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1. Introduction

Let $K$ be a nonempty subset of a real normed space $E$. A self-mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in K$ and quasi-nonexpansive if it has a fixed point in $K$ with $\|Tx - p\| \leq \|x - p\|$ for every $x \in K$ and $p$ a fixed point of $T$. $T$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for every } x, y \in K.$$  

$T$ is called uniformly $L$-Lipschitzian if there exists a real number $L > 0$ such that for every $x, y \in K$ and integers $n \geq 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [10] and the class forms an important generalization of that of nonexpansive mappings. It was proved in [10] that if $K$ is a nonempty, closed, convex and bounded subset of a real uniformly convex Banach space and $T$ is an asymptotically nonexpansive self-mapping of $K$, then $T$ has a fixed point. Strong and weak convergence theorems for nonexpansive and asymptotically nonexpansive families of mappings and for single maps have been established by many authors (see, for example, [1-3,5,9,12,13,18-20,25-28] and the references contained therein).

Definition 1.1 ([22]) A mapping $T : K \to K$ with a nonempty fixed point set $F$ in $K$ is said to satisfy condition (I) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Tx\| \geq f(d(x, F))$ for all $x \in K$, where $d(x, F) := \inf \{\|x - z\| : z \in F\}$.

In 1974, Senter and Dotson [22] studied the convergence of the Mann iteration scheme (see, [16]) defined by $x_1 \in K$,

$$x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n, \quad n \geq 1$$

in a uniformly convex Banach space, where $\{\alpha_n\}$ is a sequence satisfying $0 < a \leq \alpha_n \leq b < 1$ for all $n \geq 1$, $T$ is a nonexpansive (or a quasi-nonexpansive) mapping. They established a relation between condition (I) and demicompactness. Where $T : K \to K$ is demicompact if for every bounded sequence $\{x_n\}$ in $K$ such that $\{x_n - Tx_n\}$ converges, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ that converges strongly to some $x^*$ in $K$. Every compact operator is demicompact. They actually showed that condition (I) is weaker than demicompactness for a nonexpansive map $T$ defined on a bounded set.

In 1986, Das and Debata [7] studied an Ishikawa-like scheme (see, [11]) defined by $x_1 \in K$,

$$x_{n+1} = \alpha_n S[\beta_n Tx_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n, \quad n \geq 1$$
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences satisfying \( 0 < a \leq \alpha_n, \beta_n \leq b < 1 \) for all \( n \geq 1 \), in \([a, b]\) such that \( 0 < a < b < 1 \). They studied the scheme for two quasi-nonexpansive maps \( S \) and \( T \) and proved strong convergence of the sequence \( \{x_n\} \) to a common fixed point of \( S \) and \( T \) in a real strictly convex Banach space. Tan and Xu [26] studied the scheme (1.2) in which \( S = T \) and proved strong and weak convergence theorems in a real uniformly convex Banach space.

Recently, Khan and Fukhar-ud-din [15] studied a scheme defined by

\[
\begin{align*}
x_{n+1} &= x_n - \alpha_n T_1 x_n + \alpha_n T_2 x_n + c_n u_n, \\
y_n &= b_n x_n + a_n T_2 x_n + c_n v_n,
\end{align*}
\]

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\} \) and \( \{c'_n\} \) are sequences in \([0, 1]\) with \( 0 < \delta \leq a_n, a'_n \leq 1 - \delta, a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n \) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( K \), in a uniformly convex Banach space \( E \) for two nonexpansive mappings. They proved that \( \{x_n\} \) defined by (1.3) converges weakly to a common fixed point of \( T_1 \) and \( T_2 \) if \( E \) satisfies Opial’s condition. This excludes \( L_p \) spaces, \( 1 < p < \infty, p \neq 2 \). They also proved that \( \{x_n\} \) converges strongly to a common fixed point of \( T_1 \) and \( T_2 \) if the two maps satisfy what they called Condition \( A \) (see definition 2.1 below). We observe that by setting \( u_n = v_n = 0 \) for all \( n \geq 1 \), \( a'_n = \beta_n, T_1 \equiv S \) and \( T_2 \equiv T \), \( a_n = \alpha_n \), (1.3) reduces to (1.2). If (1.2) converges, then (1.3) is unnecessary since (1.2) is simpler. The addition of bounded error terms leads to no further generalization. Very recently, Shahzad and Al-dubiban [24] proved that the sequence defined by (1.2) converges weakly to a common fixed point of the two maps \( S \) and \( T \) if the space \( E \) is uniformly convex and its dual space \( E^* \) satisfies the Kadec-Klee property. They further proved that \( \{x_n\} \) converges strongly to a common fixed point of \( S \) and \( T \) if the maps \( S \) and \( T \) satisfy what they called condition \( B \) (see definition 2.2 below).

Remark 1. We remark that the requirement that \( E \) is uniformly convex and its dual space \( E^* \) satisfies the Kadec-Klee property is weaker than the requirement that \( E \) is uniformly convex and satisfies Opial’s condition. This is because a dual space of a reflexive Banach space with Fréchet differentiable norm or that of a reflexive Banach space satisfying Opial’s condition, also satisfies the Kadec-Klee property (see e.g.,[8] ). Furthermore, \( L_p \) spaces, \( 1 < p < \infty, p \neq 2 \) do not satisfy Opial’s property. However, their dual spaces satisfy the Kadec-Klee property.

More recently, the present authors ([4]) introduced the following scheme for a family of non-self asymptotically nonexpansive mappings

\[
\begin{align*}
x_{n+1} &= P \left[ (1 - \alpha_1) x_n + \alpha_1 T_1 (P T_1)^{n-1} y_{n+m-2} \right], \\
y_{n+m-2} &= P \left[ (1 - \alpha_2) x_n + \alpha_2 T_2 (P T_2)^{n-1} y_{n+m-3} \right] \\
&\vdots \\
y_n &= P \left[ (1 - \alpha_m) x_n + \alpha_m T_m (P T_m)^{n-1} x_n \right], \quad n \geq 1.
\end{align*}
\]
and proved the following theorems for finite families of non-self asymptotically nonexpansive mappings.

**Theorem CA 1** ([4]) Let $E$ be a real uniformly convex Banach space and $K$ be a closed convex nonempty subset of $E$ which is also a nonexpansive retract with retraction $P$. Let $T_1, T_2, \ldots, T_m : K \to E$ be asymptotically nonexpansive mappings of $K$ into $E$ with sequences (respectively) \( \{k_{in}\}_{n=1}^{\infty} \) satisfying $k_{in} \to 1$ as $n \to \infty$, $i = 1, 2, \ldots, m$ and $\sum_{n=1}^{\infty} (k_{in} - 1) < \infty$. Let $\{\alpha_{in}\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ for each $i \in \{1, 2, \ldots, m\}$ (respectively). If one of $\{T_i\}_{i=1}^{m}$ is either completely continuous or semicompact, then the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common fixed point of $\{T_i\}_{i=1}^{m}$.

**Theorem CA 2** ([4]) Let $E$ be a real uniformly convex Banach space and $K$ be a closed convex nonempty subset of $E$ which is also a nonexpansive retract with retraction $P$. Let $T_1, T_2, \ldots, T_m : K \to E$ be asymptotically nonexpansive mappings of $K$ into $E$ with sequences $\{k_{in}\}_{n=1}^{\infty}$ and $\{\alpha_{in}\}_{n=1}^{\infty}$ as in Theorem CA 1. If $E$ satisfies Opial’s condition or has a Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a common fixed point of $\{T_i\}_{i=1}^{m}$.

The following theorem was also proved for nonexpansive mappings.

**Theorem CA 3** [4] Let $E$ be a real uniformly convex Banach space whose dual space $E^*$ satisfies the Kadec-Klee property. Let $K$ be a nonempty closed convex subset of $E$. Let $T_1, T_2, \ldots, T_m : K \to K$ be nonexpansive mappings. Let the sequence $\{\alpha_{in}\}_{n=1}^{\infty}$ be as in Theorem CA 1 and $\{x_n\}$ be defined iteratively by

\[
\begin{align*}
\quad x_1 & \in K, \\
x_{n+1} & = (1 - \alpha_{1n})x_n + \alpha_{1n}T_1y_{n+m-2} \\
y_{n+m-2} & = (1 - \alpha_{2n})x_n + \alpha_{2n}T_2y_{n+m-3} \\
\vdots \quad & \\
y_n & = (1 - \alpha_{mn})x_n + \alpha_{mn}T_mx_n, \quad n \geq 1, m \geq 2.
\end{align*}
\]

(1.5)

Then, $\{x_n\}$ converges weakly to some common fixed point of $\{T_i\}_{i=1}^{m}$.

It is our purpose in this paper to prove, in a real uniformly convex Banach space $E$, (i) a weak convergence theorem for finite families of asymptotically nonexpansive mappings where the dual space $E^*$ of $E$ satisfies the Kadec Klee property; (ii) a strong convergence theorem if one member of the family of asymptotically nonexpansive maps $\{T_i\}$ satisfies a condition weaker than semicompactness.

Our theorems generalize and improve some recent important results (see remark 3). We do not require our space $E$ to satisfy Opial’s condition or to have a Fréchet differentiable norm as in Theorem CA2. Consequently, our theorems are applicable in $L_p$ spaces, $1 < p < \infty$. Our method
of proof is also of independent interest. Furthermore, theorem CA1 is improved by weakening the semicompactness condition. Finally, theorem CA3 is extended to the class of asymptotically nonexpansive mappings.

2. Preliminaries

Let $E$ be a real normed linear space. The modulus of convexity of $E$ is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{x+y}{2} : \|x\| = \|y\| = 1, \epsilon = \|x-y\| \right\}.$$ 

$E$ is called uniformly convex if and only if $\delta_E(\epsilon) > 0$ $\forall \epsilon \in (0, 2]$.

A mapping $T$ with domain $D(T)$ and range $\text{R}(T)$ in $E$ is said to be demiclosed at $p$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \to x^* \in D(T)$ and $Tx_n \to p$ then $Tx^* = p$.

A mapping $T : K \to K$ is said to be semicompact if, for any bounded sequence $\{x_n\}$ in $K$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $x^*$ in $K$. $T$ is said to be demicompact if for every bounded sequence $\{x_n\}$ in $K$ such that $\{x_n - Tx_n\}$ converges, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges strongly to some $x^*$ in $K$. Every semicompact map is demicompact.

A Banach space $E$ is said to have the Kadec-Klee property if, for every sequence $\{x_n\}$ in $E$, $x_n \to x$ and $\|x_n\| \to \|x\|$ imply $\|x_n - x\| \to 0$. Every locally uniformly convex space has the Kadec-Klee property. In particular, $L_p$ spaces, $1 < p < \infty$ have this property (see e.g., ([6]) proposition 2.8, p. 49).

**Definition 2.1** ([15]) The mappings $S, T : K \to K$ with $F(S) \cap F(T) \neq \emptyset$ are said to satisfy condition (A) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$, such that $\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, F))$ for all $x \in K$.

**Definition 2.2** ([24]) Two mappings $S, T : K \to K$ with $F(S) \cap F(T) \neq \emptyset$ are said to satisfy condition (B) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$, such that for all $x \in K$, $\max\{\|x - Tx\|, \|x - Sx\|\} \geq f(d(x, F))$.

In what follows we shall use the following results:

**Lemma 2.1** ([26]) Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \sigma_n \forall n \geq 1$, and $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n \to \infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$ then, $\lambda_n \to 0$ as $n \to \infty$.

**Lemma 2.2** ([21]) Let $E$ be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$
for all positive integers $n \geq 1$. Suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) such that
\[
\limsup_{n \to \infty} \|x_n\| \leq r, \quad \limsup_{n \to \infty} \|y_n\| \leq r \quad \text{and} \quad \lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r
\]
hold for some $r \geq 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 2.3** ([27]) Let $E$ be a real uniformly convex Banach space, $K$ a nonempty closed subset of $E$, and let $T : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$, then $(I - T)$ is demiclosed at zero.

**Lemma 2.4** ([14]) Let $E$ be a real uniformly convex Banach space whose dual $E^*$ satisfies the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in $E$ and $x^*, q^* \in \omega_\omega(\{x_n\})$ (where $\omega_\omega(\{x_n\})$ denote the weak limit set of $\{x_n\}$). Suppose
\[
\lim_{n \to \infty} \|tx_n + (1 - t)x^* - q^*\| \quad \text{exists for all} \quad t \in [0, 1].
\]
Then, $x^* = q^*$.

**Lemma 2.5** ([17]) Let $E$ be a uniformly convex Banach space and $K$ a convex subset of $E$, and let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{\alpha_n\}$. Then, there exists a strictly increasing continuous convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that
\[
\left\| T^n \left( \sum_{i=1}^{k} \lambda_i x_i \right) - \sum_{i=1}^{k} \lambda_i T^n x_i \right\| \leq \alpha_n \phi^{-1} \left( \max_{1 \leq i, j \leq k} \left[ \left\| x_i - x_j \right\| - \frac{1}{\alpha_n} \left\| T^n x_i - T^n x_j \right\| \right] \right)
\]
for any $k, n \geq 1$, any $\lambda_1, \ldots, \lambda_k \geq 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ and any $x_1, \ldots, x_n \in K$.

**Lemma 2.6** ([9]) Let $E$ be a uniformly convex Banach space and $K$ a convex subset of $E$, and let $T : K \to K$ be a Lipschitz map with a Lipschitz constant $L$. Then, there exists a strictly increasing continuous convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that
\[
\| tT x + (1 - t)T y - T(tx + (1 - t)y) \| \leq L \phi^{-1} \left( \| x - y \| - \frac{1}{L} \| T x - T y \| \right)
\]
for all $x, y \in K$ and $0 < t < 1$.

### 3. Main Results

In the sequel, we designate the set $\{1, 2, \ldots, m\}$ by $I$ and we always assume $\bigcap_{i=1}^{m} \text{Fix}(T_i) \neq \emptyset$.

Consider the following iteration scheme. For $x_1 \in K$, a closed convex nonempty subset of a uniformly convex Banach space $E$, let $\{x_n\}$ be generated for $m \geq 2$ by,

\[
\begin{cases}
  x_1 \in K, \\
  x_{n+1} = (1 - \alpha_1) x_n + \alpha_1 T^n_1 y_{n+m-2} \\
  y_{n+m-2} = (1 - \alpha_2) x_n + \alpha_2 T^n_2 y_{n+m-3} \\
  \vdots \\
  y_n = (1 - \alpha_m) x_n + \alpha_m T^n_m x_n,
\end{cases}
\]

\(n \geq 1\).
We shall need the following lemmas.

**Lemma 3.1 ([4])** Let $E$ be a real normed linear space and $K$ be a nonempty convex subset of $E$. Let $T_1, T_2, \ldots, T_m$ be asymptotically nonexpansive self mappings of $K$ with sequences $\{k_n\}_{n=1}^{\infty}$ satisfying $k_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $i = 1, 2, \ldots, m$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ for each $i \in \{1, 2, \ldots, m\}$. Let $\{x_n\}$ be a sequence defined iteratively by (3.1) and $x^* \in \bigcap_{i=1}^{m} \text{Fix}(T_i)$. Then, there exists a positive integer $M$ such that

\[(3.2) \quad \|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \delta_m M w_n \quad \forall n \geq 1,\]

$\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - x^*\|$ exists.

**Lemma 3.2 ([4])** Let $E$ be a real uniformly convex Banach space and $K$ be a closed convex nonempty subset of $E$. Let $T_1, T_2, \ldots, T_m : K \to K$ be asymptotically nonexpansive mappings with sequences $\{k_n\}_{n=1}^{\infty}$ satisfying $k_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $i = 1, 2, \ldots, m$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$, $i \in I$. Let $\{x_n\}$ be a sequence defined iteratively by (3.1). Then,

\[
\lim_{n \to \infty} \|x_n - T_1 x_n\| = \lim_{n \to \infty} \|x_n - T_2 x_n\| = \ldots = \lim_{n \to \infty} \|x_n - T_m x_n\| = 0.
\]

We now prove weak convergence theorem. We first prove the following lemma.

**Lemma 3.3** Let $E$ be a real uniformly convex Banach space, and $K$ be a nonempty closed convex subset of $E$. Let $T_1, T_2, \ldots, T_m : K \to K$ be asymptotically nonexpansive mappings with sequences $\{k_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ as in lemma 3.2. For $t \in [0, 1]$ and $p, q \in \bigcap_{i=1}^{m} \text{Fix}(T_i)$, if $\{x_n\}$ is a sequence defined by (3.1) then,

\[
\lim_{n \to \infty} \|tx_n + (1-t)p - q\| \text{ exists.}
\]

**Proof** We follow the line of proof in [19]. Considering the more general nature of our maps, we give the proof. Since $\{x_n\}$ is bounded, there exists a positive real number $r$ such that $\{x_n\} \subseteq D = \overline{B}_r(0) \cap K$, so that $D$ is a closed convex bonded and nonempty subset of $K$. Let

\[a_n(t) := \|tx_n + (1-t)p - q\|.
\]

Then,

\[
\lim_{n \to \infty} a_n(1) = \lim_{n \to \infty} \|x_n - q\| \text{ and } \lim_{n \to \infty} a_n(0) = \|p - q\| \text{ exist.}
\]
Now let $t \in (0, 1)$ and define a map $Q_n : D \to D$ by $x \in D,$
\[
\begin{cases}
Q_n x = (1 - \alpha_{1n})x + \alpha_{1n}T_1^nx^{m-2} \\
x^{m-2} = (1 - \alpha_{2n})x + \alpha_{2n}T_2^nx^{m-3} \\
\vdots \\
x^0 = (1 - \alpha_{mn})x + \alpha_{mn}T_m^nx, 
\end{cases}
\]
where $n \geq 1, m \geq 2.$

Denoting $x^{m-h}$ by $y_{n+m-h},$ $2 \leq h \leq m$ we note that $Q_n x_n = x_{n+1}.$ Observe also that $\bigcap_{i=1}^{m} Fix(T_i) \subseteq Fix(Q_n).$ Furthermore, for $x, z \in D,$
\[
\|Q_n x - Q_n z\| = \|(1 - \alpha_{1n})x + \alpha_{1n}T_1^nx^{m-2} - [(1 - \alpha_{1n})z + \alpha_{1n}T_1^nz^{m-2}]\| \\
\leq (1 - \alpha_{1n})\|x - z\| + \alpha_{1n}(1 + u_{1n})\|x^{m-2} - z^{m-2}\| \\
\leq \|x - z\| [1 + u_{1n} + u_{2n}(1 + u_{1n}) + u_{3n}(1 + u_{1n})(1 + u_{2n}) + \ldots \\
+ u_{mn}(1 + u_{1n})(1 + u_{2n}) \ldots (1 + u_{m-1n})] \\
= (1 + \beta_{nm})\|x - z\|,
\]
where
\[
\beta_{nm} = u_{1n} + u_{2n}(1 + u_{1n}) + u_{3n}(1 + u_{1n})(1 + u_{2n}) + \ldots \\
+ u_{mn}(1 + u_{1n})(1 + u_{2n}) \ldots (1 + u_{m-1n}).
\]
Observe that $\sum_{n=1}^{\infty} \beta_{nm} < \infty.$

Define the following, for an integer $d \geq 1,$
\[
S_{n,d} := Q_{n+d-1}Q_{n+d-2} \ldots Q_n
\]
and
\[
b_{n,d} := \|S_{n,d}(tx_n + (1-t)p) - (tS_{n,d}x_n + (1-t)p)\|.
\]
Then for $x, z \in D,$
\[
\|S_{n,d}x - S_{n,d}z\| = \|Q_{n+d-1}Q_{n+d-2} \ldots Q_n x - Q_{n+d-1}Q_{n+d-2} \ldots Q_n z\| \\
\leq [(1 + \beta_{n+d-1m})(1 + \beta_{n+d-2m}) \ldots (1 + \beta_{n+1m})(1 + \beta_{nm})] \|x - z\|.
\]
Denote the sequence
\[
\{(1 + \beta_{n+d-1m})(1 + \beta_{n+d-2m}) \ldots (1 + \beta_{n+1m})(1 + \beta_{nm})\}
\]
by $\{\gamma_{nmd}\}$, then $\lim_{n,d \to \infty} \gamma_{nmd} = 1.$ Also $S_{n,d}x_n = x_{n+d}$ and $S_{n,d}x^* = x^* \ \forall x^* \in \bigcap_{i=1}^{m} F(T_i).$

By lemma 2.5 and 2.6, there exists a strictly increasing, continuous and convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that
\[
b_{n,d} = \|S_{n,d}(tx_n + (1-t)p) - (tS_{n,d}x_n + (1-t)p)\| \\
\leq \gamma_{nmd}\phi^{-1}\left(\|x_n - p\| - \frac{1}{\gamma_{nmd}}\|S_{n,d}x_n - S_{n,d}p\|\right) \\
= \gamma_{nmd}\phi^{-1}\left(\|x_n - p\| - \frac{1}{\gamma_{nmd}}\|x_{n+d} - p\|\right).
\]
Thus, we have the following estimate
\[
a_{n+d}(t) = \|tx_{n+d} + (1-t)p - q\|
\]
\[
= \|tS_{n,d}x_n + (1-t)p - q\|
\]
\[
\leq \|tS_{n,d}x_n + (1-t)p - S_{n,d}(tx_n + (1-t)p)\|
\]
\[
+ \|S_{n,d}(tx_n + (1-t)p) - q\|
\]
\[
= b_{n,d} + \|S_{n,d}(tx_n + (1-t)p) - S_{n,d}q\|
\]
(3.3)
\[
\leq \gamma_{nmd}\phi^{-1}\left(\|x_n - p\| - \frac{1}{\gamma_{nmd}}\|x_{n+d} - p\|\right) + \gamma_{nmd}a_n(t).
\]
Taking the lim sup as \(d \to \infty\) and then the lim inf as \(n \to \infty\) of both sides of (3.3), and considering the fact that \(\lim_{n \to \infty} \|x_n - p\|\) exists, we have
\[
\limsup_{n \to \infty} a_n(t) \leq \liminf_{n \to \infty} a_n(t),
\]
and the proof is complete. □

**Theorem 3.4** Let \(K\) be a nonempty closed convex subset of a real uniformly convex Banach space \(E\) whose dual space \(E^*\) satisfies the Kadec-Klee property. Let \(T_1, T_2, \ldots, T_m : K \to K\) be asymptotically nonexpansive mappings with sequences \(\{k_n\}_{n=1}^\infty\) and \(\{\alpha_{in}\}_{n=1}^\infty\) as in lemma 3.2. Let \(\{x_n\}\) be defined iteratively by (3.1). Then, \(\{x_n\}\) converges weakly to some common fixed point of \(\{T_i\}_{i=1}^m\).

**Proof** Since \(\{x_n\}\) is bounded, by the reflexivity of \(E\), there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) that converges weakly to some \(x^*\) in \(K\). By lemma 2.3 and lemma 3.2 we have \(T_i x^* = x^*\) for each \(i \in I\), and this implies \(x^* \in \bigcap_{i=1}^m F(T_i)\).

Suppose we have another subsequence say, \(\{x_{n_k}\}\) of \(\{x_n\}\) converging weakly to say \(p^*\), then \(x^*, p^* \in \omega_x(\{x_n\}) \cap \bigcap_{i=1}^m F(T_i)\) (where \(\omega_x(\{x_n\})\) denotes the weak limit set of \(\{x_n\}\)). By lemma 3.3, \(\lim_{n \to \infty} \|tx_n + (1-t)x^* - p^*\|\) exists for all \(t \in [0,1]\) and by lemma 2.4 \(x^* = p^*\) and the proof is complete. □

**Remark 2.** Theorem CA3 follows as a corollary of theorem 3.4.

We shall say that a finite family of mappings \(T_i : K \to K, i = 1, 2, \ldots, m\) with \(F := \bigcap_{i=1}^m Fix(T_i) \neq \emptyset\) satisfies:

(i) **condition (A)** if there exists a nondecreasing function \(f : [0,\infty) \to [0,\infty)\) with \(f(0) = 0, f(r) > 0\) for \(r \in (0,\infty),\) such that \(\frac{1}{m} \sum_{i=1}^m \|x - T_i x\| \geq f(d(x,F))\) for all \(x \in K,\) where \(d(x,F) = \inf \{\|x - z\| : z \in F\},\)
(ii) condition (B) if there exist \(f\) and \(d\) as in (i) such that 
\[
\max_{1 \leq i \leq m} \{|x - T_i x|\} \geq f(d(x, F))
\]
for all \(x \in K\),

(iii) condition (C) if there exist \(f\) and \(d\) as in (i) such that at least one of the \(T_i\)'s satisfies
condition (I) (i.e., \(|x - T_i x| \geq f(d(x, F))\) for at least one \(T_i i = 1, 2, ..., m\)).

Clearly condition (B) reduces to condition (I) when all but one of the \(T_i\)'s are identities, and
in addition, it also contains condition (A). Furthermore, conditions (C) and (B) are equivalent.

**Theorem 3.5** Let \(E\) be a real uniformly convex Banach space and \(K\) be a closed convex nonempty
subset of \(E\). Let \(T_1, T_2, \ldots, T_m : K \rightarrow K\) be asymptotically nonexpansive mappings with sequences
\([k_n]_{n=1}^{\infty}\) and \([\alpha_n]_{n=1}^{\infty}\) as in lemma 3.2. If the family \([T_i]_{i=1}^{m}\) satisfies condition (C),
then \(\{x_n\}\) defined by (3.1) converges strongly to a common fixed point of \([T_i]_{i=1}^{m}\).

**Proof** Let one of \(T_i\)'s, say \(T_s\), \(s \in \{1, 2, ..., m\}\) satisfy condition (I) (i.e., \(|x - T_s x| \geq f(d(x, F))\)
for all \(x \in K\) ). Let \(x^* \in F\). Then by lemma 3.1, \(\lim_{n \to \infty} \|x_n - x^*\|\) exists and \(\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \delta_n M w_n\) \(\forall n \geq 1\). This implies that
\(d(x_{n+1}, F) \leq d(x_n, F) + \delta_n M w_n\) and by lemma
2.1, \(\lim_{n \to \infty} d(x_n, F)\) exists. Since \(\lim_{n \to \infty} \|x_n - T_s x_n\| = 0\) by lemma 3.2, and \(T_s\) satisfies condition (I),
we have that \(\lim_{n \to \infty} f(d(x_n, F)) = 0\). By the nature of \(f\) and the fact that \(\lim_{n \to \infty} d(x_n, F)\) exists, we have
\(\lim_{n \to \infty} d(x_n, F) = 0\). Thus, there exist a sequence, say \(\{x^*_j\}\), in \(F\) and a subsequence of \(\{x_n\}\),
say \(\{x_{n_j}\}\) such that \(\|x_{n_j} - x^*_j\| \leq 2^{-j}\) for \(j \geq 1\). By lemma 3.1 (inequality 3.2) we also have

\[
\|x_{n_{j+1}} - x^*_j\| \leq \|x_{n_j} - x^*_j\| + \delta_m M w_{n_j},
\]

for some positive real numbers \(\delta_m\), \(M\) and a subsequence \(\{w_{n_j}\}\) of the sequence \(\{w_n\}\) as in lemma
3.1. These imply that

\[
\|x^*_j - x^*_j\| \leq \|x^*_{j+1} - x^*_{j+1}\| + \|x^*_{j+1} - x^*_j\| \leq 2^{-j+1} + 2^{-j} + \delta_m M w_{n_j}.
\]

Hence, \(\{x^*_j\}\) is Cauchy and so converges to some \(x^*\) in \(K\). Since \(F\) is closed, \(x^*\) is in \(F\), and the
fact that \(\lim_{n \to \infty} \|x_n - x^*\|\) exists implies that \(\{x_n\}\) converges strongly to \(x^*\). This completes the
proof. \(\Box\)

The following corollary follows from theorem 3.5.

**Corollary 3.6** Let \(K\) be a nonempty closed convex subset of a real uniformly convex Banach
space \(E\). Let \(T_1, T_2, \ldots, T_m : K \rightarrow K\) be nonexpansive mappings. Let the sequence \(\{\alpha_n\}_{n=1}^{\infty}\) be
as in lemma 3.2. If the family \( \{T_i\}_{i=1}^m \) satisfies condition \((\overline{C})\), then the sequence \( \{x_n\} \) defined by

\[
\begin{align*}
  x_1 & \in K, \\
  x_{n+1} & = (1 - \alpha_n)x_n + \alpha_n T_1 y_{n+m-2}, \\
  y_{n+m-2} & = (1 - \alpha_2n)x_n + \alpha_2n T_2 y_{n+m-3}, \\
  \vdots \\
  y_n & = (1 - \alpha_{mn})x_n + \alpha_{mn} T_m x_n, \quad n \geq 1, m \geq 2,
\end{align*}
\]

converges strongly to a common fixed point of the family \( \{T_i\}_{i=1}^m \).

**Remark 3.** Theorems 3.3 and 3.5 generalize and improve many recent important results. Theorem 3.3 generalizes and improves theorems 3.3, 3.4 of [24] and theorem 3.5, 4.1 of [23] to asymptotically nonexpansive mappings on one hand and to finite family on the other. In the same way theorem 3.5 generalizes and improves theorems 3.5, 3.6 of [24], theorem 3.6 of [23] and theorem 2 of [15] to asymptotically nonexpansive mappings on one hand and to finite family on the other. Theorem 3.3 extends theorem 1 of [15] to the more general uniformly convex Banach spaces whose dual spaces have the Kadec-Klee property and to the more general class of operators (finite family of asymptotically nonexpansive mappings). Furthermore while the requirement that \( E \) satisfy Opial’s condition imposed in [15] excludes application of the results of [15] in \( L_p \) spaces, \( 1 < p < \infty, p \neq 2 \), our theorems are applicable in these spaces. Theorem 3.5 improves theorem CA1 by using condition \( \overline{C} \) instead of semicompactness and Theorem 3.3 extends theorem CA3 to the more general class of asymptotically nonexpansive mappings.

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**References**


