VARIATIONAL APPROACH TO SUPERSOLID HYDRODYNAMICS

Alexander Peletminskii*

Akhiezer Institute for Theoretical Physics, National Science Center
“Kharkov Institute of Physics and Technolog”, Kharkov 61108, Ukraine

and

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

On the basis of the consistent unification of the principles of non-equilibrium thermodynamics and classical mechanics, we construct a Lagrangian, describing the Hamiltonian dynamics of supersolids. This Lagrangian enables to obtain the closed algebra of the Poisson brackets for the local thermodynamic variables and to derive the Hamilton equations of motion. These equations, in the leading order in spatial gradients of the local thermodynamic variables, result in the non-dissipative supersolid hydrodynamics. The derived hydrodynamic equations do not assume any dynamic symmetry associated with Galilean or Lorentz invariance. The constraint on the thermodynamic potential related to Galilean invariance leads to Andreev-Lifshitz hydrodynamics. We also require relativistic invariance of a constructed theory and obtain a relativistically-invariant supersolid hydrodynamics.

MIRAMARE – TRIESTE

August 2007

* spelet@kipt.kharkov.ua

Junior Associate of ICTP
I. INTRODUCTION

A supersolid is a spatially ordered material with superfluid properties. This state of matter was theoretically predicted by Andreev and Lifshitz in 1969 [1]. They noticed that in crystals with a large amplitude of zero-point motion (a large value of the de-Boehr parameter [2]) exceptional quantum mechanical effects can occur. In particular, the number of lattice sites in such crystals is not equal to the number of atoms and vacancies exist even at the absolute zero of temperature. These vacancies are caused by zero-point energy, which also causes them to be mobile as waves. Since the vacancies are delocalized, they can be considered as weakly interacting quasiparticles. In a cloud of quasiparticles, there may occur a phase transition to the Bose condensed state. Therefore, a supersolid is a state with simultaneously broken continuous translational and global $U(1)$ symmetries and, therefore, it has both crystalline and superfluid order. Later on, the possibility of the supersolid phase in $^4$He was also discussed by Chester [3] and Leggett [4]. For many years, in spite of numerous experimental efforts [5], the supersolid behavior of $^4$He was not discovered. However, in 2004, Kim and Chan [6, 7] reported the possible observation of superfluidity in solid $^4$He that has stimulated an active interest in this phenomenon.

Along with the microscopic consideration of a supersolid state there were many attempts to study its macroscopic dynamics. Andreev and Lifshitz were the first, who derived the supersolid hydrodynamics [1] in close analogy to the two-fluid model (see also [8, 9]). Their consideration was based on differential conservation laws and Galilean invariance. The supersolid hydrodynamics was also derived by employing the microscopic approach [10] based on the Gibbs statistical operator and Bogoliubov’s method of quasiaverages [11]. Recently, the most general effective Lagrangian describing the low-energy dynamics of supersolids has been constructed [12]. The superfluid systems have been also studied by employing the Poisson brackets approach [13]-[15].

The present work deals with the derivation of Hamilton equations of motion as well as hydrodynamic equations (both Galilean- and relativistic-invariant) for supersolids. The consistent unification of the principles of non-equilibrium thermodynamics and classical mechanics enables to develop an effective method for obtaining the Lagrangian that results to the Hamiltonian dynamics and non-dissipative hydrodynamics of superfluid systems [14]. Namely, we identify the local thermodynamic (hydrodynamic) variables with dynamic variables in classical mechanics. Then, the Hamiltonian of the system, being a thermodynamic energy, is considered as a certain functional of hydrodynamic variables. As these variables, we choose the densities of momentum, entropy, particle number and also the quantities associated with the broken symmetries – the superfluid momentum and the crystal deformations. In order to construct a Lagrangian describing the supersolid dynamics and to formulate the variational principle, we have also to introduce a new cyclic variable as canonically conjugate to entropy. A kinematic part of Lagrangian governs a system of Poisson brackets for all dynamic variables. This system forms the closed algebra. From the principle of stationary action, we obtain the Hamilton equations of motion, which in the leading order in spatial gradients of the hydrodynamic variables, correspond to the non-dissipative supersolid hydrodynamics. When deriving the macroscopic equations of motion we do not impose any
dynamic symmetry on the thermodynamic potential density and, therefore, they are suitable for the description of systems with Galilean or Lorentz invariance. The constraint on the thermodynamic potential density associated with Galilean invariance results in Andreev-Lifshitz supersolid hydrodynamics. We also require the relativistic invariance of a constructed theory and obtain a relativistically-invariant system of hydrodynamic equations describing the supersolid phase. It should be noted that, these equations may be useful for describing the crystalline superfluid phase of quark matter [16].

II. THERMODYNAMICS

Normal state. A normal equilibrium state of the system of identical particles is specified by five independent variables. As these variables, one can chose the densities of momentum $\pi_k$, particle number $\rho$, and entropy $\sigma$. Then the energy density of the system being a function of the mentioned thermodynamic variables, $\varepsilon = \varepsilon(\pi_k, \rho, \sigma)$, determines the equation of state. One can also consider another set of the thermodynamic variables that includes temperature $T$, velocity $v_k$, and chemical potential $\mu$. These quantities are conjugate to $\sigma, \pi_k$, and $\rho$ respectively. In this case, an equilibrium state is specified by the thermodynamic potential density $\omega = \omega(T, \mu, v_k)$ ($\omega T = \omega' = -p$ is the Gibbs thermodynamic potential density and $p$ is the pressure).

Crystalline state. The description of the states with a spontaneously broken symmetry requires the introduction of the supplementary thermodynamic variables. In case of a crystalline state with the broken continuous translational symmetry, such variables characterize the crystal deformation. In order to introduce them into thermodynamics, consider a displacement of the physically infinitesimal volume under the deformation,

$$x_i = \xi_i + u_i(x_i), \quad (1)$$

where $\xi_i$ is a position of the volume in an undeformed elastic body (the Lagrangian coordinate), $x_i = x_i(\xi_k)$ is a position of the same volume in a deformed state (the Eulerian coordinate), and $u_i(x)$ is the displacement vector. Differentiation of (1) with respect to $\xi_k$ gives

$$\frac{\partial x_l}{\partial \xi_k} \lambda_{lk} = \delta_{lk}, \quad \lambda_{lk} = \delta_{lk} - \frac{\partial u_i}{\partial x_l}. \quad (2)$$

The quantity $\lambda_{lk}$ characterizes the crystal deformation. For an ordinary crystal, the number of lattice sites coincides with the number of atoms. Since the number of atoms is a constant in deformed and undeformed volumes of the body, $\rho \delta V = \rho_0 \delta V_0$, we come to the well-known relationship

$$\rho = \rho_0 \det |\lambda_{lk}|, \quad (3)$$

where $\rho_0$ is the particle number density in the undeformed volume $\delta V_0$. According to (3), the particle number density $\rho$ is determined by $\lambda_{lk}$ and, therefore, cannot be considered as independent variable. Thus, the thermodynamic state of a crystal is completely specified by the densities of momentum $\pi_k$, entropy $\sigma$, and by nine components of $\lambda_{lk}$, so that $\varepsilon = \varepsilon(\pi_k, \sigma, \lambda_{lk})$. If the energy density $\varepsilon$ depends on $\lambda_{lk}$ only through $\det |\lambda_{lk}|$, then we come back to the description of a normal equilibrium state.
Supersolid. As mentioned in the introduction, a supersolid is a state which breaks both the continuous translational and global U(1) symmetries. Therefore, while describing this state, we have to introduce along with $\lambda_{ik}$, a superfluid momentum $p_k$ as the supplementary thermodynamic variable associated with U(1) symmetry breaking ($p_k = \nabla_k \varphi(x)$, where $\varphi(x)$ is a certain scalar potential). Besides, as we have already mentioned, the density of lattice sites (see (3)) does not coincide with the particle number density for the supersolid phase and, therefore, these quantities should be considered as independent thermodynamic variables, so that $\varepsilon = \varepsilon(\pi_k, \rho, \sigma, \lambda_{ik}, p_k)$.

The equilibrium state under consideration can also be specified by another set of thermodynamic variables. As these variables, we choose temperature $T$, velocity $v_k$, chemical potential $\mu$, and also $\lambda_{ik}$ and $p_k$. Let us introduce the variables $Y_a$ ($a = 0, k, 4$), which are related to $T$, $v_k$, $\mu$ as follows [17]:

$$T = \frac{1}{Y_0}, \quad v_k = -\frac{Y_k}{Y_0}, \quad \mu = -\frac{Y_4}{Y_0}$$

and define the thermodynamic potential density $\omega = \omega(Y_0, Y_k, Y_4, \lambda_{ik}, p_k)$, so that

$$d\omega = \varepsilon dY_0 + \pi_k dY_k + \rho dY_4 + \frac{\partial \omega}{\partial \lambda_{ik}} d\lambda_{ik} + \frac{\partial \omega}{\partial p_k} dp_k.$$  

Then, it is evident that the densities of energy $\varepsilon \equiv \zeta_0$, momentum $\pi_k \equiv \zeta_k$, and particle number $\rho \equiv \zeta_4$ are related to $\omega$ by the following formulae,

$$\varepsilon = \frac{\partial \omega}{\partial Y_0}, \quad \pi_k = \frac{\partial \omega}{\partial Y_k}, \quad \rho = \frac{\partial \omega}{\partial Y_4},$$

or

$$\zeta_a = \frac{\partial \omega}{\partial Y_a}.$$  

From (5) we have

$$d\varepsilon = \frac{1}{Y_0} d\sigma - \frac{Y_k}{Y_0} d\pi_k - \frac{Y_4}{Y_0} d\rho + \frac{1}{Y_0} \frac{\partial \omega}{\partial p_k} dp_k + \frac{1}{Y_0} \frac{\partial \omega}{\partial \lambda_{ik}} d\lambda_{ik}$$

and, hence,

$$\frac{\partial \varepsilon}{\partial \sigma} = \frac{1}{Y_0}, \quad \frac{\partial \varepsilon}{\partial \pi_k} = -\frac{Y_k}{Y_0}, \quad \frac{\partial \varepsilon}{\partial \rho} = -\frac{Y_4}{Y_0},$$

$$\frac{\partial \varepsilon}{\partial p_k} = \frac{1}{Y_0} \frac{\partial \omega}{\partial p_k}, \quad \frac{\partial \varepsilon}{\partial \lambda_{ik}} = \frac{1}{Y_0} \frac{\partial \omega}{\partial \lambda_{ik}},$$

where

$$\sigma = -\omega + Y_0 \zeta_0 = -\omega + Y_0 \varepsilon + Y_k \pi_k + Y_4 \rho.$$  

Using (5), the relationship (10) is written in the form

$$d\varepsilon + d\eta = \delta Q,$$

where the differential form $d\eta$ and the quantity $\delta Q$ are given by the following expressions:

$$d\eta = -v_k d\pi_k - \mu d\rho - T \frac{\partial \omega}{\partial \lambda_{ik}} d\lambda_{ik} - T \frac{\partial \omega}{\partial p_k} dp_k, \quad \delta Q = T d\sigma.$$
The obtained two formulae reflect the first thermodynamic law: \( d\varepsilon \) is the internal energy change, \( d\eta \) is the work, which should be spent in order to change the thermodynamic variables by \( d\pi_k, dp, d\lambda_{ik}, dp_k \), and \( \delta Q \) is the released heat. The released heat \( \delta Q = d\varepsilon + d\eta \) is not a total differential, whereas the quantity \( (1/T)(d\varepsilon + d\eta) = d\sigma \) represents a total differential, where \( 1/T = Y_0 \) is the integrating multiplier and \( d\sigma \) is the differential of the entropy density. This statement is a formulation of the Caratheodori principle (the second thermodynamic law for reversible processes). Note that the above proof of the formulated principle is based on the fact that the differential form (5) is a total differential.

In conclusion of this section, we present the expressions for non-dissipative fluxes obtained within the framework of microscopic approach, which is based on the Gibbs statistical operator and Bogoliubov’s method of quasiaverages [11]. These fluxes are given by [10, 18]

\[
\zeta_{ak} = -\frac{\partial}{\partial Y_a} \frac{\omega Y_k}{Y_0} + \frac{\partial \omega}{\partial p_k} \frac{\partial Y_4 + p_i Y_i}{Y_0} + \frac{\partial \omega}{\partial \lambda_{ik}} \frac{\partial Y_a}{Y_0},
\]

(11)

where index \( a = 0, k, 4 \), so that \( \zeta_{0k} \equiv w_k \) is the energy flux density, \( \zeta_{ik} \equiv t_{ik} \) is the momentum flux density, and \( \zeta_{4k} \equiv j_k \) is the particle number flux density. We emphasize that all these fluxes are combined into the single formula (11) due to the introduction into consideration of \( \omega \) and the thermodynamic variables \( Y_a \). Below, we will show that the fluxes (11) follow from the corresponding Lagrangian studied in the present work.

III. LAGRANGIAN

In non-equilibrium statistical mechanics the locally equilibrium thermodynamic variables are assumed to describe correctly the weakly inhomogeneous states of condensed matter. Such a description is complete, i.e., there is no necessity to introduce the supplementary thermodynamic parameters in consequence of temporary evolution.

In this section, we will obtain a Lagrangian describing the supersolid dynamics (hydrodynamics). The starting point is the local statistical equilibrium principle, i.e., on hydrodynamic time scale, the thermodynamic variables are assumed to be slowly varying functions of space coordinate and time. Exactly these functions should be considered as dynamic (hydrodynamic) variables when constructing the Lagrangian. In addition, the energy of the system is a certain functional of local thermodynamic variables. In order to put the hydrodynamic variables into Lagrangian, we remind that the position of each material point of the body can be specified by the Eulerian coordinate \( x_k(\xi_i, t) \), where \( \xi_i \) is the Lagrangian (initial) coordinate of the same point, \( x_k(\xi_i, 0) = \xi_i \) (see previous section). Then, introducing the momentum density \( \tilde{\pi}_k(\xi_i, t) \) in terms of the Lagrangian coordinates, we write the system Lagrangian in the standard form, \( L = L_k - H \), where

\[
L_k = \int d^3\xi \, \tilde{\pi}_k(\xi_i, t) \dot{x}_k(\xi_i, t),
\]

(12)

is its kinematic part and \( H \) is the Hamiltonian, which we will write below.

The next step is to change in (12) from the Lagrangian variables \( \xi_i \) to the Eulerian variables \( x_i = x_i(\xi_k, t) \). To this end, similar to (1), we introduce the displacement vector \( u_k(x_i(\xi_j, t), t) \) as a function of
of course, different). (Here, we denote the Eulerian and Lagrangian quantities by the same letter; really these functions are, with \( \lambda \) and \( \sigma \) being the density of the kinematic part of the Lagrangian, we introduce a cyclic variable \( \xi \).)

The Eulerian quantity \( \tilde{\pi}_i(x) \) is related to the corresponding Lagrangian quantity \( \tilde{\pi}_i(x_k) \) through the Jacobian of the transformation \( \xi_i \rightarrow x_i = x_i(\xi_k, t) \),

\[
\tilde{\pi}_i(x) = \det \left| \frac{\partial \xi_i}{\partial \tilde{x}_k} \right| \tilde{\pi}_i(\xi)
\]

(Here, we denote the Eulerian and Lagrangian quantities by the same letter; really these functions are, of course, different).

The kinematic part of Lagrangian (16) includes \( \tilde{\pi}_i(x) \) and \( \lambda_{ij}(x) \) as dynamic variables. Since the local equilibrium state of supersolids is also specified by \( \sigma(x) \), \( \rho(x) \), \( p_k(x) \), so that

\[
H = \int d^3x \varepsilon(x; \tilde{\pi}_i(x'), \rho(x'), \sigma(x'), \lambda_{ij}(x'), p_k(x'))
\]

all these quantities should be included into \( L_k(x) \). In order to include the entropy density into the kinematic part of the Lagrangian, we introduce a cyclic variable \( \psi(x) \) as canonically conjugate to \( \sigma(x) \). In addition, since the formation of superfluidity breaks the global U(1) symmetry generated by the conserved particle number, it is natural to take the particle number density \( \rho(x) \) and the scalar field \( \varphi(x) \) (\( p_k(x) = \nabla_k \varphi(x) \) is the superfluid momentum) as canonically conjugate variables. As a result, we recast \( L_k(x) \) in the following form:

\[
L_k(x) = \tilde{q}_j(x) \dot{u}_j(x) - \sigma(x) \dot{\psi}(x) - \rho(x) \dot{\varphi}(x)
\]

where, as above, \( \tilde{q}_j(x) = \tilde{\pi}_i(x) \lambda_{ij}^{-1}(x) \).

It is known that the Poisson brackets for dynamic variables are governed by the kinematic part of the Lagrangian. For \( L_k(x) \), given by (18), the non-zero Poisson brackets are of the form

\[
\{\sigma(x), \psi(x')\} = \delta(x - x'), \quad \{\rho(x), \varphi(x')\} = \delta(x - x'),
\]

\[
\{u_i(x), \tilde{q}_j(x')\} = \delta_{ij} \delta(x - x').
\]

\[\text{(19)}\]
The brackets between other pairs of dynamic variables turn to zero.

Let us next obtain the Poisson brackets of the form \{\tilde{\pi}_i(x), \eta_a(x')\}, where \eta_a(x) is introduced to denote the whole set of dynamic variables, \tilde{\pi}_i(x), \rho(x), \sigma(x), \lambda_{ik}(x), p_i(x). Noting that \tilde{\pi}_i(x) = \tilde{q}_j(x)\lambda_{jk}(x) (see (16)) and using (19) as well as the vanishing Poisson brackets, one finds

\[
\begin{align*}
\{\tilde{\pi}_i(x), \sigma(x')\} &= 0, \\
\{\tilde{\pi}_i(x), \psi(x')\} &= 0, \\
\{\tilde{\pi}_i(x), \rho(x')\} &= 0, \\
\{\tilde{\pi}_i(x), \varphi(x')\} &= 0
\end{align*}
\]

and

\[
\begin{align*}
\{\tilde{\pi}_i(x), \tilde{\pi}_k(x')\} &= \tilde{\pi}_i(x')\nabla'_k \delta(x - x') - \tilde{\pi}_k(x)\nabla_i \delta(x - x'), \\
\{\tilde{\pi}_i(x), \lambda_{ij}(x')\} &= \lambda_{ij}(x)\nabla'_j \delta(x - x'),
\end{align*}
\]

where we have employed the definition (14) of \lambda_{ij} and the property \nabla_k \lambda_{ij}(x) = \nabla_j \lambda_{ik}(x).

The last step in constructing a Lagrangian is associated with the interpretation of \tilde{\pi}_i(x) as the momentum density of the system. The quantity \tilde{\pi}_i(x) is interpreted as the momentum density if the total momentum of the system (the generator of spatial translations)

\[\tilde{P}_i = \int d^3x \tilde{\pi}_i(x)\]

meets the relationship

\[
\{\tilde{P}_i, \eta_a(x)\} = \nabla_i \eta_a(x),
\]

where \eta_a denotes, as above, the whole set of dynamic variables. Clearly, the Poisson brackets (21) are consistent with (22), whereas (20) are inconsistent. Therefore, \tilde{\pi}_i(x) cannot be interpreted as the momentum density of the system. In order to find the true momentum density, we introduce a new dynamic variable \pi_i(x),

\[\pi_i(x) = \tilde{\pi}_i(x) + \pi_i^\sigma(x) + \pi_i^\rho(x)\]

and choose \pi_i^\sigma, \pi_i^\rho so that

\[
\{P_i, \eta_a(x)\} = \nabla_i \eta_a(x), \quad P_i = \int d^3x \pi_i(x).
\]

In doing so, it is sufficient that the quantities \pi_i^\sigma(x) and \pi_i^\rho(x) are constructed of the pair of variables \sigma(x), \psi(x) and \rho(x), \varphi(x) respectively (this statement becomes evident if one takes into account the nonzero and vanishing Poisson brackets for (18)). In addition, they must meet the following relationships:

\[
\begin{align*}
\int d^3x' \{\pi_i^\sigma(x'), \sigma(x)\} &= \nabla_i \sigma(x), \\
\int d^3x' \{\pi_i^\sigma(x'), \psi(x)\} &= \nabla_i \psi(x), \\
\int d^3x' \{\pi_i^\rho(x'), \rho(x)\} &= \nabla_i \rho(x), \\
\int d^3x' \{\pi_i^\rho(x'), \varphi(x)\} &= \nabla_i \varphi(x).
\end{align*}
\]
Next, taking into account the Poisson brackets in the first line of (19), one immediately finds
\[ \pi_i^\sigma(x) = \sigma(x) \nabla_i \psi(x), \quad \pi_i^\rho(x) = \rho(x) \nabla_i \varphi(x). \]

Therefore, making use of (23), we can recast the kinematic part of the Lagrangian (18) in the final form
\[ L_k(x) = \ddot{q}_j(x) \dot{u}_j(x) - \sigma(x) \dot{\psi}(x) - \rho(x) \dot{\varphi}(x), \tag{25} \]
where
\[ \ddot{q}_j(x) = \ddot{\pi}_i(x) \lambda_{ij}^{-1}(x), \tag{26} \]
with
\[ \ddot{\pi}_i(x) = [\pi_i(x) - \sigma(x) \nabla_i \psi(x) - \rho(x) \nabla_i \varphi(x)]. \tag{27} \]
The Lagrangian that describes the supersolid dynamics has the form
\[ L = \int d^3x L_k(x) - H, \tag{28} \]
where the Hamiltonian $H$ is determined by (17). We should remember that $\psi(x)$ is a cyclic variable.

Below, we will show that the obtained Lagrangian leads to the Hamilton equations of motion, which in the leading order in spatial gradients of the dynamic variables correspond to the supersolid hydrodynamics obtained on the basis of phenomenological [1, 9, 12] and microscopic [10, 18] approaches.

### IV. VARIATIONAL PRINCIPLE, HAMILTON EQUATIONS OF MOTION, AND POISSON BRACKETS

The equations of motion for classical fields can be derived from a Lagrangian through the principle of stationary action. Here, we will employ this principle for the obtained Lagrangian (see (25)-(28)) and will find the Hamilton equations of motion for a supersolid.

Upon varying the action functional
\[ W = \int_{t_1}^{t_2} dt \int d^3x L_k(x) - \int_{t_1}^{t_2} dt H(\pi_i(x'), \rho(x'), \sigma(x'), \varphi(x'), \lambda_{ik}(x')) \tag{29} \]
with respect to $\pi_i(x)$, one obtains the following equation of motion:
\[ \dot{\pi}_i - \lambda_{ij} \frac{\delta H}{\delta \pi_j} = 0, \tag{30} \]
which can also be recast in another form, useful in the sequel:
\[ \dot{\lambda}_{ij} + \nabla_j \left( \lambda_{ik} \frac{\delta H}{\delta \pi_k} \right) = 0. \tag{31} \]
Evaluating the variation of the action functional (29) with respect to $\psi(x)$, $\varphi(x)$, $\sigma(x)$, $\rho(x)$ and using (30), one finds respectively
\[ \dot{\sigma} + \nabla_i \left( \sigma \frac{\delta H}{\delta \pi_i} \right) = 0, \quad \dot{\rho} + \nabla_i \left( \rho \frac{\delta H}{\delta \pi_i} \right) - \frac{\delta H}{\delta \varphi} = 0, \tag{32} \]
\[ \dot{\psi} + \frac{\delta H}{\delta \pi_i} \nabla_i \psi + \frac{\delta H}{\delta \sigma} = 0, \quad \dot{\varphi} + \frac{\delta H}{\delta \pi_i} \nabla_i \varphi + \frac{\delta H}{\delta \rho} = 0. \tag{33} \]
Let us obtain finally the equation of motion for \( \pi_i(\mathbf{x}) \). To this end, we note that the variation of \( L_k(\mathbf{x}) \) with respect to \( u_i(\mathbf{x}) \) is given by

\[
\delta_u L_k = \dot{u}_j \tilde{\pi}_i \delta u_i \lambda^{-1}_{ij} - \tilde{\pi}_i \lambda^{-1}_{ij} \mathbf{d} u_j - \tilde{\pi}_i \lambda^{-1}_{ij} \delta u_j,
\]

where we have omitted the term \( d/dt(\ddot{q}, \delta u_j) \) not contributed to the equations of motion in virtue of the principle of stationary action. Next, employing the identity \( \lambda \lambda^{-1} = I \) (\( I \) is the unit matrix) and (30), one finds

\[
\delta_u \lambda^{-1}_{ij} = \lambda^{-1}_{ik}(\nabla_i \delta u_k) \lambda^{-1}_{ij}, \quad \dot{\lambda}_{ij} = \lambda^{-1}_{ik} \nabla_i \left( \lambda_{ks} \frac{\delta H}{\delta p_i} \right) \lambda^{-1}_{kj}.
\]

Taking into account that \( \delta_i H = \delta u_i H \), one can obtain, from the principle of stationary action \( \delta W = 0 \), the following equation of motion:

\[
\dot{\pi}_i \lambda^{-1}_{ij} + \tilde{\pi}_i \lambda^{-1}_{ik} \nabla_i \left( \lambda_{ks} \frac{\delta H}{\delta x_k} \right) \lambda^{-1}_{ij} + \nabla_i \left( \tilde{\pi}_i \lambda^{-1}_{ij} \frac{\delta H}{\delta \pi_i} \right) + \nabla_k \frac{\delta H}{\delta \lambda_{kj}} = 0.
\]

Finally, upon substituting the explicit expression (27) for \( \tilde{\pi}_i \) and using Eqs. (32), (33) as well as the property \( \nabla_i \lambda_{ks} = \nabla_k \lambda_{is} \), it is easy to obtain the equation of motion for the momentum density, \( \dot{\pi}_i = -\sigma \nabla_i \frac{\delta H}{\delta \sigma} - \rho \nabla_i \frac{\delta H}{\delta \rho} - p_i \nabla_k \frac{\delta H}{\delta p_k} - \nabla_k \left( \pi_i \frac{\delta H}{\delta \pi_k} \right) - \pi_k \nabla_i \frac{\delta H}{\delta \pi_k} - \lambda_{ki} \nabla_j \frac{\delta H}{\delta \lambda_{kj}} \). (34)

where \( p_i = \nabla_i \varphi \) is the superfluid momentum.

The Hamilton equations of motion (31)-(34) can also be obtained on the basis of the Poisson brackets [14]. The advantage of this method is that the derived Hamilton equations have, at once, the form of the differential conservation laws. Our next goal is to express the Poisson brackets (20), (21) through the momentum density \( \pi_i(\mathbf{x}) \). Making use of (27) and bearing in mind (19) as well as the vanishing Poisson brackets for the kinematic part (18), we can recast (20), (21) as

\[
\begin{align*}
\{ \pi_k(\mathbf{x}), \sigma(\mathbf{x}') \} &= -\sigma(\mathbf{x}) \nabla_k \delta(\mathbf{x} - \mathbf{x}'), \\
\{ \pi_k(\mathbf{x}), \psi(\mathbf{x}') \} &= \delta(\mathbf{x} - \mathbf{x}') \nabla_k \psi(\mathbf{x}), \\
\{ \pi_k(\mathbf{x}), \rho(\mathbf{x}') \} &= -\rho(\mathbf{x}) \nabla_k \delta(\mathbf{x} - \mathbf{x}'), \\
\{ \pi_k(\mathbf{x}), \varphi(\mathbf{x}') \} &= \delta(\mathbf{x} - \mathbf{x}') \nabla_k \varphi(\mathbf{x}), \\
\{ \pi_i(\mathbf{x}), \pi_k(\mathbf{x}') \} &= \pi_i(\mathbf{x}') \nabla_k \delta(\mathbf{x} - \mathbf{x}') - \pi_k(\mathbf{x}) \nabla_i \delta(\mathbf{x} - \mathbf{x}'), \\
\{ \pi_i(\mathbf{x}), \lambda_{kj}(\mathbf{x}') \} &= \lambda_{ki}(\mathbf{x}) \nabla_j \delta(\mathbf{x} - \mathbf{x}').
\end{align*}
\]

These Poisson brackets as well as the brackets standing in the first line of (19) and the vanishing Poisson brackets for the kinematic part (18) form the closed algebra. The coupled Hamilton equations of motion (31)-(34) are obtained from (A.9). We remind that \( \psi(\mathbf{x}) \) should be considered as a cyclic variable.

In order the obtained Hamilton equations have the meaning of non-dissipative hydrodynamic equations, we have to write them in the leading order in spatial gradients of the hydrodynamic variables. It is easy to see that in zeroth order in the gradients, the variational derivatives of the Hamiltonian with respect to hydrodynamic variables, are replaced by the ordinary derivatives of the energy density with respect
to corresponding variables. Moreover, the energy density depends on space coordinate and time only through the hydrodynamic variables,

$$\frac{\delta H(\eta_0(x'))}{\delta \eta_0(x)} \approx \frac{\partial \varepsilon(\eta_0(x))}{\partial \eta_0(x)} \equiv \frac{\partial \varepsilon}{\partial \eta_0}.$$ 

Consequently, the Hamilton equations of motion (31)-(34) take the form

$$\dot{\rho} = -\nabla_i j_i, \quad \dot{\pi}_i = -\nabla_k t_{ik}, \quad \dot{\sigma} = -\nabla_i \left( \sigma \frac{\partial \varepsilon}{\partial \pi_i} \right),$$

$$\dot{\lambda}_{ij} = -\nabla_j \left( \lambda_{ik} \frac{\partial \varepsilon}{\partial \lambda_{ik}} \right),$$

(36)

where \( p_i = \nabla_i \varphi \) is the superfluid momentum, \( j_i, t_{ik} \) are the particle number flux density and the momentum flux density respectively,

$$j_i = \rho \frac{\partial \varepsilon}{\partial \pi_i} + \frac{\partial \varepsilon}{\partial p_i},$$

$$t_{ik} = p \delta_{ik} + \pi_k \frac{\partial \varepsilon}{\partial \pi_k} + \rho \frac{\partial \varepsilon}{\partial \rho} + \lambda_{ji} \frac{\partial \varepsilon}{\partial \lambda_{jk}},$$

(37)

and \( p \) is the pressure,

$$p = -\varepsilon + \sigma \frac{\partial \varepsilon}{\partial \sigma} + \rho \frac{\partial \varepsilon}{\partial \rho} + \pi_k \frac{\partial \varepsilon}{\partial \pi_k}.$$ 

(38)

Note that when deriving the equation for \( \pi_i(x) \) we have used the potentiality of the superfluid flow, \( \text{rot} \, p = 0 \) (\( \nabla_k p_i = \nabla_i p_k \)) and the property \( \nabla_i \lambda_{kj} = \nabla_j \lambda_{ki} \).

Now we briefly discuss the hydrodynamic equations (36)-(38) of supersolids. The first three equations represent the differential conservation laws of the densities of particle number, momentum, and entropy. Two other equations describe the temporary evolution of the hydrodynamic variables related to the broken symmetries. Since the superfluid flow is not accompanied by the entropy transfer, the quantity \( \partial \varepsilon / \partial \pi_i = -Y_i / Y_0 \equiv \nu_{ni} \) (see (8), (4)) should be interpreted as the normal velocity. The Eulerian coordinate \( x_k(\xi, t) \), introduced by (13), represents the lattice site position of a deformed lattice and, consequently, \( \dot{x}_k(\xi_k, t) \) is the lattice velocity. From (30), (14) (equation (30) is equivalent to the last equation in (36)) we have \( \dot{x}_i(\xi_i, t) = \partial \varepsilon / \partial \pi_i = \nu_{ni} \) and, therefore, the lattice velocity coincides with the normal velocity. Notice also that when deriving (36)-(38), we have not used any dynamic symmetry (Galilean or Lorentz invariance of the system) and, hence, these equations are the most general. In case of Galilean invariance (see the next section), one can introduce the atomic mass and superfluid velocity \( v_{s1} = p_i / m \). The relativistic case will also be studied below. Finally, it is easy to show that the energy conservation law follows from (36)-(38). Indeed, noting that

$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial \rho} \dot{\rho} + \frac{\partial \varepsilon}{\partial \sigma} \dot{\sigma} + \frac{\partial \varepsilon}{\partial \pi_i} \dot{\pi}_i + \frac{\partial \varepsilon}{\partial p_i} \dot{p}_i + \frac{\partial \varepsilon}{\partial \lambda_{ik}} \dot{\lambda}_{ik}$$

and using (36)-(38) as well as \( \text{rot} \, p = 0 \), one obtains the energy conservation law,

$$\dot{\varepsilon} = -\nabla_k w_k,$$

(39)
where \( w_k \) is the energy flux density,
\[
w_k = \frac{\partial \varepsilon}{\partial \pi_k} \left( \rho \frac{\partial \varepsilon}{\partial \rho} + \sigma \frac{\partial \varepsilon}{\partial \sigma} + \pi_i \frac{\partial \varepsilon}{\partial \pi_i} \right) + \frac{\partial \varepsilon}{\partial \pi_k} \left( \frac{\partial \varepsilon}{\partial \rho} + p \frac{\partial \varepsilon}{\partial \pi} \right) + \frac{\partial \varepsilon}{\partial \lambda_{ik}} \lambda_{ij} \frac{\partial \varepsilon}{\partial \pi_j}.
\] (40)

The coupled equations (36)-(38) include the non-dissipative hydrodynamics of normal and superfluid liquids as well as the elasticity equations. The hydrodynamic equations of a superfluid liquid follow from (36)-(38) if to consider \( \lambda_{ij} \) as a cyclic variable, so that \( \varepsilon = \varepsilon(\pi_k, \sigma, \rho, p_k) \). If we take \( \lambda_{ij} \) and \( p_k \) as cyclic variables, so that \( \varepsilon = \varepsilon(\pi_k, \sigma, \rho) \), then we come to the hydrodynamic equations of a normal liquid. Finally, if \( \rho \) and \( p_k \) represent the cyclic variables, \( \varepsilon = \varepsilon(\pi_k, \sigma, \lambda_{ij}) \), then from (36)-(38) we obtain the elasticity equations. All these equations are also general, because no dynamic symmetry is imposed.

In conclusion in this section we recast the obtained hydrodynamic equations (36)-(40), describing a supersolid, in terms of the thermodynamic potential density \( \omega(Y_0, Y_k, Y_4, \lambda_{ik}, p_k) \) introduced in the second section. First, such equations have a compact form and, second, they are suitable to study the Galilean and Lorentz invariance. Indeed, employing the thermodynamic relationships (4), (6)-(10), we represent the conservation laws in a desired form:
\[
\dot{\zeta}_a = -\nabla_k \zeta_{ak}, \quad a = 0, i, 4,
\] (41)
where \( \zeta_a = \partial \omega/\partial Y_a \) (\( \zeta_0 \equiv \varepsilon, \zeta_i \equiv \pi_i, \zeta_4 \equiv \rho \)) and \( \zeta_{ak} \) are the corresponding flux densities,
\[
\zeta_{ak} = -\frac{\partial \omega}{\partial Y_a} \frac{\omega Y_k}{Y_0} + \frac{\partial \omega}{\partial p_k} \frac{\partial Y_4 + \rho Y_i}{Y_0} + \frac{\partial \omega}{\partial \lambda_{ik}} \frac{\partial \lambda_{ij} Y_j}{Y_0}
\] (42)
(\( \zeta_{0k} = w_k \) is the energy flux density, \( \zeta_{ik} \equiv t_{ik} \) is the momentum flux density, and \( \zeta_{4k} \equiv j_k \) is the particle number flux density). The hydrodynamic equations for the variables related to the broken symmetries have the form
\[
\dot{p}_k = \nabla_i \left( \frac{Y_4 + Y_k p_k}{Y_0} \right), \quad \dot{\lambda}_{ik} = \nabla_k \left( \frac{\lambda_{ij} Y_j}{Y_0} \right).
\] (43)
We note that the expression (42) for the flux densities coincides with (11) obtained within the framework of the microscopic approach [10, 18] based on the Bogoliubov method of quasievaverages [11].

V. GALILEAN INVARIANCE

As already mentioned, the obtained hydrodynamic equations for supersolids do not account for any dynamic symmetry. Here, we study their invariance under the Galilean transformation, \( x_k \to x'_k = x_k - V_k t \), where \( V_k \) is the velocity of one inertial coordinate system with respect to the other. Since the thermodynamic potential density \( \omega \) is invariant under the Galilean transformation [19],
\[
\omega(Y_a, p_k, \lambda_{ik}) = \omega(Y'_a, p'_k, \lambda'_{ik}), \quad a = 0, k, 4,
\] (44)
we address to the hydrodynamic equations (41)-(43) in terms of \( \omega \). Under the Galilean transformation, the local thermodynamic variables \( Y_a, p_k, \lambda_{ik} \) in (44) are transformed as follows:

\[
Y_0(x) \rightarrow Y'_0(x') = Y_0(x),
\]

\[
Y_k(x) \rightarrow Y'_k(x') = Y_k(x) + V_k Y_0(x),
\]

\[
Y_4(x) \rightarrow Y'_4(x') = Y_4(x) + m V_k Y_k(x) + \frac{m V^2}{2} Y_0(x).
\]

and

\[
p_k(x) \rightarrow p'_k(x') = p_k(x) - m V_k,
\]

\[
\lambda_{ik}(x) \rightarrow \lambda'_{ik}(x') = \lambda_{ik}(x).
\]

Under the Galilean transformation, the phase \( \varphi(x) \) is transformed according to the following law:

\[
\varphi(x) \rightarrow \varphi'(x') = \varphi(x) - m V_k x_k + \frac{m V^2}{2} t
\]

where \( m \) is the atomic mass. We see that the transformation law of the superfluid momentum (46) follows from (47). Now let us show that the transformation law (47) is consistent with (45). To this end, we write the first equation in (43) in the following form (\( p_0(x) = \nabla_i \varphi(x) \)):

\[
\dot{\varphi}(x) = p_0(x), \quad p_0(x) = \frac{Y_4(x) + Y_k(x) p_k(x)}{Y_0}.
\]

Upon differentiating (47) with respect to time at fixed \( x' \) (the phase \( \varphi(x) \) depends on time both explicitly and through \( x_k = x'_k + V_k t \)), one finds

\[
\dot{\varphi}'(x') = \dot{\varphi}(x) + V_k \nabla_k \varphi(x) - \frac{m V^2}{2}.
\]

Therefore, under the Galilean transformation, the quantity \( p_0(x) = \dot{\varphi}(x) \) is transformed as follows:

\[
p_0(x) \rightarrow p'_0(x') = p_0(x) + V_k p_k(x) - \frac{m V^2}{2}.
\]

This transformation law can also be obtained straightforwardly, by using the transformations (45) and the definition of \( p_0 \) through the hydrodynamic variables \( Y_a(x) \) (see (48)). Thus, (45) and (47) are consistent. Finally, the invariance of \( \lambda_{ik}(x) \) under the Galilean transformation follows from its definition (14) and from the transformation law of the displacement vector, \( u_i(x) \rightarrow u'_i(x') = u_i(x) - V_i t \).

Performing differentiation of (44) with respect to \( Y_a(x) \) and using (45), (6), we come to the well-known transformations of the densities of additive integrals of motion under the Galilean transformation,

\[
\rho(x) \rightarrow \rho'(x') = \rho(x),
\]

\[
\pi_k(x) \rightarrow \pi'_k(x') = \pi_k(x) - m V_k \rho(x),
\]

\[
\varepsilon(x) \rightarrow \varepsilon'(x') = \varepsilon(x) - V_k \pi_k(x) + \frac{m V^2}{2} \rho(x).
\]

The entropy density (10) is invariant under the Galilean transformation,

\[
\sigma(x) \rightarrow \sigma'(x') = \sigma(x)
\]
and the flux densities (42) are transformed as follows:

\[ j^j_k \rightarrow j^j_k = j_k - V_k \rho, \]
\[ t^t_{ik} \rightarrow t^t_{ik}' = t_{ik} - mV_j j_k - V_k \pi_i + mV_k \rho, \]
\[ w^w_k \rightarrow w^w_k' = w_k - V_i t_{ik} - V_k \varepsilon + V_k V_i \pi_i + \frac{mV^2}{2}(j_k - V_k \rho), \]

where, as in (44), the variables with a prime and without it should be taken at the points \( \mathbf{x}' \) and \( \mathbf{x}' \) respectively.

One can easily prove the invariance of the hydrodynamic equations (41)-(43) under the Galilean transformations. To do this, we have to use (44)-(46), (51) and the fact that

\[ \dot{\eta}_\alpha'(\mathbf{x}') \equiv \left( \frac{\partial \eta_\alpha'(\mathbf{x}')}{\partial t} \right)_{\mathbf{x}'} = \left( \frac{\partial \eta_\alpha'(\mathbf{x}')}{\partial t} \right)_{\mathbf{x}} + V_j \nabla_j \eta_\alpha'(\mathbf{x}'), \]

where \( \eta_\alpha(\mathbf{x}) \) denotes the whole set of hydrodynamic variables. Notice that the equation of motion for \( \psi(\mathbf{x}) \) (see (33)) is invariant if \( \psi(\mathbf{x}) \rightarrow \psi'(\mathbf{x}') = \psi(\mathbf{x}) \). This equation, originated from the variational principle, does not affect the hydrodynamics of a supersolid state.

Let us show that the imposed Galilean invariance on the thermodynamic potential density (44)-(46) leads to the Andreev-Lifshitz supersolid hydrodynamics [1]. To this end, consider a reference frame \( K' \) in which the superfluid velocity turns to zero (\( p'_k = 0 \)). Then, in virtue of (44), we have

\[ \omega(Y'_a, p'_k, \lambda'_{ik}) = \omega(Y'_a, 0, \lambda'_{ik}), \]

where the hydrodynamic variables in \( K \) and \( K' \) are related, according to (45)-(46), by

\[ p_k = mv_{sk} = mV_k, \quad Y_0 \rightarrow Y'_0 = Y_0, \]
\[ Y_k \rightarrow Y'_k = Y_k + v_{sk} Y_0, \quad \lambda_{ik} \rightarrow \lambda'_{ik} = \lambda_{ik}, \]
\[ Y_4 \rightarrow Y'_4 = Y_4 + mv_{sk} Y_k + \frac{mv^2}{2} Y_0, \]

with \( v_{sk} \) being the superfluid velocity. The flux densities (42) are expressed through the quantities \( \zeta_a = \partial \omega/\partial Y_a (a = 0, k, 4), \partial \omega/\partial p_k, \) and \( \partial \omega/\partial \lambda_{ik} \). Our aim is to express \( \zeta_a \) and \( \zeta_{ik} \) through their values in the reference frame \( K' \), in which the superfluid velocity is zero. Carrying out the differentiation of (52) with respect to \( Y'_a \) (see (6)) and, then, with respect to \( p_k, \lambda_{ik} \), one finds

\[ \varepsilon = \varepsilon' + v_{sk} \pi'_k + \frac{mv^2}{2} \rho', \]
\[ \pi_k = \pi'_k + mv_{sk} \rho', \quad \rho = \rho', \]

with

\[ \frac{1}{Y_0} \frac{\partial \omega}{\partial p_k} = \frac{\pi'_k}{m} + (v_{sk} - v_{nk}) \rho', \quad \frac{\partial \omega}{\partial \lambda_{ik}} = \frac{\partial \omega}{\partial \lambda'_{ik}}, \]

where \( v_{nk} \equiv v_k = -Y_k/Y_0 \) is the normal velocity. Next, for the flux densities (42), one obtains (by using (4), (53), (54))

\[ j_k = \frac{\pi'_k}{m} + v_{sk} \rho, \]
\[ t_{ik} = [T\sigma - \varepsilon' + (v_{nl} - v_{sl})\pi_i' + \mu' \rho] \delta_{ik} + v_{nk}\pi_i' + v_{si}\pi_k' + mv_{si}v_{sk}\rho + \lambda_j \frac{\partial \varepsilon}{\partial \lambda_{jk}} + v_{nk}T\sigma + v_{nk}(v_{ni}\pi_i') + \left( \mu' + \frac{mv^2}{2} \right) j_k + v_{nl} \lambda_j \rho \frac{\partial \varepsilon}{\partial \lambda_{jk}}, \]

where

\[ \mu' = \mu + mv_{nl}v_{sl} - \frac{mv^2}{2} \]

is the chemical potential in the reference frame \( K' \). When deriving \( t_{ik}, w_k \) we have eliminated the thermodynamic potential density \( \omega \) by means of (10) and employed (50). We can see that the momentum density is proportional to the particle number flux density, \( mj_k = \pi_k \), where \( m \) is the atomic mass. This fact is the consequence of the Galilean invariance. Finally, making use of (4), (56), we write (43) in the form

\[ \dot{v}_{si} + \nabla_i \left( \frac{\mu'}{m} + \frac{v^2}{2} \right) = 0, \quad \dot{\lambda}_{ik} + \nabla_k (\lambda_{ij} v_{nj}) = 0. \]

Equations (55), (57) coincide with the corresponding hydrodynamic equations (non-dissipative), obtained by Andreev and Lifshitz [1]. As was already mentioned, the second equation in (57) shows that the lattice velocity coincides with the normal velocity.

VI. LORENTZ INVARIANCE

In the previous section, we have studied the invariance of the obtained supersolid hydrodynamics under the Galilean transformation. Namely, in the case of imposed constraint (44) on the thermodynamic potential density, the general hydrodynamic equations (41)-(43) become invariant under the Galilean transformation.

Here, we will show that the general hydrodynamic equations (41)-(43) can be written in a relativistic form. Then, in case of the Lorentz-invariant thermodynamic potential density, these equations are invariant under the Lorentz transformation \( x^\mu \rightarrow x'^\mu = a_{\rho\mu}x^\rho, \) where \( a_{\rho\mu} (a_{\rho\mu}a^{\mu\nu} = \delta^\rho_\nu) \) are the coefficients of the Lorentz transformation.

To this end, let us introduce the following four-component quantities:

\[ Y_\mu = (Y_0, Y_k), \quad p_\mu = (p_0, p_k), \quad \lambda_{j\mu} = (\lambda_{j0}, \lambda_{jk}), \]

where

\[ p_0 = Y_k Y_k Y_0, \quad \lambda_{j0} = Y_k Y_k Y_0, \quad \lambda_{jk} = \frac{Y_k Y_k Y_k}{Y_0}, \]

represent the right-hand side of Eqs. (43). The constraints (58) can be written in a relativistic form,

\[ Y_\mu p^\mu = Y_4, \quad Y_\mu \lambda_{j\mu} = 0. \]

Next, consider a function \( \tilde{\omega}(Y_\mu, p_\mu, \lambda_{j\mu}) \equiv \tilde{\omega}(Y_0, Y_k; p_0, p_k; \lambda_{j0}, \lambda_{jk}) \), which we relate to the Gibbs thermodynamic potential density \( \omega' = \omega/Y_0 \) (\( \omega' = -p \), where \( p \) is the pressure) by the formula

\[ \omega'(Y_0, Y_k, Y_4, p_k, \lambda_{jk}) = \tilde{\omega}(Y_0, Y_k; p_0, p_k; \lambda_{j0}, \lambda_{jk}). \]
According to (6), (42), the densities of additive integrals of motion $\zeta$ and their fluxes $\zeta_{\alpha k}$ are expressed through the thermodynamic potential density $\omega$ that is convenient while studying the Galilean-invariant systems. However, for relativistically-invariant systems, it is reasonable to express these quantities through $\tilde{\omega} = \tilde{\omega}(Y_\mu, p_\mu, \lambda_{jk})$ (below, we will see that $\tilde{\omega}$ is a relativistic invariant). In addition, we should remember that $Y_\mu$, $p_\mu$, and $\lambda_{jk}$ meet the constraints (59). From (60) we have

$$
\frac{\partial \omega'}{\partial Y_0} = \frac{\partial \tilde{\omega}}{\partial Y_0} - p_0 \frac{\partial \omega}{\partial p_0} \frac{\lambda_{j0}}{Y_0} \frac{\partial \tilde{\omega}}{\partial \lambda_{j0}},
$$

$$
\frac{\partial \omega'}{\partial Y_k} = \frac{\partial \tilde{\omega}}{\partial Y_k} + p_k \frac{\partial \omega}{\partial p_0} \frac{\lambda_{j0}}{Y_0} \frac{\partial \tilde{\omega}}{\partial \lambda_{j0}},
$$

$$
\frac{\partial \omega'}{\partial p_k} = \frac{\partial \tilde{\omega}}{\partial p_k} + Y_k \frac{\partial \omega}{\partial p_0} \frac{1}{Y_0} \frac{\partial \tilde{\omega}}{\partial \lambda_{j0}}.
$$

Then, the densities of additive integrals of motion in terms of $\tilde{\omega}$ are given by (see (7))

$$
\zeta_4 \equiv \rho = \frac{\partial \tilde{\omega}}{\partial \rho_0}, \quad \zeta_0 \equiv \epsilon = \frac{\partial \tilde{\omega}}{\partial p_0} - p_0 \frac{\partial \omega}{\partial p_0} \frac{\lambda_{j0}}{Y_0} \frac{\partial \omega}{\partial \lambda_{j0}},
$$

$$
\zeta_k \equiv \pi_k = Y_0 \frac{\partial \tilde{\omega}}{\partial Y_k} + p_k \frac{\partial \omega}{\partial p_0} + \lambda_{jk} \frac{\partial \omega}{\partial \lambda_{j0}}
$$

(62) and their fluxes have the following form:

$$
\zeta_{4k} \equiv j_k = \frac{\partial \tilde{\omega}}{\partial p_k},
$$

$$
\zeta_{0k} \equiv w_k = -Y_0 \frac{\partial \tilde{\omega}}{\partial Y_k} - p_0 \frac{\partial \omega}{\partial p_k} - \lambda_{j0} \frac{\partial \omega}{\partial \lambda_{jk}},
$$

$$
\zeta_{ik} \equiv t_{ik} = -\frac{\partial \tilde{\omega}}{\partial Y_i} \frac{\partial \omega}{\partial Y_k} + p_i \frac{\partial \omega}{\partial p_k} + \lambda_{jk} \frac{\partial \omega}{\partial \lambda_{jk}}.
$$

(63)

After performing the differentiation we should take into account the constraints. The first formulae in (62), (63) show that the particle number density $\rho \equiv j^0$ and the particle number flux density $j^k \equiv j_k$ are combined into the four-component vector $j^\mu = (\rho, j_k)$.

$$
\begin{align*}
j^\mu &= \frac{\partial \omega}{\partial p_\mu},
\end{align*}
$$

(64)

In a similar manner, the densities of energy $\epsilon$, momentum $\pi_k$ and their fluxes $w_k$, $t_{ik}$ form the energy-momentum tensor $t^{\mu\nu}$ of rank two,

$$
\begin{align*}
t^{\mu\nu} &= \frac{\partial \tilde{\omega} Y^{\nu}}{\partial Y_\mu} - p^{\mu} \frac{\partial \tilde{\omega}}{\partial p_\nu} - \lambda_{j}^{\mu} \frac{\partial \tilde{\omega}}{\partial \lambda_{j\nu}},
\end{align*}
$$

(65)

with $t^{00} \equiv \epsilon$, $t^{0k} \equiv w_k$, $t^{k0} \equiv \pi_k$, $t^{ik} \equiv t_{ik}$. Here and below, the raising and lowering of indices are implemented by means of the metric tensor $g_{\mu\nu}$ ($g_{00} = 1$, $g_{ik} = \delta_{ik}$, $g_{0k} = 0$).

In terms of the introduced energy-momentum tensor $t^{\mu\nu}$ and the current vector $j^\mu$, the hydrodynamic equations of a supersolid have the following form:

$$
\partial_t t^{\mu\nu} = 0, \quad \partial_{\nu} j^{\mu} = 0,
$$

(66)
where $\partial_{\nu} = \partial/\partial x^\nu$. Other two hydrodynamic equations, which describe the temporary evolution of variables associated with the broken symmetries can be written as follows:

$$
\partial_{\nu} p_{\mu} - \partial_{\mu} p_{\nu} = 0, \quad \partial_{\mu} \lambda_{i\nu} - \partial_{\nu} \lambda_{i\mu} = 0.
$$

(67)

The first equation in (67) is the equation of motion for the superfluid momentum, which also includes the irrotational nature of superfluid flow, $\text{rot}\mathbf{p} = 0$. The second one, represents the equation of motion for $\lambda_{i\kappa}$ including the property $\nabla_{i} \lambda_{i\kappa} = \nabla_{\kappa} \lambda_{i\kappa}$, which we have used above. The obtained equations (66), (67) form the system of hydrodynamic equations for the supersolid phase, written in a relativistic form.

The equation of motion for the entropy density (the third equation in (36)) has, according to (8), the form

$$
\dot{\sigma} - \nabla_i \left( \sigma \frac{Y_i}{Y_0} \right) = 0.
$$

It can also be written in relativistic notations. Indeed, noting that $\sigma = -\omega + Y_{a} \zeta_{a}$ and $\zeta_{a} = \partial \omega/\partial Y_{a}$, one obtains $\sigma = Y_{a} \dot{Y}_{a} \left( \partial \omega'/\partial Y_{a} \right)$, where $\omega' = \omega / Y_{0}$. Then, making use of (61), we get $\sigma \equiv \sigma^{0} = Y_{0} \dot{Y}_{\mu} \left( \partial \tilde{\omega}/\partial Y_{\mu} \right)$. Therefore, the quantity

$$
\sigma^\mu = Y^\mu Y_{\nu} \left( \partial \tilde{\omega}/\partial Y_{\nu} \right)
$$

form a four-vector satisfying, according to the above written equation for $\sigma$, the following conservation law:

$$
\partial_{\mu} \sigma^\mu = 0.
$$

(69)

So far, we have not imposed any constraint on thermodynamic potential density associated with relativistic invariance. Indeed, when deriving (64)-(69) we just changed from the one set of variables $Y_{0}, Y_{k}, Y_{4}, p_{k}, \lambda_{ik}$ to another, that is $Y_{\mu} = (Y_{0}, Y_{k}), p_{\mu} = (p_{0}, p_{k}), \lambda_{i\mu} = (\lambda_{i0}, \lambda_{ik})$.

Within the framework of microscopic approach based on the Gibbs statistical operator [19], it can be shown that the Gibbs thermodynamic potential density $\tilde{\omega}$ represents a relativistic invariant. Thus, $\tilde{\omega}$ is a function of six invariants $J \equiv (J_{1}, J_{2}, J_{ij}, J_{i})$ and $I \equiv (I_{1}, I_{2})$,

$$
\tilde{\omega} = \tilde{\omega}(J, I),
$$

(70)

where

$$
J_{1} = (1/2)Y_{\mu}Y^{\mu}, \quad J_{2} = (1/2)p_{\mu}p^{\mu},
$$

$$
J_{ij} = (1/2)\lambda_{i\mu}\lambda_{j}^{\mu}, \quad J_{i} = p_{\mu}\lambda_{i}^{\mu},
$$

$$
I_{1} = Y_{\mu}p^{\mu}, \quad I_{2} = Y_{\mu}\lambda_{j}^{\mu},
$$

(71)

moreover, under the Lorentz transformation $x^\mu \rightarrow x'^{\mu} = a^{\mu}_{\nu} x^{\nu}$, the hydrodynamic variables are transformed as follows:

$$
Y_{\mu}(x) \rightarrow Y'_{\mu}(x') = a^{\mu}_{\nu} Y_{\nu}(x),
$$

$$
p_{\mu}(x) \rightarrow p'_{\mu}(x') = a^{\mu}_{\nu} p_{\nu}(x),
$$

$$
\lambda_{i\mu}(x) \rightarrow \lambda'_{i\mu}(x') = a^{\mu}_{\nu} \lambda_{i\nu}(x).
$$

(72)
Due to constraints (59), \( I_1 = Y_4 \) and \( I_2 = 0 \), so that \( \tilde{\omega} = \omega(J, Y_4) \). We can see that \( j^\mu \) is the Lorentz four-vector and \( t^{\mu\nu} \) represents the Lorentz four-tensor of rank two. Thus, the hydrodynamic equations (64)-(67) are invariant under the Lorentz transformations (72).

The next step is to obtain the expressions for the energy-momentum tensor \( t^{\mu\nu} \) and the current vector \( j^\mu \) taking into account that \( \tilde{\omega} = \tilde{\omega}(J, Y_4) \). From (65) we have
\[
t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial \tilde{\omega}}{\partial J_1} Y^\mu Y^\nu - \frac{\partial \tilde{\omega}}{\partial J_2} p^\mu p^\nu - \frac{\partial \tilde{\omega}}{\partial J_{kj}} \lambda^{\mu}_k \lambda^{\nu}_j - \frac{\partial \tilde{\omega}}{\partial \lambda_i} (p^\mu \lambda^{\nu}_i + p^\nu \lambda^{\mu}_i),
\]
(73)

We can see that the energy-momentum tensor is a symmetric tensor, \( t^{\mu\nu} = t^{\nu\mu} \), as it should be in relativistic theory. Due to the symmetry of \( t^{\mu\nu} \), the momentum density \( \pi_k \) coincides with the energy flux density \( \pi_k \equiv t^{k0} \equiv w_k \). For the current density \( j^\mu \), one obtains from (64)
\[
j^\mu = \frac{\partial \tilde{\omega}}{\partial J_2} p^\mu + \frac{\partial \tilde{\omega}}{\partial Y_4} Y^\mu.
\]
(74)

Equations (66), (67), (73), (74) describe the relativistic non-dissipative hydrodynamics of supersolids.

In conclusion of this section consider some particular cases.

1. **Relativistic superfluid liquid.** If \( \tilde{\omega} \) does not depend on invariants \( J_{kj} \), i.e., \( \tilde{\omega} = \tilde{\omega}(J_1, J_2, Y_4) \), then (73), (74) take the form
\[
t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial \tilde{\omega}}{\partial J_1} Y^\mu Y^\nu - \frac{\partial \tilde{\omega}}{\partial J_2} p^\mu p^\nu,
\]
j^\mu = \frac{\partial \tilde{\omega}}{\partial Y_4} Y^\mu.

The corresponding equations describe the relativistic hydrodynamics of a superfluid liquid [19].

2. **Relativistic normal liquid.** In case of \( \tilde{\omega} \) being dependent on \( J_1 \) and \( Y_4 \), \( \tilde{\omega} = \tilde{\omega}(J_1, Y_4) \), we come to the hydrodynamics of a normal liquid, for which [20]
\[
t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial \tilde{\omega}}{\partial J_1} Y^\mu Y^\nu, \quad j^\mu = \frac{\partial \tilde{\omega}}{\partial Y_4} Y^\mu.
\]

3. **Relativistic elasticity.** If \( \tilde{\omega} \) is a certain function of \( J_1, J_{kj}, Y_4 \) only, i.e., \( \tilde{\omega} = \tilde{\omega}(J_1, J_{kj}, Y_4) \), then (73), (74) result in the formulae
\[
t^{\mu\nu} = \tilde{\omega} g^{\mu\nu} + \frac{\partial \tilde{\omega}}{\partial J_1} Y^\mu Y^\nu - \frac{\partial \tilde{\omega}}{\partial J_{kj}} \lambda^{\mu}_k \lambda^{\nu}_j, \quad j^\mu = \frac{\partial \tilde{\omega}}{\partial Y_4} Y^\mu,
\]
which correspond to the equations of relativistic elasticity.

**VII. CONCLUSION**

In this work we have constructed a Lagrangian describing the Hamiltonian dynamics of supersolids. This Lagrangian, written in terms of the local thermodynamic variables, enables to derive the Hamilton equations of motion as well as the closed algebra of the Poisson brackets for basic variables. In the leading order in spatial gradients of the local thermodynamic variables, the Hamilton equations lead to the non-dissipative supersolid hydrodynamics. The obtained hydrodynamic equations do not assume neither Galilean nor Lorentz invariance and, therefore, they can be used for studying the system with any
dynamic symmetry. The Andreev-Lifshitz supersolid hydrodynamics follows from the derived equations in case of the imposed constrained on the thermodynamic potential density associated with the Galilean invariance. We have also required the relativistic invariance of a constructed theory and obtained a relativistically-invariant system of hydrodynamic equations describing the supersolid phase. Depending on what variables are cyclic, the developed approach gives also the non-dissipative hydrodynamics of normal and superfluid liquids as well as the elasticity equations.

Acknowledgments

The author would like to acknowledge the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for the support, warm hospitality, and stimulating research environment during the visit to the Centre within the framework of Associateship Scheme in summer 2007. He also gratefully acknowledges the discussions with S.V. Peletminskii, Yu.M. Poluektov, A.A. Yatsenko.

APPENDIX: CANONICAL TRANSFORMATIONS

Consider the following Lagrangian:

$$L(\eta, \dot{\eta}) = L_k(\eta, \dot{\eta}) - H(\eta) \equiv \int d^3 x F_\alpha(x; \eta) \dot{\eta}_\alpha(x) - H(\eta), \quad (A.1)$$

where $L_k(\eta, \dot{\eta})$ is its kinematic part, $F_\alpha(x; \eta(x'))$ is a certain functional of fields $\eta_\alpha(x) \equiv \eta_\alpha(x, t)$, and $H(\eta)$ is a Hamiltonian. The infinitesimal transformations of fields

$$\eta_\alpha(x, t) \to \eta'_\alpha(x, t) = \eta_\alpha(x, t) + \delta \eta_\alpha(x, t)$$

induce the variation of the action functional $W = \int_{t_1}^{t_2} dt L(\eta, \dot{\eta})$, which can be written in the form

$$\delta W = G(t_2, \eta) - G(t_1, \eta) + \int_{t_1}^{t_2} dt \int d^3 x \delta \eta_\beta(x', t) \left( \int d^3 x J_{\alpha\beta}(x', x; \eta) \dot{\eta}_\alpha(x) - \frac{\delta H}{\delta \eta_\beta(x')} \right), \quad (A.2)$$

where

$$G(t, \eta) = \int d^3 x F_\alpha(x; \eta) \delta \eta_\alpha(x, t), \quad (A.3)$$

$$J_{\alpha\beta}(x, x'; \eta) = \frac{\delta F_\beta(x'; \eta)}{\delta \eta_\alpha(x)} - \frac{\delta F_\alpha(x; \eta)}{\delta \eta_\beta(x')} \quad (A.4)$$

The principle of stationary action ($\delta W = 0$) results to the following equations of motion:

$$\ddot{\eta}_\alpha(x) = \int d^3 x' J_{\alpha\beta}^{-1}(x, x'; \eta) \frac{\delta H}{\delta \eta_\beta(x')} \quad (A.5)$$

where the inverse matrix $J_{\alpha\beta}^{-1}(x, x'; \eta)$ is defined by

$$\int d^3 x'' J_{\alpha\gamma}(x, x''; \eta) J_{\gamma\beta}^{-1}(x'', x'; \eta) = \delta_{\alpha\beta}(x - x'). \quad (A.6)$$

The Poisson brackets for two arbitrary functionals $A(\eta)$ and $B(\eta)$ we define as follows:

$$\{A, B\} = \int d^3 x d^3 x' \frac{\delta A}{\delta \eta_\alpha(x)} J_{\alpha\beta}^{-1}(x, x'; \eta) \frac{\delta B}{\delta \eta_\beta(x')} \quad (A.7)$$
Then, the equations of motion (A.5) take the form

$$\dot{\eta}_\alpha(x) = \{\eta_\alpha(x), H\}. \quad (A.8)$$

Using the identity \(\{\eta_\alpha(x_1), \eta_\beta(x_2)\} = J^{-1}_{\alpha\beta}(x_1, x_2; \eta)\), we can recast (A.8) as

$$\dot{\eta}_\alpha(x) = \int d^3x' \{\eta_\alpha(x), \eta_\beta(x')\} \frac{\delta H}{\delta \eta_\beta(x')}.$$ \quad (A.9)

The antisymmetry of \(J_{\alpha\beta}(x, x'; \eta), J_{\alpha\beta}(x, x'; \eta) = -J_{\beta\alpha}(x', x; \eta)\), enables us to prove that the Poisson brackets (A.7) possess all the necessary properties

\[\{A, B\} = -\{B, A\}, \quad \{AB, C\} = A\{B, C\} + B\{A, C\}\]

and meet the Jacobi identity

\[\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0,\]

which is satisfied due to the following relationship:

$$\frac{\delta J_{\alpha\beta}(x, x'; \eta)}{\delta \eta_\gamma(x')} + \frac{\delta J_{\eta\alpha}(x', x; \eta)}{\delta \eta_\beta(x')} + \frac{\delta J_{\beta\gamma}(x', x''; \eta)}{\delta \eta_\alpha(x)} = 0.$$ \quad (A.10)

Now we address to canonical transformations. To this end, consider the following finite transformations of fields:

$$\eta_\alpha(x) \rightarrow \eta'_\alpha(x) = \eta_\alpha(x; \eta(x')),$$ \quad (A.11)

where the transformed field \(\eta'_\alpha(x)\) is a certain functional of the initial field \(\eta_\alpha(x)\). The transformations (A.10) that meet the condition

$$\int d^3xF_\alpha(x; \eta)\delta \eta_\alpha(x) - \int d^3xF_\alpha(x; \eta')\delta \eta'_\alpha(x) = \delta Q(\eta),$$ \quad (A.12)

or

$$\frac{\delta Q(\eta)}{\delta \eta_\alpha(x)} = F_\alpha(x; \eta) - \int d^3x_1F_\beta(x_1; \eta')\frac{\delta \eta'_\beta(x_1; \eta)}{\delta \eta_\alpha(x)},$$

are referred to as canonical transformations. The quantity \(Q(\eta)\) being dependent on the structure of canonical transformations represents a certain functional of dynamic variables \(\eta_\alpha(x)\). If \(Q = \text{const}\), then we deal with the homogeneous canonical transformations, which preserve the kinematic part of the Lagrangian. Taking into account the identity \(\delta^2Q/\delta \eta_\alpha(x)\delta \eta_\beta(x') = \delta^2Q/\delta \eta_\beta(x')\delta \eta_\alpha(x)\), valid for second variational derivatives, and using (A.5), one finds

$$J_{\alpha\beta}(x, x'; \eta) = \int d^3x_1d^3x_2 \frac{\delta \eta'_\alpha(x_1; \eta)}{\delta \eta_\alpha(x)} \frac{\delta \eta'_\beta(x_2; \eta)}{\delta \eta_\beta(x')} J_{\gamma\lambda}(x_1, x_2; \eta').$$ \quad (A.13)

The Poisson brackets (A.8) are invariant under the transformations (A.10) if the following condition is satisfied:

$$J_{\alpha\beta}^{-1}(x, x'; \eta') = \int d^3x_1d^3x_2 \frac{\delta \eta'_\alpha(x; \eta)}{\delta \eta_\alpha(x_1)} \frac{\delta \eta'_\beta(x'; \eta)}{\delta \eta_\beta(x_2)} J_{\gamma\lambda}^{-1}(x_1, x_2; \eta).$$ \quad (A.14)
It can be easily proved that (A.14) is equivalent to (A.13) that reflects the condition of our transformations (A.10) to be canonical. Indeed, introducing the notation \( T_{\alpha \beta}(\mathbf{x}, \mathbf{x}') = \delta \eta'_{\alpha}(\mathbf{x}; \eta)/\delta \eta_{\beta}(\mathbf{x}') \), one can write (A.13) and (A.14) as 
\[
\tilde{T}\eta_{\alpha}(\mathbf{x}, \mathbf{x}') = \tilde{T}_{\beta \alpha}(\mathbf{x}', \mathbf{x}).
\]
Then, the first relationship results in 
\[
J^{-1}(\eta) = T^{-1}J^{-1}(\eta')\tilde{T}^{-1},
\]
whence it follows (A.14), 
\[
J^{-1}(\eta') = TJ^{-1}(\eta)\tilde{T}.
\]

The Lagrangian is determined up to a derivative with respect to time. It can be easily proved that 
\[
L = \tilde{L}(\mathbf{x}, \mathbf{x}').
\]
Then, in case of homogeneous transformations \((Q = \text{const})\), the condition (A.12) reduces to 
\[
\int d^3x' J_{\alpha \beta}(\mathbf{x}, \mathbf{x}'; \eta)\delta \eta_{\beta}(\mathbf{x}'; \eta) = \frac{\delta G}{\delta \eta_{\alpha}(\mathbf{x})},
\]

where 
\[
G = \int d^3x' F_{\beta}(\mathbf{x}'; \eta)\delta \eta_{\beta}(\mathbf{x}'; \eta).
\]
Next, using the definitions (A.6), (A.7), one immediately obtains from (A.16):
\[
\delta \eta_{\alpha}(\mathbf{x}; \eta) = \{\eta_{\alpha}(\mathbf{x}), G\}.
\]
Thus, the quantity \( G \) (see also (A.3)) should be interpreted as the generator of the infinitesimal canonical transformation (A.15). Note that \( G \) in (A.3) was interpreted by Schwinger as the generator of infinitesimal transformations when constructing the quantum action principle [21]. Note that the Hamiltonian mechanics in case of arbitrary variables was studied in [22] for the systems with finite degrees of freedom.

The generator \( G \) can be used for obtaining the Poisson brackets (19), (35), which describe the supersolid dynamics. In doing so, we note that the following infinitesimal transformations:
\[
\delta \tilde{q}_i(\mathbf{x}) = 0, \quad \delta u_i(\mathbf{x}) = f_i(\mathbf{x}), \quad \delta \sigma(\mathbf{x}) = 0,
\]
\[
\delta \psi(\mathbf{x}) = \chi(\mathbf{x}), \quad \delta \rho(\mathbf{x}) = 0, \quad \delta \varphi(\mathbf{x}) = \theta(\mathbf{x})
\]
preserve the kinematic part of the Lagrangian (25). The arbitrary real functions \( f_i(\mathbf{x}), \chi(\mathbf{x}), \theta(\mathbf{x}) \) do not depend neither on time nor on dynamic variables. Then, in accordance with (A.17), the generator \( G \) of these transformations has the form
\[
G = \int d^3x' [\tilde{q}_i(\mathbf{x}')f_i(\mathbf{x}') - \sigma(\mathbf{x}')\chi(\mathbf{x}') - \rho(\mathbf{x}')\theta(\mathbf{x}')].
\]
Therefore, making use (A.18), one gets
\[
0 = \{\tilde{q}_i(\mathbf{x}), G\}, \quad f_i(\mathbf{x}) = \{u_i(\mathbf{x}), G\}, \quad 0 = \{\sigma(\mathbf{x}), G\},
\]
\[
\chi(\mathbf{x}) = \{\psi(\mathbf{x}), G\}, \quad 0 = \{\rho(\mathbf{x}), G\}, \quad \theta(\mathbf{x}) = \{\varphi(\mathbf{x}), G\}.
\]
Next, using the arbitrariness of $f_i(x)$, $\chi(x)$, $\theta(x)$ we find
\[
\begin{align*}
\{\tilde{q}_i(x), \tilde{q}_j(x')\} &= \{\tilde{q}_i(x), \sigma(x')\} = \{\tilde{q}_i(x), \rho(x')\} = 0, \\
\{u_i(x), \sigma(x')\} &= \{u_i(x), \rho(x')\} = \{\sigma(x), \sigma(x')\} = 0, \\
\{\sigma(x), \rho(x')\} &= \{\psi(x), \tilde{q}_i(x')\} = \{\psi(x), \rho(x')\} = 0, \\
\{\rho(x), \rho(x')\} &= \{\varphi(x), \tilde{q}_i(x')\} = \{\varphi(x), \sigma(x')\} = 0 \\
\end{align*}
\]
(A.19)

and
\[
\begin{align*}
\{\sigma(x'), \psi(x)\} &= \delta(x - x'), \\
\{\rho(x'), \varphi(x)\} &= \delta(x - x'), \\
\{u_i(x), \tilde{q}_j(x')\} &= \delta_{ij}\delta(x - x'). \\
\end{align*}
\]
(A.20)
References