A NEW ITERATION PROCESS FOR FINITE FAMILIES OF GENERALIZED LIPSCHITZ PSEUDO-CONTRACTION AND GENERALIZED LIPSCHITZ ACCRETIVE MAPPINGS

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Abstract

In this paper, we introduce a new iteration process and prove that it converges strongly to a common fixed point for a finite family of generalized Lipschitz nonlinear mappings in a real reflexive Banach space $E$ with a uniformly Gateaux differentiable norm if at least one member of the family is pseudo-contractive. We also prove that a slight modification of the process converges to a common zero for a finite family of generalized Lipschitz accretive operators defined on $E$. Results for nonexpansive families are obtained as easy corollaries. Finally, our new iteration process and our method of proof are of independent interest.
1. Introduction.

Let $E$ be a real Banach space with dual $E^*$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f^*\| = \|x\|\},$$

where $\langle ., . \rangle$ denotes the pairing between elements of $E$ and elements of $E^*$. It is well known that if $E$ is smooth then $J$ is single-valued. In the sequel, single-valued normalized duality mapping will be denoted by $j$.

A mapping $T : D(T) \subset E \to E$ with domain $D(T)$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D(T)$. The mapping $T$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}_{n \geq 1} \in [1, +\infty)$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in D(T)$; and it is said to be Lipschitz if there exists $L_1 > 0$ such that for all $x, y \in D(T)$, $\|Tx - Ty\| \leq L_1 \|x - y\|$. The mapping $T$ is called pseudo-contractive if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2.$$

An important class of operators closely related to the class of pseudo-contractive ones is that of accretive mappings. A mapping $A : D(A) \subset E \to E$ is said to be accretive if for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0.$$

It is easy to see that $A$ is accretive if and only if $(I - A)$ is pseudo-contractive. The accretive operators were independently introduced by Browder [4] and Kato [23] in 1967. The importance of these operators is well known.

The main tool for approximation of fixed points of nonlinear mappings (when such fixed points exist) remains iterative technique. Numerous convergence results have been proved on the iterative methods for approximating zeros of Lipschitz accretive-type (or, equivalently, fixed points of Lipschitz pseudo-contractive-type) nonlinear mappings (see e.g., [8]-[18], [34]). Also, many authors have proved convergence theorems under the assumption that these operators have bounded range. A natural generalization of the class of Lipschitz mappings and the class of mappings with bounded range is the class of generalized Lipschitz mappings. A mapping $T : D(T) \subset E \to E$ is said to be generalized Lipschitz if there exists $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|x - y\|)$, for all $x, y \in D(T)$. Clearly, every Lipschitz map is generalized Lipschitz. Furthermore, every map with bounded range is also a generalized Lipschitz map. The following example (see e.g. [7]) shows that the class of generalized Lipschitz maps properly includes the class of Lipschitz maps and those of mappings with bounded range.
Example 1. Let $E = (-\infty, +\infty)$ and $T : E \to E$ be defined by

$$T_x = \begin{cases} 
  x - 1, & \text{if } x \in (-\infty, -1), \\
  x - \sqrt{1 - (x + 1)^2}, & \text{if } x \in [-1, 0), \\
  x + \sqrt{1 - (x - 1)^2}, & \text{if } x \in [0, 1], \\
  x + 1, & \text{if } x \in (0, +\infty).
\end{cases}$$

Clearly, $T$ is a generalized Lipschitz map which is not Lipschitz and whose range is not bounded.

Recently, the authors [14] introduced a new iteration method for approximation of fixed points (assuming existence) of single generalized Lipschitz and single generalized accretive mappings in real Banach spaces.

Markov [28] (see also Kakutani [22], 1938) showed that if a commutating family of bounded linear transformations $T_\alpha, \alpha \in \Delta, (\Delta \text{ an arbitrary index set})$ of a normed linear space $E$ into itself leaves some nonempty compact convex subset $K$ of $E$ invariant, then the family has at least one common fixed point. (The actual result of Markov is more general than this but this version is adequate for our purposes here). Motivated by this result, De Marr [27] in 1963 studied the problem of the existence of a common fixed point for a family of nonlinear maps, and proved the following theorem.

**Theorem DM.** [27] Let $E$ be a Banach space and $K$ be a nonempty compact convex subset of $E$. If $\Omega$ is a nonempty commuting family of nonexpansive mappings of $K$ into itself, then the family $\Omega$ has a common fixed point in $K$.

Browder [4] proved the result of De Marr in a uniformly convex Banach space, requiring that $K$ be only nonempty closed bounded and convex. For other fixed point theorems for families of nonexpansive mappings the reader may consult any of the following references: Belluce and Kirk [2]; Lim [24] and Bruck [5].

Within the past 10 years or so, considerable research efforts have been devoted to developing iterative methods for approximating common fixed points (assuming existence) for families of several classes of nonlinear mappings (see e.g., [1, 9, 16, 19, 20, 21, 30, 32, 33, 35]).

In this paper, we construct a new iterative sequence for the approximation of common fixed points of finite families of generalized Lipschitz pseudo-contractive and generalized Lipschitz accretive operators (assuming existence). Furthermore, our new iteration scheme and our methods of proofs are of independent interest.
2. Preliminaries.

Let \( S := \{ x \in E : \| x \| = 1 \} \) denote the unit sphere of a Banach space \( E \). The space \( E \) is said to have a Gâteaux differentiable norm if the limit
\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]
exists for each \( x, y \in S \). When this limit exists, we say that \( E \) is smooth. \( E \) is said to have a uniformly Gâteaux differentiable norm if for each \( y \in S \) the limit is attained uniformly for \( x \in S \). Furthermore, \( E \) is said to be uniformly smooth if the limit exists uniformly for \( (x, y) \in S \times S \).

It is known that if \( E \) is smooth then any duality mapping on \( E \) is single-valued; and if \( E \) has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of \( E \).

Let \( K \) be a nonempty bounded, closed and convex subset of a Banach space \( E \) and let the diameter of \( K \) be defined by \( d(K) := \sup \{ \| x - y \| : x, y \in K \} \). For each \( x \in K \), let \( r(x, K) := \sup \{ \| x - y \| : y \in K \} \) and let \( r(K) := \inf \{ r(x, K) : x \in K \} \) denote the Chebyshev radius of \( K \) relative to itself. The normal structure coefficient \( N(E) \) of \( E \) (see e.g. [6]) is defined by
\[
N(E) := \inf \left\{ \frac{d(K)}{r(K)} : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0 \right\}.
\]
A space \( E \) such that \( N(E) > 1 \) is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g. [26]).

In the sequel, we shall make use of the following lemmas.

**Lemma 2.** Let \( E \) be an arbitrary real Banach space. Then
\[
\| x + y \|^2 \leq \| x \|^2 + 2 \langle y, j(x + y) \rangle
\]
for all \( x, y \in E \) and for all \( j(x + y) \in J(x + y) \).

**Lemma 3.** (see e.g. [29]) Let \( \{a_n\}_{n \geq 1} \) be a sequence of nonnegative real numbers satisfying the following relation:
\[
a_{n+1} \leq a_n - 2\alpha_n \phi(a_{n+1}) + \sigma_n, \quad n \geq 1,
\]
where (i) \( 0 < \alpha_n < 1 \); (ii) \( \sum_{n=1}^{\infty} \alpha_n = \infty \); (iii) \( \phi : [0, \infty) \to [0, \infty) \) is a strictly increasing function with \( \phi(0) = 0 \). Suppose \( \sigma_n = o(\alpha_n) \). Then \( a_n \to 0 \) as \( n \to \infty \).

**Lemma 4.** (see e.g. [26]) Suppose \( X \) is a Banach space with uniform normal structure, \( C \) is a nonempty bounded subset of \( X \), and \( T : C \to C \) is a uniformly \( L \)-Lipschitzian mapping with \( L < N(E)^{\frac{1}{2}} \). Suppose also there exists a nonempty bounded closed convex subset \( A \) of \( C \) with the following property (P):
\[
(P) \quad x \in A \text{ implies } \omega_\omega(x) \in A,
\]
Let \( \omega(x) \) be the weak \( \omega \)-limit set of \( T \) at \( x \), that is, the set \( \{ y \in X : y = \text{ weak } \omega \lim T^n x \text{ for some } j \to \infty \} \). Then \( T \) has a fixed point in \( A \).

**Lemma 5.** (see [35], pg.202 Lemma 3) Let \( E \) be a strictly convex Banach space and \( C \) be a closed convex subset of \( E \). Let \( T_1, T_2, \ldots, T_r \) be be nonexpansive mappings of \( C \) into itself such that the set of common fixed points of \( T_1, T_2, \ldots, T_r \) is nonempty. Let \( S_1, S_2, \ldots, S_r \) be mappings of \( C \) into itself given by \( S_i = (1 - \gamma_i)I + \gamma_i T_i \) for any \( 0 < \gamma_i < 1 \), \( i = 1, 2, \ldots, r \), where \( I \) denotes the identity mapping on \( C \). Then \( S_1, S_2, \ldots, S_r \) satisfies the following:

\[
\bigcap_{i=1}^r F(S_i) = \cap_{i=1}^r F(T_i)
\]

and

\[
\bigcap_{i=1}^r F(S_i) = F(S_rS_{r-1}\ldots S_1) = F(S_1S_{r-1}\ldots S_2) = \ldots = F(S_{r-1}\ldots S_1S_r).
\]

**Lemma 6.** ([31], Proposition 2) Let \( \alpha \in \mathbb{R} \) and \( (x_0, x_1, x_2, \ldots) \in l_\infty \) be such that \( \mu_n x_n \leq \alpha \) for all Banach limits \( \mu \). If \( \limsup_{n \to \infty} (x_{n+1} - x_n) \leq 0 \), then \( \limsup_{n \to \infty} x_n \leq \alpha \).

3. **Main results.**

**Iteration process for finite families of generalized Lipschitz Pseudo-contractive mappings.**

Let \( K \) be a nonempty closed convex subset of a real normed space \( E \). Let \( T_1, T_2, \ldots, T_m : K \to K \) be \( m \) generalized Lipschitz mappings. We define the iterative sequence \( \{ x_n \} \) by

\[
\begin{align*}
  u, \ x_1 & \in K, \\
  x_{n+1} & = (1 - \alpha_n \lambda_n)x_n + \alpha_n \lambda_n T_1 y_{1n} - \lambda_n \theta_n(x_n - u) \\
  y_{1n} & = (1 - \alpha_n)x_n + \alpha_n T_2 y_{2n} \\
  & \vdots \\
  y_{(m-2)n} & = (1 - \alpha_n)x_n + \alpha_n T_{m-1} y_{(m-1)n} \\
  y_{(m-1)n} & = (1 - \alpha_n)x_n + \alpha_n T_m x_n, \ m \geq 2, \ n \geq 1,
\end{align*}
\]

where \( \{ \alpha_n \}_{n=1}^\infty \), \( \{ \lambda_n \}_{n=1}^\infty \) and \( \{ \theta_n \}_{n=1}^\infty \) are sequence in \((0, 1)\) and \( u, x_1 \) are fixed vectors in \( K \).

We now prove our main theorems.

**Theorem 7.** Let \( K \) be a nonempty closed convex subset of a real Banach space. Let \( T_1, T_2, \ldots, T_m : K \to K \) be \( m \) generalized Lipschitz mappings such that \( F := \bigcap_{i=1}^m F(T_i) = \cap_{i=1}^m \{ x \in K : T_i x = x \} \neq \emptyset \). Let \( \{ \alpha_n \}_{n=1}^\infty \), \( \{ \lambda_n \}_{n=1}^\infty \) and \( \{ \theta_n \}_{n=1}^\infty \) be real sequences in \((0, 1)\) such that \( \alpha_n = o(\theta_n) \) (i.e. \( \lim_{n \to \infty} \alpha_n = 0 \)) \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \lambda_n(\alpha_n + \theta_n) < 1 \). Let \( \{ x_n \}_{n=1}^\infty \) be iteratively generated by (1). Suppose that \( T_1 \) is pseudo-contractive. Then the sequence \( \{ x_n \}_{n=1}^\infty \) is bounded.

**Proof.** Fix \( p \in F \). If \( x_n = p \) for all integers \( n \geq 1 \), then we are done. So, let \( n_0 \in \mathbb{N} \) be the first integer such that \( x_{n_0} \neq p \). Clearly, there exist \( N^* \geq n_0, \ r > 0 \) and \( M > 0 \) such that \( x_{N^*} \in B_r(p) := \{ x \in K : ||x - p|| \leq r \}, u \in B_r(p) \) and for all \( n \geq N^*, \ \alpha_n < \min\{ \frac{1}{M}, \theta_n \}, \lambda_n < \)
Observe that for $x \in B := B_r(p)$ for all $n \geq N^*$. By construction, $x_{N^*} \in B$. Suppose that $x_n \in B$ for $n > N^*$, we prove that $x_{n+1} \in B$. Suppose for contradiction that $x_{n+1}$ is not in $B$. Then $\|x_{n+1} - p\| > r$. From (1), we have the following estimates:

$$\|y_{1n} - p\| = \|(1 - \alpha_n)(x_n - p) + \alpha_n(T_2y_{2n} - p)\|$$

$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n L(1 + \|y_{2n} - p\|)$$

$$\leq \|x_n - p\| + \alpha_n L(1 - \alpha_n)(x_n - p) + \alpha_n(T_3y_{3n} - p)\| + \alpha_n L$$

$$\leq \|x_n - p\| + \alpha_n L\|x_n - p\| + \alpha_n L(1 + \|y_{3n} - p\|) + \alpha_n L$$

$$\leq \|x_n - p\| + \alpha_n L\|x_n - p\| + \alpha_n L^2\|y_{3n} - p\| + \alpha_n L + \alpha_n^2 L^2$$

$$\leq \|x_n - p\| + \alpha_n L\|x_n - p\| + \alpha_n^2 L^2\|x_n - p\| + \alpha_n^3 L^3\|y_{4n} - p\|$$

$$+ \alpha_n L + \alpha_n^2 L^2 + \alpha_n^3 L^3$$

$$\vdots$$

$$\leq \left(1 + \alpha_n L + \alpha_n^2 L^2 + \alpha_n^3 L^3 + \ldots + \alpha_n^{m-1} L^{m-1}\right)\|x_n - p\|$$

$$+ \left(\alpha_n L + \alpha_n^2 L^2 + \alpha_n^3 L^3 + \ldots + \alpha_n^{m-1} L^{m-1}\right)\|x_n - p\|$$

$$\leq \left(1 + \alpha_n L + \alpha_n^2 L^2 + \alpha_n^3 L^3 + \ldots + \alpha_n^{m-1} L^{m-1}\right)\|x_n - p\|$$

$$+ \left(1 + \alpha_n L + \alpha_n^2 L^2 + \alpha_n^3 L^3 + \ldots + \alpha_n^{m-1} L^{m-1}\right)\|x_n - p\|$$

$$\leq \frac{1 - \alpha_n^m L^m}{1 - \alpha_n L} \|x_n - p\| + \frac{1 - \alpha_n^m L^m}{1 - \alpha_n L}$$

Again, from (1) and using Lemma 2, we get

$$\|x_{n+1} - p\|^2 = \|x_n - p - \lambda_n\left(\alpha_n x_n - \alpha_n T_1 y_{1n} + \theta_n(x_n - u)\right)\|^2$$

$$\leq \|x_n - p\|^2 - 2\lambda_n\langle \alpha_n(x_n - T_1 y_{1n}) + \theta_n(x_n - u), j(x_{n+1} - p) \rangle$$

$$= \|x_n - p\|^2 - 2\lambda_n\theta_n\|x_{n+1} - p\|^2$$

$$+ 2\lambda_n\theta_n\langle (x_{n+1} - x_n) + (u - p), j(x_{n+1} - p) \rangle$$

$$+ 2\lambda_n\alpha_n\langle (x_{n+1} - x_n) + (p - T_1 x_{n+1}) - (T_1 y_{1n} - p), j(x_{n+1} - p) \rangle$$

$$- 2\lambda_n\alpha_n\langle x_{n+1} - T_1 x_{n+1}, j(x_{n+1} - p) \rangle.$$
Thus,
\[ \langle x_{n+1} - x_n \rangle + (p - T_1 x_{n+1}) - (T_1 y_n - p), j(x_{n+1} - p) \rangle \]
\[ \leq \| x_n - p \|^2 - 2 \lambda_n \| x_{n+1} - p \|^2 \]
\[ + 2 \lambda_n (\theta_n + \alpha_n) \| x_{n+1} - x_n \| \| x_{n+1} - p \| \]
\[ + 2 \lambda_n \| u - p \| \| x_{n+1} - p \| + 2 \lambda_n \alpha_n L (1 + \| x_{n+1} - p \|) \| x_{n+1} - p \| \]
\[ + 2 \lambda_n \alpha_n L (1 + \| y_{12} - p \|) \| x_{n+1} - p \| \]
\[ \leq \| x_n - p \|^2 - 2 \lambda_n \| x_{n+1} - p \|^2 + 2 \lambda_n (\theta_n + \alpha_n) \left[ \alpha_n \| x_n - p \| + \right. \]
\[ + \alpha_n L (1 + \| y_1 - p \|) + \theta_n \| x_n - p \| + \theta_n \| u - p \| \] \[ \left. \| x_{n+1} - p \| \right] \]
\[ + 2 \lambda_n \| u - p \| \| x_{n+1} - p \| + 2 \lambda_n \alpha_n L \left[ 1 + \alpha_n \lambda_n \| x_n - p \| + \right. \]
\[ + \alpha_n \lambda_n \| x_n - p \| + \lambda_n \| x_n - p \| + \lambda_n \theta_n \| u - p \| \] \[ \left. + \| x_n - p \| \| x_{n+1} - p \| + 2 \lambda_n \alpha_n L (1 + \| y_{12} - p \|) \| x_{n+1} - p \|. \right] \]
\[ \leq \| x_n - p \|^2 - 2 \lambda_n \| x_{n+1} - p \|^2 + 2 \lambda_n \theta_n \| x_{n+1} - p \|^2 \]
\[ + 2 \lambda_n \alpha_n L (1 + \| x_n - p \| + M + \theta_n \| x_n - p \| + \theta_n \| u - p \|) \]
\[ \| x_{n+1} - p \| \]
\[ + 2 \lambda_n \| u - p \| \| x_{n+1} - p \| + 2 \lambda_n \alpha_n L \left[ 1 + \alpha_n \lambda_n \| x_n - p \| + \right. \]
\[ + \alpha_n \lambda_n \| x_n - p \| + M + \lambda_n \theta_n \| x_n - p \| + \lambda_n \theta_n \| u - p \| + \| x_n - p \| \| x_{n+1} - p \| \]
\[ + 2 \lambda_n \alpha_n L (1 + \| x_n - p \| + M) \| x_{n+1} - p \|. \right] \]

Since \( \| x_{n+1} - p \| > r \), we have
\[ 2 \lambda_n \theta_n \| x_{n+1} - p \|^2 \leq 4 \lambda_n^2 \theta_n \left( L(1 + M r + M) + r + r + \frac{r}{2} \right) + 2 \lambda_n \theta_n \frac{r}{2} \]
\[ + 2 \alpha_n \lambda_n L \left[ 1 + L(1 + M r + M) + r + r + \frac{r}{2} \right] \]
\[ + 2 \lambda_n \alpha_n L \left[ 1 + M r + M \right] . \]

Thus,
\[ \| x_{n+1} - p \| \leq 2 \lambda_n^2 \left[ L(1 + M r + M) + \frac{5}{2} r \right] + \frac{r}{2} \]
\[ + \frac{\alpha_n L}{\theta_n} \left[ 1 + 2 L(1 + M r + M) + \frac{7}{2} r \right] \]
\[ \leq \frac{r}{4} + \frac{r}{2} + \frac{r}{8} < r , \]
a contradiction. So \( x_n \in B \forall n \geq N^\ast \). Hence, \( \{ x_n \}_{n \geq 1} \) is bounded. This completes the proof. \( \Box \)

**Remark 8.** It is now easy to see that we can find \( R > 0 \) sufficiently large such that \( x_n \in B^\ast = B_R(p) \forall n \in \mathbb{N} \). Besides, the set \( K \cap B^\ast \) is a bounded closed and convex nonempty subset of \( E \).
If we define a map $\varphi : E \to \mathbb{R}$ by

$$\varphi(x) = \inf_n \|x_{n+1} - x\|^2,$$

then $\varphi$ is continuous, convex and coercive (i.e., $\lim_{\|x\| \to +\infty} \varphi(x) = +\infty$). Thus, if $E$ is a reflexive Banach space, then there exists $x^* \in K \cap B^*$ such that $\varphi(x^*) = \min_{x \in K \cap B^*} \varphi(x)$. Thus, the set

$$K^* := \{y \in K \cap B^* : \varphi(y) = \min_{x \in K \cap B^*} \varphi(x)\} \neq \emptyset.$$

**Remark 9.** For the remainder of this paper, $\{\alpha_n\}_{n \geq 1}$, $\{\lambda_n\}_{n \geq 1}$ and $\{\theta_n\}_{n \geq 1}$ will be as in Theorem 7 and in addition, $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ will be assumed.

We prove the following theorem.

**Theorem 10.** Let $K$ be a nonempty closed and convex subset of a real reflexive Banach space $E$ with uniformly Gâteaux differentiable norm. Let $T_1, T_2, \ldots, T_m : K \to K$ be $m$ generalized Lipschitz mappings such that $T_1$ is pseudo-contractive and $F := \cap_{n=1}^{m} F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}_{n \geq 1}$ be iteratively generated by (1). Suppose that $K^* \cap F \neq \emptyset$. Then $\{x_n\}_{n \geq 1}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_m$.

**Proof.** Let $x^* \in K \cap F$ and $t \in (0, 1)$. Then by the convexity of $K \cap B^*$ we have that $(1-t)x^* + tu \in K \cap B^*$. It then follows that $\varphi(x^*) \leq \varphi((1-t)x^* + tu)$. Using Lemma 2, we have that

$$\|x_{n+1} - x^* - t(u - x^*)\|^2 \leq \|x_{n+1} - x^*\|^2 - 2t \langle u - x^*, j(x_{n+1} - x^* - t(u - x^*)) \rangle,$$

i.e.,

$$\|x_n - x^* - t(u - x^*)\|^2 \leq \|x_n - x^*\|^2 - 2t \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle.$$

Thus, taking Banach limits over $n \geq 1$ gives

$$\mu_n \|x_n - x^* - t(u - x^*)\|^2 \leq \mu_n \|x_n - x^*\|^2 - 2t \mu_n \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle.$$

This implies,

$$2t \mu_n \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle \leq \varphi(x^*) - \varphi((1-t)x^* + tu) \leq 0.$$

This therefore implies that

$$\mu_n \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle \leq 0 \quad \forall \ n \geq 1.$$

Since the normalized duality mapping is norm-to-weak$^*$ uniformly continuous on bounded subsets of $E$, we obtain, as $t \to 0$, that

$$\langle u - x^*, j(x_n - x^*) \rangle - \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle \to 0.$$

Hence, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0, \delta)$ and for all $n \geq 1$,

$$\langle u - x^*, j(x_n - x^*) \rangle < \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle + \varepsilon.$$
Thus the sequence 
\[ \mu_n \langle u - x^*, j(x_n - x^*) \rangle \leq \mu_n \langle u - x^*, j(x_n - x^* - t(u - x^*)) \rangle + \varepsilon \leq \varepsilon. \]

Since \( \varepsilon \) is arbitrary, we have
\[ \mu_n \langle u - x^*, j(x_n - x^*) \rangle \leq 0. \]

On the other hand, as \( \{x_n\} \) and \( \{T_1 y_{in}\} \) are bounded, using \( \lim \lambda_n = 0 \), we have that as \( n \to \infty \),
\[ ||x_{n+1} - x_n|| \leq \lambda_n (\alpha_n ||x_n|| + \alpha_n ||T_1 y_{in}|| + \theta_n ||x_n - x_1||) \to 0. \]

Therefore, from the norm-to-weak* uniform continuity of \( j \) on bounded sets, we obtain that
\[ \lim (\langle u - x^*, j(x_{n+1} - x^*) \rangle - \langle u - x^*, j(x_n - x^*) \rangle) = 0. \]

Thus the sequence \( \{\langle u - x^*, j(x_n - x^*) \rangle\} \) satisfies the conditions of Lemma 6. Hence, we obtain that
\[ \operatorname{limsup}_{n \to \infty} \langle u - x^*, j(x_n - x^*) \rangle \leq 0. \]

Define \( \varepsilon_n := \max \{\langle u - x^*, j(x_{n+1} - x^*)\rangle, 0\} \). Then,
\[ \lim \varepsilon_n = 0, \text{ and } \langle u - x^*, j(x_{n+1} - x^*) \rangle \leq \varepsilon_n. \]

Again, using Lemma 2, the recursion formula (1) and these estimates, we obtain that
\[
||x_{n+1} - x^*||^2 = ||x_n - x^* - \lambda_n [\alpha_n x_n - \alpha_n T_1 y_{in} + \theta_n (x_n - u)]||^2 \\
\leq ||x_n - x^*||^2 - 2\lambda_n \langle \alpha_n (x_n - T_1 y_{in}) + \theta_n (x_n - u), j(x_{n+1} - x^*) \rangle \\
\leq ||x_n - x^*||^2 - 2\lambda_n \theta_n ||x_{n+1} - x^*||^2 + 2\lambda_n \theta_n ||x_{n+1} - x_n|| ||x_{n+1} - x^*|| \\
+ 2\lambda_n \alpha_n ||x_n - T_1 y_{in}|| ||x_{n+1} - x^*|| + 2\lambda_n \theta_n \langle u - x^*, j(x_{n+1} - x^*) \rangle \\
\leq ||x_n - x^*||^2 - 2\lambda_n \theta_n ||x_{n+1} - x^*||^2 + 2\lambda_n \theta_n ||x_{n+1} - x_n|| ||x_{n+1} - x^*|| \\
+ 2\lambda_n \alpha_n [||x_n - x^*|| + L(1 + M ||x_n - x^*|| + M)] ||x_{n+1} - x^*|| \\
+ 2\lambda_n \theta_n \langle u - x^*, j(x_{n+1} - x^*) \rangle \\
\leq ||x_n - x^*||^2 - 2\lambda_n \theta_n ||x_{n+1} - x^*||^2 + (\lambda_n^2 \theta_n + \lambda_n \alpha_n) Q + 2\lambda_n \theta_n \varepsilon_n \\
\leq ||x_n - x^*||^2 - 2\lambda_n \theta_n ||x_{n+1} - x^*||^2 + \delta_n
\]

where \( \delta_n := (\lambda_n^2 \theta_n + \lambda_n \alpha_n) Q + 2\lambda_n \theta_n \varepsilon_n = o(\lambda_n \theta_n) \) for some constant \( Q > 0 \). Hence, by Lemma 3, we have that the sequence \( \{x_n\}_{n \geq 1} \) converges strongly to \( p^* \in F \). This completes the proof. \( \square \)

We thus obtain the following as an easy consequence of theorem 10.

**Corollary 11.** Let \( K \) be a nonempty closed and convex subset of a real reflexive Banach space \( E \) with uniformly Gâteaux differentiable norm. Let \( T_1, T_2, ..., T_m : K \to K \) be \( m \) generalized Lipschitz pseudo-contractive mappings such that \( F := \cap_{i=1}^m F(T_i) \neq \emptyset \). Let the sequence \( \{x_n\}_{n \geq 1} \) be iteratively generated by (1). Suppose that \( K^* \cap F \neq \emptyset \). Then \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( T_1, T_2, ..., T_m \).
Proof. Boundedness of the sequence \( \{x_n\} \) follows as in the proof of Theorem 7 and the rest follows as in the proof of Theorem 10. □

We now give an example in which our condition \( K^* \cap F \neq \emptyset \) is easily satisfied.

Corollary 12. Let \( K \) be a closed convex nonempty subset of a real strictly convex Banach space \( E \) with uniformly Gâteaux differentiable norm possessing uniform normal structure and \( T_1, T_2, ..., T_m : K \to K \) be \( m \) nonexpansive mappings such that \( F := \cap_{i=1}^m F(T_i) \neq \emptyset \). For \( \gamma_i \in (0, 1) \) define \( S_i = (1 - \gamma_i)I + \gamma_iT_i, \ i = 1, 2, ..., m \) and put \( G := S_mS_{m-1}...S_1 \). Let the sequence \( \{x_n\}_{n \geq 1} \) be iteratively generated by (1). Then \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( T_1, T_2, ..., T_m \).

Proof. First observe that \( G = S_mS_{m-1}...S_1 \) is nonexpansive. Thus, the set \( K^* := \{y \in K : \phi(y) = \min_{x \in K} \phi(x)\} \) is closed and convex and has property (P) and \( K^* \cap F(G) \neq \emptyset \). (shown in [26], since \( G \) is trivially asymptotically nonexpansive). But by Lemma 5,

\[
\cap_{i=1}^m F(S_i) = \cap_{i=1}^m F(T_i)
\]

and

\[
\cap_{i=1}^m F(S_i) = F(S_mS_{m-1}...S_1) = F(G).
\]

Hence, \( K^* \cap F(G) \neq \emptyset \) and \( K^* \cap F(G) = K^* \cap F \). Let \( p^* \in K^* \cap F(G) \). Then, the rest follows as in the proof of Theorem 7 and Theorem 10. This completes the proof. □

Iteration process for finite families of generalized Lipschitz accretive operators.

Let \( A_1, A_2, ..., A_m : E \to E \) be \( m \) generalized Lipschitz accretive mappings. We introduce the following iteration process for the approximation of common zero (assuming existence) of this family of mappings:

\[
\begin{align*}
{u, x_1} \\
x_{n+1} &= (1 - \alpha_n\lambda_n)x_n + \alpha_n\lambda_ny_{1n} - \alpha_n\lambda_nA_1y_{1n} - \lambda_n\theta_n(x_n - u) \\
y_{1n} &= (1 - \alpha_n)x_n + \alpha_ny_{2n} - \alpha_nA_2y_{2n} \\
&\vdots \\
y_{(m-2)n} &= (1 - \alpha_n)x_n + \alpha_ny_{(m-1)n} - \alpha_nA_{m-1}y_{(m-1)n} \\
y_{(m-1)n} &= x_n - \alpha_nA_mx_n, \ m \geq 2, \ n \geq 1,
\end{align*}
\]

where \( \{\alpha_n\}_{n=1}^\infty, \{\lambda_n\}_{n=1}^\infty, \{\theta_n\}_{n=1}^\infty \) are as above.

Theorem 13. Let \( E \) be a real Banach space with uniformly Gâteaux differentiable norm, \( A_1, A_2, ..., A_m : E \to E \) be \( m \) generalized Lipschitz accretive operators such that \( N := \cap_{i=1}^m N(A_i) := \cap_{i=1}^m \{x \in E \} \neq \emptyset \). Let \( \{x_n\}_{n \geq 1} \) be iteratively generated by (2). Then \( \{x_n\}_{n \geq 1} \) is bounded. Moreover, if \( E \) is a reflexive Banach space with uniformly Gâteaux differentiable norm and \( K^* \cap N \neq \emptyset \)
(where $K^*$ is constructed as in Remark 8). Then $\{x_n\}_{n \geq 1}$ converges strongly to a common zero of $A_1, A_2, \ldots, A_m$.

**Proof:** Recall that $A_i$ is accretive if and only if $T_i := I - A_i$ is pseudo-contractive, $i = 1, 2, \ldots, m$. Then if we replace $A_i$ by $I - T_i$, $i = 1, 2, \ldots, m$ in (2), then (2) reduces to (1) and boundedness of $\{x_n\}$ therefore follows as in the proof of Theorem 7. Furthermore, $p^* \in K^* \cap N$ implies $A_i p^* = 0$, $i = 1, 2, \ldots, m$. This implies $T_i p^* = p^*$. Thus, $K^* \cap \left( \cap_{i=1}^{m} \overline{F}(T_i) \right) \neq \emptyset$, $i = 1, 2, \ldots, m$. The rest follows as in the proof of Theorem 10. This completes the proof. $\Box$

4. Some applications.

Let $E$ be a real Banach space. A mapping $T : D(T) \subset E \to E$ is said to be hemi-contractive if $F(T) \neq \emptyset$ and $\forall \ x \in D(T), \ x^* \in F(T)$ there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2.$$ 

A mapping $A : D(A) \subset E \to E$ is said to be quasi-accretive if $N(A) := \{x \in D(A) : Ax = 0\} \neq \emptyset$ and $\forall \ x \in D(T), \ x^* \in N(A)$ there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle Ax, j(x - x^*) \rangle \geq 0.$$ 

It is easy to see that every pseudo-contractive mapping with a fixed point is hemi-contractive and that every accretive operator such that $N(A) \neq \emptyset$ is quasi-accretive. The reverse is , however, not necessarily the case (see e.g., [5]).

Following the methods of proofs used in section 3, the following theorems are obtained.

**Theorem 14.** Let $K$ be a nonempty closed and convex subset of a real reflexive Banach space with uniformly Gâteaux differentiable norm. Let $T_1, T_2, \ldots, T_m : K \to K$ be $m$ generalized Lipschitz hemi-contractive mappings such that $F^* := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n \geq 1}$ be iteratively generated by (1). Suppose that $K^* \cap F^* \neq \emptyset$. Then $\{x_n\}_{n \geq 1}$ converges strongly to some $p^* \in F^*$.

**Proof:** Boundedness of $\{x_n\}_{n \geq 1}$ follows as in the proof of Theorem 7. The rest then follows as in the proof of Theorem 10. $\Box$

**Theorem 15.** Let $E$ be a real reflexive Banach space with uniformly Gâteaux differentiable norm. Let $A_1, A_2, \ldots, A_m : E \to E$ be $m$ generalized Lipschitz quasi-accretive operators such that $N^* := \cap_{i=1}^{m} N(A_i) \neq \emptyset$. Let $\{x_n\}_{n \geq 1}$ be iteratively generated by (2). Suppose that $K^* \cap N^* \neq \emptyset$. Then $\{x_n\}_{n \geq 1}$ converges strongly to some $x^* \in N^*$.

**Proof:** The mapping $A_i$ is quasi-accretive if and only if $T_i := (I - A_i)$ is hemi-contractive $i = 1, 2, \ldots, m$. Thus, replacing $A_i$ by $I - T_i$ in (2), the result follows as in Theorem 14. $\Box$
Remark 16. Prototypes for our iteration parameters are, for example, \( \lambda_n = \frac{1}{(n+1)^a} \), \( \alpha_n = \frac{1}{(n+1)^b} \) and \( \theta_n = \frac{1}{(n+1)^c} \), where \( a + b < 1 \).

Remark 17. The addition of bounded error terms in any of the recursion formulas (1) and (2) leads to no further generalization.

Remark 18. If \( f : K \rightarrow K \) is a contraction map and we replace \( u \) by \( f(x_n) \) in the recursion formulas of our theorems, we obtain what some authors now call viscosity iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces \( u \) by \( f(x_n) \), repeats the argument of this paper, using the fact that \( f \) is a contraction map.

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References


