q-PLANE WAVE SOLUTIONS OF q-WEYL GRAVITY

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Abstract

We give solutions of the q-deformed equations of quantum conformal Weyl gravity in terms of q-deformed plane waves.

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I. INTRODUCTION

One of the purposes of quantum deformations is to provide an alternative of the regularization procedures of quantum field theory. Applied to Minkowski space-time the quantum deformations approach is also an alternative to Connes’ noncommutative geometry [1]. The first step in such an approach is to construct a noncommutative quantum deformation of Minkowski space-time. There are several possible such deformations, cf. [2–6]. We shall follow the deformation of [6] which is different from the others, the most important aspect being that it is related to a deformation of the conformal group.

The first problem to tackle in a noncommutative deformed setting is to study the q-deformed analogues of the conformally invariant equations. Here we continue the study of hierarchies of deformed equations derived in [6–8] with the use of quantum conformal symmetry. We give now a description of our setting starting from the simplest example.

It is well known that the d’Alembert equation
\[ \Box \varphi(x) = 0 , \quad \Box = \partial^\mu \partial_\mu = (\partial_0)^2 - (\vec{\partial})^2 , \] (1)
is conformally invariant, cf., e.g., [9]. Here \( \varphi \) is a scalar field of fixed conformal weight, \( x = (x_0, x_1, x_2, x_3) \) denotes the Minkowski space-time coordinates. Not known was the fact that (1) may be interpreted as conditionally conformally invariant equation and thus may be rederived from a subsingular vector of a Verma module of the algebra \( sl(4) \), the complexification of the conformal algebra \( su(2, 2) \) [7].

The same idea was used in [7] to derive a q-d’Alembert equation, namely, as arising from a subsingular vector of a Verma module of the quantum algebra \( U_q(sl(4)) \). The resulting equation is a q-difference equation and the solution spaces are built on the noncommutative q-Minkowski space-time of [6].

Besides the q-d’Alembert equation in [7] were derived a whole hierarchy of equations corresponding to the massless representations of the conformal group and parametrized by a nonnegative integer \( r \) [7]. The case \( r = 0 \) corresponds to the q-d’Alembert equation, while for each \( r > 0 \) there are two couples of equations involving fields of conjugated Lorentz representations of dimension \( r + 1 \). For instance, the case \( r = 1 \) corresponds to the massless Dirac equation, one couple of equations describing the neutrino, the other couple of equations describing the antineutrino, while the case \( r = 2 \) corresponds to the Maxwell equations.

The construction of solutions of the q-d’Alembert hierarchy was started in [10] with the q-d’Alembert equation. One of the solutions given was a deformation of the plane wave as a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. This q-plane wave has some properties analogous to the classical one but is not an exponent or q-exponent. Thus, it differs conceptually from the classical plane wave and may serve as a regularization of the latter. In the same sense it differs from the q-plane wave in the paper [11],
which is not surprising, since there is used different \( q \)-Minkowski space-time (from [2–4] and different \( q \)-d’Alembert equation both based only on a (different) \( q \)-Lorentz algebra, and not on \( q \)-conformal (or \( U_q(sl(4)) \)) symmetry as in our case. In fact, it is not clear whether the \( q \)-Lorentz algebra of [2–4] used in [11] is extendable to a \( q \)-conformal algebra.

For the equations labelled by \( r > 0 \) it turned out that one needs a second \( q \)-deformation of the plane wave in a conjugated basis [12]. The solutions of the hierarchy in terms of the two \( q \)-plane waves were given in [12] for \( r = 1 \) and in [13] for \( r > 1 \). Later these two \( q \)-plane waves were generalized and correspondingly more general solutions of the hierarchy were given in [14].

Another hierarchy derived in [6] is the Maxwell hierarchy. The two hierarchies have only one common member - the Maxwell equations - they are the lowest member of the Maxwell hierarchy and the \( r = 2 \) member of the massless hierarchy. The compatibility of the solutions of the free \( q \)-Maxwell equations with the \( q \)-potential equations was studied in [15].

Another family contained in [8], but not explicated there, is related to the linear conformal Weyl gravity. Its study started in [16], where was written down the quantum conformal deformation of the linear conformal Weyl gravity. In the present paper we continue this study by constructing solutions of these \( q \)-deformed equations.

### II. LINEAR CONFORMAL GRAVITY

We shall consider the quantum group analogs of linear conformal gravity following the approach of [8]. We start with the \( q = 1 \) situation and we first write the Weyl gravity equations in an indexless formulation, trading the indices for two conjugate variables \( z, \bar{z} \).

Weyl gravity is governed by the Weyl tensor:

\[
C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} - \frac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\nu\tau}R_{\mu\sigma} - g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau}) + \frac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R ,
\]

where \( g_{\mu\nu} \) is the metric tensor. Linear conformal gravity is obtained when the metric tensor is written as: \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( \eta_{\mu\nu} \) is the flat Minkowski metric, \( h_{\mu\nu} \) are small so that all quadratic and higher order terms are neglected. In particular: \( R_{\mu\nu\sigma\tau} = \frac{1}{2}(\partial\mu\partial\tau h_{\nu\sigma} + \partial\nu\partial\sigma h_{\mu\tau} - \partial\mu\partial\sigma h_{\nu\tau} - \partial\nu\partial\tau h_{\mu\sigma}) \). The equations of linear conformal gravity are:

\[
\partial^{\mu}\partial^{\nu}C_{\mu\nu\sigma\tau} = T_{\mu\sigma} ,
\]

where \( T_{\mu\nu} \) is the energy-momentum tensor. From the symmetry properties of the Weyl tensor it follows that it has ten independent components. These may be chosen as follows (introducing notation for future use):

\[
C_0 = C_{0123} , \quad C_1 = C_{2121} , \quad C_2 = C_{0202} , \quad C_3 = C_{3012} , \quad C_4 = C_{2021} , \quad C_5 = C_{1012} , \quad C_6 = C_{2023} , \quad C_7 = C_{3132} , \quad C_8 = C_{2123} , \quad C_9 = C_{1213} .
\]
Furthermore, the Weyl tensor transforms as the direct sum of two conjugate Lorentz irreps, which we shall denote as $C^\pm$. The tensors $T_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric and traceless with nine independent components.

In order to be more precise we recall that the physically relevant representations $T^\chi$ of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where $n_1, n_2$ are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1 + 1)(n_2 + 1)$), and $d$ is the conformal dimension (or energy). (In the literature these Lorentz representations are labelled also by $(j_1, j_2) = (n_1/2, n_2/2)$.) The Weyl tensor transforms as the direct sum:

$$
\chi^+ \oplus \chi^-
$$

$$
\chi^+ = [4, 0; 2] , \quad \chi^- = [0, 4; 2] ,
$$
while the energy-momentum tensor and the metric transform as:

$$
\chi_T = [2, 2; 4] , \quad \chi_h = [2, 2; 0] ,
$$
as anticipated. Indeed, $(n_1, n_2) = (2, 2)$ is the nine-dimensional Lorentz representation, (carried by $T_{\mu\nu}$ or $h_{\mu\nu}$), and $(n_1, n_2) = (4,0), (0,4)$ are the two conjugate five-dimensional Lorentz representations, (carried by $C^\pm$), while the conformal dimensions are the canonical dimensions of a energy-momentum tensor ($d = 4$), of the metric ($d = 0$), and of the Weyl tensor ($d = 2$).

As we mentioned in the Introduction the case of Weyl gravity belongs together with the Maxwell case to an infinite family parametrized by $n = 1, 2, \ldots$ where the signatures analogous to (5), (6) are:

$$
\chi^+_m = [2m, 0; 2] , \quad \chi^- = [0, 2m; 2] ,
$$

$$
\chi^m_T = [m, m; 2 + m] , \quad \chi^m_h = [m, m; 2 - m] .
$$
The Maxwell case is obtained for $m = 1$, Weyl gravity for $m = 2$. (Note, however, that the representations $\chi^m_h$ are not unitary for $m > 2$.)

Further, we shall use the fact that a Lorentz irrep (spin-tensor) with signature $(n_1, n_2)$ may be represented by a polynomial $G(z, \bar{z})$ in $z, \bar{z}$ of order $n_1, n_2$, resp. More explicitly, for the Weyl gravity representations mentioned above we use [16]:

$$
C^+(z) = z^4 C^+_4 + z^3 C^+_3 + z^2 C^+_2 + z C^+_1 + C^+_0 ,
$$

$$
C^-(\bar{z}) = \bar{z}^4 C^-_4 + \bar{z}^3 C^-_3 + \bar{z}^2 C^-_2 + \bar{z} C^-_1 + C^-_0 ,
$$

$$
T(z, \bar{z}) = z^2 \bar{z}^2 T''_{22} + z^2 \bar{z} T''_{21} + z^2 T''_{20} +
+ z \bar{z}^2 T'_{12} + z \bar{z} T'_{11} + z T'_{10} +
+ \bar{z}^2 T'_{02} + \bar{z} T'_{01} + T'_{00} ,
$$
\[ h(z, \bar{z}) = z^2 \bar{z}^2 h'_{22} + z^2 \bar{z} h'_{21} + z^2 h'_{20} + \\
+ z \bar{z}^2 h'_{12} + z \bar{z} h'_{11} + z h'_{10} + \\
+ \bar{z}^2 h'_{02} + \bar{z} h'_{01} + h'_{00}, \]  
\tag{11} \]

where the indices on the RHS are not Lorentz-covariance indices, they just indicate the powers of \( z, \bar{z} \). The components \( C^\pm_k \) are given in terms of the Weyl tensor components as follows [16]:

\[
\begin{align*}
C^+_0 &= C_2 - \frac{1}{2} C_1 - C_6 + i(C_0 + \frac{1}{2} C_3 + C_7) \\
C^+_1 &= 2(C_4 - C_8 + i(C_9 - C_5)) \\
C^+_2 &= 3(C_1 - iC_3) \\
C^+_3 &= 8(C_4 + C_8 + i(C_9 + C_5)) \\
C^+_4 &= C_2 - \frac{1}{2} C_1 + C_6 + i(C_0 + \frac{1}{2} C_3 - C_7) \\
C^-_0 &= C_2 - \frac{1}{2} C_1 - C_6 - i(C_0 + \frac{1}{2} C_3 + C_7) \\
C^-_1 &= 2(C_4 - C_8 - i(C_9 - C_5)) \\
C^-_2 &= 3(C_1 + iC_3) \\
C^-_3 &= 2(C_4 + C_8 - i(C_9 + C_5)) \\
C^-_4 &= C_2 - \frac{1}{2} C_1 + C_6 - i(C_0 + \frac{1}{2} C_3 - C_7) \\
\end{align*} \tag{12} \]

while the components \( T'_{ij} \) are given in terms of \( T_{\mu\nu} \) as follows [16]:

\[
\begin{align*}
T'_{22} &= T_{00} + 2 T_{03} + T_{33} \\
T'_{11} &= T_{00} - T_{33} \\
T'_{00} &= T_{00} - 2 T_{03} + T_{33} \\
T'_{21} &= T_{01} + i T_{02} + T_{13} + i T_{23} \\
T'_{12} &= T_{01} - i T_{02} + T_{13} - i T_{23} \\
T'_{10} &= T_{01} + i T_{02} - T_{13} - i T_{23} \\
T'_{01} &= T_{01} - i T_{02} - T_{13} + i T_{23} \\
T'_{20} &= T_{11} + 2i T_{12} - T_{22} \\
T'_{02} &= T_{11} - 2i T_{12} - T_{22} \\
\end{align*} \tag{13} \]

and similarly for \( h'_{ij} \) in terms of \( h_{\mu\nu} \).

In these terms all linear conformal Weyl gravity equations (3) may be written in compact form as the following pair of equations:

\[ I^+ C^+(z) = T(z, \bar{z}) , \quad I^- C^-(\bar{z}) = T(z, \bar{z}) , \]  
\tag{14} \] 

where the operators \( I^\pm \) are given as follows:
\[ I^+ = \left( z^2 \bar{z}^2 \partial_\mu^2 + z^2 \partial_\nu^2 + \bar{z}^2 \partial_\nu^2 + \partial_\nu^2 + 2z^2 \bar{z} \partial_\nu \partial_+ + 2z \bar{z} \partial_+ \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} \partial_\nu \partial_\nu + 2z \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) \right), \] (15)

\[ I^- = \left( z^2 \bar{z}^2 \partial_\mu^2 + z^2 \partial_\nu^2 + \bar{z}^2 \partial_\nu^2 + \partial_\nu^2 + 2z^2 \bar{z} \partial_\nu \partial_+ + 2 \bar{z} \bar{z} \partial_+ \partial_\nu + 2 \bar{z} \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2 \bar{z} \bar{z} \partial_\nu \partial_\nu + 2 \bar{z} \bar{z} \partial_\nu \partial_\nu + 2 \bar{z} \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) + + 2 \bar{z} \bar{z} \partial_\nu \partial_\nu + 2 \bar{z} \bar{z} \partial_\nu \partial_\nu + 2 \bar{z} \bar{z} (\partial_+ \partial_+ + \partial_\nu \partial_\nu) \right), \] (16)

where the variables \( x_\pm, v, \bar{v} \) are expressed through the Minkowski coordinates \( x_0, x_1, x_2, x_3 \) as follows [6]:

\[ x_\pm \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar{v} \equiv x_1 + ix_2. \] (17)

These variables have, (unlike the \( x_\mu \)), definite group–theoretical interpretation as part of a six-dimensional coset of the conformal group \( SU(2, 2) \) (as explained in [6]). In terms of these variables, e.g., the d’Alembert equation (1) is:

\[ \Box \varphi = (\partial_+ \partial_+ - \partial_\nu \partial_\nu) \varphi = 0. \] (18)

To make more transparent the origin of (14) and in the same time to derive the quantum group deformation of (14), (15) we first introduce the following parameter-dependent operators:

\[ I^+ (n) = \frac{1}{2} \left( n(n - 1)I_1^2 I_2^2 - 2(n^2 - 1)I_1 I_2 I_1 + n(n + 1)I_2^2 I_1 \right), \] (19)

\[ I^- (n) = \frac{1}{2} \left( n(n - 1)I_3^2 I_2^2 - 2(n^2 - 1)I_3 I_2 I_3 + n(n + 1)I_2^2 I_3 \right), \] (20)

where

\[ I_1 \equiv \partial_\nu, \quad I_2 \equiv \bar{z} \partial_+ + z \partial_\nu + \bar{z} \partial_\nu + \partial_\nu, \quad I_3 \equiv \partial_\nu. \] (21)

It is easy to check that we have the following relation:

\[ I^\pm = I^\pm (4), \] (22)

i.e., (14) are written as:

\[ I^+ (4) C^+ (z) = T(z, \bar{z}), \quad I^- (4) C^- (\bar{z}) = T(z, \bar{z}). \] (23)
We note in passing that group-theoretically the operators $I_a$ correspond to the three simple roots of the root system of $sl(4)$, while the operators $I_n^\pm$ correspond to the two non-simple non-highest roots [17].

This is the form that is immediately generalizable to the $q$-deformed case. We first present the necessary formalism in the next Section.

Using the same operators we can write down the pair of equations which give the Weyl tensor components in terms of the metric tensor:

$$I^+ (2) \ h(z, \bar{z}) = C^- (\bar{z}) \ , \quad I^- (2) \ h(z, \bar{z}) = C^+ (z) \ .$$

(22)

We stress the advantage of the indexless formalism due to which two different pairs of equations, (21), (22), may be written using the same parameter-dependent operator expressions by just specializing the values of the parameter.

The analogues of (14), (22), for the family (7) are:

$$I^+ (4) \ C^+_m (z) = T_m (z, \bar{z}) \ , \quad I^- (4) \ C^-_m (z) = T_m (z, \bar{z}) \ ,$$

(23)

$$I^+ (2) \ h_m (z, \bar{z}) = C^-_m (\bar{z}) \ , \quad I^- (2) \ h_m (z, \bar{z}) = C^+_m (z) \ ,$$

(24)

where the operators $I^+_m (n)$ are of order $m$, and can be found in [18] also in the $q$-deformed case.

### III. $q$-DEFORMED SETTING

In the $q$-deformed case we use the noncommutative $q$-Minkowski space-time of [6] which is given by the following commutation relations (with $\lambda \equiv q - q^{-1}$):

$$x_\pm v = q^{\pm 1} v x_\pm \ , \quad x_\pm \bar{v} = q^{\pm 1} \bar{v} x_\pm \ , \quad x_+ x_- - x_- x_+ = \lambda v \bar{v} \ , \quad \bar{v} v = v \bar{v} \ ,$$

(25)

with the deformation parameter being a phase: $|q| = 1$. Relations (25) are preserved by the anti-linear anti-involution $\omega$:

$$\omega (x_\pm) = x_\pm \ , \quad \omega (v) = \bar{v} \ , \quad \omega (\bar{v}) = q^{-1} \ , \quad (\omega (\lambda) = -\lambda) \ .$$

(26)

The solution spaces consist of formal power series in the $q$-Minkowski coordinates (which we give in two conjugate bases):

$$\varphi = \sum_{j, n, \ell, m \in \mathbb{Z}_+} \mu_{jntm} \ \varphi_{jntm} \ , \quad \varphi_{jntm} = \hat{\varphi}_{jntm} , \ \tilde{\varphi}_{jntm} \ ,$$

(27)

$$\hat{\varphi}_{jntm} = v^j x^n_+ x^\ell_+ \bar{v}^m \ ,$$

$$\tilde{\varphi}_{jntm} = \bar{v}^m x^n_+ x^\ell_+ v^j = \omega (\hat{\varphi}_{jntm}) \ .$$

(28)

The solution spaces (27) are representation spaces of the quantum algebra $U_q (sl(4))$. For the latter we use the rational basis of Jimbo [19]. The action of $U_q (sl(4))$ on $\hat{\varphi}_{jntm}$ was given in [20],
and on $\varphi_{jn\ell m}$ in [12]. Because of the conjugation $\omega$ we are actually working with the conformal quantum algebra which is a deformation of $U(su(2,2))$.

Further we suppose that $q$ is not a nontrivial root of unity.

In order to write our $q$-deformed equations in compact form it is necessary to introduce some additional operators. We first define the operators:

$$\hat{M}_\kappa^\pm \varphi = \sum_{j,n,\ell,m \in \mathbb{Z}_{\pm}} \mu_{jn\ell m} \hat{M}_\kappa^\pm \varphi_{jn\ell m}, \quad \kappa = \pm, v, \bar{v}, \quad (30)$$

$$T_\kappa^\pm \varphi = \sum_{j,n,\ell,m \in \mathbb{Z}_{\pm}} \mu_{jn\ell m} T_\kappa^\pm \varphi_{jn\ell m}, \quad \kappa = \pm, v, \bar{v}, \quad (31)$$

and $\hat{M}_\kappa^\pm$, $\hat{M}_v^\pm$, $\hat{M}_{\bar{v}}^\pm$, resp., acts on $\varphi_{jn\ell m}$ by changing by $\pm 1$ the value of $j, n, \ell, m$, resp., while $T_\kappa^\pm$, $T_v^\pm$, $T_{\bar{v}}^\pm$, resp., acts on $\varphi_{jn\ell m}$ by multiplication by $q^{\pm j}, q^{\pm n}, q^{\pm \ell}, q^{\pm m}$, resp. We shall use also the 'logs' $N_\kappa$ such that $T_\kappa = q^{N_\kappa}$. Now we can define the $q$-difference operators:

$$\hat{D}_\kappa \varphi = \frac{1}{\lambda} \hat{M}_\kappa^{-1} \left( T_\kappa - T_\kappa^{-1} \right) \varphi = \frac{1}{\lambda} \hat{M}_\kappa^{-1} \left( q^{N_\kappa} - q^{-N_\kappa} \right) \varphi. \quad (32)$$

Note that when $q \to 1$ then $\hat{D}_\kappa \to \partial_\kappa$. Using (30) and (31) the $q$-d’Alembert equation may be written as [7], [12], respectively,

$$\left( q \hat{D}_- \hat{D}_+ T_v T_{\bar{v}} - \hat{D}_v \hat{D}_{\bar{v}} \right) T_v T_- T_+ T_{\bar{v}} \varphi = 0, \quad (33)$$

Note that when $q \to 1$ both equations (32), (33) go to (17). Note that the operators in (30), (31), (32), (33) for different variables commute, i.e., we have passed to commuting variables. However, keeping the normal ordering it is straightforward to pass back to noncommuting variables.

Using results from [8] we have for the $q$-analogue of (18):

$$q^{I^+}(n) = \frac{1}{2} \left( [n]_q [n-1]_q q I_1^2 I_3^2 - [2]_q [n-1]_q [n+1]_q q I_1 I_2 I_3 + [n]_q [n+1]_q q I_2^2 I_3 \right), \quad (34)$$

$$q^{I^-}(n) = \frac{1}{2} \left( [n]_q [n-1]_q q I_1^2 I_3^2 - [2]_q [n-1]_q [n+1]_q q I_1 I_3 I_2 + [n]_q [n+1]_q q I_2^2 I_3 \right), \quad (35)$$

where the $q$-deformed versions $qI_a$ of (19) in the basis (28) are:

$$qI_1 = \hat{D}_z T_z T_v T_+ (T_- T_{\bar{v}})^{-1}$$

$$qI_2 = (q \hat{M}_z \hat{D}_v T_z^2 + \hat{M}_z \hat{M}_z \hat{D}_+ T_v T_{\bar{v}}^{-1} + \hat{D}_- T_- + q^{-1} \hat{M}_v \hat{D}_{\bar{v}} - \lambda \hat{M}_v \hat{M}_z \hat{D}_- \hat{D}_+ T_{\bar{v}}) T_v T_{\bar{v}}^{-1}$$

$$qI_3 = \hat{D}_z T_z. \quad (36)$$

(For comparison, note that in the $q$-Maxwell operators are used the following expressions: $qI_n^+ = \frac{1}{2}([n+2]_q q I_1 I_2 - [n+3]_q q I_2^2 I_1), \quad qI_n^- = \frac{1}{2}([n+2]_q q I_1 I_3 I_2 - [n+3]_q q I_2 I_3).$)
Then the $q$-Weyl equations are (cf. (21)):

\[ qI^+(4) \ C^+(z) = T(z, \bar{z}) \ , \quad qI^-(4) \ C^-(z) = T(z, \bar{z}) \ , \quad (36) \]

while $q$-analogues of (22) are:

\[ qI^+(2) \ h(z, \bar{z}) = C^-(z) \ , \quad qI^-(2) \ h(z, \bar{z}) = C^+(z) \ . \quad (37) \]

We shall look for solutions of the $q$-Weyl gravity equations in terms of a deformation of the plane wave given in [14]. This deformation is given in the basis (28):

\[ \hat{\exp}_q(k, x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \hat{h}_s \ , \quad (38) \]

\[ [s]_q! = [s]_q[s-1]_q \cdots [1]_q \ , \quad [0]_q! = 1 \ , \quad [n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \]

\[ \hat{h}_s = \beta^s \sum_{a, b, n \in \mathbb{Z}_+} \frac{(-1)^s a - b \ q^n(s-2a-2b+2n) + a(s-a-1) + b(-s+a+b+1) + P_s(a, b)}{\Gamma_q(a - n + 1)\Gamma_q(b - n + 1)\Gamma_q(s - a - b + n + 1)[n]_q!} \times \]

\[ \times k_v^{s-a-b+n}k_-^{b-n}k_+^{a-n}k_0^{n}x_-^{s-a-b+n}, \quad (39) \]

\[ (\beta^s)^{-1} = \sum_{p=0}^{s} \frac{s-p}{[p]_q!} \frac{[s-p]_q!}{[s-p+1]} \]

where the momentum components ($k_v, k_-, k_+, k_0$) are supposed to be non-commutative between themselves (obeying the same rules (25) as the $q$-Minkowski coordinates), and commutative with the coordinates. Further, $\Gamma_q$ is a $q$-deformation of the $\Gamma$-function, of which here we use only the properties: $\Gamma_q(p) = [p-1]_q!$ for $p \in \mathbb{N}$, $1/\Gamma_q(p) = 0$ for $p \in \mathbb{Z}_- \ ; P_s(a, b)$ is a polynomial in $a, b$. Note that $(\hat{h}_s)|_{q=1} = (k \cdot x)^s$ and thus $(\hat{\exp}_q(k, x))|_{q=1} = \exp(k \cdot x)$. This $q$-plane wave has some properties analogous to the classical one but is not an exponent or $q$-exponent, cf. [21]. This is enabled also by the fact (true also for $q = 1$) that solving the equations may be done in terms of the components $\hat{h}_s$. This deformation of the plane wave generalizes the original one from [10] which is obtained by setting $P_s(a, b) = 0$. Each $\hat{h}_s$ satisfies the $q$-d’Alembert equation (32) on the momentum $q$-cone:

\[ \mathcal{L}_q^k \equiv k_- k_+ - q^{-1} k_v k_0 = k_+ k_- - q k_v k_0 = 0 \ . \quad (40) \]

### IV. SOLUTIONS OF $q$-WEYL GRAVITY

We shall use the basis (28). The solutions of the first equation in (36) in the homogeneous case ($T = 0$) are:

\[ qC^+_0 = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \hat{C}^+_s \ , \quad (41) \]
\[ \hat{C}^+_s = \sum_{m=0}^{4} \hat{\gamma}^+_m \left( \prod_{i=0}^{-m+3} (k_+ - q^i B_+ s + 4 k_\pm z) \right) \left( \prod_{j=-m+4}^{3} (k_\mp - q^j B_- s + 4 k_\mp z) \right) \hat{h}_s^+ , \quad (42) \]

where \( \hat{h}_s^+ \) is \( \hat{h}_s \) with:

\[ P_s (a, b) = P^+_s (a, b) \equiv R_s (a) + B_s b , \quad (43) \]

\( \hat{\gamma}^+_m, B_s \) are arbitrary constants, \( R_s (a) \) is an arbitrary polynomial in \( a \). Note that the factors preceding \( \hat{h}_s^+ \) depend on \( B_s \) but not on \( R_s (a) \). The check that (41) is a solution is done for commutative Minkowski coordinates and noncommutative momenta on the q-cone. In order to be able to write the above solution in terms of the deformed plane wave we have to suppose that the \( \hat{\gamma}^+_m, B_s \) for different \( s \) coincide: \( \hat{\gamma}^+_m = \hat{\gamma}^+_m, \) e.g., we can make the choice \( B_s = B' - s - 4 \). Then we have:

\[ qC^+_0 = \sum_{m=0}^{4} \hat{\gamma}^+_m \left( \prod_{i=0}^{-m+3} (k_+ - q^i B' k_\mp z) \right) \left( \prod_{j=-m+4}^{3} (k_\mp - q^j B' k_\mp z) \right) \exp_q^+(k, x) , \quad (44) \]

where \( \exp_q^+(k, x) \) is \( \exp_q(k, x) \) with the choice (43).

The solutions of the second equation in (36) are:

\[ qC^-_0 = \sum_{s=0}^{\infty} \frac{1}{|s| q} \hat{C}^-_s \]

\[ \hat{C}^-_s = \sum_{m=0}^{4} \hat{\gamma}^-_m \left( \prod_{i=0}^{-m+2} (k_+ - q^i D_+ k_\mp z) \right) \left( \prod_{j=-m+3}^{2} (k_\mp - q^j D_- k_\pm z) \right) \hat{h}_s^- \quad (46) \]

where \( \hat{h}_s^- \) is \( \hat{h}_s \) with:

\[ P_s (a, b) = P^-_s (a, b) \equiv D_s a + Q_s (b) , \quad (47) \]

\( \hat{\gamma}^-_m,D_s \) are arbitrary constants, and \( Q_s (b) \) is an arbitrary polynomial. In order to be able to write this solution in terms of the deformed plane wave we have to suppose that the \( \hat{\gamma}^-_m,D_s \) for different \( s \) coincide: \( \hat{\gamma}^-_m = \hat{\gamma}^-_m, D_s = D \). Then we have:

\[ qC^-_0 = \sum_{m=0}^{4} \hat{\gamma}^-_m \left( \prod_{i=0}^{-m+2} (k_+ - q^i D k_\mp z) \right) \left( \prod_{j=-m+3}^{2} (k_\mp - q^j D k_\pm z) \right) \exp_q^-(k, x) , \quad (48) \]

where \( \exp_q^-(k, x) \) is \( \exp_q(k, x) \) with the choice (47).
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