STRONG CONVERGENCE THEOREMS FOR UNIFORMLY L-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract

Let $E$ be a real reflexive Banach space with uniform Gâteaux differentiable norm, $K$ be a nonempty bounded closed and convex subset of $E$, $T : K \to K$ be a uniformly L-Lipschitzian mapping such that $F(T) := \{x \in K : Tx = x\} \neq \emptyset$, $u \in K$ be fixed and let $\{\alpha_n\}_{n \geq 0}$ and $\{\gamma_n\}_{n \geq 0} \subset (0, 1)$ be such that $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \gamma_n$ and $\lim_{n \to \infty} \frac{\beta_n - 1}{\alpha_n} = 0$, where $\beta_n = \frac{1}{(n + 1)} \sum_{j=0}^{n} \lambda_j$ and $\lambda_j = 1 + \alpha_j \gamma_j L$. Let $S_n := (1 - \alpha_n \gamma_n)I + \alpha_n \gamma_n T^n$. It is proved that there exists some integer $N_0 > 1$, such that for each $n \geq N_0$, there exists unique $x_n \in K$ such that $x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{(n + 1)} \sum_{j=0}^{n} S_j x_n$. If $\phi : E \to \mathbb{R}$ is defined by $\phi(y) := \text{LIM}_n \|x_n - y\|^2 \forall y \in E$ where LIM denotes a Banach limit, $\|x_n - T x_n\| \to 0$ as $n \to \infty$ and $K_{\text{min}} \cap F(T) \neq \emptyset$, where $K_{\text{min}} := \{x \in E : \phi(x) = \min_{u \in K} \phi(u)\}$, then it is proved that $\{x_n\}$ converges strongly to a fixed point of $T$. As an application, it is proved that the iterative process, $z_0 \in K$, $z_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{(n + 1)} \sum_{j=0}^{n} S_j z_n$, $n \geq 0$, under suitable conditions on the iteration parameters, converges strongly to a fixed point of $T$.

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1 Introduction

Let $E$ be a real Banach space with dual space $E^*$, and let $K$ be a nonempty subset of $E$. Let $J : E \to 2^{E^*}$ denote the normalized duality mapping defined by

\[ J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \|^2, \| f \| = \| x \|, \ x \in E \}. \]

A mapping $T : K \to K$ is called uniformly $L$-Lipschitzian ($L > 0$) if

\[ \| T^n x - T^n y \| \leq L \| x - y \|, \ \forall \ x, \ y \in K, \]

and for all integers $n \geq 0$. $T$ is called asymptotically nonexpansive if there exists a sequence $\{ k_n \}_{n \geq 0} \subset [1, +\infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

\[ \| T^n x - T^n y \| \leq k_n \| x - y \|, \ \forall \ x, \ y \in K, \ \forall \ n \geq 0. \]

The asymptotically nonexpansive mappings are important generalizations of nonexpansive mappings where a mapping is said to be nonexpansive if $\| T x - T y \| \leq \| x - y \| \ \forall \ x, \ y \in K$. It is clear that every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian for some constant $L > 0$. The asymptotically nonexpansive mappings were introduced by Goebel and Kirk [9] and, they proved that if $K$ is a nonempty bounded closed and convex subset of a uniformly convex Banach space $E$, and $T : K \to K$ is an asymptotically nonexpansive mapping, then $T$ has a fixed point. In [10], they extended this result to the broader class of uniformly $L$-Lipschitzian maps with $L < \gamma$, where $\gamma$ is sufficiently near 1. Several authors have investigated iterative methods for approximating fixed points of asymptotically nonexpansive mappings (e. g., Bruck et al. [2], Chang [3], Chidume [4], Chidume et al. [7], Chidume et al. [8], Lim and Xu [11], Shimizu and Takahashi [12], Shioji and Takahashi [13, 14], Schu [15, 16].

In [12], Shimizu and Takahashi proved that in Hilbert spaces, the approximating sequence $x_n = \alpha_n \omega + (1 - \alpha_n) \frac{1}{n} \sum_{j=1}^{n} T^j x_n, \ n \geq 1$ for an asymptotically nonexpansive mapping $T$ converges strongly to a fixed point of $T$ which is nearest to $\omega$ (where $\alpha_n = \frac{\beta_n - 1}{\beta_n - 1 + a}, \ 0 < a < 1$) and $\beta_n = \frac{1}{n} \sum_{j=1}^{n} (1 + |1 - k_j| + e^{-j})$. Shioji and Takahashi [14] extended this result to uniformly convex Banach spaces with uniformly Gâteaux differentiable norm.

It is our purpose in this paper to extend the results of Shimizu and Takahashi [12] and that of Shioji and Takahashi [14] to the more general class of uniformly $L$-Lipschitzian mappings and to the more general real reflexive Banach space with uniformly Gâteaux differentiable norm.
2 Preliminaries

Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space $E$. $E$ is said to have a 
Gâteaux differentiable norm if the limit $\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$ exists for each $x, y \in S$, and we call $E$ smooth when this is the case. $E$ is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, $E$ is said to be uniformly smooth if the limit exists uniformly for $(x, y) \in S \times S$. It is well known (see e.g., [5]) that if $E$ is smooth then the duality mapping on $E$ is single-valued, and if $E$ has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak$^*$ uniformly continuous on bounded subsets of $E$.

Let $K$ be a nonempty closed convex and bounded subset of a Banach space $E$ and let the diameter of $K$ be defined by $d(K) = \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) = \sup\{\|x - y\| : y \in K\}$ and let $r(K) = \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of $K$ relative to itself. The normal structure coefficient $N(E)$ of $E$ (see e.g. [1]) is defined by $N(E) := \inf\{\frac{d(K)}{r(K)} : K$ is a closed convex and bounded subset of $E$ with $d(K) > 0\}$. A space $E$ such that $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g., [8, 11]).

We shall let $LIM$ be a Banach limit. Recall that $LIM \in (l^\infty)^*$ such that $\|LIM\| = 1$, $\liminf a_n \leq LIMa_n \leq \limsup a_n$ and $LIMa_n = LIMa_{n+1}$ for all $\{a_n\}_{n \geq 0} \in l^\infty$ (see e.g., [5, 8]).

In the sequel, we shall need the following Lemmas.

Lemma 2.1. (see e. g., [11]) Suppose $X$ is a Banach space with uniform normal structure, $C$ is a nonempty bounded subset of $X$, and $T : C \to C$ is a uniformly $L$-Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset $A$ of $C$ with the following property $(P)$:

$$(P) \quad x \in A \text{ implies } \omega_\omega(x) \in A,$$

(where $\omega_\omega(x)$ is the weak $\omega - \lim$ set of $T$ at $x$, that is, the set $\{y \in X : y = \text{ weak } \omega - \lim T^nx \text{ for some } n_j \to \infty\}$.) Then $T$ has a fixed point in $A$.

Lemma 2.2. Let $E$ be an arbitrary real Banach space. Then

$$\|x + y\| \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

for all $x, y \in E$ and $j(x + y) \in J(x + y)$.

Lemma 2.3. (see e. g., [5]) Let $\{a_n\}_{n \geq 0}$, $\{\alpha_n\}_{n \geq 0}$ and $\{\sigma_n\}_{n \geq 0}$ be real sequences of nonnegative numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \ n \geq 0$$
where (i) \(0 < \alpha_n < 1\); (ii) \(\sum_{n=1}^{\infty} \alpha_n = \infty\). Suppose that \(\sigma_n = o(\alpha_n)\) Then \(\alpha_n \to 0\) as \(n \to \infty\).

3 Main results

We begin with the following lemma whose proof is immediate and is, therefore, omitted.

Lemma 3.1. Let \(K\) be a closed convex nonempty subset of a real normed linear space \(E\) and let \(T : K \to K\) be a uniformly \(L\)-Lipschitzian mapping. Let \(\{\alpha_n\}_{n \geq 0}, \{\gamma_n\}_{n \geq 0} \subset (0,1)\) be real sequences. For all \(n \in \mathbb{N}\) define \(S_n : K \to K\) by \(S_n = (1 - \alpha_n \gamma_n)I + \alpha_n \gamma_n T^n\). Then, \[\|S_n x - S_n y\| \leq \lambda_n \|x - y\|\] for all \(x, y \in E\), where \(\lambda_n = 1 + \alpha_n \gamma_n L\).

Remark 3.1. In the remaining part of this paper, \(S_n := (1 - \alpha_n \gamma_n)I + \alpha_n \gamma_n T^n\) and \(\beta_n := \frac{1}{n + 1} \sum_{j=0}^{n} \lambda_j\), \(\lambda_j = 1 + \alpha_j \gamma_j L\).

Theorem 3.2. Let \(E\) be a real Banach space, \(K\) be a nonempty closed convex and bounded subset of \(E\); \(T : K \to K\) be a uniformly \(L\)-Lipschitzian map. Let \(\{\alpha_n\}_{n \geq 0}\) and \(\{\gamma_n\}_{n \geq 0} \subset (0,1)\) be such that \(\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \gamma_n\) and \(\lim_{n \to \infty} \beta_n - \frac{1}{\alpha_n} = 0\). Let \(u \in K\) be fixed. Then there exists an integer \(N_0 > 1\) such that for each integer \(n \geq N_0\), there exists unique \(x_n \in K\) such that

\[x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n.\]  

(3.1)

Proof. Define \(f_n : K \to K\) by \(f_n x = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} S_j x\), \(x \in K\). Then for all \(x, y \in K\),

\[\|f_n x - f_n y\| = (1 - \alpha_n) \left( \frac{1}{n + 1} \sum_{j=0}^{n} S_j x - \frac{1}{n + 1} \sum_{j=0}^{n} S_j y \right) \leq (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} \lambda_j \|x - y\| = (1 - \alpha_n) \beta_n \|x - y\|.

By the hypothesis, \(\lim_{n \to \infty} \beta_n - \frac{1}{\alpha_n} = 0\). Thus, there exists a positive integer \(N_0 > 1\) such that \((1 - \alpha_n) \beta_n < 1\) \(\forall n \geq N_0\). So for all \(n \geq N_0\), \(f_n\) is clearly a strict contraction map from \(K\) into itself. Hence, \(f_n\) has a unique fixed point \(x_n \in K\). This completes the proof. \(\square\)

Remark 3.2. Observe that if we define a mapping \(\phi : E \to \mathbb{R}\) by

\[\phi(y) = LIM_n \|x_n - y\|^2 \forall y \in K, \ n \geq N_0,\]

then \(\phi(y) \to \infty\) as \(\|y\| \to \infty\) and \(\phi\) is continuous and convex, so that if \(E\) is a real reflexive Banach space, then there exists \(x^* \in K\) such that \(\phi(x^*) = \inf_{y \in K} \phi(y)\). Thus, the set

\[K_{\text{min}} := \{x \in K : \phi(x) = \inf_{y \in K} \phi(y)\} \neq \emptyset.\]
We now prove the following theorem.

**Theorem 3.3.** Let $E$ be a real reflexive Banach space with uniformly Gâteaux differentiable norm. Let $K$ be a bounded closed convex and nonempty subset of $E$. Let $T : K \rightarrow K$ be a uniformly L-Lipshitzian map such that $F(T) := \{ x \in K : Tx = x \} \neq \emptyset$. Let $\{ \alpha_n \}_{n \geq 0}$ and $\{ \gamma_n \}_{n \geq 0} \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} \alpha_n = 0 = \lim_{n \rightarrow \infty} \gamma_n$ and $\lim_{n \rightarrow \infty} \frac{\beta_n - 1}{\alpha_n} = 0$. Let $u \in K$ be fixed. Let the sequence $\{ x_n \}_n$ be given by

$$x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n.$$ 

Suppose that $K_{\min} \cap F(T) \neq \emptyset$ and that $\| x_n - T x_n \| \rightarrow 0$ as $n \rightarrow \infty$. Then $\{ x_n \}_n$ converges strongly to a fixed point of $T$.

**Proof** Let $x^* \in K_{\min} \cap F(T)$ and let $t \in (0, 1)$. Then $(1 - t)x^* + tu \in K$. It follows that $\phi(x^*) \leq \phi((1 - t)x^* + tu)$. By Lemma 2.2 we obtain,

$$\frac{\phi((1 - t)x^* + tu) - \phi(x^*)}{t} \leq -2 \text{LIM}_n \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle.$$ 

This implies that $\text{LIM}_n \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle \leq 0$. Using the fact that $j$ is norm-to-weak* uniformly continuous on bounded subsets of $E$ and taking limit as $t \rightarrow 0$, we have that

$$\text{LIM}_n \langle u - x^*, j(x_n - x^*) \rangle \leq 0. \quad (3.2)$$ 

Again, by the definition of the sequence $\{ x_n \}_{n \geq 0}$ we have that

$$\alpha_n (x_n - u) = (1 - \alpha_n) \left[ \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n - x_n \right],$$ 

so that

$$\langle x_n - u, j(x_n - x^*) \rangle = \frac{1 - \alpha_n}{\alpha_n} \left[ \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n - x_n, j(x_n - x^*) \right]$$ 

$$\leq \frac{1 - \alpha_n}{\alpha_n} \left[ \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n - \frac{1}{n + 1} \sum_{j=0}^{n} S_j x^*, j(x_n - x^*) \right]$$ 

$$+ \frac{1 - \alpha_n}{\alpha_n} \langle x^* - x_n, j(x_n - x^*) \rangle$$ 

$$\leq \frac{1 - \alpha_n}{\alpha_n} \left( \frac{1}{n + 1} \sum_{j=0}^{n} \lambda_j \| x_n - x^* \|^2 - \| x_n - x^* \|^2 \right)$$ 

$$= \frac{(1 - \alpha_n)(\beta_n - 1)}{\alpha_n} \| x_n - x^* \|^2 \leq \frac{\beta_n - 1}{\alpha_n} \| x_n - x^* \|^2.$$ 

Thus,

$$\text{LIM}_n \langle x_n - u, j(x_n - x^*) \rangle \leq 0. \quad (3.3)$$ 

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From inequalities (3.2) and (3.3) we deduce that $\lim_{n \to \infty} \langle x_n - u, j(x_n - x^*) \rangle + \lim_{n \to \infty} \langle u - x^*, j(x_n - x^*) \rangle = \lim_{n \to \infty} \langle x_n - x^*, j(x_n - x^*) \rangle = \lim_{n \to \infty} \|x_n - x^*\|^2 \leq 0$. Hence, there exists a subsequence $\{x_{n_j}\}_{j \geq 0}$ of $\{x_n\}_{n \geq 0}$ such that $\{x_{n_j}\}_{j \geq 0}$ converges strongly to $x^*$. Suppose there exists another subsequence $\{x_{n_k}\}_{k \geq 0}$ of $\{x_n\}_{n \geq 0}$ which converges strongly to (say) $y^*$. We see that $y^* \in F(T)$ since $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. It then follows from (3.2) and (3.3) that $\langle u - x^*, j(y^* - x^*) \rangle \leq 0$ and $\langle y^* - u, j(y^* - x^*) \rangle \leq 0$. Thus, adding these inequalities we get $\langle y^* - x^*, j(y^* - x^*) \rangle \leq 0$ which implies $\|y^* - x^*\|^2 \leq 0$. So, $y^* = x^*$. Hence, $\{x_n\}_{n \geq 0}$ converges strongly to $x^* \in F(T)$. This completes the proof. □

We observe that if $T$ is asymptotically nonexpansive then the condition $K_{min} \cap F(T)$ is trivially satisfied, and we have the following corollary.

**Corollary 3.4.** Let $E$ be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure; $K$ a nonempty closed convex and bounded subset of $E$, $T : K \to K$ an asymptotically nonexpansive mapping with $\{k_n\}_{n \geq 0} \subset [1, +\infty)$, $\lim_{n \to \infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n \geq 0}$, $\{\gamma_n\}_{n \geq 0} \subset (0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \gamma_n$ and $\lim_{n \to \infty} \frac{\beta_n - 1}{\alpha_n} = 0$. Let $u \in K$ be fixed. Let the sequence $\{x_n\}_{n \geq 0}$ be given by

$$x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n.$$  

Suppose that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Then $\{x_n\}_n$ converges strongly to a fixed point of $T$.

**Proof** We observe that $E$ is a reflexive Banach space and that since $N(E) > 1$ and $\lim_{n \to \infty} k_n = 1$, there exist a natural number $N_1 > 1$ and $L > 0$ such that $k_n < L < N(E)^{\frac{1}{n}} \forall n \geq N_1$. Again, $K_{min}$ has property (P) (shown in [11]). Hence $K_{min} \cap F(T) \neq \emptyset$. The result, therefore, follows from theorem 3.2. This completes the proof. □

As an application, we prove the following convergence theorem for an iterative sequence.

**Theorem 3.5.** Let $E$ be a real Banach space with uniformly Gâteaux differentiable norm; $K$ a nonempty closed convex and bounded subset of $E$, $T : K \to K$ a uniformly $L$-Lipschitzian map such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n \geq 0}$ and $\{\gamma_n\}_{n \geq 0} \subset (0, 1)$ be such that $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \gamma_n$, $\lim_{n \to \infty} \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \frac{\beta_n - 1}{\alpha_n} = 0$; let $u \in K$ be fixed and let $\{x_n\}_n$ be given by

$$x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} S_j x_n.$$  

From arbitrary $z_0 \in K$ define the sequence $\{z_n\}_{n \geq 0}$ by

$$z_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} S_j z_n, \ n \geq 0.$$  

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Suppose $K_{\text{min}} \cap F(T) \neq \emptyset$, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ and $\|z_n - \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n\| = o(\alpha_n)$. Then the sequence $\{z_n\}_{n \geq 0}$ converges strongly to a fixed point of $T$.

**Proof.** Observe that $x_n - z_n = \alpha_n (u - z_n) + (1 - \alpha_n)(\frac{1}{(n+1)} \sum_{j=0}^{n} S_j x_n - z_n)$ Using Lemma 2.2, we estimate as follows:

$$\|x_n - z_n\|^2 \leq (1 - \alpha_n)^2 \frac{1}{(n+1)} \sum_{j=0}^{n} S_j x_n - z_n\|^2 + 2\alpha_n \langle u - z_n, j(x_n - z_n) \rangle$$

$$\leq (1 - \alpha_n)^2 \frac{1}{(n+1)} \sum_{j=0}^{n} \|S_j x_n - S_j z_n\|$$

$$+ \| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\|^2 + 2\alpha_n \langle u - z_n, j(x_n - z_n) \rangle$$

$$\leq (1 - \alpha_n)^2 \langle \beta_n, x_n - z_n\rangle + \| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\|^2$$

$$+ 2\alpha_n \langle u - z_n, j(x_n - z_n) \rangle$$

$$= (1 - \alpha_n)^2 \langle \beta_n, x_n - z_n\rangle + 2\beta_n \|x_n - z_n\| \times$$

$$\| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\| + \| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\|^2$$

$$+ 2\alpha_n \langle u - z_n, j(x_n - z_n) \rangle$$

$$\leq (1 - \alpha_n)^2 \langle \beta_n, x_n - z_n\rangle + 2\beta_n \|x_n - z_n\| \times$$

$$\| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\| + \| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\|^2$$

$$+ 2\alpha_n \langle u - x_n, j(x_n - z_n) \rangle + \beta_n \|x_n - z_n\|^2$$

$$\leq (1 + \alpha_n) \beta_n \|x_n - z_n\|^2 + \| \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n - z_n\| M$$

$$+ 2\alpha_n \langle u - x_n, j(x_n - z_n) \rangle$$

for some $M > 0$. Thus,

$$\langle u - x_n, j(z_n - x_n) \rangle \leq \left( \frac{\beta_n - 1}{\alpha_n} \right) \beta_n + \|x_n - z_n\|^2 + \frac{\| \sum_{j=0}^{n} S_j z_n - z_n\| M}{\alpha_n}.$$

Thus, since $\{z_n\}$, $\{x_n\}$ and $\left\{ \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n \right\}$ are bounded and $\|z_n - \frac{1}{(n+1)} \sum_{j=0}^{n} S_j z_n\| = o(\alpha_n)$, it follows from the last inequality that,

$$\limsup_{n \to \infty} \langle u - x_n, j(z_n - x_n) \rangle \leq 0. \quad (3.4)$$

Moreover, by Theorem 3.2, we have that $\{x_n\}$ converges to $x^* \in F(T)$ as $n \to \infty$. But,

$$\langle u - x_n, j(z_n - x_n) \rangle = \langle u - x^*, j(z_n - x^*) \rangle + \langle u - x^*, j(z_n - x_n) - j(z_n - x^*) \rangle$$

$$+ \langle x^* - x_n, j(z_n - x_n) \rangle. \quad (3.5)$$
Now, \( \langle x^*-x_n,j(z_n-x_n) \rangle \to 0 \) as \( n \to \infty \). (since \( \{z_n\} \) is bounded). Also \( \langle u-x^*,j(z_n-x_n)-j(z_n-x^*) \rangle \to 0 \) as \( n \to \infty \). (since \( j \) is norm-to-weak* uniformly continuous on bounded subsets of \( E \)). Thus, from (3.4) and (3.5), we obtain \( \limsup_{n \to \infty} \langle u-x^*,j(z_n-x^*) \rangle \leq 0 \). Put

\[
\sigma_n := \max\{\langle u-x^*,j(z_n-x^*) \rangle, 0\}.
\]

Then \( \sigma_n \geq 0 \forall n \geq 1 \). We show that \( \sigma_n \to 0 \) as \( n \to \infty \). Now, since \( \limsup_{n \to \infty} \langle u-x^*,j(z_n-x^*) \rangle \leq 0 \), we have that for all \( \epsilon > 0 \), there exists a positive integer \( N_0 \) such that \( \langle u-x^*,j(z_n-x^*) \rangle < \epsilon \forall n \geq N_0 \). Thus, \( 0 \leq \sigma_n < \epsilon \forall n \geq N_0 \). Thus, since \( \epsilon > 0 \) is arbitrary we have that \( \lim_{n \to \infty} \sigma_n = 0 \).

Again, from the iteration scheme and Lemma 2.2, we have

\[
z_{n+1} - x^* = \alpha_n(u-x^*) + (1-\alpha_n)\left( \frac{1}{n+1} \sum_{j=0}^{n} S_j z_n - x^* \right)
\]

and

\[
\|z_{n+1} - x^*\|^2 = \|(1-\alpha_n)(z_n - x^*) + \alpha_n(u-x^*) + (1-\alpha_n)\left( \frac{1}{n+1} \sum_{j=0}^{n} S_j z_n - z_n \right)\|^2
\]

\[
\leq (1-\alpha_n)^2\|z_n - x^*\|^2 + 2\alpha_n \langle u-x^*,j(z_{n+1} - x^*) \rangle + 2(1-\alpha_n)\| \frac{1}{n+1} \sum_{j=0}^{n} S_j z_n - z_n \| \|z_{n+1} - x^*\|
\]

\[
\leq (1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n \sigma_n + \| \frac{1}{n+1} \sum_{j=0}^{n} S_j z_n - z_n \| M_0
\]

\[
\leq (1-\alpha_n)\|z_n - x^*\|^2 + \delta_n
\]

where \( \delta_n = 2\alpha_n \sigma_n + \| \frac{1}{n+1} \sum_{j=0}^{n} S_j z_n - z_n \| M_0 = o(\alpha_n) \) for some \( M_0 > 0 \). Since \( \sum \alpha_n = \infty \), it therefore, follows from Lemma 2.3 that \( z_n \to x^* \) as \( n \to \infty \). This completes the proof. \( \Box \)

**Remark 3.3.** The Parameters in our iteration process are easy to choose. One may choose \( \alpha_n = \frac{1}{(n+1)^2} \) and \( \gamma_n = \frac{1}{(n+1)^2} \), \( \forall n \geq 0 \) and for arbitrary \( a \in (0,1) \).

**Remark 3.4.** Our theorems extend the results of Shioji and Takahashi [14] (and a host of other authors) in two directions. In the first place, we extend the results of Shioji and Takahashi from the class of asymptotically nonexpansive mappings to the more general class of uniformly L-Lipschitzian mappings. Secondly, we also extend their results from uniformly convex Banach spaces to the more general reflexive Banach spaces.

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**References**


