ON DIFFERENTIABLE JACOBIAN MAPS OF THE PLANE

Roland Rabanal∗

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We say that $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an almost–area–preserving map if: (a) $F$ is a topological embedding, not necessarily surjective; and (b) there exist a constant $s > 0$ such that for every measurable set $B$, $\mu(F(B)) = s\mu(B)$ where $\mu$ is the Lebesgue measure. We give a condition for a differentiable map whose jacobian determinant is nonzero constant (Jacobian map), to be an almost–area–preserving map whose image is a convex set. The map $F$ is supposed to be just differentiable with the $B$–condition: “there does not exist a sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $F((x_k, y_k)) \to p \in \mathbb{R}^2$ and $DF_{(x_k, y_k)}$ has a real eigenvalue $\lambda_k$ satisfying $x_k \lambda_k \to 0$.” In particular, if for all $z$, the eigenvalues of the jacobian matrix $DF_z$ are both one (Unipotent map), $F$ is an almost–area–preserving map with convex image.

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∗rrabanal@ictp.it
Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be a differentiable (not necessarily \( C^1 \)) map for which its jacobian determinant at any point is different from zero. We will denote by \( \text{Spc}(F) \) the set of all eigenvalues of the jacobian matrix \( DF_z \), when \( z \in \mathbb{R}^2 \). Such global function shall be called \textbf{Jacobian map}, if for all \( z \in \mathbb{R}^2 \), its jacobian determinant \( \det(DF_z) \) is nonzero constant; in this way, a jacobian planar map \( F = (f, g) \) with \( f \) and \( g \) polynomials and \( \det(DF_z) = 1 \), shall be refereed as a \textbf{Keller map}. A global differentiable function \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) shall be called \textbf{Unipotent map}, if the eigenvalues of \( DF_z \) are equal one, that is, \( \text{Spc}(F) = \{1\} \). Thus, unipotent maps are always jacobian maps. It is important to observe that if \( F \) is any planar Keller map then, \( \text{Spc}(F) \subset S^1 \cup (\mathbb{R} \setminus \{0\}) \).

Of course, we can speak about jacobian and unipotent maps in other algebraic fields, for instance, the recent book [23] is devoted to explain about polynomial maps, the principal them is the by now notorious \textbf{Keller Jacobian Conjecture}, that is: “Any keller map from \( \mathbb{C}^n \) to \( \mathbb{C}^n \), is injective”(see also [1]). It has been shown that if all the unipotent polynomial maps are invertible in all dimensions, then the Keller Jacobian Conjecture is true.

In order to state the results of this article, we need the following.

**Definition 1.1.** We say that \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is an \textit{almost–area–preserving} map if:

(a) \( F \) is a topological embedding; that is, a globally injective local homeomorphism (but, \( F(\mathbb{R}^2) \) may be a proper subset of \( \mathbb{R}^2 \) not necessarily convex);

(b) there exist a constant \( s > 0 \) such that for every measurable set \( B \),

\[
\mu(F(B)) = s\mu(B)
\]

where, \( \mu \) is the Lebesgue measure.

**Remark 1.1.** If the graph of some almost–area–preserving map is an algebraic set by using the principle “Injectivity \( \Rightarrow \) Bijectivity”(which asserts that every continuous injective mapping of \( \mathbb{R}^n \) into itself whose graph is algebraic must be surjective, [17] and [19]) we obtain that such almost–area–preserving map will be bijective.

Before to presenting the results, let us give some examples of \textit{almost–area–preserving} maps. Of course, the standard area–preserving diffeomorphism of class \( C^1 \) give many examples [18, 14] for instance, the injective Keller maps.

1.–\textbf{Unipotent maps}: In [5] Chamberland proves that a real–analytic map of \( \mathbb{R}^2 \) into itself has a inverse if it is unipotent (see also [11] and [24]). Then, real–analytic–unipotent maps are almost–area–preserving maps. A proof for \( C^1 \)--maps of the plane appears in [2], in this paper Campbell gives a normal form for \( C^1 \)--unipotent maps of the plane into itself, this impressively result may be paraphrased as follows: “A \( C^1 \)--planar map \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is unipotent, if and only if, it has the form

\[
F(x, y) = (x + b\phi(ax + by) + c, y - a\phi(ax + by) + d)
\]
for all \( x, y \in \mathbb{R} \), where \( a, b, c, d \) are real constants and \( \phi \) is a function on a single variable which is of class \( C^1 \). Thus, any \( C^1 \)-unipotent map of the plane is bijective. Therefore, \( the \ C^1 \)-unipotent maps of the plane are also almost–area–preserving.

The approach used by Campbell to obtain this result, is highly dependent on the fact that the eigenvalue of every jacobian matrix both are equal one, so this proof admits no obvious modification if the eigenvalues are different. Specifically, Campbell subtracts the identity map from the original map to give a non–constant function whose jacobian matrix is nilpotent and proves that this new function is constant over straight lines. Nevertheless, as the author points out in [2, example 5] such normal form exist, in general, for maps that are defined everywhere, Campbell in that example gives a map \( H(x, y) \) from the (open) first quadrant of \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) which has a nilpotent jacobian matrix and it is constant on those parts of rays from the origin lying in that quadrant, but \( F(x, y) = H(x, y) + (x, y) \) does not have the normal form above.

2.–Maps with constant eigenvalues: In [6] Chamberland is devoted to study the \( C^1 \)-maps \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) whose jacobian matrices have constant eigenvalues, that is \( \text{Spc}(F) \) has at most two elements. These maps are completely characterized in two cases: (a) \( F \) is a \( C^1 \)-unipotent map, then \( F \) is of the form (1) and (b) \( F \) is a polynomial map, then \( F \) takes the form

\[
F(x, y) = (ax + by + \beta \varphi(\alpha x + \beta y) + e, cx + dy - \alpha \varphi(\alpha x + \beta y) + f)
\]

for some real constant \( a, b, c, d, e, f, \alpha, \beta \) and some polynomial \( \varphi \) of one variable. Furthermore, if \( F \) is a planar polynomial map and \( \text{Spc}(F) \) is bounded, Lemma 2.1 of [8] implies that, \( \text{Spc}(F) \) has at most two elements. Therefore, \( polynomial \ planar \ maps \ whose \ bounded \ \text{Spc}(F) \ does \ not \ meet \ the \ zero \ are \ almost–area–preserving \ maps. \)

This second characterization is present in [6, Theorem 1.2] as a result of Cima et. al [9]. Unfortunately, this normal form can not be extended to the class of real—analytic maps of \( \mathbb{R}^2 \) because, [6, Theorem 1.3] gives a real—analytic diffeomorphism of \( \mathbb{R}^2 \) into itself whose jacobian eigenvalue are \(-\frac{1}{2}\) and \( \frac{1}{2} \), but it does not have the form (2).

We first present the following theorem. It supports Conjecture 3 of [2], which reads as follows: “Any \( C^1 \)-unipotent map of \( \mathbb{R}^n \) to \( \mathbb{R}^n \), is injective”.

**Theorem 1.1.** Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be a differentiable map with \( \det(DF_z) \neq 0 \), for all \( z \in \mathbb{R}^2 \). If \( \text{Spc}(F) \) has at most two elements then, \( F \) is an almost–area–preserving map with convex image.

Notice that, Theorem 1.1 extends the last set of examples described in [2] and [6] to almost–area–preserving maps whose derivative must be discontinuous.

As we saw, the more natural examples of almost–area–preserving maps of class \( C^1 \) are the injective \( C^1 \)-Jacobian maps. However, it is well know that in the non–polynomial case, there exist Jacobian maps which are not injective an so, they are not almost–area–preserving maps. This
is shown by the following simple example $F(x, y) = \sqrt{2}(e^{\frac{x}{2}} \cos(\frac{y}{2} e^{-x}), e^{\frac{x}{2}} \sin(\frac{y}{2} e^{-x}))$. This non-injective jacobian map is mentioned in [7]. Consequently, the goal is to give sufficient conditions on differentiable maps to insure that is an almost–area–preserving map. In this context, we present the following condition on the real eigenvalues of $DF_z$.

**Definition 1.2.** We say that $F$ satisfies the $B$–condition if there does not exist an unbounded sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $F((x_k, y_k)) \to p \in \mathbb{R}^2$ and $DF_{(x_k, y_k)}$ has a real eigenvalue $\lambda_k$ satisfying $x_k \lambda_k \to 0$.

Notice that $F$ is has the property of this definition if and only if $-F$ satisfies the $B$–condition, because $\text{Spc}(F) \cap (\mathbb{C} \setminus \mathbb{R}) = \text{Spc}(-F) \cap (\mathbb{C} \setminus \mathbb{R})$ and for each $\lambda \in \mathbb{R}$ holds that $\lambda \in \text{Spc}(F)$ if and only if $-\lambda \in \text{Spc}(-F)$.

The $B$–condition was inspired in the assumption presented in [15] called (*) which claim “there does not exist a sequence $\mathbb{R}^2 \ni (x_k, y_k) \to \infty$ such that $F((x_k, y_k)) \to p \in \mathbb{R}^2$ and $DF_{(x_k, y_k)}$ has a real eigenvalue $\lambda_k$ satisfying $\lambda_k \to 0$.”

In the next theorem we also study the intrinsic relation between the asymptotic behavior of the real eigenvalues of the differential $DF_z$ and the global injectivity of the local diffeomorphism given by $F$.

**Theorem 1.2.** Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable jacobian map. If $F$ satisfies the $B$–condition, then $F$ is an almost–area–preserving map with convex image.

The next corollary was suggested by C. Gutierrez in the “xiv Brazilian Meeting of Topology”. For the proof we refer the reader to the end of Section 2.

**Corollary 1.1.** If $F$ is as in Theorem 1.2 and $\text{Spc}(F) \subset \{z \in \mathbb{C} : ||z|| < 1\}$, then $F$ has at most one fixed point.

If Theorem 1.2 is valid for maps $F$, it remains true for $-F$. In fact, if in such theorem we change the pair $\{F, \text{Spc}(F)\}$ by $\{-F, \text{Spc}(-F)\}$ we may see that its conclusion remains valid. Also, for each $A : \mathbb{R}^2 \to \mathbb{R}^2$ any arbitrary invertible linear map, we have that if $F$ is as in Theorem 1.2 then $A \circ F \circ A^{-1}$ is also an almost–area–preserving map with convex image.

Let us proceed to give an idea of the proof of Theorem 1.2, it shall be present in Section 2. First it will be used that the assumptions imply that the Local Inverse Function Theorem is true [4, 3] (see also [21]). As a consequence, the level curves $\{f = \text{constant}\}$ make up a $C^0$–foliation $\mathcal{F}(f)$ on the plane, without singularities. Moreover, if $L$ is any leaf of $\mathcal{F}(f)$, then it is a differentiable curve and $g|_L$ is strictly monotone; in particular $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are transversal to each other.

To prove Theorem 1.2, it will be seen that the foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) is topologically equivalent to the foliation, on the $(x, y)$-plane, induced by the form $dx$ — this foliation is made up by all the vertical straight lines—. The injectivity of $F$ will follow from the fact that $\mathcal{F}(f)$
and $\mathcal{F}(g)$ are topologically transversal everywhere, and the last result will obtain by the study of the geometrical behavior of $F$.

The study of differentiable almost–area–preserving maps on $\mathbb{R}^2$ is closer related to the problem of characterizing the injectivity of differentiable maps of $\mathbb{R}^2$. This characterization in terms of spectral conditions for maps of $\mathbb{R}^n$ has been studied, for instance, [7], [22], [16] and [13]. These works are related to the Keller Jacobian Conjecture, which in the plane is equivalent to the following question: is every Keller map of $\mathbb{R}^2$ to $\mathbb{R}^2$, an almost–area–preserving map?

Remark 1.2. Another interesting property of the Keller maps as in Corollary 1.1 (i.e with $\text{Spc}(F) \subset \{z \in \mathbb{C} : ||z|| < 1\}$), is obtained from Theorem B of [9]. This prove the existence of a unique fixed point of $F$ which is a global attractor for the discrete dynamical system generated by $F$. i.e for each $p \in \mathbb{R}^2$ the standard $\omega$–limit set of the orbit $(F^k(p))_{k \geq 0}$ is the one point set $\{z \in \mathbb{R}^2 : F(z) = z\}$.

This dynamical property is false for all the maps of class $C^1$ as shown the global diffeomorphism given in Theorem E of [9]. This planar map $G_a$ satisfies $G_a(0) = 0$ and $\text{Spc}(G_a) \subset \{z \in \mathbb{C} : ||z|| < 1\}$ but it has a periodic orbit. However, $G_a$ is not a jacobian map, this motives to present the next

Open question. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a jacobian map of class $C^1$ with $F(0) = 0$ and $\text{Spc}(F) \subset \{z \in \mathbb{C} : ||z|| < 1\}$. Does it follow that 0 is a global attractor for $F$?

Observe that Theorem 1.1 follows directly of Theorem 1.2.

This paper is organized as follows: In Section 2 it is show that the maps of Theorem 1.2 are injective maps whose image is convex, by proving a strong version about the injectivity. At the end of this second section we prove Corollary 1.1. In Section 3 we conclude the proof of Theorem 1.2.

2. Global injective result

This section is devoted to prove the following theorem. It also proves, in a strong way, the so–called Chamberland Conjecture [7] in dimension two (see also [10]).

**Theorem 2.1.** Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism such that, for some $\overline{D}_a = \{z \in \mathbb{R}^2 : ||z|| \leq \sigma\}$ with $\sigma > 0$, the restriction $F|_{\mathbb{R}^2 \setminus \overline{D}_a}$ is differentiable. If $F$ satisfies the $B$–condition, then $F$ is injective and $F(\mathbb{R}^2)$ is convex.

Theorem 2.1 improves the main injective results of [15, 10] and [12].

In order to prove the result of this section, we orient $\mathcal{F}(f)$ in agreement that if $L_p$ is an oriented leaf (or trajectory) of $\mathcal{F}(f)$ thought the point $p$, then the restriction $g|_{L_p}$ is an increasing function

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\(^1\)The orbit $(F^k(p))_{k \geq 0}$ is the planar set given by the iterations of the map: $\{F^k(p) : k \in \mathbb{N} \cup \{0\}\}$, where $F^0$ denotes the identity map.
in conformity with the orientation of $L_p$. This is possible because the function $g|_{L_p}$ is strictly monotone. Similarly, we can orient $\mathcal{F}(g)$. In this context, we consider the set

$$B = \{(x,y) \in [0,2] \times [0,2] : 0 < x + y \leq 2\},$$

and the map $h_0 : \mathbb{R}^2 \to \mathbb{R}$ given by $h_0(x,y) = xy$. Thus, for every $h \in \{f, g\}$, we will say that $A \subset \mathbb{R}^2$ is a **half-Reeb component** for $\mathcal{F}(h)$ if there is a homeomorphism $H : B \to A$ which is a topological equivalence between $\mathcal{F}(h)|_A$ and $\mathcal{F}(h_0)|_B$ such that:

(a) The segment $\{(x,y) \in B : x + y = 2\}$ is sent by $H$ onto a transversal section for the foliation $\mathcal{F}(h)$ in the complement of the point $H(1,1)$; this section is called the **compact edge** of $A$.

(b) Both segments $\{(x,y) \in B : x = 0\}$ and $\{(x,y) \in B : y = 0\}$ are sent by $H$ onto full half-trajectories of $\mathcal{F}(h)$. These two semi-trajectories of $\mathcal{F}(h)$ are called the **non-compact edges** of $A$.

![Figure 1. A half-Reeb component.](image)

Observe that $A$ may not be a closed subset of $\mathbb{R}^2$. The homeomorphism in its definition, does not need to be extended to infinity.

**Proposition 2.1.** Let $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism for which $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are well defined. If at least one of these foliations has no a Half Reeb component, then the following holds:

(a) The global map $F = (f,g)$ is injective.

(b) The image $F(\mathbb{R}^2)$ is a convex set.

**Proof.** We only consider the case related to $\mathcal{F}(f)$. In order to prove (a) we proceed by contradiction. Thus, we assume that there exist two points $p, q \in \mathbb{R}^2$ such that $F(p) = F(q) = (c,d)$. By using that the restriction $g|_L$ to any leaf of $\mathcal{F}(f)$ is strictly monotone, we must have that $p$ and $q$ belong to different connected components of $\{f = c\}$. Therefore, we conclude the proof of (a) from the next

**Assertion 1**: If some level curve $\{f = c\}$ is disconnected, then $\mathcal{F}(f)$ has a Half-Reeb component.
In fact, let \( \alpha_1 \) and \( \alpha_2 \) be two different connected components of \( \{ f = c \} \) and take two points \( p_1 \in \alpha_1 \) and \( p_2 \in \alpha_2 \). Let \( \Omega(p_1, p_2) \) be the set of all the compact arcs \( \Gamma_1 \) embedded in the plane such that: for \( i = 1, 2 \) \( \Gamma_1 \) meets \( \alpha_i \) transversally at \( p_i \).

Take \( \Gamma \in \Omega(p_1, p_2) \) which minimizes the tangent points with \( F(f) \). If we consider \( q \notin \{ p_1, p_2 \} \) a tangent point of \( F(f) \) with \( \Gamma \). Then, the trajectories of \( F(f) \) around \( q \) defines subintervals \( [p, q], [q, Tp] \) of \( \Gamma \) with \( [p, q] \cap [q, Tp] = \{ q \} \), and a homeomorphism \( T : (p, q) \to [q, Tp] \) such that,

(a.1) \( Tq = q \) and, for every \( x \in (p, q) \), there is an arc of trajectory \( [x, Tx]f \) of \( F(f) \), starting at \( x \), ending at \( Tx \) and meeting \( \Gamma \) exactly and transversally at \( \{ x, Tx \} \),

(a.2) the family \( \{(x, Tx)f : x \in (p, q)\} \) depends continuously on \( x \) and tends to the one point set \( \{ q \} \) as \( x \to q \).

Consider the arc \( [p, Tp] \subset \Gamma \) which is maximal with respect to properties (a.1) and (b.2) above. If one of the endpoints of \( [p, Tp] \) is tangent, we can perturb this \( [p, Tp] \subset \Gamma \) and meet an arc in \( \Omega(p_1, p_2) \) whose number of tangencies is smaller than of the number of tangencies of \( \Gamma \), which is a contradiction with our selection of \( \Gamma \). Therefore,

(b) if \( \Gamma \in \Omega(p_1, p_2) \) minimizes the tangent points. There exist an arc \( [p, Tp] \subset \Gamma \) which is maximal with respect to properties (a.1) and (a.2) and transversal to \( F(f) \) at its endpoints.

Since this arc \( [p, Tp] \) contain the compact edge of some Half-Reeb component of \( F(f) \), Assertion 1 is true.

For each \( \theta \in \mathbb{R} \) let \( R_\theta \) denote linear rotation

\[
R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

The next assertion can be deduced from [15].

**Assertion 2:** If \( F(f) \) has no a Half Reeb component, then \( F(\mathbb{R}^2) \) is convex.

In fact, let \( p, q \in F(\mathbb{R}^2) \) and let \( [p, q] = \{(1-t)p+tw : 0 \leq t \leq 1\} \). Take \( \theta \in \mathbb{R} \) so that \( R_\theta([p, q]) \) is contained in the vertical line \( x = c \). Assertion 1 implies that the level curve \( \{f_\theta = c\} \) is a connected subset of the straight line \( x = c \) connecting \( R_\theta(p) \) with \( R_\theta(q) \); that is \( R_\theta([p, q]) \subset F_\theta(\mathbb{R}^2) \) which implies that \( [p, q] \subset F(\mathbb{R}^2) \) and proves Assertion 2. This conclude the proof \( \square \)

The following proposition will be need. For the proof we refer the reader to [12].

**Lemma 2.1.** Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a map whose jacobian determinant at any point is non—zero and \( F(f) \) has a Half Reeb component, \( \mathcal{A} \). Let \( (f_\theta, g_\theta) = R_\theta \circ F \circ R_{-\theta} \) with \( \theta \in \mathbb{R} \). If \( \Pi(\mathcal{A}) \) is bounded, where \( \Pi : \mathbb{R}^2 \to \mathbb{R} \) is given by \( \Pi(x, y) = x \). Then, there is an \( \varepsilon > 0 \) such that, for all \( \theta \in (-\varepsilon, 0) \cup (0, \varepsilon) \) the foliation \( F(f_\theta) \) has a hrc \( \mathcal{A}_\theta \) for which \( \Pi(\mathcal{A}_\theta) \) is an interval of infinite length.

**Remark 2.1.** Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a map whose jacobian determinant at any point is different form zero.
(a) Suppose that $\mathcal{F}(f)$ has a half-Reeb component $\mathcal{A}$ whose projection $\Pi(\mathcal{A})$ is an interval of finite length, where $\Pi(x,y) = x$. Let $\mathcal{T}(x,y) = (-x,y)$ and $(f_T, g_T) = T \circ X \circ (T)^{-1}$. If $\Pi(\mathcal{A})$ is bounded from above, then $\mathcal{F}(f_T)$ has a half-Reeb component $\mathcal{A}_T$ such that $[a_0, +\infty) \subseteq \Pi(\mathcal{A}_T)$ for some $a_0 \in \mathbb{R}$.

(b) Let $\mathcal{T}(x,y) = (x,-y)$ and $(f_T, g_T) = T \circ X \circ (T)^{-1}$. If $\mathcal{F}(f)$ has a half-Reeb component $\mathcal{A}$ whose projection $\Pi(\mathcal{A})$ satisfies $[a_0, +\infty) \subseteq \Pi(\mathcal{A})$ for some $a_0 \in \mathbb{R}$. Then $\mathcal{F}(f_T)$ also has a half-Reeb component $\mathcal{A}_T$ with $[\tilde{a}_0, +\infty) \subseteq \Pi(\mathcal{A}_T)\tilde{T}$ for some $\tilde{a}_0 > \sigma$.

This remark is not difficult to check, because the linear map $T$ (resp. $\tilde{T}$) send the leaves of the foliation $\mathcal{F}(f)$ in leaves of $\mathcal{F}(f_T)$ (resp. $\mathcal{F}(f_{\tilde{T}})$). Moreover, by change of the respective compact edge we can assume that $a_0 > \sigma$ and $\tilde{a}_0 > \sigma$.

**Proof. Proof of Theorem 2.1.** From Proposition 2.1, we shall have established that $F = (f,g)$ will be injective with convex image if we prove that $\mathcal{F}(f)$ has no any unbounded half-Reeb component.

Suppose by contradiction that the foliation given by the level curves has an unbounded half-Reeb component. By Lemma 2.1 we can take a half-Reeb component whose projection is an unbounded interval; thus there are $a_0 > \sigma$ and a half-Reeb component $\mathcal{A}$ of $\mathcal{F}(f)$ such that $[a_0, +\infty) \subseteq \Pi(\mathcal{A})$, where $\Pi(x,y) = x$ (see the last remark). Then, if $a > a_0$ is large enough for any $x \geq a$, the line $\Pi^{-1}(x)$ intersects the two non-compact edges of $\mathcal{A}$, and so the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subseteq \mathcal{A}$ of $\mathcal{F}(f)|\mathcal{A}$ such that $\Pi(\alpha_x) \cap [x, +\infty) = \{x\}$. In other words, $x$ is the maximum for the restriction $\Pi|_{\alpha_x}$. The leaf $\alpha_x$ is a continuous curve, it follows that; if $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a compact subset of $\mathcal{A}$. So we can define two functions $H : (a, +\infty) \to \mathbb{R}$ by

$$H(x) = \sup \{y : (x,y) \in \alpha_x \cap \Pi^{-1}(x)\},$$

and $\varphi : (a, +\infty) \to \mathcal{A}$ by $\varphi(x) = f(x, H(x))$.

As proved in [12], $\varphi$ is bounded and a strictly monotone function such that, for some full measure subset $M \subset (a, +\infty)$ such function $\varphi$ is differentiable on $M$ and for all $x \in M$

$$DF_{(x,H(x))} = \begin{pmatrix} \varphi'(x) & 0 \\ g_x(x,H(x)) & g_y(x,H(x)) \end{pmatrix}.$$ 

In other words, if $x \in M$, then $\varphi'(x) = f_x'(x,H(x)) \in \text{Spc}(X)$.

To proceed we shall only consider the case in which $\varphi'(x) \geq 0$, because in the other case we can use $\lim_{x \to -\infty} x \varphi'(x)$.

If $\lim_{x \to -\infty} x \varphi'(x) = 0$, there exists a sequence $(x_k, H(x_k)) \to \infty$ such that the jacobian matrix $DF_{(x_k, H(x_k))}$ has a real eigenvalue $\lambda_k = \varphi'(x_k)$ for which $x_k \lambda_k = 0$ and $F(x_k, H(x_k))$ tends to a finite value in the closure $\overline{\mathcal{F}(A)}$ (which is compact). This contradicts the $B$-condition.

If $\lim_{x \to -\infty} x \varphi'(x) \neq 0$, then $\lim_{x \to -\infty} x \varphi'(x) > 0$, this implies that there are constants $a_0 \geq a$ and $\ell > 0$ such that $\ell \leq x \varphi'(x)$ if $x \geq a_0$. As $f|_{\mathcal{A}}$ is bounded, $\varphi$ is bounded too. Hence, there
is a constant $K > 0$ such that for all $x > \alpha_0$, $0 \leq \varphi(x) - \varphi(\alpha_0) \leq K$. Take $c_0 > \alpha_0$ so that

$$K < \int_{\alpha_0}^{c_0} \frac{\ell}{x} \, dx.$$ 

Then

$$K < \int_{\alpha_0}^{c_0} \frac{\ell}{x} \, dx \leq \int_{\alpha_0}^{c_0} \varphi'(x) \, dx \leq \varphi(c_0) - \varphi(\alpha_0) \leq K.$$ 

This contradiction proves the theorem.

**Corollary 2.1.** If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is as in Theorem 1.2, then $F$ is injective and $F(\mathbb{R}^2)$ is convex.

**Proof.** PROOF OF COROLLARY 1.1. Consider $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by $G(z) = F(z) - z$ for all $z \in \mathbb{R}^2$. This map has no positive eigenvalues because $\text{Spc}(G)$ is contained in $\{z \in \mathbb{R}^2 : \Re(z) < 0\}$ so it satisfies Theorem 2.1 because $\text{Spc}(G)$ avoid an open real neighborhood of the origin. Since $G$ is injective we conclude the proof.

3. Differentiable almost–area–preserving maps

In this section we conclude the proof of Theorem 1.2 which implies Theorem 1.1. This proof shall be completed at the end of this section. By [21, Theorem 3.4] if $\det(DF_z) \neq 0$ for all $z \in \mathbb{R}^2$, the non—connected function $z \mapsto \det(DF_z)$ has constant sign, so in the rest of this section we may assume that $\det(DF_z) > 0$.

**Lemma 3.1.** Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective differentiable (not necessarily $C^1$) map and let $s > 0$ a constant such that for all $z \in \mathbb{R}^2$, $|\det(DG_z)| = s$. If $B \subset \mathbb{R}^2$ is a measurable set and $G(B)$ has finite Lebesgue measure then

$$\mu(G(B)) = s\mu(B)$$

where $\mu$ is the Lebesgue-measure in $\mathbb{R}^2$.

**Proof.** If follows directly of applying [20, Corollary 10.6.10] (change of variables); because $\mu(G(B)) < \infty$, $G$ is a differentiable injective map and the Jacobian determinant $\det(DG_z)$ is constant.

**Proof.** PROOF OF THEOREM 1.2. By Corollary 2.1, $F$ is globally injective. Then, it is sufficient to prove that Statement (b) in the definition of almost–area–preserving maps is true.

Let $B \subset \mathbb{R}^2$ be any measurable set. We claim that

(a.1) If at least $\mu(F(B)) < \infty$ or $\mu(B) < \infty$ holds, then $\mu(F(B)) = s\mu(B)$ where $s = |\det(DF_z)|$.

In fact, by Lemma 3.1 (a.1) holds when $\mu(F(B)) < \infty$. In the other case, we have $\mu(B) < \infty$. We consider $A = F(B)$ and $G = F^{-1}$. As $\mu(G(A)) = \mu(B) < \infty$ we may apply Lemma 3.1 and obtain that $\mu(G(A)) = \frac{1}{s}\mu(A)$ because $|\det(DG_z)| = \frac{1}{s}$. From this, (a.1) holds.

(a.2) If $B$ is a measurable set of the plane $\mu(F(B)) = s\mu(B)$.

In fact, if we does not have the condition of (a.1) $\mu(F(B))$ and $\mu(B)$ have no finite measure, so (a.2) is true.

Therefore, $F$ is an almost–area–preserving map and conclude the proof.
Corollary 3.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be as in Theorem 1.2. If $(B_n)_n$ is an infinite sequence of measurable sets for which there exist a constant $\delta > 0$, such that for all $N \in \mathbb{N}$, $\mu(B_{N+1} \setminus \bigcup_{n=1}^{N} B_n) > \delta$. Then, $F(\bigcup_n B_n)$ has no finite measure.

Proof. Suppose, by contradiction, that $\mu(F(\bigcup_n B_n)) < \infty$. Consider, $A_1 = B_1$, and for every $N$ natural greater than one set $A_{N+1} = B_{N+1} \setminus \bigcup_{n=1}^{N} B_n$. Given $K \in \mathbb{N}$, by using the definition of almost–area–preserving, we have that

$$\mu\left(F\left(\bigcup_n B_n\right)\right) > \mu\left(\bigcup_{N=2}^{K+1} F(A_N)\right) = \sum_{N=2}^{K+1} \mu(A_N) = s\delta K.$$ 

Then, the natural numbers $N$ is bounded. This contradiction conclude the proof. \qed

Remark 3.1. Corollary 3.1 remains true if take $F$ as any almost–area–preserving map whose image may be non–convex. For instance, $F(x, y) = (\exp(x), y \exp(-x))$ for which $F(\mathbb{R}^2) \neq \mathbb{R}^2$.

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