CONVERGENCE OF PATH AND AN ITERATIVE METHOD
FOR FAMILIES OF NONEXPANSIVE MAPPINGS

C.E. Chidume¹
*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy*

and

Bashir Ali ²
*Department of Mathematical Sciences, Bayero University, Kano, Nigeria*

and

*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.*

**Abstract**

Let $E$ be a real $q$–uniformly smooth Banach space with $q \geq 1 + d_q$. Let $K$ be a closed, convex and nonempty subset of $E$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of nonexpansive self-mappings of $K$. For arbitrary fixed $\delta \in (0, 1)$ define a family of nonexpansive maps $\{S_i\}_{i=1}^{\infty}$ by $S_i := (1 - \delta)I + \delta T_i$ where $I$ is the identity map of $K$. Let $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume either at least one of the $T_i's$ is demicompact or $E$ admits weakly sequentially continuous duality map. It is prove that the fixed point sequence $\{z_{i_n}\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{\infty}$, where

$$z_{i_n} = t_n u + \sum_{i \geq 1} \sigma_{i,n} S_i z_{i_n},$$

and $\{t_n\}$ is a sequence in $(0, 1)$, satisfying appropriate conditions. As an application, it is prove that the iterative sequence $\{x_n\}$ defined by: $x_0 \in K$,

$$x_{n+1} = \alpha_n u + \sum_{i \geq 1} \sigma_{i,n} S_i x_n, \quad n \geq 0$$

converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{\infty}$ where $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ are sequences in $(0, 1)$ satisfying appropriate conditions.

**MIRAMARE – TRIESTE**

July 2007

¹chidume@ictp.it
²bashiralik@yahoo.com
1. Introduction

Let $E$ be a real normed space and $E^*$ be its dual space. For some real number $q$ $(1 < q < \infty)$, the generalized duality mapping $J_q : E \rightarrow 2^{E^*}$ is defined by

$$J_q(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \},$$

where $\langle ., . \rangle$ denotes the pairing between elements of $E$ and elements of $E^*$. The normalized duality mapping from $E$ to $2^{E^*}$ denoted by $J$ is defined by

$$J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \}.$$

Markov [14] (see also Kakutani [11]) showed that if a commuting family of bounded linear transformations $T_\alpha, \alpha \in \Delta$ ($\Delta$ an arbitrary index set) of a normed linear space $E$ into itself leaves some nonempty compact convex subset $K$ of $E$ invariant, then the family has at least one common fixed point in the set $K$. Motivated by this result, De Marr [5], studied the problem of the existence of a common fixed point for a family of nonlinear maps, and proved the following theorem.

**Theorem 1.1.** ([5]) Let $E$ be a Banach space and $K$ be a nonempty compact convex subset of $E$. If $\mathcal{F}$ is a nonempty commuting family of nonexpansive mappings of $K$ into itself, then the family $\mathcal{F}$ has a common fixed point in $K$.

Browder [3] proved the result of De Marr in a uniformly convex Banach space, requiring that $K$ be only nonempty closed bounded and convex. He proved the following theorem.

**Theorem 1.2.** ([3]) Let $E$ be a uniformly convex Banach space, $K$ a nonempty closed convex and bounded subset of $E$, $\{T_\lambda\}$ a commuting family of nonexpansive self-mappings of $K$. Then, the family $\{T_\lambda\}$ has a common fixed point in $K$.

Belluce and Kirk [2] proved the existence of common fixed point of finite family of nonexpansive mappings in a reflexive Banach space with normal structure. This result was generalized by Lim [12] to infinite families of nonexpansive maps.

Bauschke [1] studied the Halpern-type [8] iterative process for approximating a common fixed point for finite family of $r$ nonexpansive self-mappings, where for an operator $T$, $Fix(T) := \{ x \in D(T) : Tx = x \}$. He proved the following theorem.

**Theorem 1.3.** ([1]) Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T_1, T_2, ..., T_r$ be a finite family of nonexpansive mappings of $K$ into itself with $F := \cap_{i=1}^{r} Fix(T_i)$ and

$$F = Fix(T_rT_{r-1}...T_1) = Fix(T_1T_{r-1}...T_2) = ... = Fix(T_{r-1}T_{r-2}...T_1T_r) \neq \emptyset.$$ 

Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ which satisfies $C1 : \lim \alpha_n = 0$; $C2 : \sum \alpha_n = \infty$ and $C3 : \sum |\alpha_{n+r} - \alpha_n| < \infty$. Given points $u, x_0 \in K$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, n \geq 0,$$
where $T_n = T_{n \mod r}$. Then, $\{x_n\}$ converges strongly to $P_F u$, where $P_F : H \to F$ is the metric projection

Let $\{T_i\}$ be countable family of nonexpansive mappings. We denote by a set $\mathcal{N}_T := \{i \in \mathbb{N} : T_i \neq I\}$ ($I$ being the identity mapping on $E$).

Various authors have studied iterative schemes similar to that of Bauschke in more general Banach spaces on one hand and using various conditions on the sequence $\{\alpha_n\}$ on the other hand (see, for example, [4, 9, 10, 15, 20]). All the results in these references are proved for finite families of nonexpansive mappings. Very recently, Maingé [13] studied the Halpern-type scheme for approximation of a common fixed point of countable infinite family of nonexpansive mappings in a Hilbert space. He proved the following theorems.

**Theorem 1.4.** ([13]) Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_i\}$ be countable family of nonexpansive self-mappings of $K$, $\{t_n\}$ and $\{\sigma_{i,t_n}\}$ be sequences in $(0,1)$ satisfying the following conditions: (i) $\lim t_n = 0$, (ii) $\sum_{i \geq 1} \sigma_{i,t_n} = (1 - t_n)$, (iii) $\forall i \in \mathcal{N}_T$, $\lim \frac{t_n}{\sigma_{i,t_n}} = 0$. Define a fixed point sequence $\{x_{t_n}\}$ by

$$
(1.1) \quad x_{t_n} = t_n C x_{t_n} + \sum_{i \geq 1} \sigma_{i,t_n} T_i x_{t_n}
$$

where $C : K \to K$ is a strict contraction. Assume $F := \cap_i F(T_i) \neq \emptyset$, then $\{x_{t_n}\}$ converges strongly to a unique fixed point of the contraction $P_F \circ C$, where $P_F$ is a metric projection from $H$ onto $F$.

**Theorem 1.5.** ([13]) Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_i\}$ be countable family of nonexpansive self-mappings of $K$, $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ be sequences in $(0,1)$ satisfying the following conditions:

(i) $\sum \alpha_n = \infty$, $\sum_{i \geq 1} \sigma_{i,n} = (1 - \alpha_n)$,

(ii) 

\[
\begin{align*}
\left| \frac{1}{\sigma_{i,n}} \right| \frac{1}{\alpha_n} &\to 0, \text{ or } \sum_n \frac{1}{\sigma_{i,n}} |\alpha_{n-1} - \alpha_n| < \infty \\
\left| \frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}} \right| &\to 0, \text{ or } \sum_n \left| \frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}} \right| < \infty \\
\frac{1}{\sigma_{i,n}} \sum_{k \geq 0} |\sigma_{k,n} - \sigma_{k,n-1}| &\to 0, \text{ or } \sum_n \frac{1}{\sigma_{i,n}} \sum_{k \geq 0} |\sigma_{k,n} - \sigma_{k,n-1}| < \infty.
\end{align*}
\]

(iii) $\forall i \in \mathcal{N}_T$, $\lim \frac{\alpha_n}{n \to \infty \sigma_{i,n}} = 0$.

Then the sequence $\{x_n\}$ defined iteratively by $x_1 \in K$,

$$
(1.2) \quad x_{n+1} = \alpha_n C x_n + \sum_{i \geq 1} \sigma_{i,n} T_i x_n
$$

converges strongly to the unique fixed point of $P_F \circ C$, where $P_F$ is a metric projection from $H$ onto $F$.

It is our purpose in this paper to prove theorems, with recursion formulas simpler than (1.1) and (1.2), that extend theorems 1.4 and 1.5 to real Banach spaces more general than Hilbert spaces. Moreover, in our more general setting, some of the conditions on the sequences $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ imposed in Theorem 1.5 will be dispensed with /weakened.
2. Preliminaries

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of the real Banach space $E$. $E$ is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; and $E$ is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$, the limit is attained uniformly for $x \in S$. Let $E$ be a normed space with $\dim E \geq 2$. The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\| - 1}{2} : \|x\| = 1; \|y\| = \tau \right\}.$$ 

The space $E$ is called uniformly smooth if and only if $\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0$. For some positive constant $q$, $E$ is called $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$, $t > 0$.

$L_p(or l_p)$ spaces are

$$\left\{\begin{array}{ll}
2 - \text{ uniformly smooth, if } 1 < p \leq 2 \\
p - \text{ uniformly smooth, if } 2 \leq p < \infty.
\end{array}\right.$$ 

It is well known that if $E$ is smooth then the duality mapping is single-valued, and if $E$ has uniformly Gâteaux differentiable norm then the duality mapping is norm-to-$\text{weak}^*$ uniformly continuous on bounded subset of $E$. A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demiclosed at $p$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx^* = p$. A mapping $T : K \to K$ is said to be demicompact if, for any bounded sequence $\{x_n\}$ in $K$ such that $\{x_n - Tx_n\}$ converges, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $x^*$ in $K$.

A Banach space $E$ is called an Opial space [16] if for all sequence $\{x_n\}$ in $E$ such that $x_n \rightharpoonup x \in E$ the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for all $y \neq x$. For a normed linear space $E$, the existence of weakly sequentially continuous duality map implies $E$ is an Opial space (see, e.g., [7]). If $E$ is an Opial space and $T$ is a nonexpansive map define on $E$, then $(I - T)$ is demiclosed at 0 (see, e.g., [6],[16]). Thus in particular, if $E = l_p$ $(1 < p < \infty)$, it is known that it admits a weakly sequentially continuous duality map and so if $K \subset E$ and $T : K \to K$ is nonexpansive, then $(I - T)$ is demiclosed at 0.

We shall make use of the following well known result.

**Lemma 2.1.** Let $E$ be a real normed linear space. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \forall x, y \in E, \forall j(x + y) \in J(x + y).$$

In the sequel, we shall also make use of the following lemmas.
Lemma 2.2. ([17]) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( E \) and let \( \{\beta_n\} \) be a sequence in \( [0, 1] \) with \( 0 < \lim \inf \beta_n \leq \lim \sup \beta_n < 1 \). Suppose \( x_{n+1} = \beta_n y_n + (1 - \beta_n) x_n \) for all integers \( n \geq 0 \) and \( \lim \sup (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim \|y_n - x_n\| = 0 \).

Lemma 2.3. ([18]) Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following relation:

\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0,
\]

where, (i) \( \{\alpha_n\} \subset [0, 1] \), \( \sum \alpha_n = \infty \); (ii) \( \lim \sup \sigma_n \leq 0 \); (iii) \( \gamma_n \geq 0 \); (\( n \geq 0 \)), \( \sum \gamma_n < \infty \). Then, \( a_n \to 0 \) as \( n \to \infty \).

Lemma 2.4. ([19]) Let \( E \) be a real \( q \)-uniformly smooth Banach space for some \( q > 1 \), then there exists some positive constant \( d_q \) such that

\[
\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + d_q\|y\|^q
\]

for all \( x, y \in E \), \( j_q(x) \in J_q(x) \).

Remark 2.5. If \( E = L_q(\text{or} l_q) \) space, \( 2 \leq q < \infty \), it is known that \( q \geq 1 + d_q \). In a real Hilbert space, \( p = 2 \), \( d_q = 1 \) so that \( q = 1 + d_q \).

3. Path convergence theorems

Let \( K \) be a nonempty closed and convex subset of a real Banach space \( E \). Let \( \{T_i\}_{i=1}^{\infty} \) be a family of self-mappings of \( K \). For a fixed \( \delta \in (0, 1) \), define a family of mappings, \( S_i : K \to K \) by

\[
S_i x := (1 - \delta)x + \delta T_i x \quad \forall \ x \in K, \quad \text{and} \quad i \in \mathbb{N}.
\]

For \( t \in (0, 1) \), let \( \{\sigma_{i,t}\}_{i=1}^{\infty} \) be a sequence in \( (0, 1) \) such that \( \sum_{i \geq 1} \sigma_{i,t} = (1 - t) \). For arbitrary fixed \( u \in K \), define a map \( T_t : K \to K \) by

\[
T_t x = tu + \sum_{i \geq 1} \sigma_{i,t} S_i x \quad \forall x \in K.
\]

Then, \( T_t \) is a strict contraction on \( K \). For, if \( x, y \in K \), we have

\[
\|T_t x - T_t y\| = \left\| \sum_{i \geq 1} \sigma_{i,t} \left( (1 - \delta)(x - y) + \delta(T_i x - T_i y) \right) \right\|
\leq \sum_{i \geq 1} \sigma_{i,t} \left( (1 - \delta)\|x - y\| + \delta\|T_i x - T_i y\| \right)
\]

\[
= (1 - t)\|x - y\|.
\]

Thus, for each \( t \in (0, 1) \), there is a unique \( z_t \in K \) satisfying

\[
z_t = tu + \sum_{i \geq 1} \sigma_{i,t} S_i z_t.
\]

Lemma 3.1. Let \( E \) be a real Banach space. Let \( K \) be a closed, convex and nonempty subset of \( E \). For \( t \in (0, 1) \), let \( \{z_t\} \) be a sequence satisfying (3.2) and assume \( F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Then, \( \{z_t\} \) is bounded and admits unique accumulation point as \( t \to 0 \).
Proof. Let \( x^* \in \mathcal{F} \). Then, using (3.2), we have

\[
\| z_t - x^* \|_2 = \left\langle t(u - x^*) + \sum_{i \geq 1} \sigma_{i,t}(S_i z_t - x^*), j(z_t - x^*) \right\rangle \\
\leq t(u - x^*, j(z_t - x^*)) + \sum_{i \geq 1} \sigma_{i,t}\| z_t - x^* \|^2 \\
= t(u - x^*, j(z_t - x^*)) + (1 - t)\| z_t - x^* \|^2
\]

which implies

\[
\| z_t - x^* \| \leq \| u - x^* \|.
\]

Thus \( \{ z_t \} \) is bounded. Assume for contradiction that \( x' \) and \( x^* \) are two distinct accumulation points of \( \{ z_t \} \). Then, it is clear from the above argument that

\[
\| x' - x^* \|^2 \leq \langle u - x^*, j(x' - x^*) \rangle
\]

so that

\[
\| x' - x^* \|^2 \leq \langle u - x^*, j(x' - x^*) \rangle.
\]

These inequalities imply

\[
2\| x^* - x' \|^2 \leq \| x^* - x' \|^2,
\]

a contradiction, and thus \( x' = x^* \).

\[ \square \]

Lemma 3.2. Let \( E \) be a real \( q \)-uniformly smooth Banach space with \( q \geq 1 + d_q \). Let \( K \) be a closed, convex and nonempty subset of \( E \). Let \( \{ t_n \} \) be a sequence in \( (0,1) \) such that \( \lim t_n = 0 \) and \( \lim_{n \to \infty} \frac{\| x_n \|}{\| x_n \|^q} = 0 \forall \ i \in \mathbb{N} \). Let \( \{ z_{t_n} \} \) be a sequence satisfying (3.2) and let \( \mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Then, \( \lim_{n \to \infty} \| z_{t_n} - T_i z_{t_n} \| = 0 \forall \ i \in \mathbb{N} \).

Proof. For \( i \in \mathbb{N} \) and \( x^* \in \mathcal{F} \), we have the following estimates (using lemma 2.4)

\[
\| S_i z_{t_n} - z_{t_n} \|^q = \| S_i z_{t_n} - x^* + x^* - z_{t_n} \|^q \\
\leq \| x^* - z_{t_n} \|^q + q\langle S_i z_{t_n} - x^*, j_q(x^* - z_{t_n}) \rangle + d_q\| S_i z_{t_n} - x^* \|^q \\
\leq (1 + d_q)\| x^* - z_{t_n} \|^q + q\langle S_i z_{t_n} - x^*, j_q(x^* - z_{t_n}) \rangle \\
= (1 + d_q - q)\| x^* - z_{t_n} \|^q + q\langle S_i z_{t_n} - z_{t_n}, j_q(x^* - z_{t_n}) \rangle \\
\leq q(z_{t_n} - S_i z_{t_n}, j_q(z_{t_n} - x^*))
\]

(3.3)
Using (3.2), we have
\[
(z_{tn} - x^*, j_q(z_{tn} - x^*)) = t_n(u - x^*, j_q(z_{tn} - x^*)) + \sum_{i \geq 1} \sigma_{i,n} \left< S_i z_{tn} - z_{tn} + z_{tn} - x^*, j_q(z_{tn} - x^*) \right>
\]
\[
= t_n(u - x^*, j_q(z_{tn} - x^*)) + \sum_{i \geq 1} \sigma_{i,n} \left< S_i z_{tn} - z_{tn}, j_q(z_{tn} - x^*) \right>
\]
\[
+ (1 - t_n) \langle z_{tn} - x^*, j_q(z_{tn} - x^*) \rangle
\]
which implies
\[
\sum_{i \geq 1} \sigma_{i,n} \left< z_{tn} - S_i z_{tn}, j_q(z_{tn} - x^*) \right> = t_n(u - z_{tn}, j_q(z_{tn} - x^*)).
\]
Using this and (3.3), we get
\[
\frac{1}{q} \sum_{i \geq 1} \sigma_{i,n} \| S_i z_{tn} - z_{tn} \|^q \leq t_n \langle u - z_{tn}, j_q(z_{tn} - x^*) \rangle.
\]
Since \( \{z_{tn}\} \) is bounded, we have that \( \lim_{n \to \infty} \| S_i z_{tn} - z_{tn} \| = 0 \). This implies,
\[
\lim_{n \to \infty} \| T_i z_{tn} - z_{tn} \| = \frac{1}{\delta} \lim_{n \to \infty} \| S_i z_{tn} - z_{tn} \| = 0.
\]
This complete the proof. \( \square \)

**Theorem 3.3.** Let \( E \) be a real \( q \)-uniformly smooth Banach space with \( q \geq 1 + d_q \). Let \( K \) be a closed, convex and nonempty subset of \( E \). Let \( \{t_n\} \) be a sequence in \( (0, 1) \) such that \( \lim_{n \to \infty} t_n = 0 \) and \( \lim_{n \to \infty} \frac{1}{\sigma_{i,n}} = 0 \) \( \forall \ i \in \mathcal{N} \). Let \( \{z_{tn}\} \) be a sequence satisfying (3.2) and let \( \mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). If the duality mapping \( j_q \) of \( E \) is weakly sequentially continuous, then \( \{z_{tn}\} \) converges strongly to an element in \( \mathcal{F} \).

**Proof.** Since \( \{z_{tn}\} \) is bounded, there exists a subsequence say \( \{z_{tn_k}\} \) of \( \{z_{tn}\} \) that converges weakly to some point \( z \in K \). Using the demiclosedness property of \( (I - T_i) \) for each \( i \in \mathbb{N} \), and the fact that \( \lim_{k \to \infty} \| T_i z_{tn_k} - z_{tn_k} \| = 0 \), we get that \( z \) is a point in \( \mathcal{F} \). We also observe from (3.2) that
\[
\| z_{tn_k} - z \|^q = \left< t_{nk} (u - z) + \sum_{i \geq 1} \sigma_{i,nk} (S_i z_{tn_k} - z), j_q(z_{tn_k} - z) \right>
\]
\[
\leq t_{nk} \langle u - z, j_q(z_{tn_k} - z) \rangle + \sum_{i \geq 1} \sigma_{i,nk} \| z_{tn_k} - z \|^q
\]
\[
= t_{nk} \langle u - z, j_q(z_{tn_k} - z) \rangle + (1 - t_{nk}) \| z_{tn_k} - x^* \|^q
\]
which implies,
\[
\| z_{tn_k} - z \|^q \leq \langle u - z, j_q(z_{tn_k} - z) \rangle.
\]
Since \( j_q \) admits weak sequential continuity, this inequality implies that the subsequence \( \{z_{tn_k}\} \) converges strongly to \( z \), and since \( z_{tn} \) admits unique accumulation point, \( z_{tn} \) converges strongly to \( z \). \( \square \)

The following corollary follows from theorem 3.3.

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Corollary 3.4. Let $E$ be a real $l_p$ space, $(2 \leq p < \infty)$. Let $K$ be a closed, convex and nonempty subset of $E$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n \to \infty} t_n = 0$ and $\lim\frac{1}{n} = 0 \ \forall \ i \in \mathcal{N}_I$. Let $\{z_{tn}\}$ be a sequence satisfying (3.2) and let $\mathcal{F} := \cap_{i=1}^\infty F(T_i) \neq \emptyset$. Then, $\{z_{tn}\}$ converges strongly to an element of $\mathcal{F}$.

Theorem 3.5. Let $E$ be a real $q$–uniformly smooth Banach space with $q \geq 1 + d_q$. Let $K$ be a closed, convex and nonempty subset of $E$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n \to \infty} t_n = 0$ and $\lim\frac{1}{n} = 0 \ \forall \ i \in \mathcal{N}_I$. Let $\{z_{tn}\}$ be a sequence satisfying (3.2) and let $\mathcal{F} := \cap_{i=1}^\infty F(T_i) \neq \emptyset$. If at least one of the maps $T_i$ is demicompact, then $\{z_{tn}\}$ converges strongly to an element of $\mathcal{F}$.

Proof. For some fixed $s \in \mathbb{N}$, let $T_s$ be demicompact. Since $\lim_{n \to \infty} \|T_s z_{tn} - z_{tn}\| = 0$, there exists a subsequence say $\{z_{tn_k}\}$ of $\{z_{tn}\}$ that converges strongly to some point $z \in K$. By the continuity of $T_i, \forall i \in \mathbb{N}$, we have that $z \in \mathcal{F}$. But $z_{tn_k}$ admits unique accumulation point, so $z_{tn_k}$ converges strongly to $z$. \hfill \Box

The following corollaries follow from theorem 3.5

Corollary 3.6. Let $E$ be a real $L_p$ space, $(2 \leq p < \infty)$. Let $K$ be a closed, convex and nonempty subset of $E$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n \to \infty} t_n = 0$ and $\lim\frac{1}{n} = 0 \ \forall \ i \in \mathcal{N}_I$. Let $\{z_{tn}\}$ be a sequence satisfying (3.2) and let $\mathcal{F} := \cap_{i=1}^\infty F(T_i) \neq \emptyset$. If at least one of the maps $T_i$ is demicompact, then $\{z_{tn}\}$ converges strongly to an element of $\mathcal{F}$.

Corollary 3.7. Let $E$ be a real $q$–uniformly smooth Banach space with $q \geq 1 + d_q$. Let $K$ be a compact, convex and nonempty subset of $E$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n \to \infty} t_n = 0$ and $\lim\frac{1}{n} = 0 \ \forall \ i \in \mathcal{N}_I$. Let $\{z_{tn}\}$ be a sequence satisfying (3.2) and let $\mathcal{F} := \cap_{i=1}^\infty F(T_i)$. Then, $\{z_{tn}\}$ converges strongly to an element of $\mathcal{F}$.

Proof. Compactness of $K$ implies $\{z_{tn}\}$ has subsequence $\{z_{tn_k}\}$ which converges strongly to some $z$ in $K$. The rest follows as in the proof of Theorem 3.5. \hfill \Box

4. Iterative method and convergence theorems

Theorem 4.1. Let $E$ be a real $q$–uniformly smooth Banach space with $q \geq 1 + d_q$. Let $K$ be a closed, convex and nonempty subset of $E$. Let $\{T_i\}_{i=1}^\infty$ be a family of nonexpansive self-mappings of $K$. For arbitrary fixed $\delta \in (0,1)$ define a family of nonexpansive maps $\{S_i\}_{i=1}^\infty$ by $S_i := (1 - \delta)I + \delta T_i, \forall i \in \mathbb{N}$ where $I$ is an identity map of $K$. Assume $\mathcal{F} := \cap_{i=1}^\infty F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ be sequences in $(0,1)$ satisfying the following conditions: (i) $\lim_{n \to \infty} \alpha_n = 0, (ii) \sum_{i \geq 1} \alpha_n = \infty, (iii) \sum_{i \geq 1} \sigma_{i,n} = (1 - \alpha_n)$ and (iv) $\lim_{n \to \infty} \sum_{i \geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$. Define a sequence $\{x_n\}$ iteratively by $x_1, u \in K$,

$$x_{n+1} = \alpha_n u + \sum_{i \geq 1} \sigma_{i,n} S_i x_n, \ n \geq 1.$$
If either: (a) at least one of the $T_i$’s is demicompact or, (b) $E$ has weakly sequentially continuous duality map, then $\{x_n\}$ converges strongly to an element of $F$.

Proof. Let $x^* \in F$ be arbitrary. Then, the sequence $\{x_n\}$ defined by (4.1) satisfies

$$\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}$$

for all $n \in \mathbb{N}$. It is clear that this is true for $n = 1$. Assume it is true for $n = k$ for some $k > 1$, $k \in \mathbb{N}$. Then, using (4.1) we have

$$\|x_{k+1} - x^*\| \leq \alpha_k \|u - x^*\| + \sum_{i \geq 1} \sigma_{i,k} \left[(1 - \delta)\|x_k - x^*\| + \delta \|T_i x_k - x^*\|\right]$$

$$\leq \alpha_k \|u - x^*\| + (1 - \alpha_k)\|x_k - x^*\|$$

$$\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\},$$

and the result follows by induction. So $\{x_n\}$ is bounded and so are $\{T_i x_n\}$ and $\{S_i x_n\}$.

Define two sequences $\{\beta_n\}$ and $\{y_n\}$ by $\beta_n := (1 - \delta)\alpha_n + \delta$ and $y_n := \frac{\sum_{i=1}^{n+1} x_i + \beta_n x_n}{\beta_n}$. Then,

$$y_n = \frac{\alpha_n u + \delta \sum_{i \geq 1} \sigma_{i,n} T_i x_n}{\beta_n}.$$ Observe that $\{y_n\}$ is bounded and that

$$\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n} \|u\|$$

$$+ \frac{\delta(1 - \alpha_{n+1})}{\beta_{n+1}} \|x_{n+1} - x_n\|$$

$$+ \frac{\delta M}{\beta_{n+1}\beta_n} \sum_{i \geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| + \frac{\delta M}{\beta_{n+1}\beta_n} |\beta_n - \beta_n+1|,$$

for some positive real number $M > 0$. This implies,

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0,$$

and by lemma 2.2, $\lim_{n \to \infty} \|y_n - x_n\| = 0$. Hence,

$$\|x_{n+1} - x_n\| = \beta_n \|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{4.2}$$

From (4.1) we have $x_{n+1} - x_n = \alpha_n (u - x_n) + \sum_{i \geq 1} \sigma_{i,n} (S_i x_n - x_n)$ which implies,

$$\left\|\sum_{i \geq 1} \sigma_{i,n} (S_i x_n - x_n)\right\| \leq \|x_{n+1} - x_n\| + \alpha_n \|u - x_n\|$$

and thus $\lim_{n \to \infty} \|\sum_{i \geq 1} \sigma_{i,n} (S_i x_n - x_n)\| = 0$. Let $\{t_n\}$ be a real sequence in $(0,1)$ satisfying the following conditions:

$$\lim_{n \to \infty} t_n = 0, \sum_{i \geq 1} \sigma_{i,n} = (1 - t_n) \text{ and } \lim_{n \to \infty} \frac{\|\sum_{i \geq 1} \sigma_{i,n} (S_i x_n - x_n)\|}{t_n} = 0.$$ Let $z_{t_n} \in K$ be the unique fixed point satisfying (3.2) for each $n \in \mathbb{N}$ and let $z_{t_n} \to z \in F$ as $n \to \infty$. (This is guaranteed by either condition (a) or condition (b)). Using (3.2) and lemma 2.1
we have the following estimates
\[
\|z_{t_n} - x_n\|^2 \leq \left|\sum_{i \geq 1} \sigma_{i,n}(S_i z_{t_n} - S_i x_n + S_i x_n - x_n)\right|^2 + 2t_n \langle u - x_n, j(z_{t_n} - x_n)\rangle
\]
\[
\leq \left((1 - t_n)\|z_{t_n} - x_n\| + \left|\sum_{i \geq 1} \sigma_{i,n}(S_i x_n - x_n)\right|\right)^2 + 2t_n \langle u - x_n, j(z_{t_n} - x_n)\rangle.
\]
This implies,
\[
\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq \frac{t_n}{2} \|z_{t_n} - x_n\|^2
\]
\[
+ (1 - t_n)\|z_{t_n} - x_n\|\left(\left|\sum_{i \geq 1} \sigma_{i,n}(S_i x_n - x_n)\right|\right)
\]
\[
+ \frac{\|\sum_{i \geq 1} \sigma_{i,n}(S_i x_n - x_n)\|^2}{2t_n}
\]
and hence
\[
\lim \sup \langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq 0.
\]
Moreover,
\[
\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle = \langle u - z, j(x_n - z)\rangle
\]
\[
+ \langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle
\]
\[
+ \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle,
\]
and since \(j\) is norm-to-weak* uniformly continuous on bounded sets, we have
\[
\lim \sup \langle u - z, j(x_n - z)\rangle \leq 0.
\]
From the recursion formula (4.1) and lemma 2.1, we have the following.
\[
\|x_{n+1} - z\|^2 \leq \left|\sum_{i \geq 1} \sigma_{i,n}(S_i x_n - z)\right|^2 + 2\alpha_n \langle u - z, j(x_{n+1} - z)\rangle
\]
\[
\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n \langle u - z, j(x_{n+1} - z)\rangle,
\]
and by lemma 2.3, we have \(\{x_n\}\) converges strongly to \(z \in F\). This complete the proof. \(\square\)

The following are immediate corollaries of Theorem 4.1.

**Corollary 4.2.** Let \(E\) be a real \(q\)-uniformly smooth Banach space with \(q \geq 1 + d_q\). Let \(K\) be a compact, convex and nonempty subset of \(E\). Let \(\{T_i\}_{i=1}^\infty\) be a family of nonexpansive self-mappings of \(K\). For arbitrary fixed \(\delta \in (0,1)\) define a family of nonexpansive maps \(\{S_i\}_{i=1}^\infty\) by \(S_i := (1 - \delta)I + \delta T_i\ \forall i \in \mathbb{N}\) where \(I\) is an identity map of \(K\). Assume \(F := \bigcap_{i=1}^\infty F(T_i)\). Let \(\{\alpha_n\}\) and \(\{\sigma_{i,n}\}\) be sequences as in theorem 4.1. Let \(\{x_n\}\) be a sequence defined by (4.1). Then, \(\{x_n\}\) converges strongly to an element of \(F\).

**Corollary 4.3.** Let \(E\) be the real \(L_p\) space, \((2 \leq p < \infty)\). Let \(K\) be a closed, convex and nonempty subset of \(E\). Let \(\{T_i\}_{i=1}^\infty\) be a family of nonexpansive self-mappings of \(K\) such that at least one of the \(T_i\)’s is demicompact. For arbitrary fixed \(\delta \in (0,1)\) define a family of nonexpansive maps \(\{S_i\}_{i=1}^\infty\) by \(S_i := (1 - \delta)I + \delta T_i\ \forall i \in \mathbb{N}\) where \(I\) is an identity map of \(K\). Assume \(F := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset\).
Let \( \{\alpha_n\} \) and \( \{\sigma_{i,n}\} \) be sequences as in theorem 4.1. Define a sequence \( \{x_n\} \) iteratively by (4.1). Then, \( \{x_n\} \) converges strongly to an element of \( \mathcal{F} \).

**Corollary 4.4.** Let \( E \) be the real \( l^p \) space, \((2 \leq p < \infty)\). Let \( K \) be a closed, convex and nonempty subset of \( E \). Let \( \{T_i\}_{i=1}^{\infty} \) be a family of nonexpansive self-mappings of \( K \). For arbitrary fixed \( \delta \in (0, 1) \) define a family of nonexpansive maps \( \{S_i\}_{i=1}^{\infty} \) by \( S_i := (1 - \delta)I + \delta T_i \) \( \forall i \in \mathbb{N} \) where \( I \) is an identity map of \( K \). Assume \( \mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\sigma_{i,n}\} \) be sequences as in theorem 4.1. Define a sequence \( \{x_n\} \) iteratively by (4.1). Then, \( \{x_n\} \) converges strongly to an element of \( \mathcal{F} \).

**Corollary 4.5.** Let \( E \) be a real Hilbert space. Let \( K \) be a closed, convex and nonempty subset of \( E \). Let \( \{T_i\}_{i=0}^{\infty} \) be a family of nonexpansive self-mappings of \( K \). For arbitrary fixed \( \delta \in (0, 1) \) define a family of nonexpansive maps \( \{S_i\}_{i=0}^{\infty} \) by \( S_i := (1 - \delta)I + \delta T_i \) \( \forall i \in \mathbb{N} \) where \( I \) is an identity map of \( K \). Assume \( \mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\sigma_{i,n}\} \) be sequences as in theorem 4.1. Define a sequence \( \{x_n\} \) iteratively by (4.1). Then, \( \{x_n\} \) converges strongly to an element of \( \mathcal{F} \).

### 5. Nonself-mappings

Let \( K \) be a nonempty subset of a Banach space \( E \). For \( x \in K \), the inward set of \( x \), \( I_K(x) \), is defined by \( I_Kx := \{x + \alpha(u - x) : u \in K, \alpha \geq 1\} \). A mapping \( T : K \rightarrow E \) is called weakly inward if \( Tx \in cl[I_K(x)] \) for all \( x \in K \), where \( cl[I_K(x)] \) denotes the closure of the inward set. Every self map is trivially weakly inward.

Let \( K \subset E \) be closed convex and \( Q \) a mapping of \( E \) onto \( K \). Then \( Q \) is said to be sunny if \( Q(Qx + t(x - Qx)) = Qx \) for all \( x \in E \) and \( t \geq 0 \). A mapping \( Q \) of \( E \) into \( E \) is said to be a retraction if \( Q^2 = Q \). If a mapping \( Q \) is a retraction, then \( Qz = z \) for every \( z \in R(Q) \), range of \( Q \). A subset \( K \) is said to be sunny nonexpansive retract of \( E \) if there exists a sunny nonexpansive retraction of \( E \) onto \( K \).

Following the method of section 3 and 4, the following theorem can be proved.

**Theorem 5.1.** Let \( E \) be a real \( q \)-uniformly smooth Banach space with \( q \geq 1 + d_q \). Let \( K \) be a closed, convex and nonempty sunny nonexpansive retract of \( E \) with \( Q \) as the sunny nonexpansive retraction. Let \( T_i : K \rightarrow E \), \( i \in \mathbb{N} \) be a family of nonexpansive mappings of \( K \) into \( E \). For arbitrary fixed \( \delta \in (0, 1) \), define a family of nonexpansive maps \( \{S_i\}_{i=1}^{\infty} \) by \( S_i := (1 - \delta)I + \delta QT_i \) \( \forall i \in \mathbb{N} \) where \( I \) is the identity map of \( K \). Assume \( \mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\sigma_{i,n}\} \) be sequences in \( (0, 1) \) satisfying the following conditions: (i) \( \lim_{n \to \infty} \alpha_n = 0 \), (ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \), (iii) \( \sum_{i \geq 0} \sigma_{i,n} = 1 - \alpha_n \) and (iv) \( \lim_{n \to \infty} \sum_{i \geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| = 0 \). Define a sequence \( \{x_n\} \) iteratively by \( x_1, u \in K \)

\[
\sum_{i \geq 1} \sigma_{i,n} S_i x_n
\]

Then, \( \{x_n\} \) converges strongly to an element of \( \mathcal{F} \).
If either at least one of the $T_i$’s is demicompact or $E$ has weakly sequentially continuous duality map, then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Remark 5.2. All the other theorems of this paper also hold for non-self maps.

Remark 5.3. Prototypes of the sequences $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ in our theorems are the following:

$$\alpha_n := \frac{1}{n+1}, \quad \sigma_{i,n} := \frac{n}{2(n+1)} \quad \forall i \in \mathbb{N}.$$ 

For these choices, the recursion formulas (4.1) and (5.1) become $x_1, u \in K$,

$$x_{n+1} = \left(\frac{1}{n+1}\right) u + \frac{n}{n+1} \sum_{i \geq 1} \frac{1}{2^i} S_i x_n, \quad n \geq 1.$$  

(5.2)

Remark 5.4. We observe that corollaries 3.4 and 4.4 extend Theorems 1.4 and 1.5, respectively to $l_p$ spaces ($2 \leq p < \infty$). Furthermore, the following conditions in (ii):

$$\left\{ \frac{1}{\sigma_{i,n}} \left| 1 - \frac{\alpha_{n-1}}{\alpha_n} \right| \rightarrow 0, \text{ or } \sum_n \frac{1}{\sigma_{i,n}} |\alpha_{n-1} - \alpha_n| < \infty \right\}$$

and

$$\left\{ \frac{1}{\alpha_n} \left| \sigma_{i,n} - \frac{1}{\sigma_{i,n-1}} \right| \rightarrow 0, \text{ or } \sum_n \frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}} |< \infty \right\}$$

imposed in Theorem 1.5 are dispensed with even in our more general settings. In addition, the requirement,

$$\frac{1}{\sigma_{i,n} \alpha_n} \sum_{k \geq 0} |\sigma_{k,n} - \sigma_{k,n-1}| \rightarrow 0, \text{ or } \sum_n \frac{1}{\sigma_{i,n}} \sum_{k \geq 0} |\sigma_{k,n} - \sigma_{k,n-1}| < \infty$$

also imposed in (ii) of Theorem 1.5 is weakened in our theorems to $\lim_{n \rightarrow \infty} \sum_{i \geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$.

Finally Theorem 3.5, corollaries 3.6 and 3.7, Theorem 4.1, corollaries 4.2 and 4.3, all of which are of independent interest, are applicable in $L_p$ spaces, ($2 \leq p < \infty$).

Remark 5.5. The addition of bounded error terms to the recursion formulas (3.2), (4.1) or (5.1) leads to no further generalization.

Remark 5.6. If $f : K \rightarrow K$ is a contraction map and we replace $u$ by $f(x_n)$ in the recursion formulas (3.2), (4.1) or (5.1), we obtain what some authors now call viscosity iteration method.

We note that all the theorems and corollaries of this paper carry over trivially to the so-called viscosity process. One simply replaces $u$ by $f(x_n)$, repeats the argument of this paper, using the fact that $f$ is a contraction map.

Acknowledgments. B. Ali’s research supported by the Japanese Mori Fellowship of UNESCO at The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

References

