EXOTIC CIRCLES OF A REMARKABLE GROUP OF PIECEWISE GENERALIZED (NON LINEAR) CIRCLE HOMEOMORPHISMS

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Abstract

Let $G$ be a subgroup of $\text{Homeo}_+(S^1)$. An exotic circle of $G$ is a subgroup of $G$ which is conjugate to $SO(2)$ in $\text{Homeo}_+(S^1)$ but not conjugate to $SO(2)$ in $G$. The existence of exotic circles shows that the subgroup $G$ is far from being a Lie group. Let $r \geq 1$ be an integer, $r = +\infty$ or $r = \omega$. In this paper, we prove that the subgroup $P^r(S^1)$ of $\text{Homeo}_+(S^1)$ consisting of piecewise class $P^r$ homeomorphisms of the circle has no exotic circles. However, we show that there exist exotic circles of a particular subgroup (denoted $P^1(S^1)$) of $P^r(S^1)$ and we determine the conjugacy classes of all exotic circles in $P^1(S^1)$. In particular, for the group $PL_+(S^1)$ consisting of piecewise linear homeomorphisms we give a simple proof of Minakawa’s Theorems in [7], [6].

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1. Introduction

Let \( \text{Homeo}_+(S^1) \) denote the group of orientation-preserving homeomorphisms of the circle and \( SO(2) \) denote the group of rotations of \( S^1 \). Let \( G \) be a subgroup of \( \text{Homeo}_+(S^1) \). A topological circle of \( G \) is a subgroup of \( G \) which is conjugate to \( SO(2) \) in \( \text{Homeo}_+(S^1) \). An exotic circle of \( G \) is a topological circle of \( G \) which is not conjugate to \( SO(2) \) in \( G \). The existence of exotic circles shows that the topological subgroup \( G \) is very far from being a Lie group (cf. [6], [8]). The following Corollary is a consequence of Theorem 4 of Montgomery and Zippin (cf. [8], Theorem 4, p. 212):

**Corollary 1.1.** (cf. [6]) For every integer \( r \geq 1 \), \( r = \infty \), or \( r = \omega \), \( \text{Diff}_+^1(S^1) \) has no exotic circles.

To consider general groups of piecewise circle homeomorphisms, we prove the following more precise result. Let \( \text{Diff}_+^{1+BV}(S^1) \) denote the group of \( C^1 \)-diffeomorphisms which derivative of bounded variation on \( S^1 \). Then:

**Corollary 1.2.** \( \text{Diff}_+^{1+BV}(S^1) \) has no exotic circle.

The proof uses the following classical result.

**Theorem 1.3.** ([10]) If \( g \) is a measurable function defined on the interval \( (0,1) \), and if, for every \( \tau \in (0,1) \), \( g(t+\tau) - g(t) \) is of bounded variation on the interval \( (0,1-\tau) \) then \( g \) is of bounded variation on \( (0,1) \).

**Proof of Corollary 1.2.** Let \( S = h \circ SO(2) \circ h^{-1} \) be a topological circle of \( \text{Diff}_+^{1+BV}(S^1) \) where \( h \in \text{Homeo}_+(S^1) \). We let \( f = h \circ R_\alpha \circ h^{-1} \), \( \alpha \in S^1 \). By Corollary 1.1, \( h \in \text{Diff}_+^1(S^1) \).

Hence, \( Dh > 0 \) and \( (Df \circ h) Dh = Dh \circ R_\alpha \). So, \( \log Dh \circ R_\alpha - \log Dh = \log Df \circ h \). We let \( g = \log Dh \). We identify \( f, g \), and \( h \) to their lifts on \([0,1]\). So, \( g \) is a measurable function on \([0,1]\) and satisfies \( g(x+\alpha) - g(x) = \log Df \circ h \). Since \( Df \) is of bounded variation on \([0,1]\), and \( h \in \text{Homeo}_+(S^1) \), by Theorem 1.3, \( g \) is of bounded variation on \([0,1]\). Therefore, \( Dh \) is of bounded variation and \( h \in \text{Diff}_+^{1+BV}(S^1) \).  

Let \( PL_+(S^1) \) denote the subgroup of \( \text{Homeo}_+(S^1) \) consisting of piecewise linear homeomorphisms. Minakawa [6],[7] showed that \( PL_+(S^1) \) has exotic circles and obtained the conjugacy classes of all exotic circles of \( PL_+(S^1) \):

**Minakawa’s Theorem** ([6],[7]). Let \( \sigma \in \mathbb{R}_+^* > 0 \), \( \sigma \neq 1 \) and denote by \( h_\sigma \) the homeomorphism of \( S^1 \) defined by

\[
h_\sigma(x) = \frac{\sigma^x - 1}{\sigma - 1}, \quad x \in [0,1[.
\]

Then the topological circles \( S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1} \) are exotic circles of \( PL_+(S^1) \) and every exotic circle of \( PL_+(S^1) \) is conjugate in \( PL_+(S^1) \) to one of the \( S_\sigma \).
In this paper, we consider the general case: piecewise class \( P C^r \) (\( r \geq 1, r = +\infty \) or \( r = \omega \)) homeomorphisms of the circle with break point singularities, that is maps \( f \) that are \( C^r \) except at some singular points in which the successive derivatives until the order \( r \) on the left and on the right exist. These piecewise classes \( P C^r \) homeomorphisms of the circle form a group noted \( \mathcal{P}^r(S^1) \) which contains \( PL_+(S^1) \) (cf. [1]). The aim of this paper is to show that \( \mathcal{P}^r(S^1) \) has no exotic circles, and that, there exist exotic circles of a subgroup (denoted \( \mathcal{P}^1_1(S^1) \)) of \( \mathcal{P}^r(S^1) \). Moreover, we determine the conjugacy classes of all exotic circles in \( \mathcal{P}^r_1(S^1) \). In the case of \( PL_+(S^1) \), we give a simple proof of the classification of all exotic circles of \( PL_+(S^1) \) up to PL conjugacy obtained by Minakawa in [7], [6].

2. CLASS \( P C^r \) HOMEOMORPHISMS OF THE CIRCLE

Denote by \( S^1 = \mathbb{R}/\mathbb{Z} \) the circle and \( p : \mathbb{R} \rightarrow S^1 \) the canonical projection. Let \( f \) be an orientation preserving homeomorphism of \( S^1 \). The homeomorphism \( f \) admits a lift \( \tilde{f} : \mathbb{R} \rightarrow \mathbb{R} \) that is an increasing homeomorphism of \( \mathbb{R} \) such that \( p \circ \tilde{f} = f \circ p \). Conversely, the projection of such a homeomorphism of \( \mathbb{R} \) is an orientation preserving homeomorphism of \( S^1 \). Let \( x \in S^1 \). We call orbit of \( x \) by \( f \) the subset \( O_f(x) = \{ f^n(x) : n \in \mathbb{Z} \} \).

Historically, the dynamic study of circle homeomorphisms was initiated by H. Poincaré ([9], 1886). He introduced the rotation number of a homeomorphism \( f \) of \( S^1 \) as \( \rho(f) = \lim_{n \rightarrow +\infty} \frac{f^n(x) - x}{n} (mod 1) \).

Poincaré showed that this limit exists and does not depends on \( x \) and the lift \( \tilde{f} \) of \( f \).

We say that \( f \) is semi-conjugate to the rotation \( R_{\rho(f)} \) if there exists an orientation preserving surjective continuous map \( h : S^1 \rightarrow S^1 \) of degree one such that \( h \circ f = R_{\rho(f)} \circ h \).

Poincaré’s theorem. Let \( f \) be a homeomorphism of \( S^1 \) with rotation number \( \rho(f) \) irrational. Then \( f \) is semi-conjugate to the rotation \( R_{\rho(f)} \).

A natural question is whether the semi-conjugation \( h \) could be improved to be a conjugation, that is \( h \) to be an homeomorphism. In this case, we say that \( f \) is topologically conjugate to the rotation \( R_{\rho(f)} \). In this direction, A. Denjoy ([2]) proved the following:

Denjoy’s theorem([2]). Every \( C^2 \)-diffeomorphism \( f \) of \( S^1 \) with irrational rotation number \( \rho(f) \) is topologically conjugate to the rotation \( R_{\rho(f)} \).

Other classes of circle homeomorphisms commonly referred to as the class \( P \) homeomorphisms are known to satisfy the conclusion of Denjoy’s theorem (see [4]; [3], chapter VI).

Definition 2.1. Let \( f \) be an orientation preserving homeomorphism of the circle. The homeomorphism \( f \) is called of class \( P \) if it is derivable except in finitely or countable points called break points of \( f \) in which \( f \) admits left and right derivatives (denoted, respectively, by \( Df_- \) and \( Df_+ \)) and such that the derivative \( Df : S^1 \rightarrow \mathbb{R}_+^* \) has the following properties:
- there exist two constants $0 < a < b < +\infty$ such that:
  \[ a < Df(x) < b, \text{ for every } x \text{ where } Df \text{ exists}, a < Df_+(c) < b, \text{ and } a < Df_-(c) < b \text{ at the break point } c. \]
- \( \log Df \) has bounded variation on \( S^1 \)

Denote by
- \( \sigma_f(c) := \frac{Df(c)}{Df_+(c)} \) called the \( f \)-jump in \( c \).
- \( C(f) \) the set of break points of \( f \).
- \( \pi_s(f) \) the product of \( f \)-jumps in the break points of \( f \): \( \pi_s(f) = \prod_{c \in C(f)} \sigma_f(c) \).
- \( V = \text{Var} \log Df \) the total variation of \( \log Df \). We have
  \[ V := \sum_{j=0}^p \text{Var}_{[c_j,c_{j+1}]} \log(Df) + |\log(\sigma_f(c_j))| < +\infty \]
where \( c_0, c_1, c_2, \ldots, c_p \) are the break points of \( f \) with \( c_{p+1} := c_0 \). In this case, \( V \) is the total variation of \( \log Df, \log Df_-, \log Df_+ \).

Among the simplest examples of class \( P \) homeomorphisms, there are:
- \( C^2 \)-diffeomorphisms,
- Piecewise linear (PL) homeomorphisms. An orientation preserving circle homeomorphism \( f \) is called \( \text{PL} \) if \( f \) is derivable except in many finitely break points \( (c_i)_{1 \leq i \leq p} \) of \( S^1 \) such that the derivative \( Df \) is constant on each \( [c_i, c_{i+1}] \).

**Definition 2.2.** We say that \( f \) has the \( (D) \)-property (cf. [5], [7]) if the product of \( f \)-jumps on each orbit is equal to 1 i.e. \( \pi_s(f)(c) = \prod_{x \in C(f) \cap \mathcal{O}_f(c)} \sigma_f(x) = 1 \).

In particular, if \( f \) has the \( (D) \)-property then \( \pi_s(f) = 1 \). Conversely, if \( \pi_s(f) = 1 \) and if all break points belong to a same orbit then \( f \) has the \( (D) \)-property.

If \( f \) is a (PL) homeomorphism, always we have \( \pi_s(f) = 1 \). Therefore, a PL homeomorphism \( f \) satisfies the \( (D) \)-property if all its break points are on the same orbit.

**Proposition 2.3.** Let \( f, g \) be two circle orientation preserving \( C^1 \)-homeomorphisms. Then \( \pi_s(g \circ f) = \pi_s(g)\pi_s(f) \).

**Proof.** Let \( c \in S^1 \). We have \( \sigma_{g \circ f}(c) = \sigma_g(f(c))\sigma_f(c) \). So,
\[
\pi_s(g \circ f) = \prod_{c \in C(g \circ f)} \sigma_{g \circ f}(c) = \prod_{c \in C(g \circ f)} \sigma_g(f(c))\sigma_f(c).
\]
Since \( C(g \circ f) \subset C(f) \cup f^{-1}(C(g)) \) and \( \sigma_{g \circ f}(c) = 1 \) for every \( c \in S^1 \setminus C(g \circ f) \),
\[
\pi_s(g \circ f) = \prod_{c \in C(f)} \sigma_g(f(c))\sigma_f(c) \prod_{c \in f^{-1}(C(g)) \setminus C(f)} \sigma_g(f(c))\sigma_f(c)
\]
\[
= \pi_s(f) \prod_{c \in C(f)} \sigma_g(f(c)) \prod_{c \in f^{-1}(C(g)) \setminus C(f)} \sigma_g(f(c))
\]
\[
= \pi_s(f) \prod_{c \in f^{-1}(C(g))} \sigma_g(f(c)) = \pi_s(f) \pi_s(g).
\]

\[\square\]

**Corollary 2.4. (Invariance of \( \pi_s \) by piecewise \( C^1 \)-conjugation).** Let \( f, g \) be two circle orientation preserving \( C^1 \)-homeomorphisms. If \( f \) and \( g \) are bi-piecewise \( C^1 \) conjugated then \( \pi_s(f) = \pi_s(g) \).

**Definition 2.5.** Let \( r \geq 1 \) be an integer, \( r = +\infty \), or \( r = \omega \). A class \( P \) circle homeomorphism is called of piecewise class \( P C_r \) if \( f \) is \( C^r \) except in a finitely many points called *singular points* and in which the successive derivatives of \( f \) until the order \( r \) on the left and on the right exist.

Denote by

- \( S(f) \) the set of singular points of \( f \).
- \( \mathcal{P}^r(S^1) \) the set of class \( P C^r \) circle homeomorphisms (\( r \geq 1 \) integer, \( r = +\infty \), or \( r = \omega \)).

One can check that \( \mathcal{P}^r(S^1) \) is a group.

Notice that if \( r = 1 \), \( S(f) = C(f) \).

The set \( S(f) \) of singular points is partitioned into finite subsets \( S_i(f) \) which are supported by disjoints orbits:

\[ S(f) = \prod_{i=1}^{p} S_i(f) \]

where \( S_i(f) = S(f) \cap O_f(c_i), c_i \in S(f) \) and \( O_f(c_i)_{1 \leq i \leq p} \) are on distinct orbits.

**Definition-Notation.** The set \( M_i(f) = \{x_i, f(x_i), ..., f^{N(f,x_i)}(x_i)\} \) is called the *envelope* of \( S_i(f) \) (\( 1 \leq i \leq p \)) where \( N(f,x_i) \in \mathbb{N}, x_i, f^{N(f,x_i)}(x_i) \in S(f) \) and

\[ S(f) \cap M_i(f) = S(f) \cap O_f(x_i) = S_i(f). \]

**Definition 2.6.** Let \( r \geq 1 \) be an integer. Let \( f \in \mathcal{P}^r(S^1) \). We say that \( f \) has the \((D_r)\)-property if \( f^{N(f,x_i)+1} \) is \( C^r \) on \( x_i \) for \( i = 1, ..., p \).

Notice that if \( N(f,x_i) = 0 \) for some \( i \) then \( x_i \) is the unique singular point in its orbit and the \((D_r)\)-property is not satisfied.
Remark 1. In the case \( r = 1 \), the \((D_r)\)-property is equivalent to the \((D)\)-property. For every \( i = 1, \ldots, p \),

\[
\prod_{d \in M_i(f)} \sigma_f(d) = 1 = \prod_{d \in S_i(f)} \sigma_f(d).
\]

Indeed, \( f^{N(f, x_i)} \) is \( C^1 \) on \( x_i, \) \( i = 1, \ldots, p \) means that

\[
\sigma_{f^{N(f, x_i)}}(x_i) = 1 = \prod_{c \in S_i(f)} \sigma_f(c) = \prod_{j=0}^{N(f, x_i)} \sigma_f(f^j(x_i)),
\]

in other words: \( f \) satisfies the \((D)\)-property.

Proposition 2.7. ([1], Corollary 2.8). Let \( f, g \in \mathcal{P}^r(S^1) \) \( (r \geq 1) \) be a real, \( r = +\infty \) or \( r = \omega \) with irrational rotation numbers that are rationally independent. If \( f \circ g = g \circ f \) then \( f \) and \( g \) have \((D_r)\)-property.

Theorem 2.8. ([1], Theorem 2.1) Let \( r \geq 1 \) be a real, \( r = +\infty \) or \( r = \omega \) and \( f \in \mathcal{P}^r(S^1) \) with irrational rotation number. Then the following properties are equivalent:

i) \( f \) is conjugated in \( \mathcal{P}^r(S^1) \) to a \( C^r \)-diffeomorphism,

ii) \( f \) has the \((D_r)\)-property,

iii) \( f \) is conjugated to a \( C^r \)-diffeomorphism by a piecewise polynomial homeomorphism \( K \in \mathcal{P}^r(S^1) \)

Proposition 2.9. ([1], Lemma 5.1) Let \( f \in \text{Diff}^+(S^1) \) with irrational rotation number and let \( g \in \mathcal{P}^r(S^1) \). If \( f \circ g = g \circ f \) then \( g \in \text{Diff}^+(S^1) \).

Our main results are the following:

Theorem 2.10. Let \( r \geq 1 \) be an integer, \( r = +\infty \) or \( r = \omega \). Then \( \mathcal{P}^r(S^1) \) has no exotic circles.

Let \( \mathcal{P}^r_1(S^1) \) denote the subgroup of \( \mathcal{P}^r(S^1) \) consisting of class \( P \) \( C^r \) circle homeomorphisms \( f \) with \( \pi_s(f) = 1 \). Then:

Theorem 2.11. Let \( r \geq 1 \) be an integer, \( r = +\infty \) or \( r = \omega \), \( \sigma \in \mathbb{R}_+^* \) > 0, \( \sigma \neq 1 \) and let \( h_\sigma \in \mathcal{P}^r(S^1) \) with one break point \( c \) such that \( \sigma_{h_\sigma}(c) = \sigma \). Then:

i) \( S_\sigma = h_\sigma \circ \text{SO}(2) \circ h_\sigma^{-1} \subset \mathcal{P}^r_1(S^1) \) is an exotic circle of \( \mathcal{P}^r_1(S^1) \).

ii) Two exotic circles \( S_1 = h_1 \circ \text{SO}(2) \circ h_1^{-1}, \ S_2 = h_2 \circ \text{SO}(2) \circ h_2^{-1} \) of \( \mathcal{P}^r_1(S^1) \) are conjugated in \( \mathcal{P}^r_1(S^1) \) if and only if \( \pi_s(h_1) = \pi_s(h_2) \).

iii) Every exotic circle of \( \mathcal{P}^r_1(S^1) \) is conjugate in \( \mathcal{P}^r_1(S^1) \) to one of the \( S_\sigma \).
3. No Exotic Circle of $\mathcal{P}^r(S^1)$

Lemma 3.1. Let $S = h \circ SO(2) \circ h^{-1}$ be a topological circle of $\mathcal{P}^r(S^1)$, $h \in \text{Homeo}_+(S^1)$. Then every element of $S$ with irrational rotation number has the $(D_r)$-property.

Proof. Let $f \in S$ with irrational rotation number $\alpha \in S^1$, that is $f = h \circ R_\alpha \circ h^{-1} \in \mathcal{P}^r(S^1)$. Let $g = h \circ R_\beta \circ h^{-1}$ with $\beta$ irrational such that $\alpha, \beta$ are rationally independent. Since $f \circ g = g \circ f$, by Proposition 2.7, $f$ and $g$ have the $(D_r)$-property. \qed

Proof of Theorem 2.10.

Let $S = h \circ SO(2) \circ h^{-1} \subset \mathcal{P}^r(S^1)$ where $h \in \text{Homeo}_+(S^1)$. Take $f \in S$ with irrational rotation number $\beta \in S^1$. By Lemma 3.1, $f$ has the $(D_r)$-property. Hence, by Theorem 2.8, there exists a polynomials homeomorphism $K \in \mathcal{P}^r(S^1)$ such that $F = K \circ f \circ K^{-1} \in \text{Diff}^r_+(S^1)$. Now for every $g = h \circ R_\alpha \circ h^{-1} \in S$, $G = K \circ g \circ K^{-1} = (K \circ h) \circ R_\alpha \circ (K \circ h)^{-1} \in \mathcal{P}^r(S^1)$.

Since $G \circ F = F \circ G$, by Proposition 2.9, $G \in \text{Diff}^r_+(S^1)$. It follows by Corollary 1.1, that $K \circ h = u \in \text{Diff}^r_+(S^1)$. Hence, if $r \geq 2$, $r = +\infty$ or $r = \omega$, $h = K^{-1} \circ u \in \mathcal{P}^r(S^1)$.

If $r = 1$ then $G \in \mathcal{P}^1(S^1) \cap \text{Diff}^1_+(S^1)$, so, $G \in \text{Diff}^1_{BV}(S^1)$. By Corollary 1.2, we have $K \circ h = u \in \text{Diff}^1_{BV}(S^1)$. Hence $h = K^{-1} \circ u \in \mathcal{P}^1(S^1)$. This completes the proof. \qed

4. Existence of Exotic Circles of $\mathcal{P}^r_1(S^1)$

In this section, $r \geq 1$ is an integer, $r = +\infty$ or $r = \omega$. Let us consider the set

$$\mathcal{P}^r_1(S^1) = \{f \in \mathcal{P}^r(S^1) : \pi_s(f) = 1\}.$$ 

Lemma 4.1. $\mathcal{P}^r_1(S^1)$ is a subgroup of $\mathcal{P}^r(S^1)$.

Proof. Let us consider the map $\pi_s : \mathcal{P}^r(S^1) \longrightarrow \mathbb{R}^*; f \longmapsto \pi_s(f)$. Since $\pi_s(g \circ f) = \pi_s(g)\pi_s(f)$ by Proposition 2.3, $\pi_s$ is a group’s homomorphism. Its kernel $\text{Ker} \pi_s = \mathcal{P}^r_1(S^1)$ is then a subgroup of $\mathcal{P}^r(S^1)$. \qed

Lemma 4.2. Let $\sigma \in \mathbb{R}^*_+ \setminus \{1\}$ and $h_\sigma \in \mathcal{P}^r(S^1)$ with one break point $c$ such that $\sigma_{h_\sigma}(c) = \sigma$. Then $S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1}$ is an exotic circle of $\mathcal{P}^r_1(S^1)$.

Proof. Letting $f = h_\sigma \circ R_\alpha \circ h_\sigma^{-1} \in S_\sigma$. Then $f \in \mathcal{P}^r(S^1)$ and has exactly two break points $c_1$ and $c_2 = f(c_1)$ and the product of $f$-jumps: $\pi_s(f) = \sigma f(c_1)\sigma f(c_2) = 1$, hence $f \in \mathcal{P}^r_1(S^1)$. Therefore, $S_\sigma \subset \mathcal{P}^r_1(S^1)$. Since $\pi_s(h_\sigma) = \sigma_{h_\sigma}(c) = \sigma \neq 1$, $h_\sigma \notin \mathcal{P}^r_1(S^1)$. This completes the proof. \qed

Proof of Theorem 2.11. Assertion i) follows from Lemma 4.2.

Assertion ii): Let $S_1 = h_1 \circ SO(2) \circ h_1^{-1}$ and $S_2 = h_2 \circ SO(2) \circ h_2^{-1}$ be two exotic circles of $\mathcal{P}^r_1(S^1)$. 7
Suppose that $\pi_s(h_1) = \pi_s(h_2)$. Then $S_2 = L \circ S_1 \circ L^{-1}$ where $L = h_2 \circ h^{-1}_1$. Since $S_1$ and $S_1$ are topological circles of $P^r(S^1)$, so, by Theorem 2.10, $L \in P^r(S^1)$. Moreover, by Proposition 2.3,

$$\pi_s(L) = \frac{\pi_s(h_1)}{\pi_s(h_2)} = 1.$$  

Hence, $L \in P^r_1(S^1)$ and then $S_1$ and $S_2$ are conjugated in $P^r_1(S^1)$.

Conversely, suppose that $S_1$ and $S_2$ are conjugated in $P^r_1(S^1)$, that is $S_2 = L \circ S_1 \circ L^{-1}$ where $L \in P^r_1(S^1)$. Let $\alpha \in S^1$ be irrational. We have

$$L \circ h_1 \circ R_\alpha \circ h^{-1}_1 \circ L^{-1} = h_2 \circ R_\alpha \circ h^{-1}_2,$$

hence,

$$h^{-1}_1 \circ L^{-1} \circ h_2 \circ R_\alpha = R_\alpha \circ h^{-1}_1 \circ L^{-1} \circ h_2.$$  

Since $\alpha$ is irrational, $h^{-1}_1 \circ L^{-1} \circ h_2$ must belong to $SO(2)$, so $h^{-1}_1 \circ L^{-1} \circ h_2 = R_\beta$ for some $\beta \in S^1$. Thus, we have

$$L = h_2 \circ R_\beta \circ h^{-1}_1 = T \circ h_2 \circ h^{-1}_1$$

where $T = h_2 \circ R_\beta \circ h^{-1}_2 \in S_2$. Since $L, T \in P^r_1(S^1)$, $h_2 \circ h^{-1}_1 \in P^r_1(S^1)$, so $\pi_s(h_2 \circ h^{-1}_1) = 1$, that is $\pi_s(h_1) = \pi_s(h_2)$.

Assertion iii): Let $S = h \circ SO(2) \circ h^{-1}$ be an exotic circle of $P^r_1(S^1)$. By Theorem 2.10, $h \in P^r(S^1)$ but $h \notin P^r_1(S^1)$. Hence $\pi_s(h) = \sigma \neq 1$. Since $\pi_s(h_\sigma) = \sigma h_\sigma(c) = \sigma = \pi_s(h)$, $S$ is conjugated in $P^r_1(S^1)$ to $S_\sigma$ by Assertion ii). This completes the proof. $\square$

5. The PL case

In this section, we consider the group $PL_+(S^1)$ and we give a new proof of Minakawa classification of all exotic circles of $PL(S^1)$.

**Lemma 5.1.** Let $h \in Homeo_+(S^1)$. Then $S = h \circ SO(2) \circ h^{-1}$ is an exotic circle of $PL_+(S^1)$ if and only if there exists $\lambda \in \mathbb{R}^*$ and a subdivision $c_0, c_1, ..., c_{p-1}$ of $S^1$ such that

$$h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i, \ x \in [c_{i-1}, c_i]$$

where $\alpha_i \in \mathbb{R}_+^*$, $\beta_i \in \mathbb{R}$ are constants.

**Proof.** Suppose that $S$ is an exotic circle of $PL_+(S^1)$. Since $PL_+(S^1) \subset P^\infty(S^1)$ then by Theorem 2.10, $h \in P^\infty(S^1)$. We let $f = h \circ R_\alpha \circ h^{-1}$ with $\alpha \in S^1$ irrational. The set $h^{-1}(S(f)) \cap R_\alpha^{-1}(S(f)) \cap S(h)$ is finite and partitioned $S^1$ into segments $[c_{i-1}, c_i]$, $1 \leq i \leq p$ ($c_p = c_0$). So, $f(h(x)) = k_i$, for every $x \in [c_{i-1}, c_i]$. Differentiating the relation $f \circ h = h \circ R_\alpha$, we obtain successively $k_i Dh(x) = Dh(R_\alpha(x))$ and $k_i D^2h(x) = D^2h(R_\alpha(x))$ for every $x \in [c_{i-1}, c_i]$. Hence

$$\frac{D^2h(x)}{Dh(x)} = \frac{D^2h(R_\alpha(x))}{Dh(R_\alpha(x))}.$$
Letting 
\[ \varphi(x) = \begin{cases} \frac{D^2 h(x)}{Dh(x)} & \text{if } x \in S^1 \backslash \{c_0, ..., c_{p-1}\} \\ \frac{D^2 h_i(c_i)}{Dh_i(c_i)} & \text{if } x = c_i \end{cases} \]
then we have \( \varphi \circ R_\alpha = \varphi \) on \( S^1 \). Since \( \varphi \in L^2(S^1) \) and \( R_\alpha \) is ergodic with respect to the Haar measure \( m \) (\( \alpha \) is irrational), \( \varphi \) is constant \( m \text{-a.e.} \); that is there exists a subset \( E \) in \( S^1 \) with \( m(E) = 0 \) such that \( \varphi(x) = \lambda \) for every \( x \in S^1 \backslash E \). Since \( h \notin PL_+(S^1) \), \( \lambda \neq 0 \). We have 
\[ \frac{D^2 h(x)}{Dh(x)} = \lambda \] for every \( x \in ]c_{i-1}, c_i[ \backslash E \). Since \( \frac{D^2 h}{Dh} \) is continuous on \( ]c_{i-1}, c_i[ \) and \( ]c_{i-1}, c_i[ \backslash E \) is dense in \( ]c_{i-1}, c_i[ \), \( \frac{D^2 h}{Dh} = \lambda \) on \( ]c_{i-1}, c_i[ \) for every \( i \). The resolution of the differential equation 
\[ D^2 h(x) = \lambda Dh(x), \ x \in ]c_{i-1}, c_i[ \] implies that there exist two constants \( \alpha_i \in \mathbb{R}^+, \beta_i \in \mathbb{R} \) such that
\[ h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i, \ x \in ]c_{i-1}, c_i[. \]
Conversely, we let \( h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i, \ x \in ]c_{i-1}, c_i[ \) where \( \alpha_i \in \mathbb{R}^+, \beta_i \in \mathbb{R} \) are constants. Then for every \( \delta \in S^1, \ x \in ]c_{i-1}, c_i[ \), we have
\[ h \circ R_\delta \circ h^{-1}(x) = h \circ R_\delta\left( \frac{1}{\lambda} \log\left( \frac{\lambda}{\alpha_i}(x - \beta_i) \right) \right) \]
\[ = h\left( \frac{1}{\lambda} \log\left( \frac{\lambda}{\alpha_i}(x - \beta_i) \right) + \delta \right) = \frac{\alpha_i}{\lambda} e^{\lambda(x - \beta_i)} + \beta_j. \]
Therefore, \( S \subset PL_+(S^1) \) and since \( h \notin PL_+(S^1), S \) is an exotic circle of \( PL_+(S^1) \). This completes the proof. \( \square \)

**Remark 2.** Let \( h_\sigma \in \text{Homeo}_+(S^1) \) as in Lemma 5.1 with one break point \( 0 \) such that \( h_\sigma(0) = 0 \) and \( \sigma_{h_\sigma}(0) = \sigma \). Then
\[ h_\sigma(x) = \frac{\sigma x - 1}{\sigma - 1}, \ x \in [0, 1[. \]
Indeed, by Lemma 5.1,
\[ h_\sigma(x) = \frac{\alpha}{\lambda} e^{\lambda x} + \beta, \ x \in [0, 1[. \]
Since \( h_\sigma(0) = 0 \) and \( h_\sigma(1) = 1 \), we have \( \beta = \frac{1}{1 - e^\lambda} \) and \( \alpha = \frac{-\lambda}{1 - e^\lambda} \). Hence,
\[ h_\sigma(x) = \frac{-1}{1 - e^\lambda} e^{\lambda x} + \frac{1}{1 - e^\lambda} = \frac{e^{\lambda x} - 1}{e^\lambda - 1}. \]
Or
\[ \sigma_{h_\sigma}(0) = \frac{D(h_\sigma)_-(0)}{D(h_\sigma)_+(0)} = \frac{D(h_\sigma)_-(1)}{D(h_\sigma)_+(0)} = e^\lambda, \]
hence \( e^\lambda = \sigma \) and \( h_\sigma(x) = \frac{\sigma x - 1}{\sigma - 1} \).

**Proof of Minakawa’s Theorem.** Under the hypothesis of Minakawa’s Theorem, \( S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1} \) is an exotic circle of \( PL_+(S^1) \) by Remark 2.
Now let \( S = h \circ SO(2) \circ h^{-1} \) be an exotic circle of \( PL_+(S^1) \). By Theorem 2.10, \( h \in \mathcal{P}^\infty(S^1) \). Letting \( \pi_s(h) = \sigma \), we have \( S = L \circ S_\sigma \circ L^{-1} \) where \( L = h \circ h_\sigma^{-1} \) and \( \pi_s(L) = 1 \). Let’s show that \( L \in PL_+(S^1) \):
By Lemma 5.1, there exists $\lambda \in \mathbb{R}^*$ and a subdivision $c_0, c_1, ..., c_{p−1}$ of $S^1$ such that $h(x) = \frac{1}{\log \sigma} e^{\lambda x} + \beta_i, \ x \in [c_{i−1}, c_i]$ where $\alpha_i \in \mathbb{R}^*_+, \beta_i \in \mathbb{R}$ are constants. One can suppose that $c_0 = 0$ by replacing $h$ with $h \circ R_{c_0}$ since

$$S = h \circ SO(2) \circ h^{-1} = h \circ R_{c_0} \circ SO(2) \circ R_{c_0}^{-1} \circ h^{-1}.$$ 

For $i = 1, ..., p−1$, we have

$$\sigma_h(c_i) = \frac{Dh_-(c_i)}{Dh_+(c_i)} = \frac{\alpha_i e^{\lambda c_i}}{\alpha_{i+1} e^{\lambda c_i}} = \frac{\alpha_i}{\alpha_{i+1}},$$

and

$$\sigma_h(0) = \frac{D_-h(0)}{D_+h(0)} = \frac{D_-h(1)}{D_+h(0)} = \frac{\alpha_p e^{\lambda}}{\alpha_1}.$$ 

Hence,

$$\pi_s(h) = \sigma_h(0) \prod_{1 \leq i \leq p−1} \sigma_h(c_i)$$

$$= \frac{\alpha_p e^{\lambda}}{\alpha_1} \prod_{1 \leq i \leq p−1} \frac{\alpha_i}{\alpha_{i+1}} = \frac{\alpha_p e^{\lambda}}{\alpha_1} = e^{\lambda}$$

So, $\pi_s(h) = e^{\lambda} = \sigma$. Since $\lambda \neq 0$, $\sigma \neq 1$.

It follows that $h(x) = \frac{\alpha_i}{\log \sigma} x + \beta_i, \ x \in [c_{i−1}, c_i]$. On the other hand, we have $h_\sigma^{-1}(x) = \frac{1}{\log \sigma} \log((\sigma − 1)x + 1)$. We compute

$$h \circ h_\sigma^{-1}(x) = \frac{\alpha_i}{\log \sigma} ((\sigma − 1)x + 1) + \beta_i.$$ 

Moreover, $\frac{\alpha_i}{\log \sigma} (\sigma − 1) > 0$, hence $L \in PL_+(S^1)$. This completes the proof. $\square$

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**References**