FIRST ORDER DIFFERENTIAL OPERATOR ASSOCIATED TO
THE CAUCHY–RIEMANN OPERATOR IN A CLIFFORD ALGEBRA

Nguyen Thanh Van

Faculty of Mathematics, Mechanics and Informatics,
Ha Noi University of Science, Ha Noi, Viet Nam
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

The complex differentiation transforms holomorphic functions into holomorphic functions. Analogously, the conjugate Cauchy-Riemann operator of the Clifford algebra transforms regular functions into regular functions. This paper determines more general first order operator $L$ (matrix-type) for which $Lu$ is regular provided $u$ is regular.

For such operator $L$, the initial value problem

$$\frac{\partial u}{\partial t} = L(t, x, u, \frac{\partial u}{\partial x})$$  \hspace{2cm} (1)
$$u(0, x) = \varphi(x)$$  \hspace{2cm} (2)

is solvable for an arbitrary regular function $\varphi$ and the solution is regular in $x$ for each $t$. 

MIRAMARE – TRIESTE
December 2006

\footnote{thanhvanao@yahoo.com}
1. Preliminaries and notations

Let \( \mathcal{A} \) be a Clifford Algebra with the basis
\[
\mathcal{B} = \{e_0, e_1, \ldots, e_m, e_{12}, e_{13}, \ldots, e_{123} \ldots m\},
\]
where \( e_0 = 1 \). Denoting \( e_{12} = e_{m+1}, e_{13} = e_{m+2}, \ldots, e_{123} \ldots m = e_n, n = 2^m - 1 \), then \( \mathcal{B} \) passes into
\[
\mathcal{B} = \{e_0, e_1, e_2, \ldots, e_m, \ldots, e_n\}.
\]

Let \( a \) be an element of \( \mathcal{A} \) which can be presented as
\[
a = \sum_{i=0}^{n} a_i e_i, a_i \in \mathbb{R}.
\]

Suppose that \( \Omega \) is a bounded domain of \( \mathbb{R}^{m+1} \). We consider a function \( f \), defined in \( \Omega \) and taking values in the Clifford algebra \( \mathcal{A} \). Then
\[
f = \sum_{j=0}^{n} f_j e_j,
\]
where \( f_j(x) \) are real-valued functions.

Further, we introduce the Cauchy-Riemann operator
\[
\mu = \sum_{k=0}^{m} e_k \frac{\partial}{\partial x_k}.
\]

Definition 1. A function \( f \in C^1(\Omega, \mathcal{A}) \) is said to be regular in \( \Omega \) if \( f \) satisfies
\[
\mu f = 0.
\]

Remark 1. Let \( f \in C^2(\Omega, \mathcal{A}) \) be a regular function, then \( \Delta f = 0 \). This means that \( f \) is harmonic in \( \Omega \).

For other definitions, we refer the reader to [1].

2. Some properties

Lemma 1. Suppose that \( a, x \in \mathcal{A}, a = \sum_{i=0}^{n} a_i e_i, x = \sum_{j=0}^{n} x_j e_j \), then
\[
ax = (e_0, e_1, \ldots, e_n) \sigma(a) 
\begin{pmatrix}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\]
where \( \sigma(a) \) is a matrix of \( (2^m \times 2^m) \) and its elements belong to the set \( \{\pm a_0, \pm a_1, \ldots, \pm a_n\} \).

Proof. We have
\[
ab = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i x_j e_i e_j.
\]
Because \( e_i, e_j \in \mathcal{B} \), then for each \( a \) pair \((i, j)\) which is given there exists only one \( k_{ij} \in \{0, 1, 2, \ldots, n\} \) such that \( e_i e_j = e_{k_{ij}} \), where \( \varepsilon = 1 \) or \(-1\), and \( e_i e_j \neq \pm e_{i_2} e_j \) if \( i_1 \neq i_2 \).
Hence, for each choice of \( j \) and \( i \) run in the set \( \{0, 1, \ldots, n\} \), then \( \{e_{kij}\} \) is a basis of Clifford algebra \( \mathcal{A} \). So we can write

\[
ax = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{ij} x_j e_i, \quad \alpha_{ij} \in \{\pm a_0, \pm a_1, \ldots, \pm a_n\}
\]

\[
= e_0 \left( \sum_{j=0}^{n} \alpha_{0j} x_j \right) + e_1 \left( \sum_{j=0}^{n} \alpha_{1j} x_j \right) + \cdots + e_n \left( \sum_{j=0}^{n} \alpha_{nj} x_j \right)
\]

\[
= (e_0, e_1, \ldots, e_n) \left( \begin{array}{ccc}
\alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} \\
\alpha_{10} & \alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n0} & \alpha_{n1} & \cdots & \alpha_{nn}
\end{array} \right) \left( \begin{array}{c}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{array} \right)
\]

\[
= (e_0, e_1, \ldots, e_n) \sigma(a) \left( \begin{array}{c}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{array} \right),
\]

where \( \sigma(a) = \left( \begin{array}{cccc}
\alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} \\
\alpha_{10} & \alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n0} & \alpha_{n1} & \cdots & \alpha_{nn}
\end{array} \right) \). The lemma is proved.

Now we consider the elements \( e_k, k = \{0, 1, \ldots, m\} \). One gets

\[
e_k x = \sum_{j=0}^{n} (e_k e_j) x_j.
\]

Putting \( \sigma(e_k) = (\lambda_{ij}^{(k)}) \). By the lemma 1, we have

\[
e_k x = (e_0, e_1, \ldots, e_n) \left( \begin{array}{cccc}
\lambda_{00}^{(k)} & \lambda_{01}^{(k)} & \cdots & \lambda_{0n}^{(k)} \\
\lambda_{10}^{(k)} & \lambda_{11}^{(k)} & \cdots & \lambda_{1n}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n0}^{(k)} & \lambda_{n1}^{(k)} & \cdots & \lambda_{nn}^{(k)}
\end{array} \right) \left( \begin{array}{c}
x_0 \\
x_1 \\
\vdots \\
x_n
\end{array} \right)
\]

\[
= \sum_{j=0}^{n} \sum_{i=0}^{n} e_i \lambda_{ij}^{(k)} x_j.
\]

From (3) and (4), it follows that

\[
\sum_{j=0}^{n} (e_k e_j) x_j = \sum_{j=0}^{n} \sum_{i=0}^{n} e_i \lambda_{ij}^{(k)} x_j.
\]

Note that \( x = \sum_{j=0}^{n} x_j e_j \) is arbitrary, then the relation (5) leads to

\[
e_k e_j = \sum_{i=0}^{n} e_i \lambda_{ij}^{(k)}.
\]

The equality (6) implies

\[
(e_k e_j)^{\overline{e}} = \sum_{i=0}^{n} \overline{e_i} \overline{\lambda_{ij}^{(k)}} x_j.
\]
From (7), one gets

\[ e_k = \sum_{i=0}^{n} (e_i \overline{e_j}) \lambda_{ij}^{(k)}. \]  

(8)

It follows from (8) that

\[ \lambda_{ij}^{(k)} = \begin{cases} 1 & \text{if } e_i e_j = e_k \\ -1 & \text{if } e_i e_j = -e_k \\ 0 & \text{if } e_i e_j \neq \pm e_k. \end{cases} \]

Corollary 1. When \( k = 0 \) we obtain

\[ \lambda_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]

So \( \sigma(e_0) = E \) - unitary matrix.

Lemma 2. For all \( a, b \in A \), we get

i) \( \sigma(ab) = \sigma(a)\sigma(b) \)

ii) \( \sigma(a + b) = \sigma(a) + \sigma(b) \).

Proof.

i) By lemma 1, one gets

\[ (ab)x = (e_0, e_1, \ldots, e_n) \sigma(ab) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}. \]

On the other hand,

\[ (ab)x = a(bx) = (e_0, e_1, \ldots, e_n) \sigma(a) \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \]

where \( \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = \sigma(b) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}. \)

So we have

\[ (e_0, e_1, \ldots, e_n) \sigma(ab) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = (e_0, e_1, \ldots, e_n) \sigma(a)\sigma(b) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}. \]  

(9)

Because (9) is true for all \( (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \), then \( \sigma(ab) = \sigma(a)\sigma(b) \).

ii) Similarly, we can prove \( \sigma(a + b) = \sigma(a) + \sigma(b) \).

It is completed the proof of lemma 2.

\[ \square \]

Theorem 1. The elements \( e_1, e_2, \ldots, e_m \) are satisfied the following properties...
\[ i) \sigma(e_k)\sigma(e_k) = -E. \]
\[ ii) \sigma(e_k)\sigma(e_j) + \sigma(e_j)\sigma(e_k) = 0, \quad k \neq j, \quad k, j \in \{1, 2, \ldots, m\}. \]

Proof. Using lemma 2, we get

\[ i) \sigma(e_k)\sigma(e_k) = \sigma(e_k e_k) = \sigma(-e_0) = -E. \]
\[ ii) \sigma(e_k)\sigma(e_j) + \sigma(e_j)\sigma(e_k) = \sigma(e_k e_j + e_j e_k) = \sigma(0) = 0. \]

Theorem 1 is proved. \(\square\)

3. Sufficient conditions for associated pairs

Let \( f = \sum_{j=0}^{n} f_j e_j \) be a twice continuously differentiable function with respect to the space-like \( x_0, x_1, \ldots, x_m \). We get
\[ \mu f = \sum_{k=0}^{m} \sum_{j=0}^{n} \frac{\partial f}{\partial x_k} e_k e_j. \] By the relation (6), \( \mu f \) has the form
\[ \mu f = \sum_{k=0}^{m} \sum_{j=0}^{n} \sum_{i=0}^{n} \lambda^{(k)}_{ij} \frac{\partial f_i}{\partial x_k}. \]

Hence, \( \mu f = 0 \) if and only if
\[ \sum_{j=0}^{n} \sum_{k=0}^{m} \lambda^{(k)}_{ij} \frac{\partial f_j}{\partial x_k} = 0, \quad i = 0, 1, 2, \ldots, n. \]

This means that
\[
\begin{pmatrix}
\frac{\partial f_0}{\partial x_0} \\
\vdots \\
\frac{\partial f_n}{\partial x_0}
\end{pmatrix}
+ \sigma(e_1)
\begin{pmatrix}
\frac{\partial f_0}{\partial x_1} \\
\vdots \\
\frac{\partial f_n}{\partial x_1}
\end{pmatrix}
+ \cdots + \sigma(e_m)
\begin{pmatrix}
\frac{\partial f_0}{\partial x_m} \\
\vdots \\
\frac{\partial f_n}{\partial x_m}
\end{pmatrix}
= 0.
\]

Put \( A_i = \sigma(e_i) \) then by corollary 1 and theorem 1 we obtain
\( A_0 = E, \quad A_i^2 = -E \) and \( A_i A_j + A_j A_i = 0, \quad i \neq j, \quad i, j \in \{1, 2, \ldots, m\} \). Now, we define an operator \( \ell \) as follow
\[ \ell f = \sum_{i=0}^{m} A_i \frac{\partial f}{\partial x_i}, \]
where \( \frac{\partial f}{\partial x_i} = \begin{pmatrix}
\frac{\partial f_0}{\partial x_i} \\
\vdots \\
\frac{\partial f_n}{\partial x_i}
\end{pmatrix}, \quad i = 0, 1, 2, \ldots, m. \]

It is clear that \( \mu f = 0 \) is equivalent to \( \ell f = 0 \). Next, considering an operator \( L \) which has the form
\[ L f = \sum_{j=0}^{m} B_j \frac{\partial f}{\partial x_j} + C f + D, \quad (10) \]
where $B_j = [b_{i\alpha}^{(j)}], C = [c_{\alpha\beta}], Cf = [c_{\alpha\beta}] \left( \begin{array}{c} f_0 \\ f_1 \\ \vdots \\ f_n \end{array} \right), D = \left( \begin{array}{c} d_0 \\ d_1 \\ \vdots \\ d_m \end{array} \right)$, $b_{i\alpha}^{(j)}, c_{\alpha\beta}, d_\alpha, (\alpha, \beta = 0, 1, \ldots, n)$ are real-valued functions which are supposed to depend at least continuously on the time $t$ and the space-like variable $x_0, x_1, \ldots, x_m$. A pair of operators $\ell, L$ is said to be associated (see[7]) if $\ell f = 0$ implies $\ell(Lf) = 0$ (for each $t$ in case the coefficients of $L$ depend on $t$). Now we formulate sufficient conditions on the coefficients of operator $L$ under which $L$ is associated to the operator $\ell$ (on the other word, $L$ is associated to the Cauchy-Riemann operator of Clifford Algebra). Assume that the function $b_{i\alpha}^{(j)}, c_{\alpha\beta}, d_\alpha (j = 0, 1, \ldots, m, \alpha, \beta = 0, 1, \ldots, n)$ are continuously differentiable with respect to the space-like variable $x_0, x_1, \ldots, x_m$ and differentiable for $t$. Then we get the theorem.

**Theorem 2.** The operator $L$ is associated to the operator $\ell$ if the following conditions are satisfied.

i) The functions $h^{(\alpha)} = \sum_{i=0}^{n} c_\alpha e_i, \alpha = 0, 1, \ldots, n$, and $g = \sum_{i=0}^{n} d_i e_i$, are regular.

ii) $B_j = -A_j B_0, j = 1, 2, \ldots, m$.

iii) $\sum_{i=0}^{m} A_i \frac{\partial B_j}{\partial x_i} + A_j C = \gamma A_j, j = 0, 1, \ldots, m$, where $\gamma$ is an arbitrary real constant.

**Proof.** Suppose that $f$ is regular. This means that $\ell f = 0$. We get

$$\ell(Lf) = \sum_{i=0}^{m} A_i \frac{\partial(Lf)}{\partial x_i}$$

$$= \sum_{i=0}^{m} A_i \frac{\partial}{\partial x_i} \left( \sum_{j=0}^{m} B_j \frac{\partial f}{\partial x_j} + Cf + D \right)$$

$$= \sum_{i=0}^{m} A_i \frac{\partial}{\partial x_i} \left( \sum_{j=0}^{m} B_j \frac{\partial f}{\partial x_j} \right) + \sum_{i=0}^{m} A_i \frac{\partial(Cf)}{\partial x_i} + \sum_{i=0}^{m} A_i \frac{\partial D}{\partial x_i}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{m} A_i B_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=0}^{m} \sum_{j=0}^{m} A_i \frac{\partial B_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \left( \sum_{i=0}^{m} A_i \frac{\partial C}{\partial x_i} \right) f + \sum_{i=0}^{m} A_i C \frac{\partial f}{\partial x_i} + \sum_{i=0}^{m} A_i \frac{\partial D}{\partial x_i}.$$  \hspace{1cm} (11)

It is clear that

$$\sum_{i=0}^{m} \sum_{j=0}^{m} A_i B_j \frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_{i=0}^{m} A_i B_i \frac{\partial^2 f}{\partial x_i^2} + \sum_{0 \leq i < j \leq m} (A_i B_j + A_j B_i) \frac{\partial^2 f}{\partial x_i \partial x_j}.$$  \hspace{1cm} (12)

and

$$\sum_{i=0}^{m} A_i C \frac{\partial f}{\partial x_i} = \sum_{j=0}^{m} A_j C \frac{\partial f}{\partial x_j}.$$  \hspace{1cm} (13)
Substituting (12), (13) into (11), we obtain

\[
\ell(Lf) = \sum_{i=0}^{m} A_i B_i \frac{\partial^2 f}{\partial x_i^2} + \sum_{0 \leq i < j \leq m} (A_i B_j + A_j B_i) \frac{\partial^2 f}{\partial x_i \partial x_j} 
+ \sum_{j=0}^{m} \left( \sum_{i=0}^{m} A_i \frac{\partial B_j}{\partial x_i} + A_j C \right) \frac{\partial f}{\partial x_j} + \left( \sum_{i=0}^{m} A_i \frac{\partial C}{\partial x_i} \right) f + \sum_{i=0}^{m} A_i \frac{\partial D}{\partial x_i}. \quad (14)
\]

Note that

\[
\frac{\partial B_j}{\partial x_i} = \begin{pmatrix} \partial b^{(j)}_{\alpha \beta} \partial x_i \\ \vdots \end{pmatrix}, \quad \frac{\partial C}{\partial x_i} = \begin{pmatrix} \partial c_{\alpha \beta} \partial x_i \\ \vdots \end{pmatrix}, \quad \frac{\partial D}{\partial x_i} = \begin{pmatrix} \partial d_0 \partial x_i \\ \vdots \end{pmatrix}.
\]

Using the condition (i), we have

\[
\left( \sum_{i=0}^{m} A_i \frac{\partial C}{\partial x_i} \right) f = 0 \quad (15)
\]
\[
\sum_{i=0}^{m} A_i \frac{\partial D}{\partial x_i} = 0. \quad (16)
\]

The relation (ii) leads to

\[
\begin{cases}
A_j B_j = A_j (-A_j B_0) = -A_j^2 B_0 = B_0 \\
A_0 B_j + A_j B_0 = B_j + A_j B_0 = 0
\end{cases}
\]
\[j = 1, 2, \ldots, m,
\]

and if \( i \neq j, i, j \in \{1, 2, \ldots, m\} \), then

\[
A_i B_j + A_j B_i = A_i (-A_j B_0) + A_j (-A_i B_0) = -(A_i A_j + A_j A_i) B_0 = 0.
\]

Since \( f \) is regular then \( f \) is harmonic (Remark 1) and by (17), it implies

\[
\sum_{i=0}^{m} A_i B_i \frac{\partial^2 f}{\partial x_i^2} = B_0 \left( \sum_{i=0}^{m} \frac{\partial^2 f}{\partial x_i^2} \right) = 0. \quad (19)
\]

According to (17) and (18) one gets

\[
\sum_{0 \leq i < j \leq m} (A_i B_j + A_j B_i) \frac{\partial^2 f}{\partial x_i \partial x_j} = 0. \quad (20)
\]

At last, from (iii), it follows that

\[
\sum_{j=0}^{m} \left( \sum_{i=0}^{m} A_i \frac{\partial B_j}{\partial x_i} + A_j C \right) \frac{\partial f}{\partial x_j} = \sum_{j=0}^{m} \gamma A_j \frac{\partial f}{\partial x_j} = \gamma \left( \sum_{j=0}^{m} A_j \frac{\partial f}{\partial x_j} \right) = \gamma (\ell f) = 0. \quad (21)
\]

By equalities (15), (16), (19), (20) and (21), it implies

\[
\ell(Lf) = 0.
\]
So, the operator $L$ is associated to the operator $\ell$ (in other words $L$ is associated to the Cauchy-Riemann operator of Clifford algebra). The proof is completed. □

4. THE CONSTRUCTION OF THE OPERATOR $L$

4.1. The general case. Using the condition (iii) of theorem 2, we get

$$
\begin{aligned}
&\left\{ \begin{array}{l}
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} + A_2 \frac{\partial B_2}{\partial x_2} + \cdots + A_m \frac{\partial B_m}{\partial x_m} + A_0 C = \gamma A_0 \\
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} + A_2 \frac{\partial B_2}{\partial x_2} + \cdots + A_m \frac{\partial B_m}{\partial x_m} + A_1 C = \gamma A_1 \\
\cdots \\
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} + A_2 \frac{\partial B_2}{\partial x_2} + \cdots + A_m \frac{\partial B_m}{\partial x_m} + A_m C = \gamma A_m.
\end{array} \right.
\end{aligned}
$$

(22)

Substituting $B_j = -A_j B_0$, $j = 1, 2, \ldots, m$ (the condition (ii) of theorem 2) into (22), then the system turns to

$$
\begin{aligned}
&\left\{ \begin{array}{l}
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} + A_2 \frac{\partial B_2}{\partial x_2} + \cdots + A_m \frac{\partial B_m}{\partial x_m} = A_0 (\gamma E - C) \\
-A_0 A_1 \frac{\partial B_1}{\partial x_1} - A_1^2 \frac{\partial B_1}{\partial x_2} - A_2 A_1 \frac{\partial B_2}{\partial x_2} - \cdots - A_m A_1 \frac{\partial B_m}{\partial x_m} = A_1 (\gamma E - C) \\
\cdots \\
-A_0 A_m \frac{\partial B_0}{\partial x_0} - A_1 A_m \frac{\partial B_1}{\partial x_1} - A_2 A_m \frac{\partial B_2}{\partial x_2} - \cdots - A_m^2 \frac{\partial B_m}{\partial x_m} = A_m (\gamma E - C).
\end{array} \right.
\end{aligned}
$$

(23)

Because $A_i^2 = -E$, $i = 1, 2, \ldots, m$ (Theorem 1), then det $A_i \neq 0$. Multiplying $(k+1)^{\text{th}}$-equation of the system (23) by $A_k$, $k = 1, 2, \ldots, m$ from the left, we obtain

$$
\begin{aligned}
&\left\{ \begin{array}{l}
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} + A_2 \frac{\partial B_2}{\partial x_2} + \cdots + A_m \frac{\partial B_m}{\partial x_m} = A_0 (\gamma E - C) \\
-A_1 A_0 A_1 \frac{\partial B_1}{\partial x_1} - A_2 A_1 \frac{\partial B_1}{\partial x_2} - \cdots - A_m A_1 \frac{\partial B_m}{\partial x_m} = A_1^2 (\gamma E - C) \\
\cdots \\
-A_m A_0 A_m \frac{\partial B_0}{\partial x_0} - A_1 A_m \frac{\partial B_1}{\partial x_1} - A_2 A_m \frac{\partial B_2}{\partial x_2} - \cdots - A_m^3 \frac{\partial B_m}{\partial x_m} = A_m^2 (\gamma E - C).
\end{array} \right.
\end{aligned}
$$

(24)

Note that $A_0 = E$, $A_i^2 = -E$, and $A_i A_j + A_j A_i = 0$, $i \neq j$, $i, j = 1, 2, \ldots, m$, then one gets

$$
-A_k A_l A_k = -(A_k A_l) A_k = -(-A_l A_k) A_k = A_l (A_k^2) = -A_l, \quad \text{if} \quad k \neq l \neq 0,
$$

and

$$
-A_3^3 = -(A_3^2) A_3 = A_3.
$$

So, the system (24) is equivalent to

$$
\begin{aligned}
&\left\{ \begin{array}{l}
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} + A_2 \frac{\partial B_2}{\partial x_2} + \cdots + A_m \frac{\partial B_m}{\partial x_m} = (\gamma E - C) \\
A_0 \frac{\partial B_0}{\partial x_0} + A_1 \frac{\partial B_1}{\partial x_1} - A_2 \frac{\partial B_2}{\partial x_2} - \cdots - A_m \frac{\partial B_m}{\partial x_m} = -(\gamma E - C) \\
\cdots \\
A_0 \frac{\partial B_0}{\partial x_0} - A_1 \frac{\partial B_1}{\partial x_1} - A_2 \frac{\partial B_2}{\partial x_2} - \cdots - A_m \frac{\partial B_m}{\partial x_m} = -(\gamma E - C).
\end{array} \right.
\end{aligned}
$$

(25)

Comparing the first equation with each remaining equation, we obtain

$$
A_k \frac{\partial B_0}{\partial x_k} = -\frac{1}{m-1} (\gamma E - C), \quad k = 1, 2, \ldots, m.
$$

(26)

Substituting (26) into the first equation of (25), it implies

$$
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial B_0}{\partial x_k} = -\frac{1}{m-1} (\gamma E - C), \quad k = 1, 2, \ldots, m.
\end{array} \right.
\end{aligned}
$$
This means that

\[
\begin{align*}
\frac{\partial B_0}{\partial x_0} &= -\frac{1}{m-1} (\gamma E - C) \\
\frac{\partial B_k}{\partial x_k} &= \frac{1}{m-1} A_k^{-1} (\gamma E - C), \quad k = 1, 2, \ldots, m.
\end{align*}
\] (27)

4.2. The special case $m = 2$. In this case, $A$ turns to Quaternion algebra and the Cauchy-Riemann operator has the form

\[
\mu = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2},
\]

and the operator $\ell$ as follow

\[
\ell f = \sum_{i=0}^{2} A_i \frac{\partial f}{\partial x_i},
\]

where

\[
A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\]

From (27), when $m = 2$, we obtain

\[
\frac{\partial B_0}{\partial x_0} = -\begin{bmatrix} (\gamma - c_{00}) & -c_{01} & -c_{02} & -c_{03} \\ -c_{10} & (\gamma - c_{11}) & -c_{12} & -c_{13} \\ -c_{20} & -c_{21} & (\gamma - c_{22}) & -c_{23} \\ -c_{30} & -c_{31} & -c_{32} & (\gamma - c_{33}) \end{bmatrix} = -M,
\] (28)

and

\[
\frac{\partial B_0}{\partial x_k} = A_k^{-1} M, \quad k = 1, 2.
\] (29)

According to (28) and (29), we get

\[
\begin{align*}
\frac{\partial b^{(0)}_0}{\partial x_0} &= - (\gamma - c_{00}) = c_{00} - \gamma \\
\frac{\partial b^{(0)}_0}{\partial x_1} &= -c_{10} \\
\frac{\partial b^{(0)}_0}{\partial x_2} &= -c_{20}
\end{align*}
\] (30)

and similar expressions for the other $\frac{\partial b^{(0)}_{\alpha\beta}}{\partial x_i}$.

The system (30) has at least a solution if the following conditions are satisfied

\[
\begin{align*}
\frac{\partial (c_{00}-\gamma)}{\partial x_1} &= \frac{\partial (c_{01})}{\partial x_0} \\
\frac{\partial (c_{00}-\gamma)}{\partial x_2} &= \frac{\partial (c_{02})}{\partial x_0} \\
\frac{\partial (c_{00}-\gamma)}{\partial x_2} &= \frac{\partial (c_{02})}{\partial x_1}.
\end{align*}
\]

Hence,

\[
\begin{align*}
\frac{\partial c_{00}}{\partial x_1} &= -\frac{\partial c_{10}}{\partial x_0} \\
\frac{\partial c_{00}}{\partial x_2} &= -\frac{\partial c_{20}}{\partial x_0} \\
\frac{\partial c_{10}}{\partial x_2} &= -\frac{\partial c_{20}}{\partial x_1}.
\end{align*}
\] (31)

By an analogous method, we get 15 remaining relations which are similar to the condition (31).

So, we have a total of 16 systems which have the form as (31).
In fact, \( h^{(\alpha)} = \sum_{i=0}^{3} c_{i\alpha} e^i \) are regular functions, then it leads to

\[
\begin{align*}
\frac{\partial c_{0\alpha}}{\partial x_0} - \frac{\partial c_{1\alpha}}{\partial x_1} - \frac{\partial c_{2\alpha}}{\partial x_2} &= 0 \\
\frac{\partial c_{0\alpha}}{\partial x_1} + \frac{\partial c_{1\alpha}}{\partial x_0} + \frac{\partial c_{3\alpha}}{\partial x_2} &= 0 \\
\frac{\partial c_{0\alpha}}{\partial x_2} + \frac{\partial c_{2\alpha}}{\partial x_0} - \frac{\partial c_{3\alpha}}{\partial x_1} &= 0
\end{align*}
\]
\( \alpha = 0, 1, 2, 3. \) \hspace{1cm} (32)

From 16 above systems and from (32), after a calculation, we obtain

\[
\frac{\partial c_{\alpha\beta}}{\partial x_i} = 0, \quad \text{for all} \quad \alpha, \beta = 0, 1, 2, 3; i = 0, 1, 2.
\]

This implies \( c_{\alpha\beta} \) are real-constants.

Using this result and by (30), one gets

\[
b^{(0)}_{00} = (c_{00} - \gamma)x_0 - c_{10}x_1 - c_{20}x_2 + c^{(0)}_{00},
\]

where \( c^{(0)}_{00} \) is an arbitrary-real-constant.

Analogously, we have expressions for the other \( b^{(0)}_{\alpha\beta} \). All of \( b^{(0)}_{\alpha\beta} \) are linear functions in \( x_0, x_1, x_2 \).

So the construction of matrix \( B_0 \) is completed.

From the condition \( B_j = -A_jB_0, \ j=1,2 \), we can construct matrices \( B_1, B_2 \), easily.

**Remark 2.** If we consider the Cauchy-Fueter operator

\[
\mu = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_{12} \frac{\partial}{\partial x_3},
\]

then the equation \( \mu f = 0 \) is equivalent to

\[
\ell f = \sum_{i=0}^{3} A_i \frac{\partial f}{\partial x_i} = 0,
\]

where

\[
A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Because the Cauchy-Fueter operator has similar properties as Cauchy-Riemann operator, then the method (which is used in the Section 3 and 4) is valid for this case. (Note that we have to replace \( m = 2 \) by \( m + 1 = 3 \) into all formulae.)

Hence, we obtain the following result

\[
\begin{align*}
\frac{\partial B_0}{\partial x_0} &= -\frac{1}{2}(\gamma E - C) \\
\frac{\partial B_0}{\partial x_1} &= \frac{1}{2} A_k^{-1}(\gamma E - C) \\
B_j &= -A_jB_0, \quad k, j = 1, 2, 3.
\end{align*}
\]

In the same way as in 4.2, it leads to

\[
\frac{\partial c_{\alpha\beta}}{\partial x_i} = 0, \quad \alpha, \beta, i = 0, 1, 2, 3.
\]
After a calculation, we have

\[ b_{00}^{(0)} = \frac{1}{2}(c_{00} - \gamma)x_0 - \frac{1}{2}c_{10}x_1 - \frac{1}{2}c_{20}x_2 - \frac{1}{2}c_{30}x_3 + c_{00}^{(0)}, \]

where \( c_{00}^{(0)} \) is a real-arbitrary-constant, and similar expression for the other \( b_{\alpha\beta}^{(0)} \). All of \( b_{\alpha\beta}^{(0)} \) are linear functions in \( x_0, x_1, x_2, x_3 \).

Finally, using the formula \( B_j = -A_j B_0, \ j = 1, 2, 3 \), we can construct all remaining matrices \( B_j \).

5. Initial Value Problems with Regular Initial Functions

In view of the famous H. Lewy example (see [4]), there exist linear first order differential equations with infinitely differentiable coefficients not having any solutions. On the other hand, the technique of associated differential operator allows to construct operator \( L \) such that the initial value problem (1), (2) is solvable for each regular initial function \( \varphi \). Note that the components of regular functions are harmonic and, therefore, the necessary interior estimate is valid (see [8]) It follows from the Poisson Integral Formula.

At last, we get the following theorem.

Theorem 3. Suppose that the operator \( L \) is associated to the Cauchy-Riemann operator of the Clifford algebra. Suppose, further, that the initial function \( \varphi \) is an arbitrary regular function. Then the initial value problem (1), (2) is solvable. The solution \( u(t, x) \) is regular for each \( t \).

Acknowledgments. I would like to express my sincere thanks to Professor Le Hung Son and Professor Wolfgang Tutschke for their precious assistance and encouragement during the completion of this paper. I acknowledge my gratitude to Professor Le Dung Trang for his kind help. Support from the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, is also acknowledged.

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