ON TWO-LOOP SOLITON SOLUTION TO THE SCHÄFER-WAYNE SHORT-PULSE EQUATION USING HIROTA’S METHOD AND HODNETT-MOLONEY APPROACH

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MIRAMARE – TRIESTE
December 2006

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Abstract

A two-loop soliton solution to the Schäfer-Wayne short-pulse equation (SWSPE) is shown. The key step in finding this solution is to transform the independent variables in the equation. This leads to a transformed equation for which it is straightforward to find an explicit two-soliton solution using Hirota's method. The two-loop soliton solution to the SWSPE is then found in implicit form by means of a transformation back to the original independent variables. Following Hodnett and Moloney’s approach, some computations of the energy of the one- and two-soliton solutions are made.
1. Introduction

A remarkable development in our understanding of a certain class of nonlinear partial differential equations known as evolution equations has taken place in the past decade. The key to our present knowledge of these equations is the realization that they possess a special type of elementary solution. These special solutions take the form of localized disturbances, or pulses, that retain their shape even after interaction among themselves, and thus act somewhat like particles. This independence among elementary solutions is a well-known effect in processes governed by linear partial differential equations where a linear superposition principle applies but is quite unexpected when first observed in processes governed by nonlinear partial differential equations. These localized disturbances have come to be known as solitons.

The theory of solitons is attractive and exciting; it brings together many branches of mathematics, some of which touch on deep ideas. Several of its aspects are amazing and beautiful. The theory is nevertheless, related to even more areas of mathematics, and has more applications to the physical sciences. It has an interesting history and a promising future. Indeed, the work of Kruskal and his associates which has given us the ‘inverse scattering transform’ - a grand title for soliton theory - is a major achievement of twentieth century mathematics. Their work has been stimulated by a physical problem together with some surprising computational results.

In 1993, Rosenau and Hyman have introduced a class of solitary waves with compact support that are solutions of a KdV equation with nonlinear dispersion. Although the model is not integrable, the solutions have been found to emerge unaltered from collisions, and have thus been named compactons. Compact structures have also been found earlier in magnetic systems by Kosevich and coworkers. These findings have inspired the further investigation of the role of nonlinear dispersion, which is a common ingredient in a more realistic physical models. The novel feature is that the dispersion can vanish for a particular value of the amplitude and, as a consequence, gives rise to jumps in the slope of the solution. Apart from solutions without decaying tails, which is one way to account for solitons in systems of finite extent as found in nature, other forms of exotic solutions with discontinuous derivatives in the form of peaks and cusps have been found in various models with nonlinear dispersion. These nonanalytic solitons have been given equally exotic names like peakons, cuspons and tipons. Camassa and Holm, e.g., have derived an integrable shallow water model and found peakons, which have been proved to be stable. Nonanalytic solitons have also been found in a model of Heisenberg ferromagnets. Also, kinks with compact support have been reported and experimentally verified in a setting of coupled pendulums. Altogether, nonlinear dispersion does have an important impact on the properties of physical systems. Nonanalytical solitons can also appear in systems without nonlinear dispersion, but with nonlocal dispersion. The nonlocality can, in combination with a nonlinear term, effectively give rise to nonlinear dispersion and hence solutions with a jump in the derivative.

Among the class of solitary waves with compact support, there are loop solitons discovered in many systems such as stretched ropes, elastic beams under tension, modified KdV, elastica, and the Vakhnenko equation and recently in Schäfer-Wayne short-pulse equation. Recently, Schäfer and Wayne have derived the short pulse equation (SPE) as a model equation, an alternative to the nonlinear Schrödinger equation (NLSE), to approximate the
evolution of very short optical pulses in nonlinear media. More recently, Chung et al.\textsuperscript{26} have proved numerically that, as the pulse length shortens, the NLSE approximation becomes less accurate, whereas the SPE provides a better approximation to the solution of Maxwell’s equation. One year after, some of them have proved that the SWSPE is integrable by discovering a Lax pair of the SPE that is found to be of the Wadati-Konno-Ichikawa (WKI) type. This year, the same authors have derived some solitary waves solutions of the SPE\textsuperscript{27, 28} by using a blend of transformations that relate the SPE with the sine-Gordon equation. They have not shown some particular interest on loop-like solitary waves since the searched solutions have been shown to be single-valued. However, the SWSPE may be relevant in the sense that it may also describe a planar current-fed string\textsuperscript{29, 30} motion in an external magnetic field. The key step in showing this equivalence is to transform the independent variables in the SPE.

In section 2, the SWSPE is transformed into coupled equations that have a Hirota form. The one-loop soliton solution to the SWSPE is derived in section 3. The two-loop soliton solution to the SWSPE is constructed in section 4 where we use Hodnett and Mololey’s technique.\textsuperscript{31, 32} In section 5, some energy computations of the solitons are made. We end our paper with a conclusion.

2. Transformation of the SWSPE

The SWSPE is given by\textsuperscript{23, 24}

\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \]

(1)

where \( u \) is an observable, and \( x \) and \( t \) are space and time coordinates, respectively. The subscripts refer to the partial differentiation. Equation (1) may be written as follows

\[ \partial_x D u - u = 0, \]

(2)

where the symbol \( \partial \) denotes the partial differentiation, and

\[ D = \partial_t - \frac{1}{2} u^2 \partial_x. \]

(3)

We introduce new independent variables \( X \) and \( T \) defined by

\[ x = \Theta(X, T) := T - \frac{1}{2} \int_{-\infty}^{X} U^2 dX' + x_0, \]

(4a)

\[ t = X, \]

(4b)

where \( u(x, t) = U(X, T) \) and \( x_0 \) is a constant. From equation (4), it follows that

\[ \partial_X = D, \]

(5a)

\[ \partial_T = \phi \partial_x, \]

(5b)

where

\[ \phi = 1 - \int_{-\infty}^{X} U U_T dX', \]

(6)

so that

\[ \phi_X = -U U_T. \]

(7)
From equations (2)-(4), we obtain

\[ U_{XT} = \phi U, \quad \phi_X = -UU_T. \] (8a)

Introducing other variables \( \sigma \) and \( \tau \) such that

\[ X = \frac{1}{2}(\sigma - \tau), \quad T = \frac{1}{2}(\sigma + \tau), \] (9a)\( (9b) \)

and the ansatz \( \phi = Z_\sigma + Z_\tau \), we get

\[ U_{\sigma\sigma} - U_{\tau\tau} = U(Z_\sigma + Z_\tau), \] (10a)\( (10b) \)

\[ Z_{\sigma\sigma} - Z_{\tau\tau} = -U(U_\sigma + U_\tau), \] (10b)\( (10c) \)

where \( Z \) is an arbitrary function. We assume that \( U \) and its spatial derivatives vanish as \( |\sigma| \to +\infty \). Before looking for loop soliton solutions, we give another physical meaning of equation (10). Indeed, recently, some authors\textsuperscript{16} have investigated several dispersionless systems and derived a generalized form of these equations as follows:

\[ S_{xt} - [S_x, [G, S]] = 0, \] (11)

where the matrix field \( S(x,t) = \phi_a(x,t)T^a \) and the constant \( G = k_aT^a \) are elements of an arbitrary Lie algebra with generators \( T^a \) of the Lie group satisfying the commutation relation \([T^a, T^b] = i\epsilon^{abc}T^c\). Here, \( \epsilon^{abc} \) denotes the structure constant of the Lie algebra. Using a set of suitable transformations, equation (11) has been transformed to the following string equation\textsuperscript{16}

\[ \mathbf{r}_{\tau\tau} - \mathbf{r}_{\sigma\sigma} = \left( \mathbf{r}_\tau + \mathbf{r}_\sigma \right) \times (\mathbf{J} \times \mathbf{r}), \] (12)

where \( \mathbf{r}(X,Y,Z) \) is the position vector, \( \mathbf{J} \) the external electric current, and \( \tau = x - t \) and \( \sigma = x + t \), the time and string length, respectively. Equation (12) describes the motion of a 'charged' object in an external magnetic field \( \mathbf{B} = (\mathbf{J} \times \mathbf{r}) \). We easily see that equations (10) and (12) are equivalent if we take \( \mathbf{J} = (0,0,1), Y = 0, \) and \( U = X \).

Now, we consider the bilinearization of these equations to obtain solutions under the boundary conditions \( U \to 0 \) and \( Z \to \sigma \) as \( |\sigma| \to \infty \). The bilinear equations for the dispersionless equations (see equation (10)) have been first derived by Alagesan and Porsezian by the Painlevé analysis\textsuperscript{33,34} The other type of bilinear equations for the Pohlmeyer Lund-Regge equations have been derived by Hirota\textsuperscript{35} Unfortunately, the known solution of equation (12) does not satisfy the boundary conditions. We present the bilinear equations satisfying our physical situation and derive the soliton solutions.

Under the following transformation

\[ U = \frac{G}{F}, \] (13a)\( (13b) \)

\[ Z = \sigma + 2(\partial_\tau - \partial_\sigma) \ln F, \]
the bilinear forms of equation (10) are given by

\[ (D^2_t - D^2_\sigma + 1)(F \cdot G) = 0, \]  
\[ (D_\tau - D_\sigma)^2(F \cdot F) - \frac{1}{2}G^2 = 0, \]  

where \( D_s \) denotes the Hirota operator. According to the usual procedure, we expand \( F \) and \( G \) in a formal power series of \( \varepsilon \) as

\[ F = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \cdots, \]  
\[ G = \varepsilon g_1 + \varepsilon^3 g_3 + \cdots. \]

Indeed, the solution procedure for the SWSPE is as follows. We solve equation (14) for \( F \) and \( G \) using Hirota’s method and hence find the solution \( U(X, T) \). The solution to the SWSPE is then given in parametric form by

\[ u(x, t) = U(X, T), \]  
\[ x = \Theta(X, T), \]

where

\[ \Theta(X, T) = T + W(X, T) + x_0, \]

with

\[ W(X, T) = \frac{1}{2} \int_{-\infty}^{X} U^2 dX'. \]  

3. One-Loop Soliton Solution to the SWSPE

The solution to equation (10) corresponding to one-soliton is given by

\[ F = 1 + \exp(2\eta), \]  
\[ G = A \exp(\eta), \]

where \( \eta = k\sigma - \omega\tau + \beta \), and \( k, \omega, \beta \) and \( A \) are constants. The dispersion relation is given by

\[ \omega^2 - k^2 + 1 = 0, \]

and

\[ \eta = k(\sigma - c\tau) + \beta \quad \text{with} \quad c = \frac{\omega}{k}. \]

From equation (14), we get \( A = 4(k + \omega) \). We then derive

\[ U(\eta) = 2(\omega + k)sech(\eta), \]

and

\[ x = \frac{1}{2}(\sigma + \tau) - 2(k + \omega)[1 + \tanh(\eta)] + x_0, \]
\[ t = \frac{1}{2}(\sigma - \tau). \]
The one-loop soliton solution to equation (13) is given in the following way: since \( u_x = \phi^{-1}U_T \), together with equations (5), (6), (9), and (22), we get

\[
ux = \frac{k - \omega}{1 - \frac{k \omega}{2(k + \omega)}} U^2 U\eta. \tag{24}
\]

Thus, as \( \eta \) goes from \(-\infty\) to \(+\infty\), \( U\eta \) changes sign once and remains finite whereas \( u_x \) given by equation (24) changes sign three times and goes to infinity twice.

If we require symmetry in \( \sigma-\tau \) space; we take \( \beta = 0 \) and then, for symmetry in \( x-t \) space, we take \( x_0 = 2(k + \omega) \). In this case, the one-loop soliton solution may be written in terms of the parameter \( \eta \) through \( \sigma \) and \( \tau \) as

\[
U(\eta) = 2 \sqrt{\frac{1 + c}{1 - c}} \text{sech}(\eta), \tag{25a}
\]

\[
x = \frac{1}{2}(\sigma + \tau) - 2 \sqrt{\frac{1 + c}{1 - c}} \tanh(\eta), \tag{25b}
\]

\[
t = \frac{1}{2}(\sigma - \tau), \tag{25c}
\]

where \( \omega = ck \) and \( k = \frac{1}{\sqrt{1-c^2}} \).

Now let us look in more detail at the shape of the loop soliton. Let \( W \) be the maximum width of the loop and \( H \) be the height at which this occurs. Note that this will take place when \( \phi = 0 \). Furthermore, let \( h \) be the height at which the cross point occurs. This is all summarized in Fig. (1). We shall begin by considering \( \frac{h}{U_{\text{max}}} \). Indeed, \( U_{\text{max}} = 2 \sqrt{\frac{1 + c}{1 - c}} \). For simplicity and without loss of generality, let us consider the symmetric case as well as what happens at \( t = 0 \).
Hence, the crossover point will occur at $x = 0$. From equation (25), the solution $u(x, 0)$ can be expressed in parametric form, with parameter $\eta$, as

$$U(\eta) = U_{\text{max}} \text{sech}(\eta),$$  \hspace{1cm} (26a)

$$x = \frac{1}{2} U_{\text{max}} (\eta - 2 \tanh(\eta)).$$  \hspace{1cm} (26b)

Suppose $x = 0$ when $\eta = \eta_1$; then

$$2 \tanh(\eta_1) - \eta_1 = 0.$$  \hspace{1cm} (27)

Hence, from equation (26), when $\eta = \eta_1$,

$$\frac{h}{U_{\text{max}}} = \sqrt{1 - \frac{\eta_1^2}{4}}.$$  \hspace{1cm} (28)

We solve equation (27) numerically and then plot $h$ vs the phase velocity $c$. This is shown in Fig. 2. We shall now consider $\frac{H}{U_{\text{max}}}$ and $\frac{W}{c}$. Once again for simplicity, we shall consider the symmetric case. As already mentioned, to obtain $W$ and $H$, we have to consider $\phi = 0$ so that

$$\cosh(\eta) = \sqrt{2}.$$  \hspace{1cm} (29)

When $\eta$ satisfies equation (29), $U = H = \sqrt{\frac{(1+c)}{2(1-c)}}$. Therefore,

$$\frac{H}{U_{\text{max}}} = \frac{\sqrt{2}}{2}.$$  \hspace{1cm} (30)

A plot of $H$ vs $c$ is shown alongside $h$ vs $c$ in Fig. (2).
Finally, using equations (26) and (29), it can be shown that

\[
\frac{W}{k} = (1 + c) \ln(3 + 2\sqrt{2}).
\]  

A plot of \(W\) vs \(c\) as given by equation (31) is shown in Fig. (2).

We can see from Fig. (2) that the parameters \(H\), \(h\) and \(W\) increase with phase velocity. All of the above properties are observed in Fig. (3) as we look at the one-loop soliton solutions for \(c = -0.4, 0\) and 0.4.

4. Two-Loop Soliton Solution to the SWSPE

The solution to equation (10) corresponding to a two-soliton solution is given by

\[
F = 1 + \exp(2\eta_1) + \exp(2\eta_2) + B_{12} \exp(\eta_1 + \eta_2) + E_{12} \exp 2(\eta_1 + \eta_2),
\]

\[
G = A_1 \exp(\eta_1) + A_2 \exp(\eta_2) + C_1 \exp(2\eta_1 + \eta_2) + C_2 \exp(\eta_1 + 2\eta_2),
\]

where the coefficients are given by

\[
A_i = \pm 4(\omega_i + k_i), \quad (i = 1, 2),
\]

\[
B_{12} = \frac{A_1 A_2}{2(\omega_1 + \omega_2 + k_1 + k_2)},
\]

\[
C_i = -\frac{(\omega_i - \omega_j)^2 - (k_i - k_j)^2}{(\omega_i + \omega_j)^2 - (k_i + k_j)^2} A_j, \quad (i \neq j = 1, 2),
\]

\[
E_{ij} = \left\lfloor \frac{(\omega_i - \omega_j)^2 - (k_i - k_j)^2}{(\omega_i + \omega_j)^2 - (k_i + k_j)^2} \right\rfloor^2, \quad (i \neq j = 1, 2),
\]
and the phases by

$$\eta_i = k_i \sigma - \omega_i \tau, \quad (i = 1, 2), \quad (34)$$

with the dispersion relations

$$\omega_i^2 - k_i^2 = -1, \quad (i = 1, 2). \quad (35)$$

From equation (40), we can write

$$\omega_i = \frac{c_i}{\sqrt{1 - c_i^2}}, \quad (i = 1, 2), \quad (36a)$$

$$k_i = \frac{1}{\sqrt{1 - c_i^2}}, \quad (i = 1, 2). \quad (36b)$$

From equation (32), it is possible to use Hodnett-Moloney\textsuperscript{31,32} approach to get an approximated two-loop solution of equation (32). Indeed, according to the Moloney-Hodnett approach the scattering or the interaction between two single solitons can be approached to their "asymptotic behavior" treating each other as an "oblivious structure". To give some illustration, we consider the following expression:

$$P = 1 + \exp(2\eta), \quad (37)$$

from which the one-soliton $u$ is derived through $u = (\ln P)_\eta$. Thus, two single solutions $u_1$ and $u_2$ are equal if $P_1$ and $P_2$ are proportional.

Now, considering the following expression leading to the two-soliton solution

$$P = 1 + \exp(2\eta_1) + \exp(2\eta_2) + B \exp(\eta_1 + \eta_2) + C \exp 2(\eta_1 + \eta_2), \quad (38)$$

where $B$ and $C$ are real constants, the Hodnett-Moloney approach neglected the $B$-term in the limit $\eta_i \to \infty, \ i = 1, 2$. Hodnett and Moloney showed that equation (38) may be written as

$$P = [1 + \exp(2\eta_2)][1 + \exp(2g_1)], \ \text{for} \ \eta_2 = \text{constant and} \ \eta_1 \to \infty, \quad (39)$$

and

$$P = [1 + \exp(2\eta_1)][1 + \exp(2g_2)], \ \text{for} \ \eta_1 = \text{constant and} \ \eta_2 \to \infty, \quad (40)$$

with

$$g_i = \eta_i + \frac{1}{2} \ln \left( \frac{1 + C \exp(2\eta_j)}{1 + \exp(2\eta_j)} \right), \quad (i \neq j = 1, 2). \quad (41)$$

This result implies that the two-soliton solution approaches the following expression

$$u = u_1 + u_2, \quad (42)$$

where $u_i = 1 + \tanh(g_i), \ i = 1, 2$. It seems noteworthy to emphasize here that the Hodnett-Moloney approach is a particularly underlying method in the study of the scattering of $N$ single solitons.

Now, following this previous approach, equation (32a) is written in following form

$$F = 1 + \exp(2\eta_1) + \exp(2\eta_2) + b_{12}^2 \exp 2(\eta_1 + \eta_2), \quad (43)$$

with

$$b_{12} = -\frac{(\omega_1 - \omega_2)^2 - (k_1 - k_2)^2}{(\omega_1 + \omega_2)^2 - (k_1 + k_2)^2}. \quad (44)$$
Seemingly, it seems noteworthy to write down the expression of $Z$ following from equation (13b) as

$$ Z = \sigma - 2(\omega_1 + k_1)(1 + \tanh(g_1)) - 2(\omega_2 + k_2)(1 + \tanh(g_2)), \quad (45) $$

where

$$ g_i = \eta_i + \frac{1}{2} \ln \left( \frac{1 + b_i^2 \exp(2\eta_j)}{1 + \exp(2\eta_j)} \right), \quad (i \neq j = 1, 2). \quad (46) $$

Moreover, combining eqs. (7) and (10c), we derive

$$ U^2 = -2Z_X + \text{constant}. \quad (47) $$

Hence, Hodnett and Moloney in their approach suggest writing the two-soliton solution as

$$ U = 2 \sqrt{\frac{1 + c_1}{1 - c_1}} (\partial_X g_1)^{\frac{1}{2}} \text{sech}(g_1) + 2 \sqrt{\frac{1 + c_2}{1 - c_2}} (\partial_X g_2)^{\frac{1}{2}} \text{sech}(g_2), \quad (48) $$

where

$$ g_i = \eta_i + \frac{1}{2} \ln \left( \frac{1 + b_i^2 \exp(2\eta_j)}{1 + \exp(2\eta_j)} \right), \quad (i \neq j = 1, 2). \quad (49) $$

Coming back to the original independent variables,

$$ x = \frac{1}{2}(\sigma + \tau) - 2 \sqrt{\frac{1 + c_1}{1 - c_1}} \tanh(g_1) - 2 \sqrt{\frac{1 + c_2}{1 - c_2}} \tanh(g_2), \quad (50a) $$

$$ t = \frac{1}{2}(\sigma - \tau). \quad (50b) $$

Now, let us discuss the above two-loop soliton solution derived from the Hodnett-Moloney approach.\textsuperscript{31,32}

First, it is instructive to consider what occurs in $\sigma$-$\tau$ space. We assume that $c_1 > c_2$. Then we have the following:

$$ \sigma - c_2 \tau \to \pm \infty \quad \text{as} \quad \tau \to \pm \infty \quad \text{with} \quad \sigma - c_1 \tau \quad \text{fixed}, \quad (51) $$

and

$$ \sigma - c_1 \tau \to \mp \infty \quad \text{as} \quad \tau \to \pm \infty \quad \text{with} \quad \sigma - c_2 \tau \quad \text{fixed}, \quad (52) $$

From equations (48) and (49) with equation (51), it follows that, with $\sigma - c_1 \tau$ fixed,

$$ U_1 \sim 2 \sqrt{\frac{1 + c_1}{1 - c_1}} \text{sech}(\eta_1) \quad \text{as} \quad \tau \to -\infty, \quad (53a) $$

$$ U_1 \sim 2 \sqrt{\frac{1 + c_1}{1 - c_1}} \text{sech}(\eta_1 + \ln(b_{12})) \quad \text{as} \quad \tau \to +\infty. \quad (53b) $$

Similarly, from equations (48) and equation (49) with (52), it follows that

$$ U_2 \sim 2 \sqrt{\frac{1 + c_2}{1 - c_2}} \text{sech}(\eta_2 + \ln(b_{12})) \quad \text{as} \quad \tau \to -\infty, \quad (54a) $$

$$ U_2 \sim 2 \sqrt{\frac{1 + c_2}{1 - c_2}} \text{sech}(\eta_2) \quad \text{as} \quad \tau \to +\infty. \quad (54b) $$

Hence, it is apparent that, in the limits $\tau \to \pm \infty$, $U_1$ and $U_2$ may be identified as individual solitons moving with speeds $c_1$ and $c_2$, in the positive $\sigma$-direction, respectively.
The shifts $\Delta_i, (i = 1, 2)$, of the two solitons $U_1$ and $U_2$ in the positive $\sigma$-direction due to the interaction are

$$\Delta_1 = -\frac{\ln(b_{12})}{k_1}$$

$$\Delta_2 = \frac{\ln(b_{12})}{k_2},$$

respectively. As $\ln(b_{12}) < 0$, the smaller soliton is shifted backwards and the larger soliton is shifted forwards. There is familiarity with KdV ‘sech-squared’ solitons$^{38}$ where it is the larger soliton which overtakes the smaller one.

Now, let us consider what happens in $x$-$t$ space. First of all, it is worth showing that from the following equation

$$k_i \sigma \pm \omega_i \tau = (k_i \mp \omega_i)X + (k_i \pm \omega_i)T,$$

soliton solution $U(X, T)$ always moves to the left, i.e., in the negative $X$-direction. Now setting $v_i = \pm \frac{1}{c_i}$, and using equation (4), we have

$$x \pm v_i t = v_i (X \pm c_i T) + W(X, T) + x_0,$$

where $\pm$ refers to the positive and negative signs of $c_i$, respectively. Here, we will consider only the upper sign, and we note that taking the limits $T \rightarrow \pm \infty$ with $X + c_i T$ fixed is equivalent to taking the limits $X \rightarrow \mp \infty$. Accordingly, from equation (57) with $i = 1, 2$, we see that in the limits $t \rightarrow \pm \infty$ with $X + c_i T$ fixed, $U(X, T)$ and $x + v_i t$ are related by the parameter $X + c_i T$.

It follows that in the limits $t \rightarrow \pm \infty$, $u_1$ and $u_2$ may be identified as individual loop solitons moving with speeds $v_1$ and $v_2$ in the negative $x$-direction, respectively. The shifts, $\delta_i, (i = 1, 2)$,
Fig. 5. Interaction process for two-loop solitons with $c_1 = 0.5$ and $c_2 = 0.2$.

of the two-loop solitons $u_1$ and $u_2$ in the negative $x$-direction due to the interaction may be computed from equation (57).

Indeed,

$$X + c_i T = \frac{1}{2} [\sigma + c_i \tau + c_i (\sigma - v_i \tau)] , \ (i = 1, 2).$$

Thus, if $\sigma + c_i \tau$ is fixed, then $X + c_i T \to \mp \infty$ according to $\tau \to \pm \infty$. In addition, from the following equation

$$X + c_i T = X + c_j T + \frac{1}{2} (c_i - c_j) (\sigma + \tau) , \ (i \neq j = 1, 2),$$

since we chose $c_1 > c_2$, only $\sigma + c_2 \tau$ is fixed whereas $\sigma + c_1 \tau \to \mp \infty$ as $\tau \to \pm \infty$. This obviously leads to $X \mp \infty$ since $c_2 < 1$. Therefore, $\sigma - c_i \tau \to \mp \infty, \ (i, j = 1, 2)$ as $\tau \to \pm \infty$. Hence we get

$$x + v_1 t = v_1 (X + c_i T) \pm 2 (a_1 + a_2) \ \text{for} \ \ t \to \mp \infty.$$  

The shift $\delta_1$ is then given by

$$\delta_1 = 4 (a_1 + a_2),$$

with

$$a_1 = \sqrt{\frac{1 + c_1}{1 - c_1}},$$

$$a_2 = \sqrt{\frac{1 + c_2}{1 - c_2}}.$$ 

Equation (61) shows that the larger loop is always shifted forward during the interaction. The same result is to be retrieved if one considers instead the lower sign in equation (57).
Fig. 6. Interaction process for two-loop solitons with $c_1 = 0.5$ and $c_2 = -0.2$.

Now in some applications, we consider three cases: $c_i < 0$, $c_i > 0$ for $i = 1, 2$, and $c_2 < 0 < c_1$. We then get a two-loop soliton shape which moves to the left as presented in Figs. (4)-(6). Globally, the larger loop is shifted forward during the interaction and the smaller loop travels along the direction of the larger one but with an altered shape, before being ejected behind the larger loop. These three figures also illustrate the previous results concerning the three shape parameters of the one-loop soliton which increase with the phase velocity $c$. When we compare these three figures for a given time $t$, we see that the relative distances between the one-loop solitons are different. This is simply explained by their relative phase velocities given by the $c_i(i = 1, 2)$-values. We also note that at $t = 0$, Figs. (4) and (5) show the same shape of interaction where the small loop travels along the direction of the large one abruptly. In $\sigma-\tau$ space, this situation corresponds to solitons moving together either in the positive direction or in the negative direction. Those moving in opposite directions in this space are related to Fig. (6).

5. Energy Consideration

It is instructive to calculate the energy that may be associated with the solitons obtained here. An expression for the energy may be obtained by using the Hamiltonian density for the coupled dispersionless system. For this system, modelled by equation (11), it is shown$^{39}$ that the associated Lagrangian density is

$$\mathcal{L} = Tr \left( \frac{1}{2} S_x S_t - \frac{1}{3} G[S, [S_x, S]] \right),$$

where $\sigma = x + t$ and $\tau = x - t$. 

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Replacing $S$ and $G$ with their expressions, the Hamiltonian density $\mathcal{H}$ is then equal to

$$\mathcal{H} = -\frac{1}{3}k \cdot [\phi \times (\phi_x \times \phi)],$$  \hspace{1cm} (64)

where $\phi = \phi(X,Y,Z)$ and $k$ is a constant vector.

Now taking $Y = 0$ in our case, we get

$$\mathcal{H} = -\frac{1}{3}(U^2 Z_x - U U_x Z),$$  \hspace{1cm} (65)

where $U$ and $Z$ are given by equation (13). For the single-soliton solution given by equation (22), we find that

$$\mathcal{H} = \frac{1}{3} A [A - 2(\sigma - A) \tanh(\eta)] \text{sech}^2(\eta),$$  \hspace{1cm} (66)

with $A = 2 \sqrt{\frac{1+c}{1-c}}$ and $\eta = k \sigma - \omega \tau$. The energy $E = \int_{-\infty}^{+\infty} d\eta \mathcal{H}$ is derived to

$$E = \frac{2}{3} A \left( A - \frac{1}{k} \right),$$  \hspace{1cm} (67)

where $k = (1-c^2)^{-\frac{1}{2}}$. This expression is reduced as

$$E = \frac{4}{3} \frac{(1+c)^2}{1-c}.$$  \hspace{1cm} (68)

A plot of the energy $E$ vs $c$ is shown in Fig. (7). It shows that the energy of the single soliton increases with $c$ or instead, decreases with its phase velocity $V$ in the plane $x$-$t$. We recall here that $V$ is given by

$$V = \frac{1+c}{1-c}.$$  \hspace{1cm} (69)
Now, let us compute the energy of the two-loop soliton. We make use of the following idea: the previous system of interest is free from all dissipation. Hence, one may expect a conservation of its energy. The previous two-loop soliton solutions plots have shown that asymptotically, they behave as single moving solitons. Then following the Hodnett-Moloney approach, the Hamiltonian density of the two-soliton solution may be written as

$$\mathcal{H} = \frac{1}{3} A_1 [A_1 - 2(\sigma - A_1) \tanh(\eta_1)] \text{sech}^2(\eta_1) + \frac{1}{3} A_2 [A_2 - 2(\sigma - A_2) \tanh(\eta_2)] \text{sech}^2(\eta_2),$$  \hspace{1cm} (70)

where $A_1 = 2 \sqrt{\frac{1 + c_1}{1 - c_1}}$ and $A_2 = 2 \sqrt{\frac{1 + c_2}{1 - c_2}}$. The energy of the two-soliton solution is then obtained by

$$E = \frac{4}{3} \left(1 + c_1\right)^2 + \frac{4}{3} \left(1 + c_2\right)^2,$$ \hspace{1cm} (71)

which is a decreasing quantity with phases $V_1$ and $V_2$. In other words, to minimize the energy rate of a bulk of loop solitons that are of interest to us, their absolute velocity phases should be large enough.

6. Conclusion

We have investigated a partial differential equation that approximates solutions of Maxwell’s equations describing the propagation of ultra-short optical pulses in nonlinear media. It has been shown that this equation can also model the dynamics of a current-fed string within a magnetic field. We have found the two-loop soliton solution of the above equation by using a blend of transformations and Hirota’s method, and we have made some computations of their energy. Moloney and Hodnett’s technique has been particularly underlying both for searching for two-loop solitons and for describing the interactions between single loops. The procedure can also be used to find the $N$-loop soliton solutions for $N > 2$. This, together with a detailed investigation of the case $N = 3$, will be reported elsewhere.

Acknowledgments

The T.B.B. and T.C.K. acknowledge partial financial support from the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. This work was done within the framework of the Associateship Scheme of ICTP.

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