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LIPSCHITZ SOLUTIONS TO THE ISOMETRY RELATION
FOR PAIRS OF RIEMANNIAN METRICS

Giuseppina D’Ambra\textsuperscript{1}

\textit{Department of Mathematics, University of Cagliari, Cagliari, Italy}

and

Mahuya Datta\textsuperscript{2}

\textit{Stat-Math Unit, Indian Statistical Institute,}
\textit{203, B.T. Road, Calcutta 700018, India}

\textit{and}

\textit{The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.}

MIRAMARE – TRIESTE
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\textsuperscript{1}dambra@unica.it
\textsuperscript{2}Regular Associate of ICTP. mahuya@isical.ac.in
1. Introduction

It is a classical result, due to Nash and Kuiper, which says that a Riemannian manifold \((M, g)\) admitting a \(C^\infty\) immersion in \(\mathbb{R}^q\) also admits a \(C^1\) immersion \(f : M \to \mathbb{R}^q\) such that \(f^*h = g\) provided \(q > n\), where \(h\) is the canonical metric on \(\mathbb{R}^q\). Gromov generalised this result via the method of convex integration by showing that if there exists a strictly short immersion of \((M, g)\) into another Riemannian manifold \((N, h)\) then there exists an isometric \(C^1\) immersion \(f : M \to N\), when \(\dim N > \dim M\). He further proved that in the equidimensional case, there are almost everywhere differentiable (Lipschitz) maps whose derivatives \(df\) are isometric almost everywhere on \(M\). By an abuse of language, such maps will be referred to as the Lipschitz isometric maps; classically, the Lipschitz maps which preserve the lengths of all rectifiable curves relative to the given metrics are referred to as isometric maps. Our notion of Lipschitz isometric maps satisfy a much weaker condition; in fact, such an \(f\) may collapse a submanifold of positive codimension in \(M\) to a single point.

In this paper we generalise the above mentioned result of Gromov when both the manifolds \(M\) and \(N\) come with a pair of Riemannian metrics.

Let \(\mathbb{R}^q\) be the \(q\)-dimensional Euclidean space with two Euclidean metrics \(h_1\) and \(h_2\) which satisfy the following conditions: There exist two numbers \(0 < a < b\), such that

1. \(c^2h_1 - h_2\) is a non-degenerate indefinite form for each real number \(c\) lying in \([a, b]\),
2. \(r_+(a^2h_1 - h_2) \geq 2n\) and \(r_-(b^2h_1 - h_2) \geq 2n\), where \(r_+\) and \(r_-\) denote respectively the positive and the negative ranks of an indefinite metric.
3. We further assume that \(h_1 - h_2\) has distinct eigenvalues.

**Theorem 1.1.** Let \(M\) be a smooth manifold of dimension \(n\) with two Riemannian metrics \(g_1, g_2\) which are related by \(a^2g_1 < g_2 < b^2g_1\). Then there exists an almost everywhere differentiable (Lipschitz) map \(f : M \to \mathbb{R}^q\) satisfying \((df_x^*)h_i = g_i\) for \(i = 1, 2\) for almost all \(x \in M\). Moreover, such maps are \(C^0\) dense in the space of strictly \((g_1, g_2)\)-short maps (see Definition 5.1).

The maps \(f\) obtained in Theorem 1.1 will be referred to as Lipschitz isometric maps for pairs of metrics. We shall further observe, in Section 7, that under the above hypothesis we actually get a \(C^1\) isometric immersion when \(M\) is one dimensional.

In our earlier paper [1] we proved the existence of isometric \(C^1\)-immersions in Euclidean space \(\mathbb{R}^q\) for pairs of Riemannian metrics, generalizing the Nash-Kuiper \(C^1\)-immersion theorem when \(r_+(c^2h_1 - h_2) \geq 3n + 2\) for all \(c \in [a, b]\).

In the present paper our study of isometric \(C^1\)-immersions \(f : (M, g_1, g_2) \to (N, h_1, h_2)\) relies extensively on the convex integration theory which incorporates the essence of Kuiper’s approach [4]. The key idea of the method of convex integration can be stated as follows: If \(A\) is a connected subset of \(\mathbb{R}^q\) such that the interior of the convex hull of \(A\) contains the origin then
there is a $C^1$-map $f: S^1 \to \mathbb{R}^q$ whose derivative maps $S^1$ into $A$. This can be viewed as the convex integration over a circle.

We organize the paper as follows. We devote Section 2 to review the basic language of $h$-principle theory and convex integration techniques to deal with open first order partial differential relations. In Section 3 we introduce the notion of $(h_1, h_2)$-regularity for $C^1$-maps $f: M \to \mathbb{R}^q$ and study the geometry underlying the regularity condition which plays a crucial role in our treatment. In section 4 we prove the Main Lemma leading to theorem1.1 and in Section 5 we prove the existence of an approximate solution to our problem. The proof of the Main Theorem is given in Section 6. The one dimensional case is separately studied in Section 7 where we show that there exists, in fact, a $C^1$-solution.

2. Review of Convex Integration Techniques

In this section we recall the terminology of the theory of $h$-principle and discuss in brief the main result of convex integration technique following [2].

Let $f$ be a germ of some local $C^r$ map at $x \in M$. The $r$-jet of $f$ at $x$ is by definition the ordered tuple $(x, f(x), Df(x), \ldots, D^r f(x))$, where $D^k f$ denotes the derivative map of $f$ of order $k$. The collections of all such $r$-jets constitute the total space of a fibre bundle over $M$ which is denoted by $p^r: J^r(M, N) \to M$. The bundle is referred to as the $r$-jet bundle associated with the space of $C^r$ maps from $M$ to $N$.

If $r = 1$ then $j^1_f(x) = (x, f(x), Df(x))$ and $J^1(M, N)$ can be identified with the total space of the bundle $\text{Hom}(TM, TN)$.

A continuous map $\sigma: M \to J^1(M, N)$ is said to be a section if $p^r \circ \sigma = \text{id}_M$. If $f: M \to N$ is a $C^r$ map then its $r$-jet map $j^r_f$ defined by $j^r_f(x) = (x, f(x), Df(x), \ldots, D^r f(x))$ is a section of $p^r$.

**Definition 2.1.** An $r$-th order partial differential relation is a subset $\mathcal{R}$ of $J^r(M, N)$. A $C^r$ map $f: M \to N$ is said to be a solution of $\mathcal{R}$ if its $r$-jet map $j^r_f$ maps $M$ into $\mathcal{R}$.

A section of $p^r$ whose image is contained in $\mathcal{R}$ is called a formal solution of the differential relation. A formal solution of $\mathcal{R}$ is said to be holonomic if it is the $r$-jet map of some $C^r$ map $f: M \to N$.

A differential relation $\mathcal{R}$ is said to satisfy the $h$-principle if every formal solution $\sigma$ can be homotoped to a holonomic section in the space of all formal solutions.

**Definition 2.2.** Let $\Omega$ be an open subset of a manifold $M$. A continuous map $f: \Omega \to N$ into a manifold $N$ is said to be piecewise $C^r$ if there exists a countable system of mutually disjoint open sets $\Omega_j \subset \Omega$ which cover $\Omega$ up to a set of measure zero and the restriction of $f$ to each $\Omega_j$ is $C^r$.

Let $\mathcal{R} \subset J^r(M, N)$ be an $r$-th order differential relation. A piecewise $C^r$ map $f: M \to N$ is said to be a piecewise $C^r$-solution of $\mathcal{R}$ if $j^r_f(x) \in \mathcal{R}$ for all $x \in M$ where $f$ is differentiable.
The convex integration technique gives solutions to $h$-principle for some differential relations which satisfy certain convexity conditions. The key idea of the convex integration technique is stated in the following lemma.

**Lemma 2.3.** Let $A$ be a connected subset of $\mathbb{R}^q$ and let $0$ belong to the interior of the convex hull of $A$. Then there exists a $C^1$-map $f : [0, 1] \rightarrow \mathbb{R}^q$ such that $f'(t) \in A$ for almost all $t \in [0, 1]$. Moreover, $f$ can be made to lie in an arbitrary small neighbourhood of $0$.

If the connectivity condition on $A$ is dropped in the above lemma then it delivers a piecewise linear map $f$ such that $f(0) = f(1) = 0$ and $f'(t) \in A$ whenever $f$ is differentiable [2]. More generally we have the following.

**Proposition 2.4.** Let $R$ be an open subset of $J^1(\mathbb{R}, \mathbb{R}^q)$ and let $f : [0, 1] \rightarrow \mathbb{R}^q$ be a continuous function which is $C^1$ on $(0, 1)$. Suppose that $j^1(x)$ lies in the convex hull of $R(x)$ for all $x \in (0, 1)$, where $b(x) = (x, f(x)) \in J^0(\mathbb{R}, \mathbb{R}^q)$. Then $f$ can be homotoped to a piecewise $C^1$ solution $f_1$ of $R$ in any $C^0$ neighbourhood of $f$ such that $f_1(0) = f(0)$ and $f_1(1) = f(1)$.

**Proof.** Consider any $\varepsilon > 0$. Appealing to the one dimensional convex integration ([2, §17.3]) we can construct a piecewise linear map $f^1$ on the interval $[\varepsilon, 1 - \varepsilon]$ which coincides with $f$ at the boundary points and is a piecewise $C^1$ solution of $R$ on $(\varepsilon, 1 - \varepsilon)$. Next consider, for each $n \geq 1$, a pair of disjoint intervals $I_n = [\varepsilon/2^n, \varepsilon/2^{n-1}]$ and $J_n = [1 - \varepsilon/2^{n-1}, 1 - \varepsilon/2^n]$. The interior of these sets cover $[0, 1]$ up to a set of measure zero. Now, applying a result of [2] on the restriction of $f$ to $I_n \cup J_n$ we obtain a piecewise linear map $f^n$ on $I_n \cup J_n$ which coincides with $f$ at the endpoints and satisfies the differential relation except at the points where the derivative does not exist. Further, we can choose $f^n$ to be $\varepsilon/2^n$-close to $f$ on the set. Now all these maps patch together to give a piecewise linear map $f_1$ on $(0, 1)$. Further, this map extends continuously to the closed interval $[0, 1]$ and the extended map satisfies the desired conditions.

**Remark.** If $f$ is a solution of $R$ on a neighbourhood of some closed subset $K$, then the homotopy remains constant on some (possibly smaller) open neighbourhood of $K$.

The above result may be generalised to a parametric version following [2].

**Proposition 2.5.** Let $R$ be an open subset of $I^1 \times J^1(\mathbb{R}, \mathbb{R}^q)$ and $R_p = p \times J^1(\mathbb{R}, \mathbb{R}^q) \cap R$. Let $f : I^1 \times I \rightarrow \mathbb{R}^q$ be a continuous function which is $C^1$ in the interior of $I^1 \times I$. Let $f_p$ denote the restriction of $f$ to $p \times I$ and suppose that for each $p$, the pair $(f_p, R_p)$ satisfies the hypothesis of Proposition 2.4. Then $f$ can be homotoped to a piecewise $C^1$ map $f_1$ in any $C^0$ neighbourhood of $f$ such that

1. $(f_1)_p$ is a piecewise $C^1$ solution of $R_p$;
2. $f_1 = f$ on $I^1 \times \{0, 1\}$;
3. the first order derivatives of $f_1(p, t)$ with respect to $p$ are arbitrarily $C^0$ close to the respective derivatives of $f(p, t)$.
By the given hypothesis,\[ J \] neighbourhood of the graph of the section \( f \) fibred over \( I \) is completely determined by \( \beta(v) \in \mathbb{R}^q \). Thus relative to \( \tau \) we can slice the 1-jet space into \( q \)-dimensional affine subspaces \( P_\alpha \). \( P_\alpha \) is called the principal subspace through \((x, y, \alpha)\) corresponding to \( \tau \). The set of equivalence classes is denoted by \( J^1(\mathbb{R}^n, \mathbb{R}^q) \) and there is a canonical projection \( p : J^1(\mathbb{R}^n, \mathbb{R}^q) \to J^1(\mathbb{R}_1, \mathbb{R}^q) \) which takes a 1-jet onto its equivalence class. Let \( j_1^j(x) \) with \((j_1^j(x), df_x(v))\) we can write \( J^1(\mathbb{R}^n, \mathbb{R}^q) = J^1(\mathbb{R}_1, \mathbb{R}^q) \times \mathbb{R}^q \). Note that when \( n = 1 \), \( J^1(\mathbb{R}_1, \mathbb{R}^q) = J^0(\mathbb{R}_1, \mathbb{R}^q) = \mathbb{R} \times \mathbb{R}^q \).

**Theorem 2.6.** Let \( \mathcal{R} \) be an open subset of \( J^1(\mathbb{R}^n, \mathbb{R}^q) \). Let \( f_0 : I^n \to \mathbb{R}^q \) be a piecewise \( C^1 \) function such that \( j_1^j(x) \) lies in the convex hull of \( \mathcal{R}_b(x) \) whenever the derivative exists, where \( b(x) = j_1^j(x) \). Then there exists a piecewise \( C^1 \) solution of \( \mathcal{R} \), \( f_1 : I^n \to \mathbb{R}^q \), which is homotopic to \( f_0 \). Moreover, the homotopy can be made to lie in an arbitrary \( C^0 \) neighbourhood of \( f_0 \).

Further, if \( f_0 \) is a piecewise \( C^1 \) solution of \( \mathcal{R} \) on some open neighbourhood of a compact set \( K \subset I^n \), then the homotopy remains constant on some (possibly smaller) neighbourhood of \( K \).

For the sake of completeness we include the proof from [2].

**Proof.** Consider the splitting of the cube \( I^n \) as \( I^{n-1} \times I \). Form a relation \( \mathcal{R}^1 \subset I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q) \) fibred over \( I^{n-1} \) as follows:

For each \( x \in I^n \) let \( P(j_1^j(x)) \) denote the principal subspace through \( j_1^j(x) \) corresponding to the splitting \( I^{n-1} \times I \). Let \( \Omega(f(p,t)) \) be the subset defined by

\[
\{ j_1^j(p,t) \} \times \Omega(f(p,t)) = P(j_1^j(p,t)) \cap \mathcal{R}.
\]

By the given hypothesis, \( \partial_t f(p,t) \) belongs to the convex hull of \( \Omega(f(p,t)) \) in \( P(j_1^j(x)) \). Since \( \mathcal{R} \) is open there is an open neighbourhood \( D_\varepsilon^\mathcal{R}(f(p,t)) \) of \( f(p,t) \) in \( \mathbb{R}^q \) and an open subset \( \Omega(f(p,t)) \) contained in \( \Omega(f(p,t)) \) such that

1. \( \Omega(f(p,t)) \) contains \( \partial_t f(p,t) \) in its convex hull and
2. \( \{ (p,t) \} \times D_\varepsilon^\mathcal{R}(f(p,t)) \times \{ \partial_p f(p,t) \} \times \Omega(f(p,t)) \subset \mathcal{R} \) for all \((p,t) \in I^{n-1} \times I\).

In the above, \( \partial_t \) and \( \partial_p \) respectively denote the derivatives of the function with respect to the coordinates \( t \) and \( p \).

For each \( p \in I^{n-1} \) define a relation \( \mathcal{R}^1_p \subset J^1(\mathbb{R}, \mathbb{R}^q) \) as follows:

\[
\mathcal{R}^1_p = \{(t, y, v) \in I \times \mathbb{R}^q \times \mathbb{R}^q : y \in D_\varepsilon^\mathcal{R}(f(p,t)), v \in \Omega(f(p,t))\}
\]

\( \mathcal{R}^1 = \cup_p \{ p \} \times \mathcal{R}^1_p \) is then a fibred relation in \( I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q) \) which is defined over an open neighbourhood of the graph of the section \( f \) in \( I^n \times \mathbb{R}^q \).
Further for an appropriate choice of $\Omega'(f(p,t))$ we may assume that $\mathcal{R}^1$ is an open fibred relation in $I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$.

Also note that for a fixed $p \in I^{n-1}$, $t \mapsto f(p,t)$ is a short solution of $\mathcal{R}^1_p$.

We now apply the parametric one-dimensional convex integration to obtain a piecewise $C^1$-homotopy $f_\tau$ of fibrewise ‘short’ (see Definition in [2]) solutions of $\mathcal{R}^1$ which is $C^0$ close to $f$ and satisfies $f_\tau(p,0) = f(p,0)$ and $f_\tau(p,1) = f(p,1)$ for all $p \in I^{n-1}$. Furthermore, the first order derivatives of $f_1(p,t)$ with respect to the parameter $p$ (whenever exist) are arbitrarily $C^0$-close to the respective derivatives of $f(p,t)$. Hence, $(f_1(p,t), \partial_p f(p,t), \partial_t f_1(p,t)) \in \mathcal{R}$. Since $\mathcal{R}$ is open and since the derivatives of $f_1$ with respect to $p$ are arbitrarily close to the respective derivatives of $f$ it follows that $(f_1(p,t), \partial_p f_1(p,t), \partial_t f_1(p,t)) \in \mathcal{R}$. Thus $f_1$ is a solution of $\mathcal{R}$ with the desired properties. □

3. $(h_1, h_2)$ REGULARITY AND UNDERLYING GEOMETRY

Throughout this section $h_1$ and $h_2$ will denote two positive definite symmetric bilinear forms on $\mathbb{R}^q$. For any subspace $V$ of $\mathbb{R}^q$, we shall denote its orthogonal complement with respect to $h_i$ by $V^\perp_i$ for $i = 1, 2$.

**Definition 3.1.** A subspace $V$ of $\mathbb{R}^q$ is said to be $(h_1, h_2)$-regular if $V^\perp_1$ is transversal to $V^\perp_2$.

Observe that if $A : \mathbb{R}^q \rightarrow \mathbb{R}^q$ is the (unique) linear transformation defined by $h_2(v,w) = h_1(Av,w)$ for all $v,w \in \mathbb{R}^q$, then a subspace $V$ in $\mathbb{R}^q$ is regular if and only if $V + A(V)$ has the maximum dimension.

**Definition 3.2.** A vector $v \in \mathbb{R}^q$ is said to be $(h_1, h_2)$-regular provided the 1-dimensional subspace $\langle v \rangle$ spanned by $v$ is a $(h_1, h_2)$-regular subspace of $\mathbb{R}^q$.

**Observation 1.** A vector $v$ is $(h_1, h_2)$-regular if and only if $v$ and $Av$ are linearly independent, $A : \mathbb{R}^q \rightarrow \mathbb{R}^q$ being the unique linear map defined above. Consequently, the set of non-regular vectors precisely consists of the eigen-vectors of $A$.

The following observation brings out the underlying geometry of the $(h_1, h_2)$-regular vectors.

**Observation 2.** Let $(\mathbb{R}^q, h_1, h_2)$ be as in the above. We shall denote the norms of a vector $w \in \mathbb{R}^q$ relative to $h_1$ and $h_2$ by $\|w\|_1$ and $\|w\|_2$ respectively. Let $S_r = \{w \in \mathbb{R}^q | \|w\|_1 = r\}$ and $E_r = \{w \in \mathbb{R}^q | \|w\|_2 = r\}$ denote the spheres of radius $r$ in $\mathbb{R}^q$ relative to the two metrics. Observe that, a vector $v \in S_r \cap E_{r'}$ is $(h_1, h_2)$-regular if and only if $S_r$ and $E_{r'}$ intersect transversally at $v$. Indeed, $v$ is a regular vector if and only if $v^{\perp_1}$ is transversal to $v^{\perp_2}$. If $v \in S_r \cap E_{r'}$, then $v^{\perp_1}$ is tangent to $S_r$ at $v$ and $v^{\perp_2}$ is tangent to $E_{r'}$ at $v$. Therefore it follows that $S_r$ is transversal to $E_{r'}$ at $v$. 

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Observation 3. Let \( V \) be a \((h_1, h_2)\)-regular subspace of \( \mathbb{R}^q \) of dimension \((n - 1)\) and let \( X = V^{1-1} \cap V^{1-2} = (V \oplus A(V))^{1-1} \). Then \( X \) has codimension \( 2(n - 1) \) in \( \mathbb{R}^q \). For any vector \( w \in \mathbb{R}^q, \tau \oplus \langle w \rangle \) is a \((h_1, h_2)\)-regular subspace if and only if \( w^{1-1} \cap X \) is transversal to \( w^{1-2} \cap X \) in \( X \). Indeed, \( V \oplus \langle w \rangle \) is a \((h_1, h_2)\)-regular subspace if and only if \( (V \oplus \langle w \rangle)^{1-1} \) is transversal to \( (V \oplus \langle w \rangle)^{1-2} \), that is if and only if codim \([\langle V \oplus \langle w \rangle \rangle^{1-1} \cap \langle V \oplus \langle w \rangle \rangle^{1-2} \] = 2n. This is equivalent to saying \( X \cap w^{1-1} \cap w^{1-1} \) has codimension 2 in \( X \). Thus \( w^{1-1} \cap X \) is transversal to \( w^{1-2} \cap X \).

Let \( T \) be a translate of \( X \) through \( w \). Suppose that \( r = \|w\|_1 \) and \( r' = \|w\|_2 \). Since \( w^{1-1} \cap X \) is the tangent space of \( S_r \cap T \) at \( w \) and \( w^{1-2} \cap X \) is the tangent space of \( E_r \cap T \) at \( w \), it follows from the above that the sets \( S_r \cap T \) and \( E_r \cap T \) intersect transversally in \( T \) at \( w \).

In particular we can show that if \( w \) is in \( X \), then \( V \oplus \langle w \rangle \) is \((h_1, h_2)\)-regular if and only if then \( w \) is \((\tilde{h}_1, \tilde{h}_2)\)-regular, where \( \tilde{h}_1 \) and \( \tilde{h}_2 \) denote the restrictions of \( h_1 \) and \( h_2 \) respectively to \( X \).

Let \( \tilde{A} \) denote the unique linear transformation \( X \rightarrow X \) such that \( \tilde{h}_2(v, w) = h_1(Av, w) \) for \( v, w \in V \). If \( w \in X \) is \((\tilde{h}_1, \tilde{h}_2)\) regular then \( w \) and \( \tilde{A}(w) \) are linearly independent. Let \( A(w) = x + x^\perp \) where \( x \in X \) and \( x^\perp \in X^{1-1} \). Then \( h_2(w, v) = h_1(Aw, v) = h_1(x + x^\perp, v) = h_1(x, v) \) for all \( v \in X \). Hence \( x = \tilde{A}(w) \). This proves that \( Aw = \tilde{A}w + x^\perp \). Since \( w, \tilde{A}w \) are linearly independent in \( X \) and \( x^\perp \not\in X \) it follows that \( w \) and \( Aw \) are linearly independent and consequently, \( V \oplus \langle w \rangle \) is \((h_1, h_2)\)-regular.

Definition 3.3. Let \( N \) be a smooth manifold with two Riemannian metrics \( h_1 \) and \( h_2 \). A smooth map \( f : M \rightarrow N \) will be called \((h_1, h_2)\)-regular if for each \( x \in M \), \( df_x(T_xM) \) is a \((h_1, h_2)\)-regular subspace of \( T_{f(x)}N \).

We recall the following result from [1].

Proposition 3.4. Let \( h_1, h_2 \) be two positive definite symmetric bilinear forms on \( \mathbb{R}^q \) such that the eigen-values of \( h_1 - h_2 \) are all distinct. Then a generic map \( f : M \rightarrow \mathbb{R}^q \) is \((h_1, h_2)\)-regular if \( q > 3 \dim M - 1 \).

4. The Main Lemma

Let \( \mathbb{R}^q \) be the \( q \) dimensional Euclidean space. In what follows \( h_1 \) and \( h_2 \) will denote two Euclidean metrics on \( \mathbb{R}^q \) which satisfy the following conditions: There exist two numbers \( 0 < a < b \), such that

1. \( c^2h_1 - h_2 \) is a non-degenerate indefinite form for each real number \( c \) lying in \([a, b]\),
2. \( r_+(a^2h_1 - h_2) \geq 2n \) and \( r_-(b^2h_1 - h_2) \geq 2n \), where \( r_+ \) and \( r_- \) denote respectively the positive and the negative ranks of an indefinite metric,
3. We further assume that \( h_1 - h_2 \) has distinct eigenvalues.
Lemma 4.1. Let $g_1$ and $g_2$ be two Riemannian metrics on $M$ which are related by $a^2 g_1 < g_2 < b^2 g_2$. Let $f : M \to \mathbb{R}^q$ be a $(h_1, h_2)$-regular immersion such that
\begin{equation}
g_1 - f^* h_1 = \phi^2 d\psi^2 \quad \text{and} \quad g_2 - f^* h_2 = c^2 \phi^2 d\psi^2,
\end{equation}
where $\phi, \psi$ are smooth functions on $M$, $\phi$ has compact support contained in an open set $U$ of $V$ and $a < c < b$.

Then there exists a piecewise $C^1$ map $\tilde{f}$ which is an arbitrary $C^0$-fine approximation of $f$ and is such that $\tilde{f}^* h_i$ is arbitrarily close to $g_i$ ($\tilde{f}^* h_i \approx g_i$) for $i = 1, 2$ relative to the fine $C^0$-topology on each component where $\tilde{f}$ is $C^1$.

Remark 4.2. The hypothesis on $f$ stated above imply that the map $f$ is isometric outside a compact set contained in $U$. Hence, $\tilde{f}$ coincides with $f$ outside $U$.

Proof. Let $\mathcal{I}$ denote the subset of $J^1(M, \mathbb{R}^q)$ consisting of all $1$-jets $(x, y, \alpha)$ such that $\alpha^* h_1 = g_1$ and $\alpha^* h_2 = g_2$. Let $\tau$ be the hyperplane field over $U$ defined by ker $d\psi$. Then $\tau$ is integrable and its integral submanifolds are precisely the level sets of the function $\psi$.

Consider the bundle $p_1^1 : J^{(1)}(U, \mathbb{R}^q) \to J^1(U, \mathbb{R}^q)$ relative to the hyperplane distribution $\tau$ on $U$. An element $b$ of $J^1(M, \mathbb{R}^q)$ is of the form $b = (x, y, \beta)$, where $x \in U$, $y \in \mathbb{R}^q$ and $\beta : \tau_x \to \mathbb{R}^q$ is a linear map. The fibre over $b$ consists of all linear maps $\alpha : T_x M \to \mathbb{R}^q$ which restricts to $\beta$ on $\tau_x$.

To describe the intersection of the relation $\mathcal{I}$ with the principal subspaces of the fibration $p_1^1$, we choose a vector field $v_0$ on $TU$ such that $\|v_0\|_1 = \sqrt{g_1(v_0, v_0)} = 1$ and $g_1(v_0, \tau) = 0$ on $U \supset \text{supp} \phi$. Let $\|v_0\|_2 = \sqrt{g_2(v_0, v_0)} = r$; then $r$ is a smooth function on $U$ satisfying the inequality $a < r(x) < b$ for all $x \in U$. Let $p' : \mathcal{I} \to J^1(M, \mathbb{R}^q)$ denote the restriction of $p_1^1$ to $\mathcal{I}$. Recall that a $1$-jet $(x, y, \alpha)$ in a principal subspace $J^1_b(U, \mathbb{R}^q)$ is completely determined by its value at $v_0$. Moreover, if $(x, y, \alpha) \in \mathcal{I}_b = J^1_b(U, \mathbb{R}^q) \cap \mathcal{I}$, then $\alpha(v_0)$ is contained in the unique affine space
\begin{equation}
T_b = \{ w \in \mathbb{R}^q | h_1(w, \beta(\tau)) = 0 \text{ and } h_2(w, \beta(v)) = g_2(v_0, v) \text{ for all } v \in \tau \},
\end{equation}
where $b = (x, y, \beta) \in J^1(U, \mathbb{R}^q)$. Note that the equations $h_2(w, \beta(v)) = g_2(v_0, v)$ defines an affine subspace of $\mathbb{R}^q$ which is a translate of $\beta(\tau)^{-1}$. If $\alpha$ is a $(h_1, h_2)$-regular then, in particular, $\beta(\tau)^{-1}$ is transversal to $\beta(\tau)^{-1}$ and the same is true for any translates of these spaces. Thus $T_b$ is an affine plane of codimension $2(m - 1)$. Moreover, this is the translate of the vector subspace $X_b = \beta(\tau)^{-1} \cap \beta(\tau)^{-1}$ in $\mathbb{R}^q$.

Thus $J^{(1)}(U, \mathbb{R}^q) \cap \mathcal{I}$ is contained in an affine subbundle of codimension $2(n - 1)$ (over some open subset of $J^1(U, \mathbb{R}^q)$). Further, it follows that if $\alpha \in J^1_b(U, \mathbb{R}^q) \cap \mathcal{I}$ then $\|\alpha(v_0)\|_1 = 1$ and $\|\alpha(v_0)\|_2 = r$. Therefore we can characterise $J^1_b(U, \mathbb{R}^q) \cap \mathcal{I}$ as follows:
\begin{equation}
J^1_b(U, \mathbb{R}^q) \cap \mathcal{I} = \{ w \in T_b : \|w\|_1 = 1, \|w\|_2 = r \}.
\end{equation}
We shall now show that the pair \((f, I)\) satisfies the conditions stated in the hypothesis of Theorem 2.6 except that \(I\) is not an open relation.

**Notation:** We fix the following notations for the subsequent discussion:

\[ S = \{w \in \mathbb{R}^q : \|w\|_1 = 1\}, \quad E = \{w \in \mathbb{R}^q : \|w\|_2 = r\}. \]

**Sublemma 4.3.** \(j_f^1(x)\) lies in the convex hull of \(\mathcal{I}_{b(x)}\) if \(r_\pm(c^2h_1 - h_2) \geq 2n\). In other words, \(df_x(v_0)\) lies in the convex hull of the set

\[ \{w \in T_{b(x)} : \|w\|_1 = 1, \|w\|_2 = r\}, \]

where \(b(x) = j_f^1(x)\).

**Proof of the Lemma.** Observe that

1. \(df_x(v_0)\) lies in \(T_{b(x)}\), and
2. \(df_x(v_0)\) satisfies the equation

\[ c^2(1 - \|w\|_1^2) = r^2 - \|w\|_2^2 \]

since \(g_2 - f^*h_2 = c^2(g_1 - f^*h_1)\).

The above equation can be equivalently expressed as \((c^2h_1 - h_2)(w, w) = c^2 - r^2\). This represents a generalised hyperboloid \(H\) since \(r_\pm(c^2h_1 - h_2) \geq 2n\). It may be seen easily that \(H \cap S = E \cap S = H \cap E\).

Since \(r_\pm(c^2h_1 - h_2) \geq 2n\), \(H\) is generated by affine subspaces of dimension \(2n - 1\). [To see this, let \(h\) be a non-degenerate symmetric bilinear form on \(\mathbb{R}^q\) of signature \((q_+, q_-)\). Let \(v \in H\) be a such that \(h(v, v) = d \neq 0\) and let \(V\) denote the \(h\)-orthogonal complement of the subspace generated by \(v\). Then \(V\) has dimension \(n - 1\) and \(r_+(h|_V) \geq q_+ - 1, r_-(h|_V) \geq q_- - 1\). Consequently, \(V\) admits a regular \(h\)-isotropic subspace \(I\) of dimension \(\min(q_+ - 1, q_- - 1)\). Here regularity means that \(I\) does not intersect the kernel of \(h|_V\). Consider the affine subspace \(W = I + v\). It is easy to see that \(h(w, w) = d\) for every \(w \in W\). This proves the above assertion.]

Let \(A_x\) be an affine subspace in \(H\) which passes through \(df_x(v_0)\). Since \(\text{codim} T_x = 2(n - 1) < 2n - 1 = \dim A_x\), the intersection \(T_x \cap A_x\) is an affine subspace of dimension at least 1. Since \(df_x(v_0) \in T_x \cap A_x\) and \(\|df_x(v_0)\|_1 < 1\), \(T_x \cap A_x \cap S\) contains at least two points and \(df_x(v_0)\) lies in the convex hull of this intersection. Noting that \(T_x \cap A_x \cap S \subset T_x \cap E \cap S\), we conclude that \(df_x(v_0)\) lies in the convex hull of \(T_x \cap E \cap S\). This completes the proof of the sublemma. \(\square\)

Now we conclude the proof of the Main Lemma. Since \(I\) is not an open relation we cannot directly apply Theorem 2.6 to the pair \((f, I)\). We take an arbitrary small open neighbourhood \(\tilde{I}\) of \(I\) and apply Theorem 2.6 to the pair \((f, \tilde{I})\). Thus we obtain a fine \(C^0\) approximation of \(f\) by a piecewise \(C^1\) solution \(\tilde{f}\) of \(\tilde{I}\). Choosing \(\tilde{I}\) sufficiently small, we can make \(\tilde{f}^*h_1\) and \(\tilde{f}^*h_2\) arbitrarily \(C^0\) close to the pair \((g_1, g_2)\) as desired. This completes the proof. \(\square\)
5. APPRROXIMATE SOLUTION

We recall the definition of short maps from [1].

**Definition 5.1.** Let $M$ be a manifold with two Riemannian metrics $g_1$ and $g_2$. A $C^1$-map $f_0 : V \to (\mathbb{R}^q, h_1, h_2)$ is $(g_1, g_2)$-short if the metrics $g_1 - f_0^*(h_1)$ and $g_2 - f_0^*(h_2)$ on $M$ are positive definite. This will be expressed by $g_i - f_0^*(h_i) > 0$ or $g_i > f_0^*(h_i)$, $i = 1, 2$.

**Proposition 5.2.** Let $M$ be a $C^\infty$-manifold with two Riemannian metrics $g_1$ and $g_2$ which are related by $a^2 g_1 < g_2 < b^2 g_1$. Then there exists a $(g_1, g_2)$-short $C^\infty$-immersion $f_0 : V \to (\mathbb{R}^q, h_1, h_2)$ which also satisfies the following inequalities:

\[
\begin{align*}
(a^2 r_1 - f_0^* h_1) &< (g_2 - f_0^* h_2) < b^2 (g_1 - f_0^* h_1) \\
(a^2 f_0^* h_1) &< f_0^* h_2 < b^2 f_0^* h_1.
\end{align*}
\]

**Proof.** For any number $c$, $a < c < b$, consider the non-degenerate form $\tilde{h} = c^2 h_1 - h_2$. By the hypothesis of Theorem 1.1, $r_+ (\tilde{h}) \geq 2n$ and $r_- (\tilde{h}) \geq 2n$. Therefore, there exists a $C^1$-immersion $f : V \to W$ such that $f^* (\tilde{h}) = 0$ [3, 2.4.9, Corollary (2)]. Such an $f$ clearly satisfies the relation $a^2 f^* h_1 < f^* h_2 < b^2 f^* h_1$. Moreover, without any loss of generality we may assume that the map $f$ satisfying the above inequality is smooth, because if that is not the case we replace $f$ by a $C^\infty$-immersion which is sufficiently $C^1$-close to $f$.

Now, if $M$ is a closed manifold, then starting with an $f_0$ as above we can obtain the required $f_0$ by scaling the map $f$ with a suitable scalar (see the corresponding result in [1]). To obtain such an $f_0$, in the case of open manifolds we have to employ the partition of unity techniques. \hfill \Box

Let $\mathcal{F}$ denote the set of all piecewise $C^1$ maps $f : M \to \mathbb{R}^q$ which satisfy the following conditions (almost everywhere) on $M$:

- **F1:** $f$ is $(h_1, h_2)$-regular;
- **F2:** $f$ is short relative to both $(g_1, h_1)$ and $(g_2, h_2)$;
- **F3:** $a^2 (g_1 - f^* h_1) < g_2 - f^* h_2 < b^2 (g_1 - f^* h_1)$;
- **F4:** $a^2 f^* h_1 < f^* h_2 < b^2 f^* h_1$.

**Proposition 5.3.** Let $f_0$ be a piecewise $C^1$ map $M \to \mathbb{R}^q$ in $\mathcal{F}$. Then there exists a piecewise $C^1$ map $f_1 \in \mathcal{F}$ such that the following conditions are satisfied:

1. $\varepsilon g_1 < f_1^* h_1 < g_1$ almost everywhere on $M$, where $0 < \varepsilon < 1$;
2. $f_1$ is arbitrarily close to $f_0$ in the fine $C^0$ topology.

**Proof.** Fix a locally finite open covering $\{U_i\}$ of $M$ by coordinate neighbourhoods. Since the metrics $g_1 - f^* h_1$ and $g_2 - f^* h_2$ are related by the inequalities (2) we can get simultaneous decomposition of $g_1 - f^* h_1$ and $g_2 - f^* h_2$ as follows:

\[
2\varepsilon (g_1 - f^* h_1) = \sum_i \phi_i^2 d\psi_i^2 \quad \text{and} \quad 2\varepsilon (g_2 - f^* h_2) = \sum_i \phi_i^2 d\psi_i^2
\]
where \( c_i \)'s are constants which lie between \( a \) and \( b \), and \( \phi_i \) and \( \psi_i \)'s are smooth real valued functions. Further, for each \( i \), the function \( \phi_i \) has compact support contained in \( U_i \) (see Decomposition Lemma in [1]). Let us define two sequences of Riemannian metrics \( \{ g^1_i \} \) and \( \{ g^2_i \} \) as follows:

\[
g^1_i = g^{i-1}_1 + \phi_i^2 d\psi_i^2 \quad \text{and} \quad g^2_i = g^{i-1}_2 + \psi_i^2 d\psi_i^2,
\]

where \( g^1_i = f^*h_1 \) and \( g^2_i = f^*h_2 \). Clearly, \( g^1_i < g_1 \) and \( g^2_i < g_2 \) for each \( i \). Further, since \( a^2f^*h_1 < f^*h_2 < b^2f^*h_1 \) and \( a < c_i < b \) for each \( i \),

\[
a^2g^1_i < g^2_i < b^2g^1_i \quad \text{for each} \quad i.
\]

By applying the Main Lemma (Lemma 4.1) successively (with an appropriate choice of \( \tilde{T} \) for each \( i \)) we obtain a sequence of piecewise \( C^1 \) maps such that \( \tilde{f}^i h_j \approx g^j_i \), \( j = 1, 2, i = 1, 2, \ldots \) and \( \tilde{f}_i \) lies in a given neighbourhood of \( f \) in the fine \( C^0 \) topology. Note that each \( \tilde{f}_i \) satisfies conditions \( F2 \) and \( F4 \). Since \( \text{supp} \phi_i \)'s are contained in a locally finite open covering of \( M \) the sequence \( \tilde{f}_i \) is eventually constant near any point \( x \in M \) (see Remark 4.2). Therefore the sequence converges to a piecewise \( C^1 \) map on \( V \). Let \( f_1 = \lim_{i \to \infty} \tilde{f}_i \). If \( \tilde{f}^* h_\alpha \) are sufficiently close to \( g^\alpha_i \) for \( \alpha = 1, 2 \) and for all \( i \), then \( f_1 \) can be made to satisfy \( F2 \), \( F3 \) and \( F4 \). Further, \( g_i - f^*_1 h_1 \approx g_i - (f^*h_1 + 2\varepsilon(g_i - f^*h_1)) = (1 - 2\varepsilon)(g_i - f^*h_1) < (1 - 2\varepsilon)g_i \). Hence \( f_1 \) satisfies \( \varepsilon g_i < f^*_1 h_1 < g_i \).

\[
\square
\]

6. Proof of the Main Theorem

We begin this section with some preliminaries on Lipschitz maps.

**Definition 6.1.** Let \( (X, d) \) and \( (Y, d') \) be two metric spaces and let \( f : X \to Y \) be a continuous map. The map \( f \) is said to be **Lipschitz** if there is a constant \( K > 0 \) such that \( d'(f(x), f(x')) < K d(x, x') \) for all \( x, x' \in X \). \( K \) is called the Lipschitz constant for \( f \).

A Riemannian metric \( g \) on a \( C^\infty \) manifold \( M \) induces a canonical metric space structure on \( M \). If we denote this metric by \( d_g \), then the distance \( d_g(x, x') \) between two points \( x, x' \in M \) is defined to be the infimum of the lengths of all piecewise \( C^1 \) paths in \( M \) joining \( x \) and \( x' \).

**Definition 6.2.** A continuous map \( f : (M, g) \to (N, h) \) from a Riemannian manifold \((M, g)\) into another Riemannian manifold \((N, h)\) will be called **Lipschitz** if it is a Lipschitz map relative to the metrics \( d_g \) and \( d_h \) on \( M \) and \( N \) respectively.

**Example 6.3.** A \( C^1 \) isometric map \( f : (M, g) \to (N, h) \) between Riemannian manifolds is a Lipschitz map with a Lipschitz constant equal to 1. Hence, every \( g \)-short map is also a Lipschitz map.

A Riemannian metric \( g \) on a manifold \( M \) induces a canonical volume measure which we denote by \( \mu_g \). Measurability on \((M, g)\) is therefore to be understood in terms of this \( \mu_g \). Observe that if \( g' \) is another Riemannian metric on \( M \) then a set \( A \) in \( M \) has measure zero relative to \( \mu_g \) if and only if it has measure zero relative to \( \mu_{g'} \).

We recall the following facts about Lipschitz maps between Riemannian manifolds [8].
• Every Lipschitz map between Riemannian manifolds is almost everywhere differentiable, since a Lipschitz map \( f : \Omega \rightarrow \mathbb{R}^q \) defined on some open subset of \( \mathbb{R}^n \) is almost everywhere differentiable.

• The Lipschitz functions on a Riemannian manifold are precisely those which have bounded measurable exterior derivative \( df \).

**Definition 6.4.** A Lipschitz map \( f : (M, g) \rightarrow (N, h) \) from a Riemannian manifold \( (M, g) \) into another Riemannian manifold \( (N, h) \) will be called *Lipschitz isometric* if \( df_x : T_x M \rightarrow T_{f(x)} N \) is isometric for almost all \( x \in M \).

• If \( g_1 \) and \( g_2 \) be two Riemannian metrics on a manifold \( M \) satisfying \( b^2 g_1 < g_2 < a^2 g_1 \) then a map \( f : M \rightarrow \mathbb{R}^q \) is Lipschitz with respect to the pair \( (g_1, h_1) \) if and only if it is Lipschitz with respect to the pair \( (g_2, h_2) \), where \( h_1, h_2 \) are two linear metrics on \( \mathbb{R}^q \). Therefore, there is no ambiguity when we speak of almost everywhere differentiable Lipschitz maps in the context of Theorem 1.1.

**Proof of Theorem 1.1.** Since \((h_1, h_2)\)-regular immersions are generic for \( q \geq 3 \dim V + 1 \), it follows from Proposition 5.2 that there is a \((h_1, h_2)\)-regular immersion \( f_0 : V \rightarrow \mathbb{R}^q \) which satisfies the inequalities in (2).

Let \( \mathcal{R} \) denote the set of all 1-jets \((x, y, \alpha)\) which satisfy the following properties:

1. \( \alpha \) is short relative to both \((g_1, h_1)\) and \((g_2, h_2)\);
2. \( a^2(g_1 - \alpha^* h_1) < g_2 - \alpha^* h_2 < b^2(g_1 - \alpha^* h_1) \);
3. \( a^2 \alpha^* h_1 < \alpha^* h_2 < b^2 \alpha^* h_1 \).

For every \( \eta > 0 \) define relations \( \mathcal{R}_\eta \) by

\[
\mathcal{R}_\eta = \mathcal{R} \cap \{ (x, y, \alpha) : (1 - \eta) g_1 < \alpha^* h_1 < g_1 \}. 
\]

Let \( \mathcal{I} \) denote the isometry relation

\[
\mathcal{I} = \{ (x, y, \alpha) \in J^1(M, \mathbb{R}^q) : \alpha^* h_1 = g_1, \alpha^* h_2 = g_2 \}. 
\]

• Each \( \mathcal{R}_\eta \) is then an open relation.
• The fibres of \( \mathcal{I} \) over \( J^0(M, \mathbb{R}^q) \) are compact sets. Hence, the relations \( \mathcal{R}_\eta \) are uniformly bounded over compact sets in \( M \).
• Let \( \eta_i \) be a sequence of positive numbers such that \( \eta_i \to 0 \). If \( \alpha_i \in \mathcal{R}_{\eta_i} \) and \( \alpha_i \to \alpha \), then \( \alpha \in \mathcal{I} \). (Compare with [3])

Let \( \eta_i \) be a sequence of constants converging to zero and \( \delta_i \) be a sequence of positive continuous functions on \( M \) such that the series \( \sum_i \delta_i \) converges pointwise on \( M \). By applying Proposition 5.3 we obtain a sequence of piecewise \( C^1 \) maps \( f_i : M \rightarrow \mathbb{R}^q \) for \( i = 1, 2, \ldots \) such that \( f_i \) is a piecewise \( C^1 \) solution of the relation \( \mathcal{R}_{\eta_i} \) and the distance between \( f_i(x) \) and \( f_{i+1}(x) \) is less than \( \delta_i(x) \) for all \( x \in M \). Thus the sequence \( \{f_i\} \) converges (in the \( C^0 \) compact open topology) to a continuous function \( f \) on \( M \). Since \( f_i \) is a piecewise \( C^1 \) solution of the relation \( \mathcal{R}_{\eta_i} \), it is Lipschitz (relative to \((g_1, h_1)) \) and the Lipschitz constants of \( f_i \) are uniformly bounded. Hence the limit function \( f \).
is also a Lipschitz map [8]. Consequently, $f$ is almost everywhere differentiable and the $L^\infty$ norm of $df$ is finite on any coordinate neighbourhood of $M$.

We would further like to show that the sequence $df_i$, $i = 1, 2, \ldots$, converges to $df$ in $L^1(\Omega)$ for any compact coordinate neighbourhood $\Omega$. Since $L^1$ convergence of a sequence of functions guarantees the almost everywhere convergence of a subsequence of the original sequence to $df$, this would imply that $f$ is a Lipschitz solution of $I$ on all of $M$ (by a property of $R_\eta$ discussed above).

However, to prove the desired $L^1$ convergence we need to choose the functions $\delta_i$ appropriately. First we fix a locally finite open covering of $M$ by coordinate neighbourhoods $\{\Omega_\alpha : \alpha = 1, 2, \ldots \}$. For our convenience we choose each $\Omega_\alpha$ to be compact. Suppose we have already constructed $\delta_i$ and $f_i$ for $i = 1, 2, \ldots, k$. Let $\{\varepsilon_\alpha\}$ be a sequence of positive numbers, $0 < \varepsilon_\alpha < 2^{-\alpha}$ such that
\[
\|df_i * \rho_{\varepsilon_\alpha} - df_i\|_{L^1(\Omega_\alpha)} \leq 2^{-\alpha}.
\]
The functions $\rho_{\varepsilon}$ are defined as in [5] by $\rho_{\varepsilon} = \varepsilon^{-n} \rho(x/\varepsilon)$, where $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is the mollifying kernel, (that is a smooth non-negative function supported in the open unit disc in $\mathbb{R}^n$ with $\int \rho = 1$).

Observing that there exists a positive continuous function $\varepsilon$ on $M$ which is strictly less than $\varepsilon_\alpha$ on $\Omega_\alpha$ for each $\alpha = 1, 2, \ldots$ define
\[
\delta_{i+1} = \varepsilon \delta_i.
\]
Now we apply Proposition 5.3 to obtain a piecewise $C^1$-solution of $R_{\eta_{i+1}}$ such that $|f_{i+1} - f_i| < \delta_{i+1}$. Proceeding this way we construct a sequence $\{f_i\}$, $i = 1, 2, \ldots$ which has all the desired property.

Now, arguing exactly as in [5] we can prove that $df_i$ converges to the derivative map of $f$ in the $L^1(\Omega_\alpha)$ for each $\alpha$. This completes the proof of the theorem. \hfill \Box

7. ONE DIMENSIONAL CASE

In this section we discuss the one-dimensional case which is the motivation to the general problem.

Let $M = S^1$ be the unit circle and let $g_1 = d\theta^2$ be the canonical metric on $S^1$, where $\theta$ is a canonical coordinate function. Let $g_2 = c^2 g_1$. If $f : S^1 \rightarrow \mathbb{R}^q$ is a $C^1$-immersion such that $f^* h_i = g_i$ for $i = 1, 2$ then
\[
\|\frac{\partial f}{\partial \theta}\|_1 = 1 \quad \text{and} \quad \|\frac{\partial f}{\partial \theta}\|_2 = c,
\]
where $\|\cdot\|_i$ denote the norm relative to the metric $h_i$, $i = 1, 2$.

Thus it is required to obtain a $C^1$-immersion $f$ such that $\frac{\partial f}{\partial \theta} \in A$, where $A$ is given by
\[
A = \{y = (y_1, \ldots, y_q) \in \mathbb{R}^q : \sum y_i^2 = 1 \text{ and } \sum \lambda_i^2 y_i^2 = c^2\}.
\]

Lemma 7.1. Let $h_1$ and $h_2$ be two inner products on $\mathbb{R}^q$ such that $h_1 - h_2$ is non-degenerate. Let $S_1$ and $S_2$ denote the unit spheres relative to the metrics $h_1$ and $h_2$ respectively. Then $S_1 \cap S_2$
has the same homotopy type as $S^{r_+} \times S^{r_-}$, where $r_+$ and $r_-$ are respectively the positive and the negative ranks of $h_1 - h_2$. Consequently, if $r_\pm \geq 2$ then $S_1 \cap S_2$ is connected. Further the interior of the convex hull of $S_1 \cap S_2$ contains the origin.

Proof. Let $h_1 - h_2$ is non-degenerate. Note that a non-zero vector $v$ satisfies $(h_1 - h_2) = 0$ if and only if $\lambda v$ satisfies the same equation for all $\lambda$. This means that the 1-dimensional subspace $\ell_v$ containing $v$ lies completely inside the solution space $C$ of $h_1 - h_2 = 0$. In other words, the solution space of this equation in $\mathbb{R}^q$ is a cone. Now, if $h$ is an arbitrary positive definite quadratic form on $\mathbb{R}^q$ then $\ell_v$ intersects the unit sphere relative to $h$ in exactly two points. Thus we see that $S_1 \cap S_2$ has the same homotopy type as the space of non-zero solutions of the equation $h_1 - h_2 = 0$.

Choose basis vectors in $\mathbb{R}^q$ so that $h_1 - h_2$ is in the diagonal form. The homeomorphism type of $C$ is then given by the following system of equations:

$$
\begin{align*}
  x_1^2 + \cdots + x_2^2 + y_1^2 + \cdots + y_2^2 &= 1 \\
  x_1^2 + \cdots + x_2^2 - y_1^2 + \cdots - y_2^2 &= 0
\end{align*}
$$

which is further equivalent to

$$x_1^2 + x_2^2 + \cdots + x_2^2 = 1/2 \text{ and } y_1^2 + \cdots + y_2^2 = 1/2.$$

Therefore, $S \cap E$ has the homeomorphism type of $S^{q_+ - 1} \times S^{q_- - 1}$. Hence the intersection of $S_1$ and $S_2$ is $k$-connected for $k \leq \min(q_+ - 2, q_- - 2)$. Thus if $q_\pm \geq 2$ then $S_1 \cap S_2$ is connected and nowhere flat; (note that in the lowest admissible dimension the intersection is topologically equivalent to a torus in embedded in $S^3$). Also note that if $(\bar{x}_1, \ldots, \bar{x}_q, \bar{y}_1, \ldots, \bar{y}_q) \in S_1 \cap S_2$ then $(\pm \bar{x}_1, \ldots, \pm \bar{x}_q, \pm \bar{y}_1, \ldots, \pm \bar{y}_q) \in S_1 \cap S_2$, so that the convex hull of $S_1 \cap S_2$ has non-empty interior and 0 belongs to the interior convex hull of $S_1 \cap S_2$. \qed

It follows from the above lemma that if $r_\pm(c^2 h_1 - h_2) \geq 2$, then $A$ is connected and the interior of the convex hull of $A$ contains the origin. Thus, by Lemma 2.2 there exists a $C^1$ immersion $f : S^1 \longrightarrow \mathbb{R}^q$ such that $f^* h_i = g_i, i = 1, 2$ when $r_\pm(c^2 h_1 - h_2) \geq 2$.

On the other hand there does not exist any such isometric immersion if $q \leq 3$ since Gromov observes in [3, 2.4.1(A)] that if $f : S^1 \longrightarrow \mathbb{R}^q$ is a $C^1$-map whose derivative takes the unit circle $S^1$ into a (connected) subset $A$, then the convex hull of $A$ must contain the origin. For, if there is a map $f : S^1 \longrightarrow \mathbb{R}^q$ such that $f^* h_1 = d\theta^2 = f^* h_2$, then $f^*(h_1 - h_2) = 0$. If $h_1 - h_2$ is a non-degenerate indefinite form then either $r_+ = 1, r_- = 2$ or $r_+ = 1, r_- = 2$. In either of these two cases, $A$ is a disjoint union of two circles none of which contains the origin in its convex hull, thereby ruling out the existence of $C^1$-immersion with the desired isometry property.

We like to conclude the paper with the following conjecture:

If $r_\pm(c^2 h_1 - h_2) \geq 2n + 1$, then it is possible to obtain a $C^1$ solution of the general problem.

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