to
note that the following matching relations between the fields at the top-left corner
\[ \phi^{1,\text{in}}_{-1} + \phi^{2,\text{in}}_{-2} = \phi_0^u + \phi^u_{-1} + \phi_0^l + \phi^l_{-1} \]
\[ \phi^{1,\text{in}}_{-1} - \phi^{2,\text{in}}_{-2} = \phi_0^u - \phi^u_{-1} + \phi_0^l - \phi^l_{-1} , \] (25)
are needed together with the Hamiltonians for the 1, in, u and l fields to obtain a complete de-
scription equivalent to that provided by the actions given earlier (eqn.13). From these matching
relations, it is easy to work out the commutation relation
\[ \left[ \frac{\phi_0^u + \phi^l_{-1}}{2}, (\phi^l_{-1} - \phi^l_{-2}) \right] = i(\nu_1 - \nu_2) , \] (26)
where, in the intermediate step, we have used the commutation relations for \( \phi^{1,\text{in}} \) and \( \phi^u \) fields
as being governed by the filling fractions, \( \nu_1 \) and \( \nu_2 \), respectively. It is also clear that we can
derive the equations of motion, as well as the same relation between the quantities \( f \) and \( g \), at
the other three corners in precisely the same manner from either the action or by using similar
matching relations together with the Hamiltonians. A similar program in the continuum version
of this model has, in fact, recently been carried out in Ref.(13).

IV. DISCUSSION

To summarise, we have formulated here a lattice approach to the study of TLL systems
which are connected locally at junctions. Applying these ideas to studies of the three non-
chiral TLL wire fork and the chiral box junction, we were able to identify the roles played by
symmetry, changing bulk properties (like the filling factor in the chiral box junction) as well as the requirement of charge and current continuity in determining unambiguously the form of the boundary terms in the action. In being able to construct quadratic lattice field theories, we are able to interpret the equations of motion simply in terms of local currents and densities. Remarkably, we are able to achieve this at the junction node in the various models as well. It is also simple to see that all correlation functions can also be computed easily from these quadratic theories. This underlines our construction of theories for configurations with intermediate ballistic transmissions (i.e., different from the scenarios of perfect transmission and reflection of the Kane-Fisher paradigm\textsuperscript{5}). Our findings can now be applied towards formulating theories for regular arrays of connected wire junction systems\textsuperscript{14}. This will require incorporating the backscattering of electrons from impurities into the non-chiral TLL models (or, inter-edge quasiparticle scattering in the chiral TLL models) and will be reported elsewhere\textsuperscript{24}. Another promising outlook lies in the formulation of a numerical Monte Carlo approach based on the lattice field theoretic approaches developed here, in analogy with those in lattice gauge theory approaches to QCD\textsuperscript{15}.

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