CROSS THEOREMS WITH SINGULARITIES

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Abstract

We establish extension theorems for separately holomorphic mappings defined on all sets of the form $W \setminus M$, with values in a complex analytic space which possesses the Hartogs extension property. Here $W$ is a 2-fold cross of arbitrary complex manifolds and $M$ is a set of singularities which is locally pluripolar (resp. thin) in fibers.

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1 Introduction

Let $D \subset X$ (resp. $G \subset Y$) be an open set, $A \subset \overline{D}$ (resp. $B \subset \overline{G}$), where $X$ and $Y$ are complex manifolds \(^1\), and let $M \subset (A \times (G \cup B)) \bigcup ((D \cup A) \times B)$. The set $M_a := \{ w \in G : (a, w) \in M \}$, $a \in A$, is called the vertical fiber of $M$ over $a$ (resp. the set $M^b := \{ z \in D : (z, b) \in M \}$, $b \in B$, is called the horizontal fiber of $M$ over $b$). We say that $M$ possesses a certain property in fibers over $A$ (resp. $B$) if all vertical fibers $M_a$, $a \in A$, (resp. all horizontal fibers $M^b$, $b \in B$) possess this property.

The main purpose of this work is to investigate the following PROBLEM:

Let $X$, $Y$, $A$, $B$ be as above, and let $Z$ be a complex analytic space \(^2\). Define the cross

$$W := ((D \cup A) \times B) \bigcup (A \times (G \cup B)).$$

We want to determine an “optimal” open subset of $X \times Y$, denoted by $\widehat{W}$, which is characterized by the following property:

Let $M \subset W$ be a subset which is relatively closed and locally pluripolar (resp. thin) \(^3\) in fibers over $A$ and $B$ ($M = \emptyset$ is allowed). Then there exists a new set of singularities $\widehat{M} \subset \widehat{W}$, which is, in some sense, of the same structure as $M$ and which is characterized by the following property:

For all mappings $f : W \setminus M \longrightarrow Z$ that satisfy, in essence, the following condition:

$$f(a, \cdot) \in \mathcal{C}((G \cup B) \setminus M_a, Z) \cap \mathcal{O}(G \setminus M_a, Z), \quad a \in A,$$

$$f(\cdot, b) \in \mathcal{C}((D \cup A) \setminus M^b, Z) \cap \mathcal{O}(D \setminus M^b, Z), \quad b \in B,$$

there exists an $\hat{f} \in \mathcal{O}(\widehat{W} \setminus \widehat{M}, Z)$ such that for all $(\zeta, \eta) \in W \setminus M$, $\hat{f}(z, w)$ tends to $f(\zeta, \eta)$ as $(z, w) \in \widehat{W} \setminus \widehat{M}$ tends, in some sense, to $(\zeta, \eta)$.

We briefly recall the very recent developments \(^5\) around this PROBLEM.

The case when $M = \emptyset$ has been thoroughly investigated in the work \([16, 17]\) of the first author. These articles also show that the natural “target spaces” $Z$ for obtaining a satisfactory answer to the above PROBLEM are the ones which possess the Hartogs extension property \(^6\).

The case where $X$ and $Y$ are Riemann domains (over $\mathbb{C}^n$), $A \subset D$, $B \subset G$, and $Z = \mathbb{C}$ has been completed in some joint-articles of Jarnicki and the second author (see \([8, 9, 10, 12]\)).

Therefore, it is reasonable to conjecture that a positive solution to the PROBLEM may exist when the “target space” $Z$ possesses the Hartogs extension property. As our first attempt towards

\(^1\)In this paper complex manifolds are always assumed to be of finite dimension and countable at infinity.

\(^2\)All complex analytic spaces are assumed to be reduced, irreducible, of finite dimension, and countable at infinity.

\(^3\)The notion of local pluripolarity and thinness will be recalled in Section 2 below.

\(^4\)\(\mathcal{C}(D', Z)\) (resp. \(\mathcal{O}(D', Z)\)) denotes the set of all continuous (resp. holomorphic) mappings from a topological space (resp. a complex manifold) $D'$ to $Z$.

\(^5\)For a more detailed history see \([19]\).

\(^6\)This notion will be formulated in Subsection 2.3 below.
an affirmative answer to the PROBLEM, we solve in [20] the following special case: \( X = Y = \mathbb{C} \), \( D \) and \( G \) are copies of the open unit disc in \( \mathbb{C} \), \( A \) (resp. \( B \)) is a measurable subset of \( \partial D \) (resp. \( \partial G \)) of positive one-dimensional Lebesgue measure, \( Z = \mathbb{C} \), and \( M \) is polar (resp. discrete) in fibers over \( A \) and \( B \).

The main purpose of this article is to complete the above conjecture in its full generality. Our proof is geometric in nature. Indeed, our method consists in using holomorphic discs, and it is based on the works in [9, 20, 16, 17]. Moreover, the novelty of this new approach is that it does not use the classical method of doubly orthogonal bases of Bergman type. It is worthy to note here that most of previous works in the subject of separate holomorphy make use of the latter method.

2 Preliminaries and statement of the main result

First we recall some notions developed in [17] such as system of approach regions for an open set in a complex manifold, and the corresponding plurisubharmonic measure. These will provide the framework for an exact formulation of the PROBLEM and for our solution.

2.1 Approach regions, local pluripolarity and plurisubharmonic measure

**Definition 2.1.** Let \( X \) be a complex manifold and let \( D \subset X \) be an open subset. A system of approach regions for \( D \) is a collection \( \mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_\zeta} \) (\( I_\zeta = \emptyset \) for some \( \zeta \in \partial D \) is allowed) of open subsets of \( D \) with the following properties:

(i) For all \( \zeta \in D \), the system \( (\mathcal{A}_\alpha(\zeta))_{\alpha \in I_\zeta} \) forms a basis of open neighborhoods of \( \zeta \) (i.e., for any open neighborhood \( U \) of a point \( \zeta \in D \), there is \( \alpha \in I_\zeta \) such that \( \zeta \in \mathcal{A}_\alpha(\zeta) \subset U \)).

(ii) For all \( \zeta \in \partial D \) and \( \alpha \in I_\zeta \), \( \zeta \in \mathcal{A}_\alpha(\zeta) \).

\( \mathcal{A}_\alpha(\zeta) \) is often called an approach region at \( \zeta \).

In what follows we fix an open subset \( D \subset X \) and a system of approach regions \( \mathcal{A} = (\mathcal{A}_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_\zeta} \) for \( D \).

For every function \( u : D \rightarrow [-\infty, \infty) \), let

\[
\mathcal{A}-\limsup u(z) := \begin{cases} 
\sup_{\alpha \in I_z, \mathcal{A}_\alpha(z) \ni w \rightarrow z} \limsup_{w \rightarrow z} u(w), & \text{if } z \in D \text{ or } z \in \partial D \text{ with } I_z \neq \emptyset, \\
\limsup_{D \ni w \rightarrow z} u(w), & \text{if } z \in \partial D \text{ with } I_z = \emptyset.
\end{cases}
\]

In other words,

\[
\mathcal{A}-\limsup u(z) := \begin{cases} 
\limsup_{D \ni w \rightarrow z} u(w), & \text{if } z \in D \text{ or } z \in \partial D \text{ with } I_z = \emptyset, \\
\sup_{\alpha \in I_z, \mathcal{A}_\alpha(z) \ni w \rightarrow z} \limsup_{w \rightarrow z} u(w), & \text{if } z \in \partial D \text{ with } I_z \neq \emptyset.
\end{cases}
\]
By Definition 2.1 (i), \((A-\limsup)\) coincides with the usual upper semicontinuous regularization of \(u\).

For a set \(A \subset \overline{D}\) put

\[
h_{A,D} := \sup \{ u : u \in PSH(D), \ u \leq 1 \text{ on } D, \ A-\limsup u \leq 0 \text{ on } A \},
\]

where \(PSH(D)\) denotes the cone of all functions plurisubharmonic on \(D\).

A set \(A \subset D\) is said to be pluripolar in \(D\) if there is \(u \in PSH(D)\) such that \(u\) is not identically \(-\infty\) on every connected component of \(D\) and \(A \subset \{ z \in D : u(z) = -\infty \}\). A set \(A \subset D\) is said to be nonpluripolar (resp. non locally pluripolar) if it is not pluripolar (resp. not locally pluripolar). According to a classical result of Josefson and Bedford (see [14], [3]), if \(D\) is a Riemann domain over a Stein manifold, then \(A \subset D\) is locally pluripolar if and only if it is pluripolar.

**Definition 2.2.** The relative extremal function of \(A\) relative to \(D\) is the function \(\omega(\cdot, A, D)\) defined by

\[
\omega(z, A, D) = \omega_A(z, A, D) := (A-\limsup h_{A,D})(z), \quad z \in \overline{D}.
\]

Note that when \(A \subset D\), Definition 2.2 coincides with the classical definition of Siciak’s relative extremal function for \(z \in D\).

Next, we say that a set \(A \subset \overline{D}\) is locally pluriregular at a point \(a \in \overline{A}\) if \(\omega(a, A \cap U, D \cap U) = 0\) for all open neighborhoods \(U\) of \(a\), where the system of approach regions for \(D \cap U\) is given by \(\mathcal{A}(D \cap U) := (A_a(z) \cap U)_{z \in \overline{D \cap U}, \ a \in I_z}\). Moreover, \(A\) is said to be locally pluriregular if it is locally pluriregular at all points \(a \in A\). It should be noted from Definition 2.1 that if \(a \in \overline{A} \cap D\), then the property of local pluriregularity of \(A\) at \(a\) does not depend on the system of approach regions \(\mathcal{A}\), while the situation is different when \(a \in \overline{A} \cap \partial D\): then the property does depend on \(\mathcal{A}\).

We denote by \(A^*\) the following set

\[
(A \cap \partial D) \cup \{ a \in \overline{A} \cap D : \ A \text{ is locally pluriregular at } a \}.
\]

If \(A \subset D\) is non locally pluripolar, then a classical result of Bedford and Taylor (see [3, 4]) says that \(A^*\) is locally pluriregular and \(A \setminus A^*\) is locally pluripolar. Moreover, when \(A \subset D\), \(A^*\) is locally of type \(G_\delta\), that is, for every \(a \in A^*\) there is an open neighborhood \(U \subset D\) of \(a\) such that \(A^* \cap U\) is a countable intersection of open sets.

Now we are in the position to introduce the following version of a plurisubharmonic measure.

**Definition 2.3.** For a set \(A \subset \overline{D}\), let \(\tilde{A} = \tilde{A}(A) := \bigcup_{P \in \mathcal{E}(A)} P\), where

\[
\mathcal{E}(A) = \mathcal{E}(A, A) := \{ P \subset \overline{D} : P \text{ is locally pluriregular}, \overline{P} \subset A^* \},
\]

\(\overline{\mathcal{E}(A)}\) relates to the classical definition of plurisubharmonic measure.\(^7\)
The plurisubharmonic measure of \( A \) relative to \( D \) is the function \( \tilde{\omega}(\cdot, A, D) \) defined by
\[
\tilde{\omega}(z, A, D) := \omega(z, \tilde{A}, D), \quad z \in \overline{D}.
\]

It is worthy to remark that \( \tilde{\omega}(\cdot, A, D) \mid_D \in \mathcal{PSH}(D) \) and \( 0 \leq \tilde{\omega}(z, A, D) \leq 1 \), \( z \in D \). Moreover,
\[
\tilde{\omega}(z, A, D) = 0, \quad z \in \tilde{A}. \tag{1}
\]

An example in [1] shows that in general, \( \omega(\cdot, A, D) \neq \tilde{\omega}(\cdot, A, D) \) on \( D \).

Now we compare the plurisubharmonic measure \( \tilde{\omega}(\cdot, A, D) \) with Siciak’s relative extremal function \( \omega(\cdot, A, D) \). For the moment, we only focus on the case where \( A \subset D \).

If \( A \) is an open subset of an arbitrary complex manifold \( D \), then it can be shown that
\[
\tilde{\omega}(z, A, D) = \omega(z, A, D), \quad z \in D.
\]

If \( A \) is a (not necessarily open) subset of an arbitrary complex manifold \( D \), then we have, by Proposition 7.1 in [17],
\[
\tilde{\omega}(z, A, D) = \omega(z, A^*, D), \quad z \in D.
\]

On the other hand, if, moreover, \( D \) is a bounded open subset of \( \mathbb{C}^n \) then we have (see, for example, Lemma 3.5.3 in [7]) \( \omega(z, A, D) = \omega(z, A^*, D), z \in D \). Consequently, under the last assumption,
\[
\tilde{\omega}(z, A, D) = \omega(z, A, D), \quad z \in D.
\]

The case where \( A \subset \partial D \) has been investigated in [17, 18]. Our discussion shows that, at least in the case where \( A \subset D \), the notion of the plurisubharmonic measure is a good candidate for generalizing Siciak’s relative extremal function to the manifold context in the theory of separate holomorphy.

For a good background of the pluripotential theory, see the books [7] or [15].

### 2.2 Cross and separate holomorphicity and \( \mathcal{A} \)-limit

Let \( X, Y \) be two complex manifolds, let \( D \subset X \), \( G \subset Y \) be two nonempty open sets, let \( A \subset \overline{D} \) and \( B \subset \overline{G} \). Moreover, \( D \) (resp. \( G \)) is equipped with a system of approach regions \( \mathcal{A}(D) = (A_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_\zeta} \) (resp. \( \mathcal{A}(G) = (A_\alpha(\eta))_{\eta \in \overline{G}, \alpha \in I_\eta} \)). We define a 2-fold cross \( W \), its interior \( W^\circ \) and its regular part \( \tilde{W} \) (with respect to \( \mathcal{A}(D) \) and \( \mathcal{A}(G) \)) as
\[
W = \mathbb{X}(A, B; D, G) := (D \cup A) \times (B \cup G),
\]
\[
W^\circ = \mathbb{X}^\circ(A, B; D, G) := (A \times G) \cup (D \times B),
\]
\[
\tilde{W} = \tilde{\mathbb{X}}(A, B; D, G) := ((D \cup \tilde{A}) \times (B \cup \tilde{B})).
\]

Moreover, put
\[
\omega(z, w) := \omega(z, A, D) + \omega(w, B, G), \quad (z, w) \in D \times G,
\]
\[
\tilde{\omega}(z, w) := \tilde{\omega}(z, A, D) + \tilde{\omega}(w, B, G), \quad (z, w) \in D \times G.
\]
For a 2-fold cross \( W := \mathcal{X}(A, B; D, G) \) let

\[
\widetilde{W} := \widetilde{\mathcal{X}}(A, B; D, G) = \{(z, w) \in D \times G : \omega(z, w) < 1\}.
\]

Therefore, we obtain

\[
\widetilde{W} = \widetilde{\mathcal{X}}(\tilde{A}, \tilde{B}; D, G) = \{(z, w) \in D \times G : \tilde{\omega}(z, w) < 1\}.
\]

Let \( Z \) be a complex analytic space and \( M \subset W \) a subset which is relatively closed in fibers over \( A \) and \( B \). We say that a mapping \( f : W^\omega \setminus M \to Z \) is separately holomorphic and write \( f \in \mathcal{O}_s(W^\omega \setminus M, Z) \), if, for all \( a \in A \) (resp. \( b \in B \)) the mapping \( f(a, \cdot)|_{G \setminus M_a} \) (resp. \( f(\cdot, b)|_{D \setminus M^b} \)) is holomorphic.

We say that a mapping \( f : W \setminus M \to Z \) is separately continuous and write \( f \in \mathcal{C}_s(W \setminus M, Z) \) if, for all \( a \in A \) (resp. \( b \in B \)) the mapping \( f(a, \cdot)|_{(G \cup B) \setminus M_a} \) (resp. \( f(\cdot, b)|_{(D \cup A) \setminus M^b} \)) is continuous.

In virtue of (1), for every \((\zeta, \eta) \in \tilde{W}\) and every \( \alpha \in I_\zeta, \beta \in I_\eta \), there are open neighborhoods \( U \) of \( \zeta \) and \( V \) of \( \eta \) such that

\[
\left(U \cap A_\alpha(\zeta)\right) \times \left(V \cap A_\beta(\eta)\right) \subset \tilde{W}.
\]

Let \( S \) be a relatively closed subset of \( \tilde{W} \) and let \((\zeta, \eta) \in \tilde{W}\) be an \( A \)-accumulation point\(^8\) of \( \tilde{W} \setminus S \). Then a mapping \( f : \tilde{W} \setminus S \to Z \) is said to admit the \( A \)-limit \( \lambda \) at \((\zeta, \eta)\), and one writes

\[
(A - \lim f)(\zeta, \eta) = \lambda,
\]

if, for all \( \alpha \in I_\zeta, \beta \in I_\eta \),

\[
\lim_{\tilde{W} \setminus S \ni (z, w) \to (\zeta, \eta), \ z \in A_\alpha(\zeta), \ w \in A_\beta(\eta)} f(z, w) = \lambda.
\]

We conclude this introduction with a notion we need in the sequel. Let \( \mathcal{M} \) be a topological space. A mapping \( f : \mathcal{M} \to Z \) is said to be bounded if there exists an open neighborhood \( U \) of \( f(\mathcal{M}) \) in \( Z \) and a holomorphic embedding \( \phi \) of \( U \) into a bounded polydisc of \( \mathbb{C}^k \) such that \( \phi(U) \) is an analytic set in this polydisc. \( f \) is said to be locally bounded along \( \mathcal{N} \subset \mathcal{M} \) if for every point \( z \in \mathcal{N} \), there is an open neighborhood \( U \) of \( z \) (in \( \mathcal{M} \)) such that \( f|_U : U \to Z \) is bounded. \( f \) is said to be locally bounded if it is so for \( \mathcal{N} = \mathcal{M} \). It is clear that, if \( Z = \mathbb{C} \), then the above notions of boundedness coincide with the usual ones. A subset \( M \subset \mathcal{M} \) is said to be relatively closed along \( \mathcal{N} \subset \mathcal{M} \) in \( \mathcal{M} \) if for every point \( z \in \mathcal{N} \), there is an open neighborhood \( U \) of \( z \) (in \( \mathcal{M} \)) such that \( M \cap U \) is relatively closed in \( U \).

\(^8\)accumulation point with respect to \( A = A(D) \times A(G) \).

\(^9\)Note that here \( A = A(D) \times A(G) \).
2.3 Hartogs extension property

We recall here the following notion (see, for example, Shiffman [29] and a result by Ivashkovich [6]). For $0 < r < 1$, the Hartogs figure, denoted by $H(r)$, is given by

$$H(r) := \{(z_1, z_2) \in E^2 : |z_1| < r \text{ or } |z_2| > 1 - r\},$$

where, in this article, $E$ always denotes the open unit disc of $\mathbb{C}$.

**Definition 2.4.** A complex analytic space $Z$ is said to possess the Hartogs extension property if every mapping $f \in \mathcal{O}(H(r), Z)$ extends to a mapping $\hat{f} \in \mathcal{O}(E^2, Z), r \in (0, 1)$.

We mention an important characterization due to Shiffman (see [29]).

**Theorem 2.5.** A complex analytic space $Z$ possesses the Hartogs extension property if and only if for every subdomain $D$ of any Stein manifold $\mathcal{M}$, every mapping $f \in \mathcal{O}(D, Z)$ extends to a mapping $\hat{f} \in \mathcal{O}(\hat{D}, Z)$, where $\hat{D}$ is the envelope of holomorphy\(^{10}\) of $D$.

In the light of this result, the natural “target spaces” $Z$ for obtaining satisfactory answers to the PROBLEM are the complex analytic spaces which possess the Hartogs extension property.

2.4 Statement of the main result

Recall that a subset $S$ of a complex manifold $\mathcal{M}$ is said to be thin if for every point $x \in \mathcal{M}$ there is a connected neighborhood $U = U(x) \subset \mathcal{M}$ and a holomorphic function $f$ on $U$, not identically zero such that $U \cap S \subset f^{-1}(0)$. We are now ready to state the main result.

**Main Theorem.** Let $X$, $Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two connected open sets, let $A$ (resp. $B$) be a subset of $\overline{D}$ (resp. $\overline{G}$). $D$ (resp. $G$) is equipped with a system of approach regions $(A_\alpha(\zeta))_{\zeta \in D, \alpha \in I_\zeta}$ (resp. $(A_\beta(\eta))_{\eta \in G, \beta \in I_\eta}$). Suppose in addition that $\bar{A}$, $\bar{B} \neq \emptyset$. Let $Z$ be a complex analytic space possessing the Hartogs extension property. Let $M$ be a relatively closed subset of $W$ with the following properties:

- $M$ is thin in fibers (resp. locally pluripolar in fibers) over $A$ and over $B$;
- $M \cap ((A \cap \partial D) \times B) = M \cap (A \times (B \cap \partial G)) = \emptyset$.

Then there exists an analytic (resp. a relatively closed locally pluripolar) subset $\hat{M}$ of $\hat{W}$ such that $\hat{M} \cap \hat{W} \subset M$ and that:

for every mapping $f : W \setminus M \longrightarrow Z$ satisfying the following conditions:

(i) $f \in \mathcal{O}_s(W \setminus M, Z) \cap \mathcal{O}_s(W^o \setminus M, Z)$;

\(^{10}\)For the notion of the envelope of holomorphy, see, for example, [7].
(ii) $f$ is locally bounded along $X(A \cap \partial D, B \cap \partial G; D, G) \setminus M$; \(^{11}\)

(iii) $f|_{(A \times B) \setminus M}$ is continuous at all points of $(A \cap \partial D) \times (B \cap \partial G)$,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\hat{W} \setminus \hat{M}, Z)$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in (W \setminus \hat{W}) \setminus M$.

Although our main result has been stated only for the case of a 2-fold cross, they can be formulated for the general case of an $N$-fold cross with $N \geq 2$ (see also [9, 16, 22]). It remains an open question whether $\hat{W}$ is the maximal extension region of $W$ for the family of mappings discussed in the Main Theorem (for a special case see [25]). Various applications of the Main Theorem will be given in Section 7 below. It is possible to obtain a generalization of the Main Theorem in the case where $M$ is not necessarily closed in $W$. Indeed, it suffices to make use of the works [11, 12] and combine them with our present method.

Before going further we say some words about the exposition of the paper. We only give the proof of the Main Theorem for the case where the set of singularities $M$ is locally pluripolar in fibers. It is therefore left to the interested reader to treat the easier case where $M$ is thin in fibers. On the other hand, as in any article of holomorphic extension, there are always two parts: describing the method of extension and justifying the gluing process. Since our primary aim is to make the article as compact as possible, we focus more on the way we extend the mappings than the gluing process. Throughout the paper, $Z$ always denotes a complex analytic space possessing the Hartogs extension property.

### 3 Auxiliary results

First we recall and prove some auxiliary results. From [12] we extract the following particular case of a general cross theorem with singularities which will be needed in the future.

**Theorem 3.1.** Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$, let $D \subset X$, $G \subset Y$ be two bounded domains, let $A \subset D$ and $B \subset G$ be nonpluripolar subsets. Let $M$ be a closed subset of $W$ such that $M$ is pluripolar in fibers over $A$ and $B$.

Then there exists a relatively closed pluripolar set $\hat{M} \subset \hat{W}$ such that:

- $\hat{M} \cap W \cap \hat{W} \subset M$;
- for every mapping $f \in \mathcal{O}_s(W^o \setminus M, Z)$, there exists a unique mapping $\hat{f} \in \mathcal{O}(\hat{W} \setminus \hat{M}, Z)$ such that $\hat{f} = f$ on $(W \cap \hat{W}) \setminus M$.

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\(^{11}\)It follows from Subsection 2.2 that

$$X(A \cap \partial D, B \cap \partial G; D, G) = ((D \cup A) \times (B \cap \partial G)) \cup ((A \cap \partial D) \times (G \cup B)).$$

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Proof. The special case when $D$ and $G$ are pseudoconvex, $Z = \mathbb{C}$ and $M$ is relatively closed in $W$ has been proved in [9]. However, using recent result in [11], the assumption that $D$ and $G$ are pseudoconvex and $M$ is relatively closed in $W$ can be removed. Finally, by applying Theorem 2.5, the desired conclusion of the theorem follows from its special case $Z = \mathbb{C}$ (see also [2]).

The next result was proved by the first author in [17].

**Theorem 3.2.** We keep the hypotheses and notation of the Main Theorem. Suppose in addition that $M = \emptyset$. Then the conclusion of the Main Theorem holds for $\hat{M} = \emptyset$.

The following result play an important role in the sequel.

**Theorem 3.3 ([5]).** Let $D \subset \mathbb{C}^n$ be a domain and let $\hat{D}$ be the envelope of holomorphy of $D$. Assume that $S$ is relatively closed pluripolar subset of $D$. Then there exists a relatively closed pluripolar subset $\hat{S}$ of $\hat{D}$ such that $\hat{S} \cap D \subset S$ and $\hat{D} \setminus \hat{S}$ is the envelope of holomorphy of $D \setminus S$.

In this article, let $\text{mes}$ denote the Lebesgue measure on the unit circle $\partial E$. Recall here the system of angular (or Stolz) approach regions for $E$ (see, for example, [17]). Put

$$A_{\alpha}(\zeta) := \left\{ t \in E : \left| \arg \left( \frac{\zeta - t}{\zeta} \right) \right| < \alpha \right\}, \quad \zeta \in \partial E, \quad 0 < \alpha < \frac{\pi}{2},$$

where $\arg : \mathbb{C} \rightarrow (-\pi, \pi]$ is as usual the argument function. $A = (A_{\alpha}(\zeta))_{\zeta \in \partial E, \ 0 < \alpha < \frac{\pi}{2}}$ is referred to as the system of angular (or Stolz) approach regions for $E$. In this context $A \lim$ is also called angular limit.

**Theorem 3.4.** Let $D = G = E$ and let $A \subset \partial D$ be a measurable subset such that $\text{mes}(A) > 0$, and let $B \subset G$ be an open set. Consider the cross $W := \mathbb{X}(A, B; D, G)$. Let $M$ be a relatively closed subset of $W$ such that $M_a$ is polar (resp. discrete) in $G$, $M_a \cap B = \emptyset$ for all $a \in A$ and $M^b = \emptyset$ for all $b \in B$\footnote{i.e., $M$ is polar (resp. discrete) in fibers over $A$, and $M$ is empty in fibers over $B$.} and that $M \cap A \times B = \emptyset$. Let $S$ be a relatively closed pluripolar subset (resp. analytic subset) of $D \times B$. Then there exists a relatively closed pluripolar subset (resp. an analytic subset) $T$ of $\hat{W}$ with the following property: Let $f : W \setminus (M \cup S) \rightarrow \mathbb{C}$ be a bounded function such that

- for all $a \in A$, $f(a, \cdot)|_{G \setminus M_a}$ is holomorphic;
- for all $b \in B$, $f(\cdot, b)|_{D \setminus S_b}$ is holomorphic and admits the angular limit $f(a, b)$ at all points $a \in A$;\footnote{i.e., the set of end-points of $(D \times B) \setminus S$ contains $A \times B$.}

Then there is a unique function $\hat{f} \in \mathcal{O}(\hat{W} \setminus T, \mathbb{C})$ which extends $f|_{(D \times B) \setminus S}$.

Moreover, if $M = S = \emptyset$, then $T = \emptyset$.\footnote{We assume that the set of end-points of $(D \times B) \setminus S$ contains $A \times B$.}
The proof is based on the technique of holomorphic discs. We postpone it to Section 5 below.

In the sequel for \( z \in \mathbb{C}^n \) and \( r > 0 \), let \( \Delta_0^a(r) \) denote the open polydisc centered at \( z \) with radius \( r \). We will need the following higher dimensional version of Theorem 3.4.

**Theorem 3.5.** Let \( A \) be a measurable subset of \( \partial E \) with \( \operatorname{mes}(A) > 0 \) and let \( r > 1 \). Let \( M \) be a relatively closed subset of \( A \times \Delta_0^a(r) \) such that \( M \cap (A \times \overline{E}^a) = \emptyset \) and that \( M_a := \{ w \in \Delta_0^a(r) : (a, w) \in M \} \) is pluripolar for all \( a \in A \). Then there exists a relatively closed pluripolar subset \( S \) of \( E \times \Delta_0^a(r) \) with \( S \cap E^{n+1} = \emptyset \) and with the following additional property:

Let \( f : \mathcal{K}(A, E^n; E, \Delta_0^a(r)) \setminus M \rightarrow \mathbb{C} \) be bounded such that:
- for all \( a \in A \), \( f(a, \cdot)|_{\Delta_0^a(r) \setminus M_a} \) is holomorphic;
- for all \( w \in E^n \), \( f(\cdot, w)|_E \) admits the angular limit \( f(a, w) \) at all points \( a \in A \).

Then there is a unique function \( \hat{f} \in \mathcal{O}(\mathcal{K}(A, E^n; E, \Delta_0^a(r)) \setminus S) \) which extends \( f|_{E^{n+1}} \).

We also need the following generalization of Theorem 3.5 where \( D \) need not to be a disc.

**Theorem 3.6.** Let \( X \) be a bounded open set of \( \mathbb{C}^m \) and \( Y = \mathbb{C}^n \), let \( D \subset X \) be a domain and \( G = \Delta_0^a(r) \) for some \( r > 1 \), let \( \hat{A} \) be a subset of \( \overline{D} \) and let \( B = E^n \). \( D \) is equipped with a system of approach regions \( (\mathcal{A}_a(\zeta))_{\zeta \in \overline{D}, a \in I^\zeta} \) and \( G \) is equipped with the canonical system of approach regions. Let \( A_0 \) (resp. \( B_0 \)) be a subset of \( \overline{D} \) (resp. \( \overline{G} \)) such that \( A_0 \) and \( B_0 \) are locally pluriregular and that \( \overline{A}_0 \subset A^\ast \) and \( \overline{B}_0 \subset B^\ast \). Put \( W_0 := \mathcal{K}(A_0, B_0; D, G) \). Let \( M \) be a closed subset of \( W \) with the following properties:
- \( M \) is locally pluripolar in fibers over \( A \) and over \( B \);
- \( M \cap (A \times B) = \emptyset \), \( M \cap (D \times B) = \emptyset \).

Then there exists a relatively closed locally pluripolar subset \( \hat{M} \) of \( \hat{W}_0 \) with \( \hat{M} \cap (D \times B) = \emptyset \) such that:

for every mapping \( f : W \setminus M \rightarrow Z \) satisfying the following conditions:
- \( f \in \mathcal{C}_a(W \setminus M, Z) \cap \mathcal{O}_a(W^o \setminus M, Z) \);
- \( f \) is locally bounded along \( ((A \cap \partial D) \times G) \setminus M \); there exists a unique mapping \( \hat{f} \in \mathcal{O}(\hat{W}_0 \setminus \hat{M}, Z) \) which admits the \( A \)-limit \( f(\zeta, \eta) \) at every point \( (\zeta, \eta) \in (W \cap W_0) \setminus M \).

In order to prove Theorem 3.5 and 3.6, our strategy is as follows. First, observe that Theorem 3.5 for the case \( n = 1 \) follows from Theorem 3.4. Next, we will show that Theorem 3.5 implies Theorem 3.6. Finally, Theorem 3.6 for \( n = 1 \) implies, in turn, Theorem 3.5 for arbitrary \( n \).

For \( a \in \partial E \) and \( 0 < \rho, \epsilon < 1 \), let
\[
\Delta_a(\rho, \epsilon) := \{ z \in \Delta_a(\rho) \cap E : \omega(z, A \cap \Delta_a(\rho), \Delta_a(\rho) \cap E) < \epsilon \}.
\]
Prior to the proof of Theorem 3.5 we establish the following
Proposition 3.7. For every density point $a_0 \in A$ and every $r' \in (1, r)$, there exist $0 < \rho = \rho_{r'}, \epsilon = \epsilon_{r'} < 1$ and a relatively closed pluripolar set $S = S_{r'} \subset \Delta_{a_0}(\rho, \epsilon) \times \Delta_{0}^{n}(r')$, such that any mapping $f$ satisfying the hypotheses of Theorem 3.5 extends holomorphically to $(\Delta_{a_0}(\rho, \epsilon) \times \Delta_{0}^{n}(r')) \setminus S$.

Proof. Fix a density point $a_0$ of $A$ and let $r_0'$ be the supremum of all $r' \in (0, r)$ such that $\rho_{r'}, \epsilon_{r'}$ and $S_{r'}$ exist. Note that $1 \leq r_0' \leq r$. It suffices to show that $r_0' = r$.

Suppose that $r_0' < r$. Fix $r_0' < r'' < r$ and choose $r' \in (0, r_0')$ such that $\sqrt[r''-1]{r''} > r_0'$. Let $\rho := \rho_{r''}, S := S_{r''}$.

Write $w = (w', w_n) \in \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$. Let $C$ denote the set of all $(a, b') \in (A \cap \Delta_{a_0}(\rho)) \times \Delta_{0}^{n-1}(r')$ such that the fiber $(M \cup S)(a, b', r)$ is polar. Now, by Theorem 3.6 applied to the cross

$$Y_n := \mathbb{X}(C, \Delta_{0}(r'); \Delta_{a_0}(\rho, \epsilon) \times \Delta_{0}^{n-1}(r'), \Delta_{0}(r))$$

and the set $M_n := M \cup S$, we conclude that there exists a closed pluripolar set $S_n \subset Y_n$ such that $S_n \cap Y_n \subset M_n$ and any mapping $f$ satisfying the hypotheses of Theorem 3.5 extends holomorphically to $Y_n \setminus S_n$. We need the following

Lemma 3.8. For $(z, w') \in \Delta_{a_0}(\rho, \epsilon) \times \Delta_{0}^{n-1}(r')$,

$$\omega((z, w'), C, \Delta_{a_0}(\rho, \epsilon) \times \Delta_{0}^{n-1}(r')) = \max\left\{ \frac{1}{\epsilon} \cdot \omega(z, A \cap \Delta_{a_0}(\rho), \Delta_{a_0}(\rho)), \omega(w', \Delta_{0}^{n-1}(r'), \Delta_{0}^{n-1}(r')) \right\}.$$ 

Proof. Using Proposition 5.2 in [17] we may assume without loss of generality that $\epsilon = 1$. Observe that the $(2n - 1)$-dimensional Lebesgue measure of

$$(A \cap \Delta_{a_0}(\rho)) \times \Delta_{0}^{n-1}(r') \setminus C$$

is zero and that the set $(A \cap \Delta_{a_0}(\rho)) \times \Delta_{0}^{n-1}(r')$ is living on the boundary of the smooth hypersurface $\partial E \times \Delta_{0}^{n-1}(r')$ in $\mathbb{C}^n$. Consequently, in the desired equality we may suppose that $C = (A \cap \Delta_{a_0}(\rho)) \times \Delta_{0}^{n-1}(r')$. Then the desired equality follows easily from the product property of extremal functions.
Using the above lemma, we get
\[
\hat{Y}_q = \{(z, w', w_n) \in \Delta_{\alpha_0}(\rho, \epsilon) \times \Delta_{\alpha_0}^{n-1}(r') \times \Delta_0(r) : \\
\omega((z, w'), C, \Delta_{\alpha_0}(\rho, \epsilon) \times \Delta_{\alpha_0}^{n-1}(r')) + \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \}
\]
\[
= \{(z, w', w_n) \in \Delta_{\alpha_0}(\rho) \times \Delta_{\alpha_0}^{n-1}(r') \times \Delta_0(r) : \\
\omega((z, w'), (A \cap \Delta_{\alpha_0}(\rho)) \times \Delta_0^{n-1}(r'), \Delta_{\alpha_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r')) + \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \}
\]
\[
= \{(z, w', w_n) \in \Delta_{\alpha_0}(\rho) \times \Delta_{\alpha_0}^{n-1}(r') \times \Delta_0(r) : \\
\max \left\{ \frac{1}{\epsilon}, \omega\left(z, A \cap \Delta_{\alpha_0}(\rho), \Delta_{\alpha_0}(\rho) \right), \omega(w', \Delta_{\alpha_0}^{n-1}(r'), \Delta_{\alpha_0}^{n-1}(r')) \right\} \\
+ \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \}
\]
\[
= \{(z, w', w_n) \in \Delta_{\alpha_0}(\rho) \times \Delta_{\alpha_0}^{n-1}(r') \times \Delta_0(r) : \\
\frac{\omega(z, A \cap \Delta_{\alpha_0}(\rho), \Delta_{\alpha_0}(\rho) \cap E)}{\epsilon} + \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \}. \]

Since \( r'' < r \), we find an \( \rho_n \in (0, \rho) \) such that any mapping \( f \) as in the hypotheses of Theorem 3.5 extends holomorphically to a mapping \( \hat{f}_n \) on \( \Delta_{\alpha_0}(\rho_n) \times \Delta_{\alpha_0}^{n-1}(r') \times \Delta_0(r'') \setminus S_n \). We may assume that \( S_n \) is singular with respect to the family \( \{ \hat{f}_n : f \text{ as in the hypotheses of Theorem } 3.5 \} \).

Repeating the above argument for the coordinates \( w_\nu, \nu = 1, \ldots, n-1 \), and gluing the obtained sets, we find an \( \rho_0 \in (0, \rho], \epsilon_0 \in (0, \epsilon] \) and a relatively closed pluripolar set \( S_0 := \bigcup_{j=1}^n S_n \) such that any mapping \( f \) as in the hypotheses of Theorem 3.5 extends holomorphically to a mapping \( \hat{f}_0 := \bigcup_{j=1}^n \hat{f}_j \) holomorphic in \( \Delta_{\alpha_0}(\rho_0, \epsilon_0) \times \Omega \setminus S_0 \), where
\[
\Omega := \bigcup_{j=1}^n \Delta_{\alpha_0}^{j-1}(r') \times \Delta_0(r'') \times \Delta_0^{n-j}(r').
\]

Let \( \hat{\Omega} \) denote the envelope of holomorphy of \( \Omega \). Applying the Chirka theorem (Theorem 3.3), we find a relatively closed pluripolar subset \( \hat{S}_0 \) of \( \Delta_{\alpha_0}(\rho_0, \epsilon_0) \times \hat{\Omega} \) such that any function \( f \) as in the hypotheses of Theorem 3.5 extends to a mapping \( \hat{f} \) holomorphic on \( \Delta_{\alpha_0}(\rho_0, \epsilon_0) \times \hat{\Omega} \setminus \hat{S}_0 \). Let \( r'' := \sqrt{r''} \cdot r_1^{m-1} r'' \). Observe that \( \Delta_0(r'') \subset \hat{\Omega} \). Recall that \( r''' > r_0 \). We may assume that \( \hat{M} \) is singular with respect to the family \( \{ \hat{f} : f \text{ as in the hypotheses of Theorem } 3.5 \} \).

Now we are in the position to show that Theorem 3.6 for \( n = 1 \) implies Theorem 3.5.

**Proof of Theorem 3.5.**

For a bounded mapping \( \phi \in \mathcal{O}(E, \mathbb{C}^n) \) and \( \zeta \in \partial E \), \( f(\zeta) \) denotes the angular limit value of \( f \) at \( \zeta \) if it exists. A classical theorem of Fatou says that \mes(\{ \zeta \in \partial E : \exists f(\zeta) \}) = 2\pi.\)

**Theorem 3.9.** Let \( D \) be a bounded open set in \( \mathbb{C}^n \), \( A \subset \overline{D} \), \( z_0 \in D \) and \( \epsilon > 0 \). Let \( A \) be a system of approach regions for \( D \). Suppose in addition that \( A \) is locally pluriregular (relative to \( A \)). Then there exist a bounded mapping \( \phi \in \mathcal{O}(E, \mathbb{C}^n) \) and a measurable subset \( \Gamma_0 \subset \partial E \) with the following properties:
1) $\Gamma_0$ is pluriregular (with respect to the system of angular approach regions), $\phi(0) = z_0$, $\phi(E) \subset \overline{D}$, $\Gamma_0 \subset \{ \zeta \in \partial E : \phi(\zeta) \in \overline{A} \}$, and
$$1 - \frac{1}{2\pi} \cdot \text{mes}(\Gamma_0) < \omega(z_0, A, D) + \epsilon.$$

2) Let $f \in C(D \cup \overline{A}, Z) \cap O(D, Z)$ be such that $f(D)$ is bounded. Then there exist a bounded function $g \in O(E, Z)$ such that $g(f) \in O(\zeta ; (f \circ \phi)(\zeta)$ for all $\zeta \in \Gamma_0$. Moreover, $g|_{\Gamma_0} \in C(\Gamma_0, Z)$.

This theorem was proved in [17] for the case when in Part 2) $Z = \mathbb{C}$. But using the hypothesis that $f(D)$ is bounded, Part 2) in the general case follows immediately from the above special case. Theorem 3.9 motivates the following

**Definition 3.10.** We keep the hypothesis and notation of Theorem 3.9. Then every pair $(\phi, \Gamma_0)$ satisfying the conclusions 1)–2) of this theorem is said to be an $\epsilon$-candidate for the triplet $(z_0, A, D)$.

Theorem 3.9 says that there always exist $\epsilon$-candidates for all triplets $(z, A, D)$.

The following result reduces the Main Theorem to local situations.

**Proposition 3.11.** We keep the hypotheses and notation of the Main Theorem. Suppose in addition that the following property holds:

Let $A_0$ (resp. $B_0$) be a subset of $\overline{D}$ (resp. $\overline{G}$) such that $A_0$ and $B_0$ are locally pluriregular and that $A_0 \subset A^*$ and $B_0 \subset B^*$ and that $A_0, B_0$ are compact. Then there exists a relatively closed pluripolar subset $\hat{M}$ of $\hat{X}(A_0, B_0 ; D, G)$ such that for every mapping $f : W \longrightarrow Z$ which satisfies conditions (i)–(iii) of the Main Theorem, there exists a unique mapping $\hat{f}$ defined and holomorphic on $\hat{X}(A_0, B_0 ; D, G) \setminus \hat{M}$ which admits $A$-limit $f(\zeta, \eta)$ at all points $(\zeta, \eta) \in \hat{X}(A_0, B_0 ; D, G) \setminus M$.

Then the conclusion of the Main Theorem holds.

This result permits to pass from the relative extremal functions $\omega(\cdot, A_0, D)$, where $A_0 \in E(A)$, to the plurisubharmonic measure $\hat{\omega}(\cdot, A, D)$.

**Proof.** The case where $M = \emptyset$ is treated by the first author in [17] where he starts from Theorem 8.2 therein in order to prove the main theorem in Section 9 of that article. This method also works in the present context making the obviously necessary changes.

**Proof.** First we find a subset $M \subset \hat{W}_0$ such that

- $M \cap (D \times B) = \emptyset$;
- for all $z \in D$, the vertical fibers $M_z := \{ w \in G : (z, w) \in M \}$ are relatively closed pluripolar in $(\hat{W}_0)_z := \{ w \in G : (z, w) \in \hat{W}_0 \}$;

Note here that by Part 1), $(f \circ \phi)(\zeta)$ exists for all $\zeta \in \Gamma_0$. 

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By Theorem 3.9 and Definition 3.10, there is an \( \epsilon \) such that \( z \) is well-defined at a point \((z, 0)\). We may assume without loss of generality that \( z \) is closed pluripolar. To this end fix a \( z_0 \in D \) and we want to construct the vertical fiber \( M_{z_0} \). Fix an arbitrary \( \epsilon > 0 \) such that
\[
\omega(z, A_0, D) + \epsilon < 1.
\]
By Theorem 3.9 and Definition 3.10, there is an \( \epsilon \)-candidate \((\phi, \Gamma)\) for \((z_0, A, D)\). By shrinking \( \Gamma \) (if necessary), we may assume without loss of generality that \( \phi|\Gamma \) is continuous. Moreover, using the hypotheses, we see that the mapping \( f \) is defined on \( \mathbb{X}(\Gamma, B; E, G) \setminus M_0 \),
\[
\phi(t, w) := f(\phi(t), w), \quad (t, w) \in \mathbb{X}(\Gamma, B; E, G) \setminus M_0,
\]
satisfies the hypotheses of Theorem 3.5, where \( M_0 := \{ (t, w) \in \Gamma \times G : (\phi(t), w) \in M \} \). By this theorem, let \( M_0 \) be the relatively closed pluripolar subset of \( \mathbb{K}(\Gamma, B; E, G) \) with \( M_0 \cap (E \times B) = \emptyset \) and let \( \hat{\phi} \in \mathcal{O}(\mathbb{K}(\Gamma, B; E, G) \setminus M_0, Z) \) such that
\[
(A - \lim \hat{\phi})(t, w) = f_\phi(t, w), \quad (t, w) \in \mathbb{X}(\Gamma, B; E, G) \setminus M_0.
\]
We can define \( M_{z_0} \) and the desired extension mapping \( \hat{f}(z_0, \cdot) \) on \((\hat{\mathbb{W}}_0 \setminus \mathcal{M})_{z_0} \) as follows. \( \hat{f} \) is well-defined at a point \((z_0, w) \in \hat{\mathbb{W}}_0 \) if there exist an \( \epsilon > 0 \) that satisfies (2) and an \( \epsilon \)-candidate \((\phi, \Gamma)\) for \((z_0, A, D)\) and a \( t \in E \) such that \((t, w) \in \mathbb{K}(\Gamma, B; E, G) \setminus M_0 \). If this condition is satisfied, then the value of \( \hat{f} \) at \((z, w) \) is given by
\[
\hat{f}(z, w) := \hat{\phi}(t, w),
\]
where \( \hat{\phi} \) is defined in (3)–(4). On the other hand, let
\[
M_{z_0} := \{ w \in G : \forall \phi \text{ as above}, \exists t \in E : \phi(t) = z_0 \text{ and } (t, w) \in M_0 \}.
\]
Using Lemma 4.5 in [17] it can be checked that \( \hat{f}(z_0, \cdot) \) is well-defined on \((\hat{\mathbb{W}}_0)_{z_0} \setminus M_{z_0} \). Moreover, it is easy to see that \( M \cap (D \times B) = \emptyset \) and that all vertical fibers \( M_z \) with \( z \in D \) are relatively closed pluripolar. This completes the construction of \( M \subset \hat{\mathbb{W}}_0 \).

For all \( 0 < \delta < \frac{1}{2} \) let
\[
A_\delta := \{ z \in D : \omega(z, A_0, D) < \delta \} \text{ and } G_\delta := \{ w \in G : \omega(w, B, G) < 1 - \delta \}.
\]
We are able to define a new mapping \( \hat{f}_\delta \) on \( \mathbb{X}(A_\delta, B; D, G_\delta) \setminus M \) as follows
\[
\hat{f}_\delta(z, w) := \begin{cases} \hat{f}(z, w) & (z, w) \in (A_\delta \times G_\delta) \setminus M, \\ f(z, w) & (z, w) \in D \times B. \end{cases}
\]
Using the hypotheses on \( f \) and the previous remark, we see that \( \hat{f}_\delta \in \mathcal{O}(\mathbb{X}(A_\delta, B; D, G_\delta) \setminus M, Z) \).
Observe that \( A_\delta \) is an open set in \( D \), all vertical fibers \( M_z \) with \( z \in D \) are relatively closed pluripolar and all horizontal fibers \( M^w \) with \( w \in B \) are empty. Consequently, \( \tilde{f}_\delta \) satisfies the hypotheses of Theorem 3.1. Applying this theorem yields a relatively closed pluripolar subset \( \widehat{M}_\delta \) of \( \widehat{X}(A_\delta, B; D, G_\delta) \) with \( \widehat{M}_\delta \cap (D \times B) = \emptyset \) and a mapping \( \tilde{f}_\delta \in \mathcal{O}(\widehat{X}(A_\delta, B; D, G_\delta) \setminus \widehat{M}_\delta, Z) \) such that

\[
\tilde{f}_\delta(z, w) = f_\delta(z, w), \quad (z, w) \in X(A_\delta, B; D, G_\delta) \setminus \widehat{M}_\delta.
\]

This, combined with (7), implies that \( \tilde{f} \) (given in (5)) extends holomorphically to \( (A_\delta \times G_\delta) \setminus \widehat{M}_\delta \). On the other hand, it follows from (6) that

\[
\widehat{W}_0 = \widehat{X}(A_0, B; D, G) = \bigcup_{0<\delta<1} A_\delta \times G_\delta.
\]

Now fix a sequence \((\delta_k)_{k=1}^\infty\) such that \( 0 < \delta_k < 1 \) and \( \delta_k \searrow 0^+ \). Therefore, using the last equality, we may glue \((\widehat{M}_{\delta_k})_{k=1}^\infty\) together in order to obtain a relatively closed pluripolar subset \( \widehat{M} \) of \( \widehat{W}_0 \) with the desired properties of the theorem. \( \square \)

4 A local result

The main purpose of the section is to prove the following local version of the Main Theorem.

**Theorem 4.1.** We keep the hypotheses and notation of the Main Theorem. Let \( A_0 \) (resp. \( B_0 \)) be a subset of \( \overline{D} \) (resp. \( \overline{G} \)) such that \( A_0 \) and \( B_0 \) are locally pluriregular and that \( \overline{A}_0 \subset A^* \) and \( \overline{B}_0 \subset B^* \). Put

\[
W_0 := X(A_0, B_0; D, G), \quad \overline{W}_0 := X(\overline{A}_0, \overline{B}_0; D, G).
\]

Then for every \((a, b) \in \overline{W}_0\), there exists an open neighborhood \( U \) of \( a \) in \( X \), an open neighborhood \( V \) of \( b \) in \( Y \), and a relatively closed locally pluripolar subset \( M = M_{(a, b)} \) of \( \widehat{X}(A_0 \cap U, B_0 \cap V; \overline{D} \cap U, \overline{G} \cap V) \) such that:

for every mapping \( f : W \setminus M \rightarrow Z \) satisfying conditions (i)–(iii) of the Main Theorem, there exists a unique mapping \( \tilde{f} \in \mathcal{O}(\widehat{X}(A_0 \cap U, B_0 \cap V; \overline{D} \cap U, \overline{G} \cap V) \setminus \widehat{M}, Z) \) which admits the \( A \)-limit \( f(\zeta, \eta) \) at every point

\[
(\zeta, \eta) \in (W \cap \widehat{X}(A_0 \cap U, B_0 \cap V; \overline{D} \cap U, \overline{G} \cap V)) \setminus M.
\]

**Proof.** There are four cases to consider.

**Case** \( a \in D \) and \( b \in G \). Since we have either \( a \in A^* \cap D \) and \( b \in G \), or \( a \in D \) and \( b \in B^* \cap G \), the desired conclusion follows from Theorem 3.1.

**Case** \( a \in \partial D \) and \( b \in \partial G \). Recall from the hypotheses that \( M \) is relatively closed along \( X(A \cap \partial D, B \cap \partial G; D, G) \) in \( W \) and that \( M \cap (A \cap \partial D) \times (B \cap \partial G) = \emptyset \). Therefore, there exist an open neighborhood \( U \) of \( a \) in \( X \) and an open neighborhood \( V \) of \( b \) in \( Y \) such that

\[
X(A \cap U, B \cap V; D \cap U, B \cap V) \cap M = \emptyset.
\]
Using this we are able to apply Theorem 3.2 to \( f \) restricted to \( \mathcal{X}(A \cap U, B \cap V; D \cap U, B \cap V) \).

Consequently, the desired conclusion follows.

**Case** \( a \in \partial D \) and \( b \in G \). Recall from the hypotheses that \( M \) is relatively closed along the singleton set \( \{(a, b)\} \) in \( W \). Therefore, there exist an open neighborhood \( U \) of \( a \) in \( X \) and an open neighborhood \( V \) of \( b \) in \( G \) such that

\[
\mathcal{X}(A \cap U, B \cap V; D \cap U, G \cap V) \cap M \text{ is relatively closed in } W. \tag{8}
\]

We need the following auxiliary result whose proof will be given later on.

**Lemma 4.2.** For every \( a \in \overline{A}_0 \) and \( w \in G \cap V \), there are an open neighborhood \( U \) of \( a \) in \( U \), a number \( 0 < \delta < 1 \), an open neighborhood \( V \) of \( w \) in \( V \), a relatively closed locally pluripolar subset \( \overline{M} \) of \( U_{a,\delta} \times V \) with

\[
U_{a,\delta} := \{ z \in U \cap D : \omega(z, A_0 \cap U, D \cap U) < \delta \},
\]

and a mapping \( f \in \mathcal{O}(\overline{U}_{a,\delta} \times V) \) which extends \( f \).

Now we come back to the proof of Theorem 4.1. Fix a relatively compact subdomain \( G \) of \( V \) such that \( b \in G \) and that \( B^* \cap G \neq \emptyset \). Applying Lemma 4.2 and using a compactness argument, we may find an open neighborhood \( U \) of \( a \) in \( U \), a number \( 0 < \delta < 1 \), a relatively closed locally pluripolar subset \( M' \) of \( U_{a,\delta} \times G \) and a mapping \( f' \in \mathcal{O}(\overline{U}_{a,\delta} \times G) \setminus M' \) which extends \( f \).

Consider the mapping \( \tilde{f} : \mathcal{X}(U_{a,\delta}, B \cap V; U, G) \setminus (M \cup M') \rightarrow Z \) as follows

\[
\tilde{f}(z, w) := \begin{cases} 
  f'(z, w), & (z, w) \in (U_{a,\delta} \times G) \setminus M', \\
  f(z, w), & (z, w) \in (U \times (B \cap V)) \setminus M.
\end{cases}
\]

Applying Theorem 3.1 to \( \tilde{f} \), we obtain a relatively closed locally pluripolar subset \( \overline{M} \) of \( \overline{\mathcal{X}}(U_{a,\delta}, B \cap V; U, G) \) and a mapping

\[
\tilde{f} \in \mathcal{O}\left(\overline{\mathcal{X}}(U_{a,\delta}, B \cap V; U, G) \setminus \overline{M}, Z\right)
\]

which extends \( f \). Since \( \omega(\cdot, U_{a,\delta}, D \cap U) \leq \omega(\cdot, A_0 \cap U, D \cap U) \) on \( D \cap U \), it follows that

\[
\overline{\mathcal{X}}(A_0 \cap U, B_0 \cap V; U, G) \subset \overline{\mathcal{X}}(U_{a,\delta}, B_0 \cap V; U, G).
\]

Since the the domain \( G \) satisfies \( b \in G \) and \( B^* \cap G \neq \emptyset \), the desired conclusion follows.

**Case** \( a \in D \) and \( b \in \partial G \). It is similar to the previous case. \( \square \)

**Proof of Lemma 4.2.** We may assume without loss of generality that \( U \) (resp. \( V \)) is a open polydisc of \( \mathbb{C}^m \) (resp. \( \mathbb{C}^n \)) and that \( a = 0 \in \mathbb{C}^m \) and \( b = 0 \in \mathbb{C}^n \). In the sequel if \( V = \Delta_b^m(r) \) then let \( \frac{1}{2} V := \Delta_b^n(r) \). Now we shrink \( U \) and \( V \) (if necessary) and use (8) and the hypotheses that \( M \) is locally pluripolar in fibers over \( A \) and \( B \) and that \( M \cap ((A \cap \partial D) \times B) = \emptyset \). Consequently, we may choose an open polydisc \( V' \subset \frac{1}{2} V \) centered at a point \( b' \neq b \) such that

\[
\bigcup_{z \in A_0 \cap U} M_z \cap V' = \emptyset \quad \text{and} \quad \bigcup_{w \in B_0 \cap V'} M_w \cap \overline{U} = \emptyset.
\]
Using this we are able to apply Theorem 3.2 to $f$ restricted to $X(A \cap U, B \cap V'; D \cap U, V')$. Consequently, fix an open polydisc $V'' \subset V'$ we may find $0 < \delta < 1$ such that by letting $U = \{z \in U \cap D : \omega(z, A_0 \cap U, D \cap U) < \delta\}$, $f$ extends holomorphically to $f'$ on $U \times V''$. Choosing a polydisc $V''$ with the same center as $V'$ such that $b \in V'' \subset \frac{1}{2}V$. Now we are in the position to apply Theorem 3.6 to the mapping $\tilde{f} : X(A \cap U, V'; U, V'') \setminus M \rightarrow Z$ as follows

$$\tilde{f}(z, w) := \begin{cases} f'(z, w), & (z, w) \in U \times V', \\ f(z, w), & (z, w) \in ((A \cap U) \times V'') \setminus M. \end{cases}$$

Hence, we obtain the desired conclusion. □

5 Using holomorphic discs

In this section we combine Poletsky theory of discs [26, 27], Rosay Theorem on holomorphic discs [28] and Theorem 3.1.

Let us recall some elements of Poletsky theory of discs. Let $E$ denote as usual the unit disc in $\mathbb{C}$. For a complex manifold $M$, let $\mathcal{O}(E, M)$ denote the set of all holomorphic mappings $\phi : E \rightarrow M$ which extend holomorphically to a neighborhood of $E$. Such a mapping $\phi$ is called a holomorphic disc on $M$. Moreover, for a subset $A$ of $M$, let

$$1_A(z) := \begin{cases} 1, & z \in A, \\ 0, & z \in M \setminus A. \end{cases}$$

In the work [28] Rosay proved the following remarkable result.

**Theorem 5.1.** Let $u$ be an upper semicontinuous function on a complex manifold $M$. Then the Poisson functional of $u$ defined by

$$\mathcal{P}[u](z) := \inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(\phi(e^{i\theta}))d\theta : \phi \in \mathcal{O}(E, M), \phi(0) = z \right\},$$

is plurisubharmonic on $M$.

This implies the following important consequence (see, for example, Proposition 3.4 in [16]).

**Corollary 5.2.** Let $M$ be a complex manifold and $A$ a nonempty open subset of $M$. Then $\omega(z, A, M) = \mathcal{P}[1_{M \setminus A}](z)$, $z \in M$.

Now we arrive at

**Proof of Theorem 3.4.** First observe that we only need to prove the following weaker version of the theorem.

For every $(z_0, w_0) \in \widehat{W}$, there are an open neighborhood $U \times V$ of $(z_0, w_0)$ in $\widehat{W}$ and a relatively closed locally pluripolar subset $S$ of $U \times V$ with $S \cap (U \times (B \cap V)) = \emptyset$ and a function $\hat{f} \in \mathcal{O}((U \times V) \setminus S, C)$ which extends $f|_{U \times (B \cap V)}$. Indeed, taking for granted this assertion, then the proof will follow by a routine gluing procedure.
Now we come back to the proof of the assertion. Applying Theorem 5.1 and Corollary 5.2, we may find and \( \psi \in \mathcal{O}(\mathcal{E}, G) \) such that

\[
\psi(0) = w_0, \quad \omega(z_0, A, D) + \int_0^{2\pi} 1_{G \setminus B}(\psi(e^{i\theta}))d\theta < 1.
\]

Using this and the hypotheses, we see that the mapping \( f_\psi \), defined by

\[
f_\psi(z, t) := f(z, \psi(t)), \quad (z, t) \in \mathcal{X}(A, \Gamma; D, E) \setminus M_,
\]

satisfies the hypotheses of the Main Theorem in [20], where \( \Gamma := \{ t \in \partial E : \psi(t) \in B \} \) and \( M_\psi := \{ (z, t) \in A \times E : (z, \psi(t)) \in M \} \). By this theorem, let \( \hat{M}_\psi \) be the relatively closed pluripolar subset of \( \hat{\mathcal{X}}(A, \Gamma; D, E) \) and let \( \hat{f}_\psi \in \mathcal{O}(\hat{\mathcal{X}}(A, \Gamma; D, E) \setminus \hat{M}_\psi, \mathcal{C}) \) such that

\[
(A - \lim \hat{f}_\psi)(z, t) = f_\psi(z, t), \quad (z, t) \in \mathcal{X}^o(A, \Gamma; D, E) \setminus M_\psi.
\]

Now we may define the desired extension function \( \hat{f} \) in an open neighborhood of \( (z_0, w_0) \) (except possibly a relatively closed pluripolar set) as follows

\[
\hat{f}(z, w) := \hat{f}_\psi(z, t), \quad w = \psi(t), \quad (z, t) \in \mathcal{X}(A, \Gamma; D, E) \setminus \hat{M}_\psi.
\]

This completes the above assertion. \( \square \)

The main result of this section is

**Theorem 5.3.** Let \( X, Y \) be two complex manifolds, let \( D \subset X, G \subset Y \) be two open sets, let \( A_0 \) (resp. \( B_0 \)) be an open subset of \( D \) (resp. \( G \)) and let \( A \) (resp. \( B \)) be a subset of \( A_0 \) (resp. \( B_0 \)) such that \( A_0 \setminus A \) (resp. \( B_0 \setminus B \)) is locally pluripolar. Let \( M \) be a subset of \( W \) such that \( M \) is relatively closed locally pluripolar in fibers over \( A \) and over \( B \). Then there exists a relatively closed locally pluripolar subset \( \hat{M} \) of \( \hat{W} \) such that: for every mapping \( f \in \mathcal{O}(W^o \setminus M, Z) \), there exists a unique mapping \( \hat{f} \in \mathcal{O}(\hat{W} \setminus \hat{M}, Z) \) such that \( \hat{f} = f \) on \( W \setminus M \).

**Proof.** First observe as in the proof of Theorem 3.4 that we only need to prove the following weaker version of Theorem 5.3:

For every \( (z_0, w_0) \in \hat{W} \), there are an open neighborhood \( U \times V \) of \( (z_0, w_0) \) in \( \hat{W} \) and a relatively closed locally pluripolar subset \( S \) of \( U \times V \) and a mapping \( \hat{f} \in \mathcal{O}((U \times V) \setminus S, Z) \) such that \( \hat{f} = f \) on \( W \setminus (M \cup S) \).

Now we come back to the proof of the assertion. Applying Theorem 5.1 and Corollary 5.2, we may find \( \phi \in \mathcal{O}(\mathcal{E}, D) \) and \( \psi \in \mathcal{O}(\mathcal{E}, G) \) such that

\[
\phi(0) = z_0, \quad \psi(0) = w_0, \quad \frac{1}{2\pi} \int_0^{2\pi} 1_{D \setminus A_0}(\phi(e^{i\theta}))d\theta + \int_0^{2\pi} 1_{G \setminus B_0}(\psi(e^{i\theta}))d\theta < 1. \tag{9}
\]
Using the work of Rosay [28] we may find an open neighborhood \( U \) of \( \{ \phi(t) : t \in \mathbb{F} \} \) in \( D \) and an open neighborhood \( \hat{U} \) of \( \{ \psi(t) : t \in \mathbb{F} \} \) in \( G \), an open subset \( \tilde{U} \) in \( \mathbb{C}^{\mu} \) and an open subset \( \tilde{V} \) in \( \mathbb{C}^{\nu} \) and surjective mappings \( \Phi \in \mathcal{O}(\tilde{U}, U) \) and \( \Psi \in \mathcal{O}(\tilde{V}, V) \).

Next, consider the cross
\[
\mathcal{W} := X(\Phi^{-1}(A), \Psi^{-1}(B); \tilde{U}, \tilde{V})
\]
and the set
\[
\mathcal{M} := \{(x, y) \in \tilde{U} \times \tilde{V} : (\Phi(x), \Psi(y)) \in M\}.
\]
It is clear that \( \mathcal{M} \subset \mathcal{W} \) is locally pluripolar in fibers over \( \Phi^{-1}(A) \) and \( \Psi^{-1}(B) \). Now consider the mapping \( F : \mathcal{W} \setminus \mathcal{M} \to Z \) defined by
\[
F(x, y) := f(\Phi(x), \Psi(y)), \quad (x, y) \in \mathcal{W} \setminus \mathcal{M}.
\]
Using the hypotheses of the theorem, we are able to apply Theorem 3.1 to \( F \). Consequently, we obtain a relatively closed locally pluripolar subset \( \hat{M} \) of \( \hat{W} \) and a mapping \( \hat{F} \in \mathcal{O}(\hat{W} \setminus \hat{M}, Z) \) such that \( \hat{F} = F \) on \( \mathcal{W} \setminus \hat{M} \).

Let \( C \) be the set of critical points of \( (x, y) \mapsto (\Phi(x), \Psi(y)) \). This is a proper analytic subset of \( \tilde{U} \times \tilde{V} \). Now define the set
\[
S := \{(\Phi(x), \Psi(y)) : (x, y) \in \mathcal{M} \cup C\}.
\]
It is not difficult to show that there is a sufficiently small open neighborhood \( U \times V \) of \((z_0, w_0)\) in \((U \times V) \cap \hat{W} \) such that the mapping given by
\[
\hat{f}(z, w) := \hat{F}(\Phi^{-1}(z), \Psi^{-1}(w)), \quad (z, w) \in (U \times V) \setminus S,
\]
satisfies the above assertion. This completes the proof. \( \square \)

6 Proof of the Main Theorem

In this section we deduce the global desired extension from the local ones.

**Proposition 6.1.** We keep the hypotheses and notation of the Main Theorem. Let \( A_0 \) (resp. \( B_0 \)) be a subset of \( \overline{D} \) (resp. \( \overline{G} \)) such that \( A_0 \) and \( B_0 \) are locally pluriregular and that \( \overline{A}_0 \subset A^* \) and \( \overline{B}_0 \subset B^* \) and that \( \overline{A}_0 \) is compact. Let \( b_0 \in \overline{B}_0 \) and \( K \in D \). Then there exists an open set \( D' \) of the form \( D' = U \cap D \), where \( U \) is open open neighborhood of \( \overline{A}_0 \cup K \) in \( X \) with the following property: for every mapping \( f : W \to Z \) which satisfies conditions (i)–(iii) of the Main Theorem, there exist an open neighborhood \( V \) of \( b_0 \) in \( Y \) and a unique mapping \( \hat{f} \in \mathcal{O}(\overline{X}(A_0, B_0 \cap V; D', V \cap G), Z) \) which admits \( A \)-limit \( f(\zeta, \eta) \) at all points \( (\zeta, \eta) \in \overline{X}(A_0, B_0 \cap V; D', V \cap G) \).

\[15\] In general, \( \mu \gg \dim(D) \) and \( \nu \gg \dim(G) \).
Proof of Proposition 6.1. Since \( \overline{A}_0 \) is compact, we may apply Theorem 4.1 to all pairs \((a,b_0)\), \( a \in \overline{A}_0 \). Consequently, we may find a finite number of points \( a_1, \ldots, a_M \), their respective open neighborhoods \( U_{a_1}, \ldots, U_{a_M} \) in \( X \) and an open neighborhood \( V \) of \( b_0 \) in \( Y \) with the following properties:

- \( \overline{A}_0 \subset \bigcup_{j=1}^{M} U_{a_j} \);
- there exist a relatively closed locally pluripolar subset \( \widehat{M}_{a_j} \) of \( \mathcal{K}(A_0 \cap U_{a_j}, B_0 \cap V; D \cap U_{a_j}, G \cap V) \) and a mapping \( \hat{f}_j \in \mathcal{O}(\mathcal{K}(A_0 \cap U_{a_j}, B_0 \cap V; D \cap U_{a_j}, G \cap V) \setminus \widehat{M}_{a_j}, Z) \) which admits the \( \mathcal{A} \)-limit \( f(\zeta, \eta) \) at every point \((\zeta, \eta) \in \mathcal{K}(A_0 \cap U_{a_j}, B_0 \cap V; D \cap U_{a_j}, G \cap V) \setminus M \).

By shrinking \( V \) (if necessary), we may find a finite number of points \( z_1, \ldots, z_N \in D \), their respective open neighborhoods \( U_{z_1}, \ldots, U_{z_N} \) in \( X \) with the following properties:

- \( K \subset \bigcup_{k=1}^{N} U_{z_k} \);
- there exist a relatively closed locally pluripolar subset \( \widehat{M}_{z_k} \) of \( \mathcal{K}(A_0 \cap U_{z_k}, B_0 \cap V; D \cap U_{z_k}, G \cap V) \) and a mapping \( \hat{f}_{z_k} \in \mathcal{O}(\mathcal{K}(A_0 \cap U_{z_k}, B_0 \cap V; D \cap U_{z_k}, G \cap V) \setminus \widehat{M}_{z_k}, Z) \) which admits the \( \mathcal{A} \)-limit \( f(\zeta, \eta) \) at every point \((\zeta, \eta) \in \mathcal{K}(A_0 \cap U_{z_k}, B_0 \cap V; D \cap U_{z_k}, G \cap V) \setminus M \).

Now put

\[ D' := (D \cap \bigcup_{j=1}^{M} U_{a_j}) \cup \bigcup_{k=1}^{N} U_{z_k}. \]

For any \( 0 < \delta < 1 \) put

\[
B_{\delta} := \{ w \in G \cap V : \omega(w, B_0 \cap V; G \cap V) < \delta \}, \quad V_{\delta} := B_{1-\delta},
\]

\[
U_{a_j, \delta} := \{ z \in D \cap U_{a_j} : \omega(z, A_0 \cap U_{a_j}; D \cap U_{a_j}) < \delta \}, \quad j = 1, \ldots, M;
\]

\[
U_{z_k, \delta} := \{ z \in D \cap U_{z_k} : \omega(z, A_0 \cap U_{z_k}; D \cap U_{z_k}) < \delta \}, \quad k = 1, \ldots, N;
\]

\[
A_{\delta} := \left( \bigcup_{j=1}^{M} U_{a_j, \delta} \cup \bigcup_{k=1}^{N} U_{z_k, \delta} \right) \cap D',
\]

\[
D_{\delta} := \left( \bigcup_{j=1}^{M} U_{a_j, 1-\delta} \cup \bigcup_{k=1}^{N} U_{z_k, 1-\delta} \right) \cap D'.
\]

Observe that \( A_{\delta} \) and \( D_{\delta} \) are open subsets of \( D' \), and \( B_{\delta} \) and \( V_{\delta} \) are open subsets of \( V \). Moreover, \( D_{\delta} \not\subset D' \), \( V_{\delta} \not\subset V \) and

\[ \omega(\cdot, A_0, D_{\delta}) \searrow \omega(\cdot, A_0, D') \quad \text{and} \quad \omega(\cdot, B_0 \cap V, V_{\delta}) \searrow \omega(\cdot, B_0, V) \quad (10) \]

as \( \delta \searrow 0 \). Using the above constructions, we may glue the mappings \( (\hat{f}_{a_j})_{j=1}^{M} \cup (\hat{f}_{z_k})_{k=1}^{N} \) together in order to obtain a relatively closed locally pluripolar subset \( M_{\delta} \) of \( \mathcal{K}(A_{\delta}, B_{\delta}; D_{\delta}, V_{\delta}) \) and a mapping \( \tilde{f}_{\delta} \in \mathcal{O}_{\delta}(\mathcal{K}(A_{\delta}, B_{\delta}; D_{\delta}, V_{\delta}) \setminus M_{\delta}, Z) \). Applying Theorem 5.3 to \( \tilde{f}_{\delta} \), we get a relatively closed locally...
Similarly, we may find a finite number of points $b$ such that for every mapping $(\hat{f})_0^{<\delta<1}$ together in order to obtain a relatively closed locally pluripolar subset $\hat{M}$ of $\hat{X}(A_0, B_0 \cap V; D', V)$ and a mapping $\hat{f} \in \mathcal{O}(\hat{X}(A_0, B_0 \cap V; D', V) \setminus \hat{M}, Z)$.

**Proposition 6.2.** We keep the hypotheses and notation of the Main Theorem. Let $A_0$ (resp. $B_0$) be a subset of $\overline{D}$ (resp. $\overline{G}$) such that $A_0$ and $B_0$ are locally pluriregular and that $\overline{A_0} \subset A^*$ and $\overline{B_0} \subset B^*$ and that $\overline{A_0}, \overline{B_0}$ are compact. Then there exists a relatively closed pluripolar subset $\hat{M}$ of $\hat{X}(A_0, B_0; D, G)$ such that for every mapping $f : W \to Z$ which satisfies conditions (i)–(iii) of the Main Theorem, there exists a unique mapping $\hat{f}$ defined and holomorphic on $\hat{X}(A_0, B_0; D, G) \setminus \hat{M}$ which admits $A$-limit $f(\zeta, \eta)$ at all points $(\zeta, \eta) \in \hat{X}(A_0, B_0; D, G) \setminus M$.

**Proof of Proposition 6.2.** It is divided into two steps.

**Step 1.** There exists an open neighborhood $U_0$ of $\overline{A_0}$ in $X$ (resp. $V_0$ of $\overline{B_0}$ in $Y$) with the following property: Given $K \subset D$ and $L \subset G$, then there exists an open set $D'$ (resp. $G'$) which contains $(U_0 \cap D) \cup K$ (resp. $(V_0 \cap G) \cup L$) and a relatively closed pluripolar subset $\hat{M}$ of $\hat{X}(A_0, B_0; D', G')$ such that: for every mapping $f : W \to Z$ which satisfies conditions (i)–(iii) of the Main Theorem, there exists a unique mapping $\hat{f}$ defined and holomorphic on $\hat{X}(A_0, B_0; D', G') \setminus \hat{M}$ which admits $A$-limit $f(\zeta, \eta)$ at all points $(\zeta, \eta) \in \hat{X}(A_0, B_0; D', G') \setminus M$.

**Proof of Step 1.** Since $\overline{B_0}$ is compact, we may apply Proposition 6.1 to all $b \in \overline{B_0}$. Consequently, we may find a finite number of points $b_1, \ldots, b_N \in \overline{B_0}$, their respective open neighborhoods $V_{b_1}, \ldots, V_{b_N}$ in $Y$, and an open set $D'$ of the form $D' = U \cap D$, where $U$ is open open neighborhood of $\overline{A_0} \cup K$ in $X$, with the following properties:

- $\overline{B_0} \subset \bigcup_{k=1}^N V_{b_k}$;
- there exist a relatively closed locally pluripolar subset $\hat{M}_{b_k}$ of $\hat{X}(A_0, B_0 \cap V_{b_k}; D', G \cap V_{b_k})$ and a mapping $\hat{f}_{b_k} \in \mathcal{O}(\hat{X}(A_0, B_0 \cap V_{b_k}; D', G \cap V_{b_k}) \setminus \hat{M}_{b_k}, Z)$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in \hat{X}(A_0, B_0 \cap V_{b_k}; D', G \cap V_{b_k}) \setminus \hat{M}_{b_k}$.

Similarly, we may find a finite number of points $a_1, \ldots, a_M \in \overline{A_0}$, their respective open neighborhoods $U_{a_1}, \ldots, U_{a_M}$ in $X$, and an open set $G'$ of the form $G' = V \cap G$, where $V$ is open open neighborhood of $\overline{B_0} \cup L$ in $Y$, with the following properties:

- $\overline{A_0} \subset \bigcup_{j=1}^M U_{a_j}$;
- there exist a relatively closed locally pluripolar subset $\hat{M}_{a_j}$ of $\hat{X}(A_0 \cap U_{a_j}, B_0; D \cap U_{a_j}, G')$ and a mapping $\hat{f}_{a_j} \in \mathcal{O}(\hat{X}(A_0 \cap U_{a_j}, B_0; D \cap U_{a_j}, G') \setminus \hat{M}_{a_j}, Z)$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in \hat{X}(A_0 \cap U_{a_j}, B_0; D \cap U_{a_j}, G') \setminus \hat{M}_{a_j}$.
Observe that $A_δ$ and $D_δ'$ are open subsets of $D'$, and $B_δ$ and $G_δ'$ are open subsets of $G'$. Moreover, $D_δ' \not\supset D'$, $G_δ' \not\supset G'$ and
\[
\omega(\cdot, A_0, D_δ') \not\supset \omega(\cdot, A_0, D') \quad \text{and} \quad \omega(\cdot, B_0, G_δ') \not\supset \omega(\cdot, B_0, G')
\]
(11) as $\delta \searrow 0$. Using the above constructions, we may glue the mappings $(\hat{f}_n)_0 \leq \delta < 1$ together in order to obtain a relatively closed locally pluripolar subset $M_δ$ of $\mathcal{X}(A_δ, B_δ; D_δ', G_δ')$ and a mapping $\hat{f}_δ \in \mathcal{O}_δ(\mathcal{X}(A_δ, B_δ; D_δ', G_δ') \setminus M_δ, Z)$. Applying Theorem 5.3 to $\hat{f}_δ$, we get a relatively closed locally pluripolar subset $\hat{M}_δ$ of $\hat{\mathcal{X}}(A_δ, B_δ; D_δ', G_δ')$ and a mapping $\hat{f}_δ \in \mathcal{O}(\hat{\mathcal{X}}(A_δ, B_δ; D_δ', G_δ') \setminus \hat{M}_δ, Z)$. Now using (11), we may glue the mappings $(\hat{f}_δ)_0 \leq \delta < 1$ together in order to obtain a relatively closed locally pluripolar subset $\hat{M}$ of $\mathcal{X}(A_0, B_0; D', G')$ and a mapping $\hat{f} \in \mathcal{O}(\hat{\mathcal{X}}(A_0, B_0; D', G') \setminus \hat{M}, Z)$.

\textbf{Step 2. End of the proof.}

\textit{Proof of Step 2.}

Fix an increasing sequence $(K_n)_{n=1}^{\infty}$ (resp. $(L_n)_{n=1}^{\infty}$) of relatively compact subsets of $D$ (resp. $G$) such that $K_n \not\supset D$ (resp. $L_n \not\supset G$) as $n \not\to \infty$. Applying Step 1, we may find an increasing sequence $(D_n)_{n=1}^{\infty}$ (resp. $(G_n)_{n=1}^{\infty}$) of open subsets of $D$ (resp. $G$) such that $U_0 \cap D \subset D_n \subset D$ and $D_n \not\supset D$ (resp. $V_0 \cap G \subset G_n \subset G$ and that $G_n \not\supset G$) as $n \not\to \infty$. Moreover, there exists a relatively closed pluripolar subset $\hat{M}_n$ of $\hat{\mathcal{X}}(A_0, B_0; D_n, G_n)$ such that for every mapping $f : W \to Z$ which satisfies conditions (i)–(iii) of the Main Theorem, there exists a unique mapping $\hat{f}_n$ defined and holomorphic on $\hat{\mathcal{X}}(A_0, B_0; D_n, G_n) \setminus \hat{M}_n$ which admits $A$-limit $f(\zeta, \eta)$ at all points $(\zeta, \eta) \in \mathcal{X}(A_0, B_0; D_n, G_n) \setminus M$. Using the above mentioned properties of the sequences $(D_n)_{n=1}^{\infty}$ and $(G_n)_{n=1}^{\infty}$, it can be checked that on $D$ and on $G$,
\[
\omega(\cdot, A_0, D_n) \not\supset \omega(\cdot, A_0, D) \quad \text{and} \quad \omega(\cdot, B_0, G_n) \not\supset \omega(\cdot, B_0, G)
\]
as $n \not\to \infty$. Using this we may glue the sequence $(\hat{f}_n)_{n=1}^{\infty}$ in order to obtain a new mapping $\hat{f} \in \mathcal{O}(\hat{\mathcal{X}}(A_0, B_0; D, G) \setminus \hat{M}, Z)$, where $\hat{M}$ is a relatively pluripolar subset of $\hat{\mathcal{X}}(A_0, B_0; D, G)$. □

Now we arrive at

\textbf{Proof of the Main Theorem.} Combining Proposition 6.2 and Proposition 3.11, the Main Theorem follows. □
7 Applications

In [17] the first author gives various applications of the Main Theorem for the case where $M = \emptyset$ using three systems of approach regions. These are the canonical one, the system of angular approach regions and the system of conical approach regions. We only give here some applications of the system of conical approach regions. We leave the reader to treat the two first cases, that is, to translate Theorem 10.2 and 10.3 of [17] into the new context of the Main Theorem.

Let $X$ be an arbitrary complex manifold and $D \subset X$ an open subset. We say that a set $A \subset \partial D$ is locally contained in a generating manifold if there exist an (at most countable) index set $J \neq \emptyset$, a family of open subsets $(U_j)_{j \in J}$ of $X$ and a family of generating manifolds $(M_j)_{j \in J}$ such that $A \cap U_j \subset M_j$, $j \in J$, and that $A \subset \bigcup_{j \in J} U_j$. The dimensions of $M_j$ may vary according to $j \in J$.

Suppose that $A \subset \partial D$ is locally contained in a generating manifold. Then we say that $A$ is of positive size if under the above notation $\sum_{j \in J} \text{mes}_{M_j}(A \cap U_j) > 0$, where $\text{mes}_{M_j}$ denotes the Lebesgue measure on $M_j$. A point $a \in A$ is said to be a density point of $A$ if it is a density point of $A \cap U_j$ on $M_j$ for some $j \in J$. Denote by $A'$ the set of density points of $A$.

Suppose now that $A \subset \partial D$ is of positive size. We equip $D$ with the system of conical approach regions supported on $A$. Using the work of B. Jöricle (see, for example, Theorem 3, pages 44–45 in [13]), one can show that $A$ is locally pluriregular at all density points of $A$ and $A' \subset \tilde{A}$.

Consequently, it follows from Definition 2.3 that

$$\tilde{\omega}(z, A, D) \leq \omega(z, A', D), \quad z \in D.$$ 

This estimate, combined with the Main Theorem, implies the following result.

**Theorem 7.1.** Let $X$, $Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two open sets, and let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$). $D$ (resp. $G$) is equipped with a system of conical approach regions $(A_\alpha(\zeta))_{\zeta \in D, \alpha \in I_\zeta}$ (resp. $(A_\beta(\eta))_{\eta \in G, \beta \in I_\eta}$) supported on $A$ (resp. on $B$). Suppose in addition that $A$ and $B$ are of positive size. Define

$$W' := \mathbb{X}(A', B'; D, G),$$
$$\tilde{W'} := \{(z, w) \in D \times G : \omega(z, A', D) + \omega(w, B', G) < 1\},$$

where $A'$ (resp. $B'$) is the set of density points of $A$ (resp. $B$). Let $M$ be a subset of $W$ with the following properties:

- $M$ is thin in fibers (resp. relatively closed locally pluripolar in fibers) over $A$ and over $B$;
- $M$ is relatively closed in $W$;

$\text{mes}_{M_j}$ denotes the Lebesgue measure on $M_j$. A differentiable submanifold $M$ of a complex manifold $X$ is said to be a generating manifold if for all $\zeta \in M$, every complex vector subspace of $T_{\zeta}X$ containing $T_{\zeta}M$ coincides with $T_{\zeta}X$. A complete proof will be available in [18].

\[23\]
Then there exists an analytic (resp. a relatively closed locally pluripolar) subset \( \hat{M} \) of \( \hat{W'} \) such that:

for every mapping \( f : W \setminus M \rightarrow Z \) satisfying the following conditions:

(i) \( f \in C_s(W \setminus M, Z) \cap O_s(W^o \setminus M, Z) \);

(ii) \( f \) is locally bounded along \( X(A, B; D, G) \setminus M \);

(iii) \( f |_{(A \times B)} \) is continuous,

there exists a unique mapping \( \hat{f} \in O(\hat{W'} \setminus \hat{M}, Z) \) which admits the \( A \)-limit \( f(\zeta, \eta) \) at every point \((\zeta, \eta) \in (W \cap W') \setminus M \).

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References


