Abstract

It was shown in ref.[1], only for scalar conformal fields, that the Moyal-Weyl star product can introduce the quantum effect as the phase factor to the ordinary product.

In this paper we show that, even on the same complex structure, the Moyal-Weyl star product of two j-differentials (conformal fields of weights (j,0)) does not vanish but it generates the quantum effect at the first order of its perturbative series.

More generally, we get the explicit expression of the Moyal-Weyl star product of j-differentials defined on any complex structure of a bi-dimensional Riemann surface Σ. We show that the star product of two j-differentials is not a j-differential and does not preserve the conformal covariance character.

This can shed some light on the Moyal-Weyl deformation quantization procedure connection’s with the deformation of complex structures on a Riemann surface. Hence, the situation might relate the star products to the Moduli and Teichmüller spaces of Riemann surfaces.
Introduction

There are several different procedures to quantize a classical system like, geometric quantization, asymptotic quantization and deformation quantization. On the other hand, the basic underlying principle of any quantization scheme is that classical and quantum systems are just different realizations of the same abstract object. The two fundamental components of this object are the phase space ("space of states") and the algebra of observables "physical observables"). At the classical level, the phase space is a Poisson manifold and the algebra of observables is the algebra of smooth functions on the manifold. In the quantum system, these concepts vary depending on the quantization procedure. In the deformation quantization scheme, one keeps the nature of observables and simply deforms the algebraic structure. This is very much related to the modern point of view of non-commutative geometry: at the classical level of a physical system, we have standard, classical geometry, but when we change scale and go to the quantum level, the non-commutativity of nature reveals itself as the parameter measuring this non-commutativity is the Planck’s constant $\hbar$.

Deformation quantization was born with the paper [2] and grew with the work of Weyl, Moyal and Vey. It has become a large research area covering several algebraic theories like the formal deformation of associative algebras and the more recent theory of operades, as well as geometric theories like the theory of symplectic and (more generally) Poisson manifolds, and physical theories like string theory and non commutative gauge theory.

The theory of deformation quantization has culminated with the recent work of Kontsevitch [3] proving that every Poisson manifold admits a non-trivial deformation quantization.

Two-dimensional conformal field theories on Riemann surfaces without boundaries are relevant models in string theory [4]. The dependence of these bidimensional conformal models on the background geometry turns out to be useful for the construction of effective actions for two-dimensional gravity [5]. This geometrical dependence is well exhibited using the Beltrami parametrization of the bidimensional world sheet metric of the bosonic string [6].

In the Riemannian manifold approach, Beltrami coefficients parametrize conformal classes of the metrics [7]. However, in the Riemannian surface formalism, they parametrize complex structures of the Riemann surface [6] and satisfy the ellipticity condition:

$$\text{Sup}_\Sigma |\mu| < 1, \mu \in C^\infty (\Sigma)$$

(1)

where, $\Sigma$ is a Riemann surface.

In a reference complex structure $C_0 (z, \overline{z})$ of $\Sigma$, the conformal classes of the metric are characterized by

$$ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = \rho^2 |dz + \mu d\overline{z}|^2$$

(2)

where, $\rho (z, \overline{z})$ is the conformal factor.

The Beltrami parametrization scheme is the most natural framework which exhibits the holomorphic factorization of Green’s functions of bidimensional conformal models. Moreover, in this geometrical formulation, the degree of freedom of the Weyl symmetry is eliminated from the very beginning and the remaining deffeomorphism symmetry is kept as the basic local invariance of the conformal model [6].
Riemann surface

A Riemann surface $\Sigma$ is a $C^\infty$-differentiable manifold equipped with an atlas of compatible complex analytic coordinates system. On each patch, the transition functions are holomorphic. It is endowed with a reference complex structure which is fixed by an analytic complex structure $(z, \overline{z})$.

Beltrami differential

The Riemann surface $\Sigma$ can be endowed with a $C^\infty$-Riemann metric $(g_{\alpha\beta})$ which can be written in the reference complex structure as follows [8]:

$$ds^2 = \rho^2(z, \overline{z}) |dz + \mu d\overline{z}|^2.$$  \hspace{1cm} (3)

where $\mu \in C^\infty(\Sigma)$ is called the Beltrami differential and satisfies the ellipticity condition:

$$\text{Sup}_{\Sigma} |\mu(z, \overline{z})| < 1$$  \hspace{1cm} (4)

and $\rho(z, \overline{z})$ is called the conformal factor.

The Beltrami differential can be seen as a $(-1, 1)$-differential and written locally as:

$$\mu = \mu d\overline{z} \otimes \partial, \quad \partial \equiv \frac{\partial}{\partial z}.$$  \hspace{1cm} (5)

It can be defined in terms of a quadratic differential $\phi = \phi_{zz} dzdz$ as follows:

$$\langle \mu, \phi \rangle = \int_{\Sigma} dz \wedge d\overline{z} 2i \mu \phi_{zz}.$$  \hspace{1cm} (6)

Now, on this Riemann surface $\Sigma$, we consider a Beltrami differential $\mu(z, \overline{z})$ which induces another complex structure on $\Sigma$ that is parametrized by local coordinates $(Z, \overline{Z})$. These coordinates are $C^\infty$-diffeomorphisms of the reference variables $(z, \overline{z})$:

$$z \rightarrow Z(z, \overline{z})$$  \hspace{1cm} (7)

and satisfy, in each map, the Beltrami equation

$$(\overline{\partial} - \mu \partial) Z = 0.$$  \hspace{1cm} (8)

This equation can be rewritten as

$$W_0 Z = 0$$  \hspace{1cm} (9)

The diffeomorphisms on the Riemann surface $\Sigma$ that satisfy the Beltrami equation are called quasiconformal mappings.

Then, the set $\text{Beltr} (\Sigma)$ of Beltrami differentials parametrizes the set of all conformal structures on the Riemann surface $\Sigma$. Indeed, any Beltrami differential in $\text{Beltr} (\Sigma)$ is associated to a conformal structure $C_\mu$ whose generic coordinate $Z(z, \overline{z})$ is defined by:

$$\overline{\partial} Z = \mu \partial Z.$$  \hspace{1cm} (10)
Diffeomorphisms on $\Sigma$

Each diffeomorphism on $\Sigma$ is characterized by a constraint satisfied by the Schwarzian derivative $[9]$

$\zeta(Z) \equiv \partial^2 \ln \partial Z - \frac{1}{2} (\partial \ln \partial Z)^2$.

(11)

For any diffeomorphism $z \rightarrow Z(z, \bar{z})$ one can verify that the function $\zeta$ satisfies the constraint

$\left( \partial^2 + \frac{1}{2} \zeta(Z) \right) \left( (\partial Z)^{-2} \right) = 0$.

(12)

If the diffeomorphism is quasiconformal, the Schwarzian derivative satisfies

$W_2 \zeta(Z) = \partial^4 \mu$.

(13)

where, $W_2 \equiv \partial - \mu \partial - 2 \partial \mu$ is the Ward operator.

This is the same equation (up to a central charge) that is satisfied by the quantum energy-momentum tensor of an effective bidimensional conformal model. It expresses the conformal anomaly of the model.

For a holomorphic diffeomorphism ($\mu = 0$) one can verify that the corresponding Schwarzian derivative is also holomorphic:

$\overline{\partial} \zeta(Z) = 0$.

(14)

Finally, the corresponding Schwarzian derivative of a projective mapping vanishes:

$\zeta(Z) = 0$.

(15)

$\mu$-holomorphic $j$-differentials on $\Sigma$

As we will consider $j$-differentials on a Riemann surface, let us introduce their essential properties.

A $\mu$-holomorphic $j$-differential in the conformal structure $C_\mu$ is a conformal field $f_j$ of weights $(j, 0)$ that satisfies the following $\mu$-holomorphy condition:

$W_j f_j = 0$.

(16)

where

$W_j \equiv \overline{\partial} - \mu \partial - j \partial \mu$

(17)

is the generalized Ward operator.

In particular, by putting $j = 0$ in equation (16), this later reduces to the Beltrami equation (9) that is satisfied by the scalar field $f_{j=0}$.

Moreover, it was shown in ref. [9] that (16) is equivalent to the following Beltrami equation:

$W_0 (\lambda^{-j} f_j) = 0$.

(18)
where \( \lambda \equiv \partial Z \) is called the scale factor.

This latter equation tells us that the function

\[
Q_Z \equiv \lambda^{-j} f_j,
\]

which depends holomorphically on the complex coordinates system \((Z, \overline{Z})\), is the solution of the \( \mu \)-holomorphy equation (16) which can be written as:

\[
f_j = \lambda^j Q_Z.
\]

Any \( \mu \)-holomorphic \( j \)-differential can be associated to a conformal structure \( C_\mu \) by a map, that is, a holomorphic section of a vector bundle whose base manifold is the set of all conformal structures and whose fiber at a given conformal structure \( C_\mu \) is the complex vector space of holomorphic \( j \)-differentials \( f_j[\mu] \) in \( C_\mu \) depending holomorphically on the Beltrami differential:

\[
\frac{\delta f_j[\mu]}{\delta \mu} = 0.
\]

**Moyal-Weyl star product on \( \Sigma \)**

The operator of the star product is chosen to generate the Poisson bracket in the canonical form. Then, for any two diffeomorphisms \( f \) and \( g \) on a Riemann surface \( \Sigma \), the Poisson bracket, in the reference complex structure \( C_0(z, \overline{z}) \) of \( \Sigma \), is defined by:

\[
\{f, g\}_{C_0} = \partial f \overline{\partial} g - \overline{\partial} f \partial g.
\]

The star product of two diffeomorphisms \( f \) and \( g \) defined on \( \Sigma \) is a function

\[
w(z, \overline{z}) = f(z, \overline{z}) \ast g(z, \overline{z})
\]

where the star operator is defined by:

\[
\ast \equiv \exp \left[ ih \left( \frac{\overline{\partial}}{\partial} \overline{\partial} - \frac{\overline{\partial}}{\partial} \overline{\partial} \right) \right].
\]

It is associative: for three functions \((f_i)_{i=1,2,3}\) defined on \( \Sigma \) we have

\[
(f_1 \ast f_2 \ast f_3) = f_1 \ast (f_2 \ast f_3),
\]

but it does not commute:

\[
f_1 \ast f_2 \ast f_3 \neq f_2 \ast f_3 \ast f_1.
\]

On the other hand, the star product is often interpreted as an expansion of power series of the Planck constant \( h \):

\[
w = w^{(0)} + hw^{(1)} + h^2w^{(2)} + \ldots
\]
In particular, if $f$ and $g$ are scalars (quasiconformal mappings) parametrizing different complex structures $C_{\mu_f}$ and $C_{\mu_g}$:

$$W^f_0 f = 0 \quad (28)$$
$$W^g_0 g = 0 \quad (29)$$

where, $W^f_0 = \overline{j} - \mu f \partial$, the Moyal-weyl star product (23) is given (at the first order) by:

$$w = fg + i\hbar \partial f \partial g (\mu_g - \mu_f). \quad (30)$$

For example, if $f$ and $g$ are given respectively by:

$$f(z, \overline{z}) = e^{i \alpha_f z} e^{i \beta_f \overline{z}} \quad (31)$$
$$g(z, \overline{z}) = e^{i \alpha_g z} e^{i \beta_g \overline{z}}, \quad (32)$$

the Moyal-Weyl star product is a phase factor to the ordinary product:

$$w = e^{i\hbar(\alpha_f \beta_g - \beta_f \alpha_g)} fg. \quad (33)$$

It reduces to the simple product if the diffeomorphisms $f$ and $g$ define the same complex structure that is; $\mu_g = \mu_f$ and then, for this example, we have the following relation:

$$\frac{\beta_f}{\alpha_f} = \frac{\beta_g}{\alpha_g}. \quad (34)$$

Moreover, as the Beltrami differential satisfies the ellipticity condition, the Moyal-Weyl star product can be developed in terms of the Beltrami differential in the following form:

$$w(z, \overline{z}) = \tilde{w}^{(0)} + \mu_f \tilde{w}^{(1)}_f + \mu_g \tilde{w}^{(1)}_g + \mu^2 \tilde{w}^{(2)}_f + \mu^2 \tilde{w}^{(2)}_g + \mu_f \mu_g \tilde{w}^{(2)}_{fg} + ... \quad (35)$$

**Moyal-Weyl star product of j-differentials**

First, let us consider two j-differentials say $F_j$ and $G_j$ defined on the same complex structure $C_{\mu}$ of the Riemann surface $\Sigma$, that is, parametrized by the scalar field $Z$:

$$W_0 Z = 0. \quad (36)$$

Then, let $F_j$ and $G_j$ satisfy the same $\mu$-holomorphy equation:

$$W_j F_j = 0, \quad (37)$$
$$W_j G_j = 0. \quad (38)$$

Hence, we get their Poisson Bracket as follows:

$$\{F_j, G_j\}_{C_0} = -j \partial \mu (F_j \partial G_j - G_j \partial F_j). \quad (39)$$
This Poisson bracket vanishes if \( F_j \) and \( G_j \) are scalar fields \((j = 0)\) defined on the complex structure \( C_\mu \):

\[
F_j = 0 \equiv F, \quad G_j = 0 \equiv G,
\]

\[
\{F, G\}_C = 0 \tag{40}
\]

or if the Beltrami differential defining the \( j \)-differentials is a constant function:

\[
\partial \mu = 0. \tag{41}
\]

On the other hand, equation (39) tells us that, even on the same complex structure, two \( j \)-differentials do not commute with respect to the Poisson bracket and the star product, at the first order, is not given as a phase factor:

\[
F_j \ast G_j = F_j G_j - i\hbar j \partial \mu (F_j \partial G_j - G_j \partial F_j). \tag{42}
\]

As the Poisson Bracket (39) can be rewritten as

\[
\{F_j, G_j\}_C = j \partial \mu \partial \ln \left( \frac{G_j}{F_j} \right) F_j G_j, \tag{43}
\]

one can verify that the star product (42) can become a phase factor to the ordinary product \( F_j G_j \) if the two \( j \)-differentials satisfy the following equation

\[
\partial \ln \left( \frac{G_j}{F_j} \right) = C \tag{44}
\]

where \( C \) is a constant and then they are related by the relation

\[
G_j = K \exp \left[ C \left( z + \int_{\Sigma} \mu (z, \overline{z}) \, d\overline{z} \right) \right] F_j \tag{45}
\]

with \( K \) being another constant.

As equations (37) and (38) can be rewritten as Beltrami differentials

\[
W_0 (\lambda^{-j} F_j) = 0, \tag{46}
\]

\[
W_0 (\lambda^{-j} G_j) = 0
\]

the Poisson Bracket (39) can be expressed as

\[
\{\lambda^{-j} F_j, \lambda^{-j} G_j\} = 0 \tag{47}
\]

where \( \lambda \equiv \partial Z \) is called the conformal factor.

Now, by using this latter expression for the Poisson Bracket we show that the function

\[
R_j \equiv G_j \partial F_j - F_j \partial G_j \tag{48}
\]

is a \((2j + 1)\)-differential and then it satisfies the following \( \mu \)-holomorphy condition:

\[
W_{2j+1} (F_j \partial G_j - G_j \partial F_j) = 0 \tag{49}
\]
Moreover, this function can be rewritten in terms of the conformal factor as:

$$F_j G_j \partial \ln \left( \frac{F_j}{G_j} \right) = \lambda^{2j+1} Q_Z$$

where $Q_Z$ is a holomorphic function of the scalar $Z$.

On the other hand, equation (39) with the help of (49) enables us to show that the Poisson bracket of two $j$-differentials (corresponding to the same complex structure) is not a $k$-differential. Indeed, we get

$$W_{2j+1} \left( \{F_j, G_j\}_{C_0} \right) = j \left( \partial \overline{\partial} \mu - \mu \partial^2 \mu \right) (F_j \partial G_j - G_j \partial F_j)$$

and then the Poisson bracket preserves the holomorphic character only of scalar fields.

**General case of Moyal-Weyl star product**

Now let $F_j$ and $G_j$ be two $j$-differentials satisfying different $\mu$-holomorphy conditions:

$$\left( \overline{\partial} - \mu_F \partial - j \partial \mu_F \right) F_j = 0,$$

$$\left( \overline{\partial} - \mu_G \partial - j \partial \mu_G \right) G_j = 0$$

where, $\mu_F$ and $\mu_G$ are respectively the corresponding Beltrami coefficients.

The corresponding Poisson Bracket, in the reference complex structure, is given by:

$$\{F_j, G_j\}_{C_0} = (\mu_G - \mu_F) \partial F_j \partial G_j + j \left( \partial \mu_G G_j \partial F_j - \partial \mu_F F_j \partial G_j \right).$$

On the other hand, the star product, at the first order, of these two $j$-differentials can be written as:

$$F_j \ast G_j = (F_j \ast G_j)^{(0)} + i \hbar \left( \{F_j, G_j\}_{C_0} \right) + O \left( \hbar^2 \right)$$

Then, by using (54) we get the general expression of the star product of any two $j$-differentials as follows:

$$F_j \ast G_j = F_j G_j + i \hbar \left[ [\mu_G - \mu_F] \partial F_j \partial G_j + j \left( \partial \mu_G G_j \partial F_j - \partial \mu_F F_j \partial G_j \right) \right].$$

Moreover, if we consider the star product of two $j$-differentials $F_j$ and $G_j$ as a $j$-differential:

$$\overline{\partial} (F_j \ast G_j) = \mu_{(F \ast G)} \partial (F_j \ast G_j) + j \partial \mu_{(F \ast G)} \left( F_j \ast G_j \right),$$

the corresponding Beltrami coefficient is the solution of the equation

$$W_j^* (F_j G_j) = i \hbar W_j^* \left( \{F_j, G_j\}_{C_0} \right)$$

where

$$W_j^* \equiv \overline{\partial} - \mu_{(F \ast G)} \partial - j \partial \mu_{(F \ast G)}$$

is the generalized Ward operator corresponding to the star product (56).
Conformal covariance and the star product

One can verify that the $\mu$-holomorphy equation is conformally covariant. Indeed, with respect to conformal transformation

\begin{align}
  z &\to z' = w(z), \\
  \bar{z} &\to \bar{z}' = \overline{w(z)},
\end{align}

we have the following transformation laws respectively for the Beltrami differential and a $j$-differential:

\begin{align}
  \mu^z_i &= (\overline{\partial w})^{-1} (\partial w) \mu^z_i, \\
  F^i_j &= (\partial w)^{-j} F^i_j.
\end{align}

Then we get the following transformation law:

\begin{equation}
  W^i_j F^i_j = (\overline{\partial w})^{-j} (\partial w)^{-j} W_j F_j
\end{equation}

and hence, the $\mu$-holomorphy equation is conformally covariant.

On the other hand, as one can verify from equation (56), the star product is not a $j$-differential and does not verify the $\mu$-holomorphy equation:

\begin{equation}
  W_j^* (F_j * G_j) \neq 0
\end{equation}

and then is not conformally covariant.

However, at the beginning we considered the associativity of the star product. Then, if the associativity of the star product is maintained, we lose its conformal covariance character. On the other hand, if we impose the star product to be a $j$-differential, one can verify that the associativity of the star product is lost:

\begin{equation}
  (F_j * G_j) * H_j \neq F_j * (G_j * H_j).
\end{equation}

**Examples of $j$-differentials**

Let $F_j$ and $G_j$ be $j$-differentials defining different complex structures (corresponding to different generalized Ward operator):

\begin{align}
  W_j^{\mu \sigma} F_j &= 0, \\
  W_j^{\mu \sigma} G_j &= 0
\end{align}

and are respectively given by:

\begin{align}
  F_j &= e^{ia_F z} e^{ib_F \tau}, \\
  G_j &= e^{ia_G z} e^{ib_G \tau}
\end{align}

with $(a_F, a_G, b_F, b_G \in \mathbb{C}, a_F, a_G \neq 0)$. 

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Then the corresponding Beltrami differentials satisfy the following differential equations

\begin{equation}
ib_{(F)G} = i a_{(F)G} \mu_{(F)G} + j \partial \mu_{(F)G}
\end{equation}

and are expressed as:

\begin{equation}
\mu_{(F)G} = \frac{b_{(F)G}}{a_{(F)G}} + C_{(F)G}(z) e^{-ia_{(F)G}z^2}
\end{equation}

such that the criterion of ellipticity restricts the Beltrami coefficients to satisfy

\begin{equation} \left| \mu_{(F)G} \right| < 1. \end{equation}

This generalizes the results of the scalar fields given in \([1]\).

It is easy to show that the Poisson Bracket of these two \(j\)-differentials is

\begin{equation}
\{ F, G \}_{C_0} = a_F a_G \left( \frac{b_F}{a_F} - \frac{b_G}{a_G} \right) FG.
\end{equation}

In particular we get, at the first order, their star product as follows:

\begin{align}
F \ast G &= (F \ast G)^{(0)} + i \hbar \left( \{ F, G \}_{C_0} \right) + O(\hbar^2), \\
F \ast G &= e^{-i\hbar(b_F a_G - b_G a_F)} FG.
\end{align}

This latter formula shows that, in this particular case, the effect of the deformation quantization is introduced as a phase factor to the product \(FG\).

Moreover, the star product of two \(j\)-differentials defining different complex structures on a Riemann surface \(\Sigma\), but expressed by (68) and (69), takes the similar functional structure with these \(j\)-differentials. Then, in this example, the star product is also a \(j\)-differential.

If we consider the star product as a \(j\)-differential, that is

\begin{equation} \mathcal{J}(F \ast G) = \mu_{(F \ast G)} \partial (F \ast G) + j \partial \mu_{(F \ast G)} (F \ast G), \end{equation}

we get the following relation:

\begin{equation}
(\mu_F - \mu_{(F \ast G)}) a_F + (\mu_G - \mu_{(F \ast G)}) a_G = -ij \partial \left( \mu_{(F \ast G)} - \mu_F - \mu_G \right).
\end{equation}

which generalizes the expression given in \([1]\). Indeed, by putting \(j = 0\) in (77) we find the following equation

\begin{equation}
\mu_{(F \ast G)} = \frac{a_F \mu_F + a_G \mu_G}{a_F + a_G}.
\end{equation}

**Conclusion**

Here we have developed the Moyal-Weyl star product of any two \(j\)-differentials defining different complex structures and we have given its general expression.

We have shown that the star product reduces to a phase factor to the simple product in a particular case.
We stress that the star product is not a $j$-differential and does not preserve the conformal covariance. If we release the associativity, the star product can become a conformal field of weights $(j, 0)$. Inversely, if we release the conformal covariance, the associativity is maintained.

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References