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DIFFERENTIAL OPERATORS ASSOCIATED TO THE CAUCHY-RIEMANN OPERATOR IN A QUATERNION ALGEBRA

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Abstract

This paper deals with the initial value problem of the type

\[
\frac{\partial w}{\partial t} = L(t, x, w, \frac{\partial w}{\partial x_i})
\]

(1)

\[w(0, x) = \varphi(x)
\]

(2)

where \( t \) is the time, \( L \) is a linear first order operator (matrix-type) in a Quaternion algebra and \( \varphi \) is a regular function. The article proves necessary and sufficient conditions on the coefficients of operator \( L \) under which \( L \) is associated to the Cauchy-Riemann operator of Quaternion algebra.

This criterion makes it possible to construct the operator \( L \) for which the initial problem (1),(2) is solvable for an arbitrary initial regular function \( \varphi \) and the solution is also regular for each \( t \).

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1. Preliminaries and notations

Let $\mathcal{H}$ be a Quaternion algebra with the basis is formed by $e_0, e_1, e_2, e_3$ where $e_0 = 1, e_3 = e_1 e_2 = e_{12}$.

Suppose that $\Omega$ is a bounded domain of $\mathbb{R}^3$. A function $f$ defines in $\Omega$ and takes values in the Quaternion algebra $\mathcal{H}$ which can be presented as

$$f = \sum_{j=0}^{3} f_j e_j,$$

where $f_j(x)$ are real-valued functions.

We introduce the Cauchy-Riemann operator

$$\mu = \sum_{k=0}^{2} e_k \frac{\partial}{\partial x_k}.$$

Definition 1. A function $f \in C^1(\Omega, \mathcal{H})$ is said to be regular in $\Omega$ if $f$ satisfies

$$\mu f = 0.$$

Remark 1. If $f \in C^2(\Omega, \mathcal{H})$ is a regular function, then $f$ is harmonic in $\Omega$.

2. Necessary and sufficient conditions for associated pairs

Suppose that $f = \sum_{j=0}^{3} f_j e_j$ be a twice continuously differentiable function with respect to the space-like $x_0, x_1, x_2$. Now assume $f$ is regular. This means that $\mu f = 0$. It is easy to verify that the condition $\mu f = 0$ is equivalent to

$$\sum_{i=0}^{2} A_i \frac{\partial f}{\partial x_i} = 0,$$

where

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

$$\frac{\partial f}{\partial x_i} = \begin{bmatrix} \frac{\partial f_0}{\partial x_i} \\ \frac{\partial f_1}{\partial x_i} \\ \frac{\partial f_2}{\partial x_i} \\ \frac{\partial f_3}{\partial x_i} \end{bmatrix}.$$

We define an operator $\ell$ as follow

$$\ell f = \sum_{i=0}^{2} A_i \frac{\partial f}{\partial x_i}.$$

(3)
It is clear that $\mu f = 0$ if and only if $\ell f = 0$. Next, we identify the function $f$ with $f := \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$ and introduce a differential operator $L$ as follows

$$Lf = \sum_{j=0}^{2} B_j \frac{\partial f}{\partial x_j} + Cf + D,$$

(4)

where $B_j = [b^{(j)}_{\alpha\beta}]$, $C = [c_{\alpha\beta}]$, $D = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}$, $b^{(j)}_{\alpha\beta}$, $c_{\alpha\beta}$, $d_\alpha$, $(\alpha, \beta = 0, 1, 2, 3)$ are real-valued functions which are supposed to depend at least continuously on the time $t$ and the space-like $x_0, x_1, x_2$.

A pair of operators $\ell, L$ is said to be associated (see [8]) if $\ell f = 0$ implies $\ell (Lf) = 0$ (for each $t$ in case the coefficient of $L$ depend on $t$). Now we formulate necessary and sufficient conditions on the coefficients of operator $L$ under which $L$ is associated to the operator $\ell$ (on the other word, $L$ is associated to the Cauchy-Riemann operator of Quaternion algebra). Assume that the functions $b^{(j)}_{\alpha\beta}$, $c_{\alpha\beta}$, $d_\alpha$ $(j = 0, 1, 2, \alpha, \beta = 0, 1, 2, 3)$ are continuously differentiable with respect to the space-like variable $x_0, x_1, x_2$ and differentiable on $t$.

Putting

$$P_j = [p^{(j)}_{\alpha\beta}] = A_j B_j, \quad j = 0, 1, 2$$

(5)

$$Q_{ij} = [q^{(ij)}_{\alpha\beta}] = A_i B_j + A_j B_i, \quad 0 \leq i < j \leq 2$$

(6)

$$R_j = [r^{(j)}_{\alpha\beta}] = \sum_{i=0}^{2} A_i \frac{\partial B_j}{\partial x_i} + A_j C, \quad j = 0, 1, 2, \quad \alpha, \beta = 0, 1, 2, 3.$$  

(7)

Then we get following theorem

**Theorem 1.** The operator $L$ is associated to the operator $\ell$ if and only if the following conditions are satisfied

i) The functions $h^{(\alpha)} = \sum_{i=0}^{3} c_{\alpha i} e_i$, $\alpha = 0, 1, 2, 3$, and $g = \sum_{i=0}^{3} d_i e_i$ are regular.

\[
\begin{align*}
\left\{ \begin{array}{l}
 r_{00}^{(1)} = r_{10}^{(0)}, \quad r_{00}^{(2)} = r_{12}^{(0)}, \\
r_{11}^{(1)} = -r_{00}^{(0)}, \quad r_{11}^{(2)} = -r_{13}^{(0)}, \\
r_{12}^{(1)} = r_{13}^{(0)}, \quad r_{12}^{(2)} = -r_{10}^{(0)}, \\
r_{13}^{(1)} = -r_{12}^{(0)}, \quad r_{13}^{(2)} = r_{11}^{(0)}
\end{array} \right. \\
\left\{ \begin{array}{l}
 q_{00}^{(1)} = p_{10}^{(0)} - p_{11}^{(1)}, \quad q_{02}^{(0)} = p_{20}^{(0)} - p_{22}^{(2)}, \quad q_{10}^{(1)} = -p_{13}^{(1)} + p_{13}^{(2)}, \\
 q_{10}^{(2)} = p_{10}^{(0)} + p_{11}^{(1)}, \quad q_{12}^{(0)} = -p_{13}^{(0)} + p_{13}^{(2)}, \quad q_{12}^{(1)} = p_{10}^{(1)} + p_{12}^{(2)}, \\
 q_{12}^{(0)} = p_{13}^{(0)} - p_{13}^{(1)}, \quad q_{12}^{(2)} = p_{10}^{(0)} + p_{10}^{(2)}, \quad q_{12}^{(1)} = p_{10}^{(1)} - p_{12}^{(2)}. 
\end{array} \right.
\]

ii) $i = 0, 1, 2, 3.$

iii) $i = 0, 1, 2, 3.$

Proof. We get

$$\ell (Lf) = \sum_{i=0}^{2} A_i \frac{\partial (Lf)}{\partial x_i}.$$
\[
\sum_{i=0}^{2} A_i \frac{\partial}{\partial x_i} \left( \sum_{j=0}^{2} B_j \frac{\partial f}{\partial x_j} + C f + D \right)
\]

\[
= \sum_{i=0}^{2} A_i \frac{\partial}{\partial x_i} \left( \sum_{j=0}^{2} B_j \frac{\partial f}{\partial x_j} \right) + \sum_{i=0}^{2} A_i \frac{\partial(C f)}{\partial x_i} + \sum_{i=0}^{2} A_i \frac{\partial D}{\partial x_i}
\]

\[
= \sum_{i=0}^{2} \sum_{j=0}^{2} A_i B_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=0}^{2} \sum_{j=0}^{2} A_i C \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=0}^{2} A_i \frac{\partial D}{\partial x_i}
\]

By (5), (6) and (7), then (8) can be rewritten as follow

\[
l(Lf) = \sum_{i=0}^{2} P_i \frac{\partial^2 f}{\partial x_i^2} + \sum_{0 \leq i < j \leq 2} Q_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{j=0}^{2} R_j \frac{\partial f}{\partial x_j}
\]

\[
+ \left( \sum_{i=0}^{2} A_i \frac{\partial C}{\partial x_i} \right) f + \sum_{i=0}^{2} A_i \frac{\partial D}{\partial x_i}.
\]

Denote

\[
M = \sum_{i=0}^{2} P_i \frac{\partial^2 f}{\partial x_i^2} + \sum_{0 \leq i < j \leq 2} Q_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix}
\]

\[
N = \sum_{j=0}^{2} R_j \frac{\partial f}{\partial x_j} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{pmatrix}
\]

\[
S = \left( \sum_{i=0}^{2} A_i \frac{\partial C}{\partial x_i} \right) f, \quad T = \sum_{i=0}^{2} A_i \frac{\partial D}{\partial x_i}
\]

Then we obtain

\[
l(Lf) = M + N + S + T.
\]
We get

\[ m_i = p^{(0)} \frac{\partial^2 f_0}{\partial x_0^2} + p^{(1)} \frac{\partial^2 f_1}{\partial x_1^2} + p^{(2)} \frac{\partial^2 f_2}{\partial x_2^2} + p^{(3)} \frac{\partial^2 f_3}{\partial x_3^2} \]

\[ + r^{(0)} \frac{\partial f_0}{\partial x_0} + r^{(1)} \frac{\partial f_1}{\partial x_1} + r^{(2)} \frac{\partial f_2}{\partial x_2} + r^{(3)} \frac{\partial f_3}{\partial x_3} \]

\[ + q^{(0)} \frac{\partial^2 f_0}{\partial x_0 \partial x_1} + q^{(1)} \frac{\partial^2 f_1}{\partial x_1 \partial x_1} + q^{(2)} \frac{\partial^2 f_2}{\partial x_2 \partial x_2} + q^{(3)} \frac{\partial^2 f_3}{\partial x_3 \partial x_3}. \tag{11} \]

Similarly, one gets

\[ n_i = r^{(0)} \frac{\partial f_0}{\partial x_0} + r^{(1)} \frac{\partial f_1}{\partial x_1} + r^{(2)} \frac{\partial f_2}{\partial x_2} + r^{(3)} \frac{\partial f_3}{\partial x_3} \]

\[ + r^{(0)} \frac{\partial^2 f_0}{\partial x_0 \partial x_0} + r^{(1)} \frac{\partial^2 f_1}{\partial x_1 \partial x_1} + r^{(2)} \frac{\partial^2 f_2}{\partial x_2 \partial x_2} + r^{(3)} \frac{\partial^2 f_3}{\partial x_3 \partial x_3}. \tag{12} \]

Suppose that f is a regular function, then

\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} &= 0 \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} &= 0 \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_3}{\partial x_2} &= 0 \\
\frac{\partial f_2}{\partial x_1} - \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_2} &= 0.
\end{align*} \tag{13} \]

It follows from (13) that

\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\
\frac{\partial f_0}{\partial x_1} &= \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_2} \\
\frac{\partial f_0}{\partial x_2} &= \frac{\partial f_1}{\partial x_0} - \frac{\partial f_3}{\partial x_1} \\
\frac{\partial f_1}{\partial x_0} &= \frac{\partial f_0}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\
\frac{\partial f_1}{\partial x_1} &= \frac{\partial f_0}{\partial x_0} + \frac{\partial f_3}{\partial x_1} \\
\frac{\partial f_1}{\partial x_2} &= \frac{\partial f_0}{\partial x_2} - \frac{\partial f_3}{\partial x_0}.
\end{align*} \tag{14} \]

and

\[
\begin{align*}
\frac{\partial^2 f_0}{\partial x_0^2} &= \frac{\partial^2 f_1}{\partial x_0 \partial x_1} + \frac{\partial^2 f_2}{\partial x_0 \partial x_2} \\
\frac{\partial^2 f_0}{\partial x_1^2} &= \frac{\partial^2 f_1}{\partial x_1^2} - \frac{\partial^2 f_2}{\partial x_0 \partial x_1} - \frac{\partial^2 f_3}{\partial x_1 \partial x_2} \\
\frac{\partial^2 f_0}{\partial x_2^2} &= -\frac{\partial^2 f_1}{\partial x_2 \partial x_1} + \frac{\partial^2 f_3}{\partial x_1 \partial x_2}.
\end{align*} \tag{15} \]

and similar expression for the other \( \frac{\partial^2 f_i}{\partial x_j^2}, i = 1, 2, 3; \quad j = 0, 1, 2. \)

Hence we get 3 remaining systems having the form of (15). Thus, one has a total of 12 equations.

Substituting above 12 equations into (11), and after a calculation, we obtain
\[ m_i = \left( -p_{i1}^{(0)} + p_{i1}^{(1)} + q_{i0}^{(0)} \right) \frac{\partial^2 f_0}{\partial x_0 \partial x_1} + \left( -p_{i2}^{(0)} + p_{i2}^{(2)} + q_{i0}^{(0)} \right) \frac{\partial^2 f_0}{\partial x_0 \partial x_2} \\
+ \left( p_{i3}^{(1)} - p_{i3}^{(2)} + q_{i0}^{(12)} \right) \frac{\partial^2 f_0}{\partial x_1 \partial x_2} + \left( p_{i4}^{(0)} - p_{i4}^{(1)} + q_{i0}^{(01)} \right) \frac{\partial^2 f_1}{\partial x_0 \partial x_1} \\
+ \left( p_{i5}^{(0)} - p_{i5}^{(2)} + q_{i1}^{(02)} \right) \frac{\partial^2 f_1}{\partial x_0 \partial x_2} + \left( p_{i6}^{(1)} - p_{i6}^{(2)} + q_{i1}^{(12)} \right) \frac{\partial^2 f_1}{\partial x_1 \partial x_2} \\
+ \left( p_{i7}^{(0)} + p_{i7}^{(2)} + q_{i2}^{(01)} \right) \frac{\partial^2 f_2}{\partial x_0 \partial x_2} + \left( p_{i8}^{(0)} - p_{i8}^{(1)} + q_{i2}^{(02)} \right) \frac{\partial^2 f_2}{\partial x_0 \partial x_2} \\
+ \left( p_{i9}^{(1)} + p_{i9}^{(2)} + q_{i3}^{(02)} \right) \frac{\partial^2 f_3}{\partial x_0 \partial x_2} + \left( -p_{i0}^{(1)} + p_{i0}^{(2)} + q_{i3}^{(12)} \right) \frac{\partial^2 f_3}{\partial x_1 \partial x_2}. \tag{16} \]

Analogously, substituting the relation (14) into (12), one gets
\[ n_i = -r_{i1}^{(0)} + r_{i1}^{(1)} \frac{\partial f_0}{\partial x_1} + (-r_{i2}^{(0)} + r_{i2}^{(2)}) \frac{\partial f_0}{\partial x_2} + (r_{i0}^{(0)} + r_{i0}^{(1)}) \frac{\partial f_1}{\partial x_1} \\
+ (r_{i3}^{(1)} + r_{i3}^{(2)}) \frac{\partial f_1}{\partial x_2} + (-r_{i1}^{(0)} + r_{i1}^{(1)}) \frac{\partial f_2}{\partial x_1} + (r_{i0}^{(0)} + r_{i0}^{(2)}) \frac{\partial f_2}{\partial x_2} \\
+ (r_{i2}^{(0)} + r_{i2}^{(1)}) \frac{\partial f_3}{\partial x_1} + (-r_{i1}^{(0)} + r_{i1}^{(2)}) \frac{\partial f_3}{\partial x_2}. \tag{17} \]

\textbf{(*)Sufficient condition}

Suppose that the conditions (i), (ii), and (iii) of theorem are satisfied. From the relation (i), it follows that \( S = T = 0 \). Because (ii) it leads to \( n_i = 0, i = 0, 1, 2, 3 \). Using the condition (iii) it implies \( m_i = 0, i = 0, 1, 2, 3 \). This means that \( M = N = 0 \).

Hence \( l(Lf) = M + N + S + T = 0 \) for all regular functions \( f \).

The sufficient conditions is proved.

\textbf{(*)Necessary condition}

Assume that a \((l,L)\) is an associated pair, i.e., if \( lf = 0 \), then \( l(Lf) = 0 \). We will choose 22 regular functions as follow

First, choose \( f^{(1)} = 0 \), then (10) passes into \( T \). Because \( l(Lf) = 0 \), then \( T = 0 \). This means that \( g = \sum_{i=0}^{3} d_i e_i \) is a regular function. Thus the term \( T \) can be omitted in (10). Next, we choose \( f^{(2)} \) is arbitrary Quaternion constant, \( f^{(2)} \neq 0 \). For this choice (10) implies \( S = 0 \). Since \( f^{(2)} \) is arbitrary, then \( \sum_{i=0}^{2} A_i \frac{\partial C}{\partial x_i} = 0 \). On another word \( h^{(\alpha)} = \sum_{i=0}^{3} C_{\alpha} e_i, \alpha = 0, 1, 2, 3 \) are regular functions. Hence \( S \) vanished in (10). Now, choose \( f^{(3)} = x_0 + x_1 e_1 \), then (10) leads to \( N = 0 \), so \( n_i = 0, i = 0, 1, 2, 3 \). But in fact \( n_i = r_{i0}^{(0)} + r_{i1}^{(1)} \). Therefore, we get \( r_{i1}^{(1)} = -r_{i0}^{(0)} \).

Note that the equality is the same the condition \( 3^{rd} \) of the relation (i).

By similar method, choose
\[ f^{(4)} = x_1 - x_0 e_1, \quad f^{(5)} = x_0 e_2 + x_1 e_3, \quad f^{(6)} = x_1 e_2 - x_0 e_3, \]
\[ f^{(7)} = x_0 + x_2 e_2, \quad f^{(8)} = x_0 e_1 - x_2 e_3, \quad f^{(9)} = x_2 - x_0 e_2, \]
\[ f^{(10)} = x_2 e_1 + x_0 e_3 \]
and substituting these functions into (10) we obtain \( N = 0 \) for all \( f^{(i)}, \ i = 4, \ldots, 10 \). From this, we have remaining equalities which are contained in the condition (ii). Hence \( N \) can be omitted in (10).

Now we choose \( f^{(11)} = (x_0^2 - x_1^2) + 2x_0x_1e_1 \) and replace \( f \) in (10) by \( f^{(11)} \), it follows that \( M = 0 \). This means
\[
m_i = -p_i^{(0)} + p_i^{(1)} + q_i^{(01)} = 0, \ i = 0, 1, 2, 3.
\]
The equality leads to
\[
q_i^{(01)} = p_i^{(0)} - p_i^{(1)}.
\] (18)

Note that (18) is the same the first condition of (iii). Similarly, choose
\[
\begin{align*}
f^{(12)} &= (x_0^2 - x_2^2) + 2x_0x_2e_2, \quad f^{(13)} = (x_1^2 - x_2^2) - 2x_1x_2e_3 \\
f^{(14)} &= -2x_0x_1 + (x_0^2 - x_1^2)e_1 \\
f^{(15)} &= (x_0^2 - x_1^2)e_1 - 2x_0x_1e_2 \\
f^{(16)} &= (x_1^2 - x_2^2)e_1 - 2x_1x_2e_2 \\
f^{(17)} &= (x_0^2 - x_1^2)e_2 + 2x_0x_1e_3 \\
f^{(18)} &= -2x_0x_2 + (x_0^2 - x_1^2)e_2 \\
f^{(19)} &= 2x_1x_2e_1 + (x_1^2 - x_2^2)e_2 \\
f^{(20)} &= -2x_0x_1 + (x_0^2 - x_1^2)e_3 \\
f^{(21)} &= 2x_0x_2e_1 + (x_0^2 - x_2^2)e_3 \\
f^{(22)} &= 2x_1x_2 + (x_1^2 - x_2^2)e_3,
\end{align*}
\]
and substituting \( f = f^{(j)}, j = 12, \ldots, 22 \) into (10) one obtains \( M = 0 \). By similar arguments we get all remaining equalities of the condition (iii). This completed the proof of necessary condition. \( \square \)

**Remark 2.** If we replace the Cauchy-Riemann operator by the Cauchy-Fueter operator

\[
\mu = \sum_{k=0}^{3} e_k \frac{\partial}{\partial x_k},
\]

and consider the operators \( l, L \) which are given by

\[
\ell f = \sum_{i=0}^{3} A_i \frac{\partial f}{\partial x_i}
\]

where
\[
A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\]
\[
A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\]
and
\[ Lf = \sum_{j=0}^{3} B_j \frac{\partial f}{\partial x_j} + C f + D, \]
and putting
\[ P_j = [p_{\alpha \beta}^{(j)}] = A_j B_j, \quad j = 0, 1, 2, 3 \]
\[ Q_{ij} = [q_{\alpha \beta}^{(ij)}] = A_i B_j + A_j B_i, \quad 0 \leq i < j \leq 3 \]
\[ R_j = [r_{\alpha \beta}^{(j)}] = \sum_{i=0}^{3} A_i \frac{\partial B_i}{\partial x_i} + A_j C, \quad j = 0, 1, 2, 3, \quad \alpha, \beta = 0, 1, 2, 3. \]

Then by analogously method which used in the section 2, we obtain the following theorem

**Theorem 2.** The operator \( L \) is associated to the operator \( \ell \) if and only if the following conditions are satisfied

i) The functions \( h^{(\alpha)} = \sum_{i=0}^{3} c_{i\alpha} e_i, \quad \alpha = 0, 1, 2, 3, \) and \( g = \sum_{i=0}^{3} d_i e_i \) are regular.

\[ \begin{align*}
q_{i0}^{(1)} &= p_{i1}^{(0)} - p_{i1}^{(1)}, & q_{i0}^{(2)} &= p_{i2}^{(0)} - p_{i2}^{(2)}, & q_{i0}^{(03)} &= p_{i3}^{(0)} - p_{i3}^{(03)} \\
q_{i0}^{(12)} &= -p_{i1}^{(1)} + p_{i3}^{(2)}, & q_{i0}^{(13)} &= p_{i2}^{(1)} - p_{i3}^{(3)}, & q_{i0}^{(23)} &= -p_{i1}^{(2)} + p_{i3}^{(3)} \\
q_{i0}^{(01)} &= p_{i1}^{(0)} + p_{i0}^{(1)}, & q_{i0}^{(02)} &= p_{i2}^{(0)} + p_{i0}^{(2)}, & q_{i0}^{(03)} &= p_{i3}^{(0)} + p_{i0}^{(03)} \\
q_{i2}^{(1)} &= -p_{i1}^{(1)} + p_{i0}^{(2)}, & q_{i2}^{(13)} &= -p_{i3}^{(1)} + p_{i3}^{(3)}, & q_{i2}^{(23)} &= -p_{i1}^{(2)} + p_{i3}^{(3)} \\
q_{i3}^{(1)} &= -p_{i1}^{(1)} + p_{i2}^{(2)}, & q_{i3}^{(13)} &= -p_{i3}^{(1)} + p_{i3}^{(3)}, & q_{i3}^{(23)} &= -p_{i1}^{(2)} + p_{i3}^{(3)} \\
q_{i3}^{(01)} &= p_{i1}^{(0)} - p_{i1}^{(1)}, & q_{i3}^{(02)} &= -p_{i0}^{(0)} + p_{i0}^{(2)}, & q_{i3}^{(03)} &= p_{i1}^{(0)} - p_{i1}^{(03)} \\
q_{i3}^{(12)} &= p_{i1}^{(1)} - p_{i1}^{(2)}, & q_{i3}^{(13)} &= p_{i0}^{(1)} - p_{i1}^{(3)}, & q_{i3}^{(23)} &= p_{i0}^{(2)} - p_{i1}^{(3)}.
\end{align*} \]
3. Example

3.1. Operator L is associated to the Cauchy-Riemann operator. First, we choose $c_{\alpha\beta}$ are arbitrary real-constants, $g = \sum_{i=0}^{3} d_i e_i$ is arbitrary regular function and choose the elements $b_{\alpha\beta}^{(0)}$, $\alpha, \beta = 0, 1, 2, 3$ of the matrix $B_0$ as follow

\[
\begin{align*}
    b_{00}^{(0)} &= - (\gamma - c_{00}) x_0 - c_{10} x_1 - c_{20} x_2 + \delta_{00}^{(0)} \\
    b_{01}^{(0)} &= c_{01} x_0 + (\gamma - c_{11}) x_1 - c_{21} x_2 + \delta_{01}^{(0)} \\
    b_{02}^{(0)} &= c_{02} x_0 - c_{12} x_1 + (\gamma - c_{22}) x_2 + \delta_{02}^{(0)} \\
    b_{03}^{(0)} &= c_{03} x_0 - c_{13} x_1 - c_{23} x_2 + \delta_{03}^{(0)} \\
    b_{10}^{(0)} &= c_{10} x_0 - (\gamma - c_{00}) x_1 + c_{30} x_2 + \delta_{10}^{(0)} \\
    b_{11}^{(0)} &= - (\gamma - c_{11}) x_0 + c_{01} x_1 + c_{31} x_2 + \delta_{11}^{(0)} \\
    b_{12}^{(0)} &= c_{12} x_0 + c_{02} x_1 + c_{32} x_2 + \delta_{12}^{(0)} \\
    b_{13}^{(0)} &= c_{13} x_0 + c_{03} x_1 - (\gamma - c_{33}) x_2 + \delta_{13}^{(0)} \\
    b_{20}^{(0)} &= c_{20} x_0 - c_{30} x_1 - (\gamma - c_{00}) x_2 + \delta_{20}^{(0)} \\
    b_{21}^{(0)} &= c_{21} x_0 - c_{31} x_1 + c_{01} x_2 + \delta_{21}^{(0)} \\
    b_{22}^{(0)} &= - (\gamma - c_{22}) x_0 - c_{32} x_1 + c_{02} x_2 + \delta_{22}^{(0)} \\
    b_{23}^{(0)} &= c_{23} x_0 + (\gamma - c_{33}) x_1 + c_{03} x_2 + \delta_{23}^{(0)} \\
    b_{30}^{(0)} &= c_{30} x_0 + c_{20} x_1 - c_{10} x_2 + \delta_{30}^{(0)} \\
    b_{31}^{(0)} &= c_{31} x_0 + c_{21} x_1 + (\gamma - c_{11}) x_2 + \delta_{31}^{(0)} \\
    b_{32}^{(0)} &= c_{32} x_0 - (\gamma - c_{22}) x_1 - c_{12} x_2 + \delta_{32}^{(0)} \\
    b_{33}^{(0)} &= -(\gamma - c_{33}) x_0 + c_{23} x_1 - c_{13} x_2 + \delta_{33}^{(0)},
\end{align*}
\]

where $\gamma, \delta_{\alpha\beta}^{(0)}, \alpha, \beta = 0, 1, 2, 3$ are arbitrary real-constants.

Second, choose $B_1 = -A_1 B_0$ and $B_2 = -A_2 B_0$. Then it is easy to verify that all the conditions of theorem 1 are satisfied. By this way one obtains a class of differential operators $L$ which are associated to the Cauchy-Riemann operator of Quaternion algebra.

3.2. Operator L is associated to the Cauchy-Fueter operator. Choosing $c_{\alpha\beta}$ are arbitrary real-constants, $g = \sum_{i=0}^{3} d_i e_i$ is arbitrary regular function. The elements $b_{\alpha\beta}^{(0)}$, $\alpha, \beta = 0, 1, 2, 3$ of the matrix $B_0$ are given by
problem is solvable provided

Next, choose \( L \) operators

Then we can see that all the conditions of theorem 2 hold. So one gets a class of differential equations with infinitely differentiable coefficients not having any solutions. On the other hand, in view of the H.Lewy example (see [4]), there exist linear first order differential equations with infinitely differentiable coefficients not having any solutions. On the other hand, by the criterion which is given in theorem 1 (and theorem 2, respectively), we can construct

\[
\begin{align*}
\mathcal{b}_{00}^{(0)} &= \frac{1}{2} \left[-(\gamma - c_{00}) x_0 - c_{10} x_1 - c_{20} x_2 - c_{30} x_3\right] + \delta_{00}^{(0)} \\
\mathcal{b}_{01}^{(0)} &= \frac{1}{2} \left[c_{01} x_0 + (\gamma - c_{11}) x_1 - c_{21} x_2 - c_{31} x_3\right] + \delta_{01}^{(0)} \\
\mathcal{b}_{02}^{(0)} &= \frac{1}{2} \left[c_{02} x_0 - c_{12} x_1 + (\gamma - c_{22}) x_2 - c_{32} x_3\right] + \delta_{02}^{(0)} \\
\mathcal{b}_{03}^{(0)} &= \frac{1}{2} \left[c_{03} x_0 - c_{13} x_1 - c_{23} x_2 + (\gamma - c_{33}) x_3\right] + \delta_{03}^{(0)} \\
\mathcal{b}_{10}^{(0)} &= \frac{1}{2} \left[c_{10} x_1 - (\gamma - c_{00}) x_0 + c_{30} x_2 - c_{20} x_3\right] + \delta_{10}^{(0)} \\
\mathcal{b}_{11}^{(0)} &= \frac{1}{2} \left[-(\gamma - c_{11}) x_0 + c_{01} x_1 + c_{31} x_2 - c_{21} x_3\right] + \delta_{11}^{(0)} \\
\mathcal{b}_{12}^{(0)} &= \frac{1}{2} \left[c_{12} x_0 + c_{02} x_1 + c_{32} x_2 + (\gamma - c_{22}) x_3\right] + \delta_{12}^{(0)} \\
\mathcal{b}_{13}^{(0)} &= \frac{1}{2} \left[c_{13} x_0 + c_{03} x_1 - (\gamma - c_{33}) x_2 - c_{23} x_3\right] + \delta_{13}^{(0)} \\
\mathcal{b}_{20}^{(0)} &= \frac{1}{2} \left[c_{20} x_2 - c_{03} x_1 - (\gamma - c_{00}) x_0 + c_{10} x_3\right] + \delta_{20}^{(0)} \\
\mathcal{b}_{21}^{(0)} &= \frac{1}{2} \left[c_{21} x_0 - c_{31} x_1 + c_{01} x_2 - (\gamma - c_{11}) x_3\right] + \delta_{21}^{(0)} \\
\mathcal{b}_{22}^{(0)} &= \frac{1}{2} \left[-(\gamma - c_{22}) x_0 - c_{32} x_1 + c_{02} x_2 + c_{12} x_3\right] + \delta_{22}^{(0)} \\
\mathcal{b}_{23}^{(0)} &= \frac{1}{2} \left[c_{23} x_2 + (\gamma - c_{33}) x_1 + c_{03} x_2 + c_{13} x_3\right] + \delta_{23}^{(0)} \\
\mathcal{b}_{30}^{(0)} &= \frac{1}{2} \left[c_{30} x_0 + c_{20} x_1 - c_{10} x_2 + (\gamma - c_{00}) x_3\right] + \delta_{30}^{(0)} \\
\mathcal{b}_{31}^{(0)} &= \frac{1}{2} \left[c_{31} x_0 + c_{21} x_1 + (\gamma - c_{11}) x_2 + c_{01} x_3\right] + \delta_{31}^{(0)} \\
\mathcal{b}_{32}^{(0)} &= \frac{1}{2} \left[c_{32} x_0 - (\gamma - c_{22}) x_1 - c_{12} x_2 + c_{02} x_3\right] + \delta_{32}^{(0)} \\
\mathcal{b}_{33}^{(0)} &= \frac{1}{2} \left[-(\gamma - c_{33}) x_0 + c_{23} x_1 - c_{13} x_2 + c_{03} x_3\right] + \delta_{33}^{(0)},
\end{align*}
\]

where \( \gamma, \delta_{a,b}^{(0)}, \alpha, \beta = 0, 1, 2, 3 \) are arbitrary real-constants.

Next, choose

\[
\begin{align*}
B_1 &= -A_1 B_0 \\
B_2 &= -A_2 B_0 \\
B_3 &= -A_3 B_0.
\end{align*}
\]

Then we can see that all the conditions of theorem 2 hold. So one gets a class of differential operators \( L \) which are associated to the Cauchy-Fueter operator of Quaternion algebra.

4. Initial value problems with regular initial functions

The classical Cauchy-Kovalevskaya theorem (in Complex analysis) shows that the initial value problem is solvable provided \( L \) has holomorphic coefficients and the initial function is holomorphic, but in view of the H.Lewy example (see [4]), there exist linear first order differential equations with infinitely differentiable coefficients not having any solutions. On the other hand, by the criterion which is given in theorem 1 (and theorem 2, respectively), we can construct
operator $L$ such that the initial value problem (1), (2) is solvable for each regular initial function $\varphi$. Because the components of regular functions are harmonic so the necessary interior estimate (see [9]) follows from the Poisson Integral Formula.

Finally, we get the following theorem

Theorem 3. Suppose that the operator $L$ is associated to the Cauchy-Riemann operator (the Cauchy-Fueter, respectively) of Quaternion algebra. Then the initial value problem (1), (2) is solvable for any arbitrary initial regular function $\varphi$ and the solution $u(t, x)$ is regular for each $t$.

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