ON A SINGULAR QUADRATIC INTEGRAL EQUATION
OF VOLTERRA TYPE WITH SUPRENUM

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Abstract
We present an existence theorem of monotonic solutions for a singular quadratic integral equation of Volterra type with supremum in C[0, 1]. The concept of measure of noncompactness and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

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1 Introduction

In this paper, we establish the existence of the solution, defined on $[0, 1]$ for a singular quadratic integral equation of Volterra type with supremum, namely

$$x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{k(t, s)}{(t-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds,$$  \hspace{1cm} (1.1)

$t \in [0, 1]$, $0 < \alpha \leq 1$. Let us recall that the function $f = f(t, x)$ involved in Eq.(1.1) generates the superposition operator $F$ defined by

$$(Fx)(t) = f(t, x(t)), \hspace{1cm} (1.2)$$

where $x = x(t)$ is an arbitrary function defined on $[0, 1]$ (cf. [1], [13]).

Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, and in the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can very often be encountered in many applications (cf.[3, 4, 6, 9, 10, 15]). On the other hand, integral equations of Volterra type with supremum have been studied in [2, 14], among others. These equations can be considered with connection to the Cauchy problem [8]:

$$x'(t) = f(t) \cdot \max_{[0,t]} |x(\tau)|, \hspace{0.5cm} x(0) = 0.$$  

More recently, Caballero et al [8] investigated the so-called quadratic integral equations of Volterra type with supremum and proved the existence of monotonic solutions in $C[0, 1]$.

Using the technique associated with measures of noncompactness we show that equation (1.1) has solutions belonging to $C[0, 1]$ and is nondecreasing on the interval $[0, 1]$. In fact, our paper is motivated by the extension and generalization of the work of Caballero et al [8] and the work of the author [11] based on the measure of noncompactness and fixed point theorem due to Darbo.

2 Auxiliary facts and results

This section is devoted to collect some definitions and results which will be used throughout this paper.

Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element $0$. Let $B(x, r)$ denote the closed ball centered at $x$ and with radius $r$. The symbol $B_r$ stands for the ball $B(0, r)$. If $X$ is a subset of $E$, then $X$ and $\text{Conv} X$ denote the closure and convex closure of $X$, respectively. The symbols $\lambda X$ and $X + Y$ denote the usual algebraic operators on sets. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [7]:
Definition 2.1 A mapping \( \mu : \mathcal{M}_E \rightarrow [0, +\infty) \) is said to be a measure of noncompactness in \( E \) if it satisfies the following conditions:

1) The family \( \text{Ker} \mu = \{ X \in \mathcal{M}_E : \mu(X) = 0 \} \) is nonempty and \( \text{Ker} \mu \subset \mathcal{N}_E \).

2) \( X \subset Y \Rightarrow \mu(X) \leq \mu(Y) \).

3) \( \mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X) \).

4) \( \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y) \) for \( 0 \leq \lambda \leq 1 \).

5) If \( X_n \in \mathcal{M}_E, X_n = \overline{X}_n, X_{n+1} \subset X_n \) for \( n = 1, 2, 3, ... \) and \( \lim_{n \to \infty} \mu(X_n) = 0 \) then \( \cap_{n=1}^{\infty} X_n \neq \emptyset \).

The family ker \( \mu \) described above is called the kernel of the measure of noncompactness \( \mu \).

In what follows we will work in the Banach space \( C[0,1] \) consisting of all real functions defined and continuous on \([0,1]\). For convenience, we write \( I \) and \( C(I) \) instead of \([0,1]\) and \( C[0,1] \), respectively. The space \( C(I) \) is equipped with the standard norm

\[
\| x \| = \max\{|x(t)| : t \geq 0\}
\]

Now, we recollect the construction of the measure of noncompactness in \( C(I) \) which will be used in the next section (see [4], [5]).

Let us fix a nonempty and bounded subset \( X \) of \( C(I) \). For \( x \in X \) and \( \varepsilon \geq 0 \) denoted by \( \omega(x, \varepsilon) \), the modulus of continuity of the function \( x \), i.e.,

\[
\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}
\]

Further, let us put

\[
\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}, \quad \omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).
\]

Define

\[
d(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I, t \leq s\}
\]

and

\[
d(X) = \sup\{d(x) : x \in X\}.
\]

Observe that, all functions belonging to \( X \) are nondecreasing on \( I \) if and only if \( d(X) = 0 \).

Now, let us define the function \( \mu \) on the family \( \mathcal{M}_{C(I)} \) by the formula

\[
\mu(X) = \omega_0(X) + d(X).
\]

The function \( \mu \) is a measure of noncompactness in the space \( C(I) \) [5].

We will make use of the following fixed point theorem due to Darbo [12]. To quote this theorem, we need the following definition
**Definition 2.2** Let $M$ be a nonempty subset of a Banach space $E$, and $\mathcal{P} : M \rightarrow E$ be a continuous operator that transforms bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with constant $k \geq 0$) with respect to a measure of noncompactness $\mu$ if for any bounded subset $X$ of $M$ we have

$$\mu(\mathcal{P}X) \leq k \mu(X).$$

If $\mathcal{P}$ satisfies the Darbo condition with $k < 1$ then it is called a contraction operator with respect to $\mu$.

**Theorem 2.3** [12]
Let $Q$ be a nonempty, bounded, closed and convex subset of the space $E$ and let

$$\mathcal{P} : Q \rightarrow Q$$

be a contraction with respect to the measure of noncompactness $\mu$.

Then $\mathcal{P}$ has a fixed point in the set $Q$.

**Remark 2.4** [7]
Under the assumptions of the above theorem it can be shown that the set $\text{Fix } \mathcal{P}$ of fixed points of $\mathcal{P}$ belonging to $Q$ is a member of $\text{Ker } \mu$.

Finally, we will need the following two lemmas

**Lemma 2.5** Suppose $x \in C(I)$ and we define

$$(Gx)(t) = \max_{[a(t), b(t)]} |x(\tau)| \text{ for } t \in I.$$ 

Then $Gx \in C(I)$.

**Lemma 2.6** Let $(x_n), x \in C(I)$. Suppose that $x_n \rightarrow x$ in $C(I)$. Then $Gx_n \rightarrow Gx$ uniformly on $I$.

The proof of the above two lemmas are given in [8].

**3 Main Theorem**

In this section, we will study Eq.(1.1) assuming that the following hypotheses are satisfied:

(a1) $a : I \rightarrow \mathbb{R}$ is a continuous, nondecreasing and nonnegative function on $I$.

(a2) $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a nonnegative constant $c$ such that

$$|f(t, x) - f(t, y)| \leq c |x - y|$$

for all $t \in I$ and $x, y \in \mathbb{R}$. Moreover $f : I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$
Let the hypothesis (\(y \in C(I)\)) be satisfied.

Proof: Let \(\mathcal{K}\) and \(\mathcal{F}\) be the two operators defined on the space \(C(I)\) by

\[
(\mathcal{K}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds
\]

and

\[
(\mathcal{F}x)(t) = a(t) + f(t,x(t)) (\mathcal{K}x)(t),
\]

respectively. Solving Eq.(1.1) is equivalent to finding a fixed point of the operator \(\mathcal{F}\) defined on the space \(C(I)\).

First, we prove that \(\mathcal{F}\) transforms the space \(C(I)\) into itself. To do this it suffices to show that if \(x \in C(I)\) then \(\mathcal{K}x \in C(I)\). Fix \(\varepsilon > 0\), let \(x \in C(I)\) and \(t_1, t_2 \in I\) such that \(t_2 \geq t_1\) and \(|t_2 - t_1| \leq \varepsilon\). Then

\[
|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \right|
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds
\]

\[
\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \right|
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds
\]

\[
+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \right|
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds
\]

\[
+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \right|
\]

\[
\int_0^{t_1} \frac{k(t_1,s)}{(t_2-s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \]

Theorem 3.1 Let the hypothesis \((a_1)-(a_6)\) be satisfied. Then Eq.(1.1) has at least one solution \(y \in C(I)\) being nondecreasing on the interval I.
Thus, the operator $F$ transforms the ball $B_{\varepsilon}$ into itself.

In what follows we will consider the operator $F$ on the subset $B_{r_0}^+$ of the ball $B_{r_0}$ defined by

$$B_{r_0}^+ = \{ x \in B_{r_0} : x(t) \geq 0, \text{ for } t \in I \}.$$
Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. In view of these facts and assumptions $(a_1)$, $(a_3)$ and $(a_5)$, we deduce that $\mathcal{F}$ transforms the set $B_{r_0}^+$ into itself.

Next, we prove that the operator $\mathcal{F}$ is continuous on $B_{r_0}^+$. To do this, let us fix $\{x_n\}$ to be a sequence in $B_{r_0}^+$ such that $x_n \to x$ and we will prove that $\mathcal{F}x_n \to \mathcal{F}x$. In fact, for each $t \in I$ we have

$$
|\mathcal{F}(x_n)(t) - (\mathcal{F}x)(t)| = \left| \int_0^t \frac{k(t,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x_n(\tau)| \right) ds 
- \int_0^t f(t,x(t)) \frac{k(t,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| \int_0^t \frac{k(t,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x_n(\tau)| \right) ds 
- \int_0^t f(t,x(t)) \frac{k(t,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
+ \left| \int_0^t f(t,x(t)) \frac{k(t,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^t f(t,x(t)) \frac{k(t,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \frac{1}{\Gamma(\alpha)} \left| f(t,x_n(t)) - f(t,x(t)) \right| \int_0^t \frac{k(t,s)}{\Gamma(\alpha)(t - s)^{1-\alpha}} \left( \max_{[b,b]} |x_n(\tau)| - \max_{[b,b]} |x(\tau)| \right) ds
+ \frac{1}{\Gamma(\alpha)} \int_0^t \left| \frac{k(t,s)}{(t - s)^{1-\alpha}} \left( \max_{[b,b]} |x_n(\tau)| - \max_{[b,b]} |x(\tau)| \right) ds \right|
$$

In virtue of Lemma 2.6, we get

$$
||\mathcal{F}x_n - \mathcal{F}x|| \leq \frac{c \|k\| r_0}{\Gamma(\alpha + 1)} \|x_n - x\| + \frac{\|k\| (c r_0 + m)}{\Gamma(\alpha + 1)} \|x_n - x\|. \quad (3.4)
$$

This proves that $\mathcal{F}$ is continuous in $B_{r_0}^+$.

Now, let us take a nonempty set $X \subset B_{r_0}^+$. Fix arbitrarily the number $\varepsilon > 0$ and choose $x \in X$ and $t_1$, $t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may assume that $t_2 \geq t_1$. Then, in view of our assumptions, we obtain

$$
|\mathcal{F}(x)(t_2) - (\mathcal{F}x)(t_1)| = \left| a(t_2) + \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- a(t_1) + \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq \left| a(t_2) - a(t_1) \right| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_2} \frac{k(t_2,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds 
- \int_0^{t_1} \frac{k(t_1,s)}{\Gamma(\alpha)} \left( \max_{[b,b]} |x(\tau)| \right) ds \right|
on $I$ from the last inequality, we then have

$$\sup_{[0,b(s)]} |x(\tau)| = \max_{[0,b(s)]} |x(\tau)| = \max_{[0,b(s)]} |x(\tau)|.$$

Consequently,

$$Hence,$$
In what follows, fix arbitrary \(x \in X\) and \(t_1, t_2 \in I\) with \(t_2 > t_1\). Then, taking into account our assumptions, we have

\[
|\mathcal{F}(t_2) - \mathcal{F}(t_1)| - |\mathcal{F}(t_2) - \mathcal{F}(t_1)| = a(t_2) + f(t_2, x(t_2)) \int_0^{t_2} k(t_2, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- a(t_1) - f(t_1, x(t_1)) \int_0^{t_1} k(t_1, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- \left[ a(t_2) + f(t_2, x(t_2)) \int_0^{t_2} k(t_2, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- a(t_1) - f(t_1, x(t_1)) \int_0^{t_1} k(t_1, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \right] \\
\leq \{|a(t_2) - a(t_1)| - |a(t_2) - a(t_1)| \} \\
+ \left[ \int_0^{t_2} k(t_2, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- f(t_1, x(t_1)) \int_0^{t_1} k(t_1, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
+ \int_0^{t_1} k(t_1, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
n \leq \{|f(t_2, x(t_2)) - f(t_1, x(t_1))| - |f(t_2, x(t_2)) - f(t_1, x(t_1))| \} \\
\times \frac{1}{ \Gamma(\alpha) } \int_0^{t_2} k(t_2, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
+ f(t_1, x(t_1)) \left[ \int_0^{t_2} k(t_2, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- \int_0^{t_1} k(t_1, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- \int_0^{t_2} k(t_2, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \\
- \int_0^{t_1} k(t_1, s) \left( \frac{1}{ \Gamma(\alpha) } \right)_{t_1}^{t_2} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \right] \\
\right) (3.6)
\]

Now, we will prove that

\[
\int_0^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds - \int_0^{t_1} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds \geq 0.
\]

In fact, we have

\[
\int_0^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds - \int_0^{t_1} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{[0,b(s)]} |x(\tau)| \, ds
\]
Therefore, this, together with (3.6), yields

\[
\int_0^{t_1} k(t_1, s) \max_{|b(s)|} x(\tau) \, ds - \int_0^{t_1} k(t_1, s) \max_{|b(s)|} x(\tau) \, ds = 0.
\]

Finally, (3.7) and (3.8) imply

\[
\int_0^{t_2} \frac{k(t_2, s)}{(t_2 - s)^{1-\alpha}} \max_{|b(s)|} x(\tau) \, ds - \int_0^{t_1} \frac{k(t_1, s)}{(t_1 - s)^{1-\alpha}} \max_{|b(s)|} x(\tau) \, ds \geq 0.
\]

This, together with (3.6), yields

\[
|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| - |\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} k(t_2, s) \max_{|b(s)|} x(\tau) \, ds \right| \leq \frac{\|k\| r_0}{\Gamma(\alpha + 1)} d(F x).
\]

The above estimate implies

\[
d(\mathcal{F} x) \leq \frac{\|k\| r_0}{\Gamma(\alpha + 1)} d(F x).
\]

Therefore,

\[
d(\mathcal{F} x) \leq c \frac{\|k\| r_0}{\Gamma(\alpha + 1)} d(x).
\]
and consequently,

$$d(\mathcal{F}X) \leq \frac{c \|k\| r_0}{\Gamma(\alpha + 1)} d(X).$$  \hspace{1cm} (3.9)$$

Finally, from (3.5) and (3.9) and the definition of the measure of noncompactness $\mu$, we obtain

$$\mu(\mathcal{F}X) \leq \frac{c \|k\| r_0 q}{\Gamma(\alpha + 1)} \mu(X).$$  \hspace{1cm} (3.10)$$

Now, the above obtained inequality together with the fact that $c \|k\| r_0 < \Gamma(\alpha + 1)$ enable us to apply Theorem 2.3, then Eq.(1.1) has at least one solution $x \in C(I)$. This completes the proof.

4 Example

Consider the following singular quadratic integral equation of Abel type with supremum, $\alpha = \frac{1}{2}$,

$$x(t) = t^2 + \frac{t}{(t^2 + 3)\Gamma(\frac{1}{2})} x(t) \int_0^t \frac{e^t}{\sqrt{t-s}} \max_{[0,\sqrt{t}]} |x(\tau)| \, ds.$$  \hspace{1cm} (4.1)$$

In this example, we have that $a(t) = t^2$ and this function satisfies assumption $(a_1)$ and $\|a\| = 1$. Moreover, $k(t,s) = e^t$ verifies assumption $(a_4)$ and $|k| = e$. The function $r$ is defined by $b(s) = \sqrt{s}$ and satisfies assumption $(a_5)$. Also, $f(t,x) = \frac{x^2 - x}{t^2 + 3}$ and satisfies assumption $(a_2)$ since $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ and

$$|f(t,x) - f(t,y)| \leq \frac{1}{4} |x - y|,$$

for all $x, y \in \mathbb{R}$ and $t \in I$. Moreover, the function $f$ satisfies assumption $(a_3)$. Indeed, taking an arbitrary nonnegative function $x \in C(I)$ and $t_1, t_2 \in I$ such that $t_2 \geq t_1$, we obtain

$$|\langle Fx \rangle(t_2) - \langle Fx \rangle(t_1)| - |\langle Fx \rangle(t_2) - \langle Fx \rangle(t_1)|$$

$$= [f(t_2,x(t_2)) - f(t_1,x(t_1))] - [f(t_2,x(t_2)) - f(t_1,x(t_1))]$$

$$= \left| \frac{t_2}{t_2^2 + 3} x(t_2) - \frac{t_1}{t_1^2 + 3} x(t_1) \right| - \left| \frac{t_2}{t_2^2 + 3} x(t_2) - \frac{t_1}{t_1^2 + 3} x(t_1) \right|$$

$$\leq \frac{t_2}{t_2^2 + 3} |x(t_2) - x(t_1)| + \frac{t_2}{t_2^2 + 3} - \frac{t_1}{t_1^2 + 3} |x(t_1)|$$

$$- \frac{t_2}{t_2^2 + 3} \frac{t_1}{t_1^2 + 3} |x(t_2) - x(t_1)|$$

$$\leq \frac{t_2}{t_2^2 + 3} d(x) \leq \frac{1}{4} d(x).$$

In this case the inequality (3.1) has the form

$$\Gamma(\frac{3}{2}) + \frac{1}{4} e r_0 \leq r_0 \Gamma(\frac{3}{2})$$

or

$$\Gamma(\frac{1}{2}) + \frac{e}{2} r_0 \leq r_0 \Gamma(\frac{1}{2})$$

and this admits

$$r_0 = \frac{2 \Gamma(\frac{1}{2})}{2 \Gamma(\frac{1}{2}) - e}$$
as a positive solution since $\Gamma(\frac{1}{2}) \approx 1.77245$. Moreover, as

$$e r_0 \|k\| = \frac{e}{4} \left[ \frac{2 \Gamma(\frac{1}{2})}{2 \Gamma(\frac{1}{2}) - e} \right]$$

$$= \frac{e \Gamma(\frac{2}{2})}{2 \Gamma(\frac{1}{2}) - e}$$

$$< \Gamma(\frac{3}{2}).$$

Theorem 3.1 guarantees that equation (4.1) has a nondecreasing solution.

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**References**


