United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

QUATERNIONIC CONTACT EINSTEIN STRUCTURES
AND THE QUATERNIONIC CONTACT YAMABE PROBLEM

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MIRAMARE – TRIESTE
December 2006
Abstract

The paper is a study of the conformal geometry of quaternionic contact manifolds with the associated Biquard connection. We give a partial solution of the quaternionic contact Yamabe problem on the quaternionic sphere. It is shown that the torsion of the Biquard connection vanishes exactly when the trace-free part of the horizontal Ricci tensor of the Biquard connection is zero and this occurs precisely on 3-Sasakian manifolds. In particular, the scalar curvature of the Biquard connection with vanishing torsion is a global constant. We consider interesting classes of functions on hypercomplex manifold and their restrictions to hypersurfaces. We show a ’3-Hamiltonian form’ of infinitesimal automorphisms of quaternionic contact structures and transformations preserving the trace-free part of the horizontal Ricci tensor of the Biquard connection. All conformal deformations sending the standard flat torsion-free quaternionic contact structure on the quaternionic Heisenberg group to a quaternionic contact structure with vanishing trace-free part of the horizontal Ricci tensor of the Biquard connection are explicitly described.
1. Introduction

The ideas and the solutions surrounding the Riemannian [1] and CR Yamabe problems [1, 2, 3, 4] have been a fruitful subject in geometry and analysis. A major step in the solution is the understanding of the conformally flat cases. A model for this setting is given by the corresponding spheres, or equivalently, the Heisenberg groups with, respectively, 0-dimensional and 1-dimensional centers. The equivalence is established through the Cayley transform [K], [CDKR1] and [CDKR2], which in the Riemannian case is the usual stereographic projection.

In the present paper we consider the Yamabe problem on the quaternionic Heisenberg group (three dimensional center), which turns out to be equivalent to the quaternionic contact Yamabe problem on the unit $(4n+3)$-dimensional sphere in the quaternionic space due to the quaternionic Cayley transform which is a quaternionic contact conformal deformation (see the proof of Theorem 1.2).

The central notion is the quaternionic contact structure (QC structure for short) introduced by O. Biquard in [Biq1, Biq2] which appears naturally as the conformal boundary at infinity of quaternionic hyperbolic space, see also [GL] and [FG]. Namely, a QC structure $(\eta, Q)$ on a $(4n+3)$-dimensional smooth manifold $M$ is a codimension 3 distribution $H$, such that, at each point $p \in M$ the nilpotent Lie algebra $H_p \oplus (T_p M/H_p)$ is isomorphic to the quaternionic Heisenberg algebra $\mathbb{H}^m \oplus Im \mathbb{H}$, which is equivalent to the existence of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, such that, $H = Ker \eta$ and the three 2-forms $d\eta|_H$ are the fundamental 2-forms of a quaternionic structure $Q$ on $H$. A special phenomena here, noted by Biquard [Biq1], is that the 3-contact form $\eta$ determines the quaternionic structure as well as the metric on the horizontal bundle in a unique way. Of crucial importance is the existence of a distinguished linear connection, see [Biq1], preserving the QC structure and their Ricci tensor and scalar curvature $Scal$, defined in (3.35), and called correspondingly qc-Ricci tensor and qc-scalar curvature. The Biquard connection will play a role similar to the Tanaka-Webster connection [W] and [T] in the CR case.

The quaternionic contact Yamabe problem, in the considered setting, is about the possibility of finding in the conformal class of a given QC structure one with constant qc-scalar curvature.

The question reduces to the solvability of the Yamabe equation (5.14). As usual if we take the conformal factor in a suitable form the gradient terms in (5.14) can be removed and
one obtains the more familiar form of the Yamabe equation. In fact, taking the conformal factor of the form \( \bar{\eta} = u^{1/(n+1)} \eta \) reduces (5.14) to the equation
\[
\mathcal{L} u \equiv 4 \frac{n+2}{n+1} \Delta_H u - u \text{Scal} = -u^{2^* - 1} \text{Scal},
\]
where \( \Delta_H \) is the horizontal sublaplacian and \( \text{Scal} \) and \( \bar{\text{Scal}} \) are the qc-scalar curvatures correspondingly of \((M, \eta)\) and \((M, \bar{\eta})\), and
\[
2^* = \frac{2Q}{Q - 2},
\]
with \( Q = 4n + 6 \). In the case of the quaternionic Heisenberg group, cf. Section 4.1, the equation is
\[
(1.1) \quad \mathcal{L} u \equiv \sum_{\alpha=1}^{n} (T^2_\alpha u + X^2_\alpha u + Y^2_\alpha u + Z^2_\alpha u) = -\frac{n+1}{4(n+2)} \bar{S} u^{2^* - 1}.
\]
This is also, up to a scaling, the Euler-Lagrange equation of the non-negative extremals in the \( L^2 \) Folland-Stein embedding theorem \[FG\] and \[FS\], see \[GV1\] and \[Va2\]. On the other hand, on a compact quaternionic contact manifold \( M \) with a fixed conformal class \( [\eta] \) the Yamabe equation characterizes the non-negative extremals of the Yamabe functional defined by
\[
\Upsilon([\eta]) = \inf \{ \int_M (4 \frac{n+2}{n+1} |\nabla_H u|^2 + S u^2) dv_g : \int_M |u|^{2^*} dv_g = 1 \}.
\]

When the Yamabe constant is less than that of the sphere the existence of solutions can be constructed with the use of suitable normal coordinates see \[Wei\] and \[JL2, JL3, JL4\].

Our goal is to solve the Yamabe problem in the most difficult case when the Yamabe constant \( \Upsilon([\eta]) \) is equal to the Yamabe constant of the unit sphere, with its standard quaternionic contact structure, in the quaternion \((n+1)\)-dimensional space. Since here we are concerned mainly with the case of the sphere or the quaternionic Heisenberg group, let us note that according to \[GV2\] the extremals of the above variational problem are \( \mathcal{C}^\infty \) functions, so we will not consider regularity questions in this paper. Furthermore, according to \[Va1\] or \[Va2\] the infimum is achieved, and it is a solution of the Yamabe equation.

In this paper we provide a partial solution of the Yamabe problem on the quaternionic sphere with its standard contact quaternionic structure. Let us observe that \[GV2\] solves the same problem in a more general setting, but under the assumption that the solution is invariant under a certain group of rotation. If one is on the flat models, i.e., the groups of Iwasawa type \[ODKRI\] the assumption in \[GV2\] is equivalent to the a-priori assumption
that, up to a translation, the solution is radial with respect to the variables in the first layer. The proof goes on by using the moving plane method and showing that the solution is radial also in the variables from the center, after which a very non-trivial identity is used to determine all cylindrical solutions. In this paper the a-priori assumption is of different nature, see further below, and the method has the potential of solving the general problem.

Our strategy, following the steps of [LP] and [JL3] is to solve the Yamabe problem on the quaternionic sphere by replacing the non-linear Yamabe equation by an appropriate geometrical system of equations which could be solved.

Our first observation is that if the qc-Ricci tensor is trace-free (qc-Einstein condition) then the qc-scalar curvature is constant (Theorem 4.9). Studying conformal deformations of QC structures preserving the qc-Einstein condition, we describe explicitly all global functions on the quaternionic Heisenberg group sending conformally the standard flat QC structure to another qc-Einstein structure. Our second main result is the following Theorem.

**Theorem 1.1.** Let $\Theta = \frac{1}{2h} \tilde{\Theta}$ be a conformal deformation of the standard qc-structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $G(\mathbb{H})$. If $\Theta$ is also qc-Einstein, then up to a left translation the function $h$ is given by

$$h = c \left[ (1 + \nu |q|^2)^2 + \nu^2 (x^2 + y^2 + z^2) \right],$$

for any $c$ and $\nu$ positive constants.

The crucial observation reducing the Yamabe equation to the system preserving the qc-Einstein condition is Proposition 8.2 which asserts that, under some "extra" conditions, QC structure with constant qc-scalar curvature obtained by a conformal transformation of a qc-Einstein structure on compact manifold must be again qc-Einstein. The prove of this relies on detailed analysis of the Bianchi identities for the Biquard connection. Using the quaternionic Cayley transform combined with Theorem 1.1 lead to our main result.

**Theorem 1.2.** Let $\tilde{\eta} = \frac{1}{2h} \eta$ be a conformal deformation of the standard qc-structure $\tilde{\eta}$ on the quaternionic sphere $S^{4n+3}$. Suppose $\eta$ has constant qc-scalar curvature. Then any one of the following two conditions

i) the vertical space of $\eta$ is integrable,

ii) the function $\frac{1}{h}$ is the real part of an anti-CRF function,
implies that $\eta$ is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism.

The solutions we find agree with those conjectured in [GV1]. We hope to remove the "extra" assumptions in Theorem 1.2 in a subsequent paper. The results of the present paper will be instrumental for the completion of the project.

Studying the geometry of the Biquard connection, our main geometrical tool towards understanding the geometry of the Yamabe equation, we show that the qc-Einstein condition is equivalent to the vanishing of the torsion of Biquard connection and we characterize locally these spaces in our third main result.

**Theorem 1.3.** Let $(M^{4n+3}, \eta, Q)$ be a QC manifold with non-zero qc scalar curvature $\text{Scal} \neq 0$. The next conditions are equivalent:

- a) $(M^{4n+3}, \eta, Q)$ is qc-Einstein manifold;
- b) $M$ is locally 3-Sasakian in the sense that locally there exists a $\text{SO}(3)$-matrix $\Psi$ with smooth entries, such that, the local QC structure $(\frac{16n(n+2)}{\text{Scal}} \Psi \cdot \eta, Q)$ is 3-Sasakian;
- c) The torsion of the Biquard connection is identically zero.

In particular, a qc-Einstein manifold is Einstein manifold with positive Riemannian scalar curvature and if complete it is compact with finite fundamental group.

In the paper we also develop useful tools necessary for the understanding geometry and analysis of QC structures.

Another theme concerns some special functions, which will be relevant in the geometric analysis on quaternionic contact and hypercomplex manifolds as well as properties of infinitesimal automorphisms of QC structures.

**Organization of the paper:** In the subsequent two chapters we describe in details the notion of a quaternionic contact manifold, abbreviate sometimes to QC-manifold, and the Biquard connection, which is central to the paper.

In Chapter 4 we write explicitly the Bianchi identities and derive a system of equations satisfied by the divergences of some important tensors. As a result we are able to show that qc-Einstein manifolds, i.e., manifolds for which the restriction to the horizontal space of the qc-Ricci tensor is proportional to the metric, have constant scalar curvature, see Theorem 4.9. The proof uses Theorem 4.8 in which we derive a relation between the horizontal divergences of certain $Sp(n)Sp(1)$-invariant tensors. By introducing an integrability condition
on the horizontal bundle we define hyperhermitian contact structures, see Definition 4.14, and with the help of Theorem 4.8 we prove Theorem 1.3.

Chapter 5 describes the conformal transformations preserving the qc-Einstein condition. Note that here a conformal quaternionic contact transformation between two quaternionic contact manifold is a diffeomorphism $\Phi$ which satisfies

$$\Phi^* \eta = \mu \Psi \cdot \eta,$$

for some positive smooth function $\mu$ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$. One defines in an obvious manner a point-wise conformal transformation. Let us note that the Biquard connection does not change under rotations as above, i.e., the Biquard connection of $\Psi \cdot \eta$ and $\eta$ coincides. In particular, when studying conformal transformations we can consider only transformations with $\Phi^* \eta = \mu \cdot \eta$. We find all conformal transformations preserving the qc-Einstein condition on the quaternionis Heisenberg group or, equivalently, on the quaternionic sphere with their standard contact quaternionic structures proving Theorem 1.1.

Chapter 6 concerns a special class of functions, which we call anti-regular, defined respectively on the quaternionic space, real hyper-surface in it, or on a quaternionic contact manifold, cf. Definitions 6.6 and 6.15 as functions preserving the quaternionic structure. The anti-regular functions play a role somewhat similar to those played by the CR functions, but the analogy is not complete. The real parts of such functions will be also of interest in connection with conformal transformation preserving the qc-Einstein tensor and should be thought of as generalization of pluriharmonic functions. Let us stress explicitly that regular quaternionic functions have been studied extensively, see [S] and many subsequent papers, but they are not as relevant for the considered geometrical structures. Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in [CL1, CL2, LZ] in a different context, namely in connection with minimal surfaces and quaternionic maps between quaternionic Kähler manifolds. The notion of hypercomplex contact structures will appear in this section again since on such manifolds the real part of anti-CRF functions, see (6.27) for the definition, have some interesting properties, cf. Theorem 6.20.

In Chapter 7 we study infinitesimal automorphisms of QC structures (QC-vector fields) and show that they depend on three functions satisfying some differential conditions thus established '3-hamiltonian’ form of the QC-vector fields (Proposition 7.8) which takes especially simple expression on 3-Sasakian manifolds. We characterized the vanishing of the
torsion of Biquard connection in terms of existence of three vertical vector fields whose flow preserves the metric and the quaternionic structure. Among them, 3-Sasakian are exactly those admitting three transversal QC-vector fields.

In the last section we complete the proof of our main result Theorem 1.2.

2. Quaternionic contact structures and Biquard connection

The notion of Quaternionic Contact Structure has been introduced by O. Biquard in [Biq1] and [Biq2]. Namely, a quaternionic contact structure (QC structure for short) on a \((4n+3)\)-dimensional smooth manifold \(M\) is a codimension 3 distribution \(H\), such that, at each point \(p \in M\) the nilpotent step two Lie algebra \(H_p \oplus (T_p M / H_p)\) is isomorphic to the quaternionic Heisenberg algebra \(\mathbb{H}^n \oplus Im \mathbb{H}\). The nilpotent Lie algebra structures on \(H_p \oplus (T_p M / H_p)\) is defined by

\[
[V_1, V_2] = \begin{cases} 
\pi_{T_p M / V_p} [\tilde{V}_1, \tilde{V}_2], & \text{if } V_1, V_1 \in H_p \\
0, & \text{otherwise},
\end{cases}
\]

where \(\tilde{V}_1, \tilde{V}_2\) are two vector fields, such that, \(\tilde{V}_j(p) = V_j, j = 1, 2\). The quaternionic Heisenberg algebra structure on \(\mathbb{H}^n \oplus Im \mathbb{H}\) is obtained by the identification of \(\mathbb{H}^n \oplus Im \mathbb{H}\) with the algebra of the left invariant vector fields on the quaternionic Heisenberg group, see Section 5.2. In particular, the Lie bracket is given by the formula

\[
[(q_o, \omega_o), (q, \omega)] = 2 Im \ q_o \cdot \bar{q},
\]

where \(q = (q^1, q^2, \ldots, q^n)\), \(q_o = (q_o^1, q_o^2, \ldots, q_o^n) \in \mathbb{H}^n\) and \(\omega, \omega_o \in Im \mathbb{H}\) with

\[
q_o \cdot \bar{q} = \sum_{\alpha=1}^{n} q_o^\alpha \cdot \bar{q}^\alpha,
\]

see Section 6.1.1 for notations concerning \(\mathbb{H}\). It is important to observe that if \(M\) has a quaternionic contact structure as above then the definition implies that the distribution \(H\) and its commutators generate the tangent space at every point.

One can see that the above considerations are equivalent to the following Definition.

**Definition 2.1.** [Biq2] Let \(M\) be a smooth manifold of dimension \(4n + 3\). A quaternionic contact structure (QC structure for short) is a distribution \(H\) of codimension three locally given by a 1-form \(\eta = (\eta_1, \eta_2, \eta_3)\) with values in \(\mathbb{R}^3\), \(H = Ker \eta\), such that,
i) there exists a metric tensor $g$ on $H$; 
ii) a quaternionic structure $Q = (I_s), s = 1, 2, 3$ on $H$, i.e. there are three local almost complex structures $I_s : H \hookrightarrow H$, $s = 1, 2, 3$, satisfying the commutation relations of the imaginary quaternions $I_1I_2 = -I_2I_1 = I_3$ and $I_s^2 = -1$, such that, 

\[(2.1) \quad g(I_sX,Y) = \frac{1}{2} d\eta_s(X,Y), \quad s = 1, 2, 3, \quad X, Y \in H.\]

**Remark 2.2.** Let us note explicitly, that in this paper for a one form $\theta$ we use 

\[d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta([X,Y]).\]

**Remark 2.3.** If $H$ is formally integrable with respect to $I_1, I_2, I_3$ we have the notion of local hyperhermitian contact structure (see Definition 4.1). 

If, in definition 2.1 the 1-form $\eta$ (the contact form for short) and the almost complex structures $I_s, s = 1, 2, 3$ are globally defined on $M$ and $H$, respectively, we have the notion of almost hyperhermitian contact structure. If in addition $H$ is formally integrable with respect to each $I_s, s = 1, 2, 3$ then we have hyperhermitian contact structure. In particular, the standard QC structures on the quaternionic sphere $S^{4n+3}$ and on the quaternionic Heisenberg group $G(\mathbb{H})$ are hyperhermitian contact. In the quaternionic contact Yamabe problem on $S^{4n+3}$ and on $G(\mathbb{H})$ we actually deal with hyperhermitian contact structures.

Clearly, if we rotate the $\mathbb{R}^3$-valued contact form and the almost complex structures by the same rotation we obtain again a contact form, almost complex structures and a metric (the latter is unchanged) satisfying the above conditions. On the other hand, it is important to observe that given a contact form the almost complex structures and the horizontal metric are unique if they exist. Finally, if we are given the horizontal bundle and a metric on it, there exists at most one sphere of associated contact forms with a corresponding sphere $Q$ of almost complex structures. This is the content of the next Lemma.

**Lemma 2.4.** [Biq1]

a) If $(\eta, I_s, g)$ and $(\eta, I'_s, g')$ are two QC structures on $M$, then $I_s = I'_s$, $s = 1, 2, 3$ and $g = g'$.

b) If $(\eta, g)$ and $(\eta', g')$ are two QC structures on $M$ with $\text{Ker}(\eta) = \text{Ker}(\eta') = H$ then $Q = Q'$ and $\eta' = \Psi \eta$ for some matrix $\Psi \in SO(3)$ with smooth functions as entries.
Proof. a) Let us fix a basis \( \{e_1, \ldots, e_{4n}\} \) of \( H \). Suppose the tensors \( g, d\eta_1|_H, d\eta_2|_H, d\eta_3|_H, I_1, I_2, I_3, g', I_1', I_2', I_3' \) (tensors on \( H \)) are given in local coordinates, respectively, by the matrices \( G, N_1, N_2, N_3, J_1, J_2, J_3, G', J_1', J_2', J_3' \in GL(4n) \). From (2.1) it follows

\[
GJ_s = N_s = G'J'_s, \quad s = 1, 2, 3.
\]

Let \((i, j, k)\) be any cyclic permutation of \((1, 2, 3)\). Using (2.2) we compute that

\[
J_k = J_i J_j = -J_i^{-1} G^{-1} G J_j = -(GJ_i)^{-1}(GJ_j) = -N_i^{-1} N_j = \quad (G'J_i')^{-1}(G'J_j') = J_i'J_j' = J_k'.
\]

Hence \( I_s = I'_s, s = 1, 2, 3, \) and \( g = g' \).

b) The condition \( \text{Ker}(\eta) = \text{Ker}(\eta') = H \) implies that

\[
\eta'_k = \sum_{l=1}^{3} \Psi_{kl} \eta_l, \quad k = 1, 2, 3
\]

for some matrix \( \Psi \in GL(3) \) with smooth functions \( \Psi_{ij} \) as entries. Applying the exterior derivative in (2.4) we find

\[
d\eta'_k = d\eta_{kl} \wedge \eta_l + \Psi_{kl} d\eta_l, \quad k = 1, 2, 3.
\]

Let the \( H \) tensors \( I_k \) and \( I'_k \) be defined as usual with (2.1) using respectively \( \eta \) and \( \eta' \). Restricting the equation (2.5) to \( H \) and using the metric tensor \( g \) on \( H \) we have

\[
g(I_k X, Y) = \Psi_{kl} g(I_l X, Y), \quad X, Y \in H
\]
or the equivalent equations \( I_k = \Psi_{kl} I_l \) on \( H \). It is easy to see that this is possible if and only if \( \Psi \in SO(3) \).

\[\square\]

Besides the non-uniqueness due to the action of \( SO(3) \), the 1-form \( \eta \) can be changed by a conformal factor, in the sense that if \( \eta \) is a form for which we can find associated almost complex structures and metric \( g \) as above, then for any \( \Psi \in SO(3) \) and a positive function \( \mu \), the form \( \mu \Psi \eta \) also has an associated complex structures and metric. In particular, when \( \mu = 1 \) we obtain a whole unit sphere of contact forms, and we shall denote, as already mentioned, by \( Q \) the corresponding sphere bundle of associated triples of almost complex structures.

We recall the definition of the Lie group \( Sp(n)Sp(1) \). Identify \( \mathbb{H}^n = \mathbb{R}^{4n} \) and let \( \mathbb{H} \) acts on \( \mathbb{H}^n \) by right multiplications. This defines antihomomorphism
\[ \lambda : \{ \text{unit quaternions} \} \to SO(4n) \]

with the convention that \( SO(4n) \) acts on \( \mathbb{H}^n \) on the left. The image is the Lie group \( Sp(1) \) and let \( \lambda(i) = I_0, \lambda(j) = J_0, \lambda(k) = K_0 \). The Lie algebra of \( Sp(1) \) is \( sp(1) = \text{span} \{ I_0, J_0, K_0 \} \). The group \( Sp(n) = \{ O \in SO(4n) : OB = BO \text{ for all } B \in Sp(1) \} \).

Denote \( Sp(n)Sp(1) \) the product of the two groups in \( SO(4n) \). Abstractly, \( Sp(n)Sp(1) = (Sp(n) \times Sp(1))/\mathbb{Z}_2 \). The Lie algebra of the group \( Sp(n)Sp(1) \) is \( sp(n) \oplus sp(1) \).

Any endomorphism \( \Psi \) of \( H \) can be decomposed with respect to the quaternionic structure \( (\mathbb{Q}, g) \) uniquely into \( Sp(n) \)-invariant parts as follows

\[ \Psi = \Psi^{+++} + \Psi^{++} - \Psi^{-+} + \Psi^{--}, \]

where \( \Psi^{+++} \) commutes with all three \( I_i \), \( \Psi^{++} \) commutes with \( I_1 \) and anti-commutes with the others two and etc. Explicitly, we have,

\[
\begin{align*}
4\Psi^{+++} &= \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3, \\
4\Psi^{++} &= \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3, \\
4\Psi^{-+} &= \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3, \\
4\Psi^{--} &= \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3.
\end{align*}
\]

The two \( Sp(n)Sp(1) \)-invariant components are given by

\[ \Psi_3 = \Psi^{+++}, \quad \Psi_{-1} = \Psi^{++} - \Psi^{-+} + \Psi^{--}. \]

Denoting the corresponding \((0,2)\) tensor via \( g \) by the same letter one gets that the \( Sp(n)Sp(1) \)-invariant components are the projections on the eigenspaces of the Kasimir operator

\[ \Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3 \]

corresponding, respectively, to the eigenvalues 3 and \(-1\), see [CSal].

There exists a canonical connection compatible with a given quaternionic contact structure. This connection was discovered by O. Biquard [Biq1] when the dimension \((4n+3) > 7\) and by D. Duchemin [D] in the 7-dimensional case. In this paper we consider only the case of dimensions strictly bigger then 7. The next result due to O. Biquard is crucial in the quaternionic contact geometry.

**Theorem 2.5.** [Biq1] Let \((M^{4n+3}, \eta, \mathbb{Q})\) be a quaternionic contact structure on a \(4n+3 \) \( > \) 7-dimensional smooth manifold with a fixed metric \( g \) on \( H \) in the conformal class \([g]\). Then there exists a unique connection \( \nabla \) with torsion \( T \) on \( M^{4n+3} \) and a unique supplementary subspace \( V \) to \( H \) in \( TM \), such that:
i) $\nabla$ preserves the decomposition $H \oplus V$ and the metric $g$;

ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_V$;

iii) $\nabla$ preserves the $Sp(n)Sp(1)$-structure on $H$, i.e., $\nabla g = 0$ and $\nabla Q \subset Q$;

iv) for $\xi \in V$, the endomorphism $T(\xi, \cdot)|_H$ of $H$ lies in $(sp(n) \oplus sp(1))^\perp \subset so(4n)$;

v) the connection on $V$ is induced by the natural identification $\varphi$ of $V$ with the subspace $sp(1)$ of the endomorphisms of $H$, i.e. $\nabla \varphi = 0$.

In (iv) the inner product on $End(H)$ is given by

\begin{equation}
(2.10) \quad g(A, B) = tr(B^*A) = \sum_{a=1}^{4n} g(A(e_a), B(e_a)),
\end{equation}

where $A, B \in End(H)$, $\{e_1, ..., e_{4n}\}$ is some $g$-orthonormal basis of $H$.

We shall call the above connection the Biquard connection and its torsion endomorphism $T(\xi, \cdot)|_H$ evaluated on $H$ we call the torsion of the quaternionic contact structure. Biquard [Biq1] also described the supplementary subspace $V$ explicitly, namely $V$ is generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

\begin{equation}
(2.11) \quad \eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \cdot d\eta_s)|_H = 0,
\end{equation}

\begin{equation}
(\xi_s \cdot d\eta_k)|_H = -(\xi_k \cdot d\eta_s)|_H.
\end{equation}

The vector fields $\xi_1, \xi_2, \xi_3$ are called Reeb vector fields or fundamental vector fields.

Notice that equations (2.11) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend $g$ to a metric on $M$ by requiring

\begin{equation}
(2.12) \quad \text{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H \text{ and } g(\xi_s, \xi_k) = \delta_{sk}.
\end{equation}

The extended metric does not depend on the action of $SO(3)$ on $V$, but it changes in an obvious manner if $\eta$ is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on $TM, \nabla g = 0$. We shall also extend the quaternionic structure by setting $I_s|_V = 0$.

Suppose $\{\xi_1, \xi_2, \xi_3\}$ are fixed. The restriction of the torsion of the Biquard connection to the vertical space $V$ satisfies [Biq1]

\begin{equation}
(2.13) \quad T(\xi_i, \xi_j) = \lambda \xi_k - [\xi_i, \xi_j]|_H,
\end{equation}

where $\lambda$ is a smooth function on $M$. Here and further $\{i, j, k\}$ denote a cyclic permutation of $\{1, 2, 3\}$. 

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The properties of the Biquard connection are encoded in the properties of the torsion of the quaternionic contact structure, i.e., the torsion endomorphism \( T_\xi = T(\xi, \cdot) : H \to H, \quad \xi \in V \). We decompose the endomorphism \( T_\xi \in (sp(n) + sp(1))^\perp \) into symmetric part \( T^0_\xi \) and skew-symmetric part \( b_\xi \),

\[
T_\xi = T^0_\xi + b_\xi.
\]

We summarize the description of the torsion due to O. Biquard in the following Proposition.

**Proposition 2.6.** [Biq1] The torsion \( T_\xi \) is completely trace-free,

\[
tr T_\xi = \sum_{a=1}^{4n} g(T_\xi(e_a), e_a) = 0, \quad tr T_\xi \circ I = \sum_{a=1}^{4n} g(T_\xi(e_a), I e_a) = 0, \quad I \in Q,
\]

where \( e_1 \ldots e_{4n} \) is an orthonormal basis of \( H \).

The symmetric part of the torsion has the properties:

\[
T^0_\xi I_i = -I_i T^0_\xi, \quad i = 1, 2, 3;
\]

\[
I_i (T^0_\xi)^{++} = I_i (T^0_\xi)^{--},
\]

\[
I_i (T^0_\xi)^{+0} = I_i (T^0_\xi)^{-0},
\]

\[
I_i (T^0_\xi)^{00} = I_i (T^0_\xi)^{00}.
\]

The skew-symmetric part can be represented in the following way

\[
b_\xi I_i = I_i u, \quad i = 1, 2, 3,
\]

where \( u \) is a traceless symmetric \((1,1)\)-tensor on \( H \) which commutes with \( I_1, I_2, I_3 \).

3. **The torsion and curvature of the Biquard connection**

Let \((M^{4n+3}, \eta, Q)\) be a quaternionic contact structure on a \(4n+3 > 7\)-dimensional smooth manifold with fixed \( V = \text{span}\{\xi_1, \xi_2, \xi_3\}\) satisfying the Biquard conditions \([2.11]\). The fundamental 2-forms \( \omega_i, i = 1, 2, 3 \) of the quaternionic structure \( Q \) are defined by

\[
\omega_i|_H = \frac{1}{2} d\eta_i|_H, \quad \xi \omega_i = 0, \quad \xi \in V.
\]
Define three 2-forms \( \theta_i, i = 1, 2, 3 \) by the formulas

\[
\theta_i = \frac{1}{2} \left\{ d((\xi_j \lrcorner d\eta_k)_H) + (\xi_j \lrcorner d\eta_j) \wedge (\xi_k \lrcorner d\eta_k) \right\}_H
\]

\[
= \frac{1}{2} \left\{ d(\xi_j \lrcorner d\eta_k) + (\xi_j \lrcorner d\eta_j) \wedge (\xi_k \lrcorner d\eta_k) \right\}_H - d\eta_k(\xi_j, \xi_k)\omega_k + d\eta_k(\xi_i, \xi_j)\omega_i,
\]

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\). Define, further, the corresponding \((1, 1)\) tensors \(A_i\) determined by

\[
g(A_i(X), Y) = \theta_i(X, Y), \, X, Y \in H.
\]

3.1. **The torsion.** Due to (3.1), the torsion restricted to \(H\) has the form

\[
T(X, Y) = -[X, Y]_V = 2 \sum_{s=1}^{3} \omega_s(X, Y)\xi_s, \quad X, Y \in H.
\]

The next two Lemmas provide some useful technical facts.

**Lemma 3.1.** Let \(D\) be any differentiation of the tensor algebra of \(H\). Then we have

\[
D(I_i)I_i = -I_i D(I_i), \quad i = 1, 2, 3,
\]

\[
I_1 D(I_1)^{++} = I_2 D(I_2)^{++},
\]

\[
I_1 D(I_1)^{++} = I_2 D(I_2)^{++},
\]

\[
I_1 D(I_1)^{++} = I_2 D(I_2)^{++}.
\]

**Proof.** The proof is straightforward consequence of the next identities

\[
0 = I_2(D(I_1) - I_2 D(I_1)I_2) + I_1(D(I_2) - I_1 D(I_2)I_1) = I_2 D(I_1)^{++} + I_1 D(I_2)^{++},
\]

\[
0 = D(-Id_V) = D(I_i I_i) = D(I_i)I_i + I_i D(I_i).
\]

With \(L\) denoting the Lie derivative, we set \(L' = L\lrcorner H\).
Lemma 3.2. The following identities hold true.

\[(3.5)\quad L'_{\xi_1}I_1 = -2T_0^{\xi_1}I_1 + d\eta_1(\xi_1, \xi_2)I_2 + d\eta_1(\xi_1, \xi_3)I_3,\]

\[(3.6)\quad L'_{\xi_1}I_2 = -2T_0^{\xi_1}I_2 - 2I_3\bar{u} + d\eta_1(\xi_2, \xi_1)I_1 + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2))I_3\]

\[(3.7)\quad L'_{\xi_2}I_1 = -2T_0^{\xi_2}I_1 + 2I_3\bar{u} + d\eta_1(\xi_1, \xi_2)I_2 - \frac{1}{2}(-d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2))I_3,\]

where the symmetric endomorphism \(\bar{u}\) on \(H\) is defined by

\[(3.8)\quad 2\bar{u} = I_3((L'_{\xi_1}I_2)^{++, -}) + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2))Id_H\]

and it commutes with \(I_1, I_2, I_3\). In addition, we have six more identities, which can be obtained with a cyclic permutation of \((1, 2, 3)\).

Proof. For all \(k, l = 1, 2, 3\) we have

\[(3.9)\quad L_{\xi_k}\omega_l(X,Y) = L_{\xi_k}g(I_lX,Y) + g((L_{\xi_k}I_l)X,Y)\]

Cartan’s formula yields

\[(3.10)\quad L_{\xi_k}\omega_l = \xi_k \lrcorner (d\omega_l) + d(\xi_k \lrcorner \omega_l).\]

A direct calculation using \((3.1)\) gives

\[(3.11)\quad 2\omega_l = (d\eta_l)|_H = d\eta_l - \sum_{s=1}^{3}\eta_s \wedge (\xi_s \lrcorner d\eta_l) + \sum_{1 \leq s < t \leq 3} d\eta_l(\xi_s, \xi_t)\eta_s \wedge \eta_t.\]

Combining \((3.11)\) and \((3.10)\) we obtain after a short calculation the following identities

\[(3.12)\quad (L_{\xi_1}\omega_1)|_H = (d\eta_1(\xi_1, \xi_2)\omega_2 + d\eta_1(\xi_1, \xi_3)\omega_3)|_H\]

\[(3.13)\quad (L_{\xi_1}\omega_2)|_H = \frac{1}{2}(d(\xi_1 \lrcorner d\eta_2) - (\xi_1 \lrcorner d\eta_3) \wedge (\xi_3 \lrcorner d\eta_2))|_H\]

\[(3.14)\quad (L_{\xi_1}\omega_1)|_H = \frac{1}{2}(d(\xi_2 \lrcorner d\eta_1) - (\xi_2 \lrcorner d\eta_3) \wedge (\xi_3 \lrcorner d\eta_1))|_H\]

Clearly, \((3.2)\) and \((3.4)\) imply \((3.5)\).
Furthermore, if we use (2.11) and add (3.13) to (3.14) we come to
\[
(3.15) \quad (\mathcal{L}_{\xi_1}\omega_2 + \mathcal{L}_{\xi_2}\omega_1)|_H = \frac{1}{2} \left( d(\xi_1 \omega_2) + d(\xi_2 \omega_1) \right)|_H
= \eta_1(\xi_2, \xi_1)\omega_1 + \eta_2(\xi_1, \xi_2)\omega_2 + (\eta_1(\xi_2, \xi_3) + \eta_2(\xi_1, \xi_3))\omega_3.
\]

On the other hand, (3.9) implies
\[
(3.16) \quad T^0_{\xi_1}I_2 + L'_{\xi_2}I_2 + 2T^0_{\xi_2}I_1 + L'_{\xi_1}I_1
= \eta_1(\xi_2, \xi_1)I_1 + \eta_2(\xi_1, \xi_2)I_2 + (\eta_1(\xi_2, \xi_3) + \eta_2(\xi_1, \xi_3))I_3.
\]

Let us decompose (3.16) into $Sp(n)$-invariant components:
\[
(3.17) \quad (\mathcal{L}_{\xi_1}I_2)^{++} = -2T^0_{\xi_1}I_2 + \eta_1(\xi_2, \xi_1)I_1,
(\mathcal{L}_{\xi_2}I_1)^{++} = -2T^0_{\xi_2}I_1 + \eta_2(\xi_1, \xi_2)I_2,
(\mathcal{L}_{\xi_1}I_2 + \mathcal{L}_{\xi_2}I_1)^{--} = (\eta_1(\xi_2, \xi_3) + \eta_2(\xi_1, \xi_3))I_3.
\]

Using (3.18) and (3.8), we obtain
\[
2\tilde{u} = -I_3((\mathcal{L}_{\xi_1}I_2)^{--}) + \frac{1}{2}(\eta_1(\xi_2, \xi_3) + \eta_2(\xi_1, \xi_3) - \eta_3(\xi_1, \xi_2))Id_H.
\]

The latter together with (3.8) tells us that $\tilde{u}$ commutes with all $I \in Q$. Now, Lemma 3.1 with $D = \mathcal{L}'$ implies (3.6) and (3.7).

The vanishing of the symmetric part of the left hand side in (3.9) for $k = 1, \ l = 2$, combined with (3.20) and (3.21) yields
\[
0 = -2g(I_3 \tilde{u}X, Y) - 2g(I_3 \tilde{u}Y, X).
\]

As $\tilde{u}$ commutes with all $I \in Q$ we conclude that $\tilde{u}$ is symmetric.

The rest of the identities can be obtained through a cyclic permutation of (1,2,3). \(\square\)

In the next Proposition we describe the properties of the quaternionic contact torsion more precisely.

**Proposition 3.3.** The torsion of the Biquard connection satisfies the identities:
\[
(3.19) \quad T^i_{\xi_i} = T^0_{\xi_i} + I_iu, \quad i = 1, 2, 3,
(3.20) \quad T^0_{\xi_i} = \frac{1}{2} L_{\xi_i}g, \quad i = 1, 2, 3,
(3.21) \quad u = \tilde{u} - \frac{\text{tr}(\tilde{u})}{4n} Id_H,
\]
where the symmetric endomorphism \( \tilde{u} \) on \( H \) commuting with \( I_1, I_2, I_3 \) satisfies

\[
\tilde{u} = \frac{1}{2} I_1 A_1^{-} - \frac{1}{4} \left( -d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2) \right) Id_H \\
= \frac{1}{2} I_2 A_2^{-} - \frac{1}{4} \left( d\eta_1(\xi_2, \xi_3) - d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2) \right) Id_H \\
= \frac{1}{2} I_3 A_3^{-} + \frac{1}{4} \left( d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2) \right) Id_H.
\]

Proof. Express the Lie derivative in terms of Biquard connection to get for \( X, Y \in H \) that

\[
\mathcal{L}_\xi, g(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + g(\nabla T_\xi, X, Y) + g(T_\xi, Y, X) = 2g(T_\xi^0, X, Y),
\]

since \( \nabla \) preserves the splitting \( H \oplus V \) which proves (3.20).

To show that \( \tilde{u} \) satisfies (3.22), insert (3.13) into (3.2) to get

\[
\theta_3 = (\mathcal{L}_\xi, \omega_2)_{|H} = d\eta_3(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1), \xi_3) \omega_3.
\]

Substitute (3.9) and (3.6) into (3.23) to obtain

\[
A_3 = 2T_{\xi_1}^0 - I_2 - 2I_3 \tilde{u}
\]

\[
+ d\eta_1(\xi_2, \xi_3) I_1 - d\eta_2(\xi_3, \xi_1) I_2 + \frac{1}{2} (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) - d\eta_3(\xi_1, \xi_2)) I_3.
\]

Now, compare the (++) component on both sides of (3.24) to see the last equality of (3.22). The rest of the identities can be obtained with a cyclic permutation of (1,2,3).

Denote with \( \Sigma^2 \) and \( \Lambda^2 \), respectively, the subspaces of symmetric and skew-symmetric endomorphisms of \( H \). Let \( skew : \text{End}(H) \rightarrow \Lambda^2 \) be the natural projection with kernel \( \Sigma^2 \). We have

\[
4[T_{\xi_1}]_{(\Sigma^2 \oplus \text{sp}(n))} = 3skew(T_{\xi_1}) + I_1skew(T_{\xi_1})I_1 + I_2skew(T_{\xi_1})I_2 + I_3skew(T_{\xi_1})I_3
\]

\[
= \sum_{s=1}^{3} (skew(T_{\xi_1}) + I_s skew(T_{\xi_1}) I_s).
\]

According to Theorem 2.5, \( T_\xi X \in H \) for \( X \in H, \xi \in V \). Hence,

\[
T(\xi, X) = \nabla_\xi X - [\xi, X]_H = \nabla_\xi X - \mathcal{L}_\xi(X).
\]

An application (3.25) gives

\[
g(4[T_{\xi_1}]_{(\Sigma^2 \oplus \text{sp}(n))} X, Y) = -\sum_{s=1}^{3} g((\nabla_\xi I_s) X, I_s Y)
\]

\[
+ \frac{1}{2} \sum_{s=1}^{3} \{ g((\mathcal{L}_\xi I_s) X, I_s Y) - g((\mathcal{L}_\xi I_s) Y, I_s X)\}.
\]
The metric tensor $g$ on the bundle $H$ determines in a natural way a metric structure on the bundle $\text{End}(H)$, see (2.11). Let $B(H)$ be the orthogonal complement of $\Sigma^2 \oplus sp(n) \oplus sp(1)$ in $\text{End}(H)$. Obviously, $B(H) \subset \Lambda^2$ and we have the following splitting of $\text{End}(H)$ into mutually orthogonal components

$$(3.27) \quad \text{End}(H) = \Sigma^2 \oplus sp(n) \oplus sp(1) \oplus B(H).$$

If $\Psi$ is an arbitrary section of the bundle $\Lambda^2$ of $M$, the orthogonal projection of $\Psi$ into $B(H)$ is given by

$$[\Psi]_{B(H)} = \Psi^{++} + \Psi^{+-} + \Psi^{-+} - [\Psi]_{sp(1)},$$

where $[\Psi]_{sp(1)}$ is the orthogonal projection of $\Psi$ onto $sp(1)$. We have also

$$[\Psi]_{sp(1)} = \frac{1}{4n} \sum_{s=1}^{3} \sum_{a=1}^{4n} g(\Psi e_{a}, I_{s} e_{a}) I_{s}.$$  

Theorem 2.3 - (iv) and the splitting (3.27) yield

$$(3.28) \quad T_{\xi_{i}} = [T_{\xi_{i}}]_{(sp(n) \oplus sp(1))^\perp} = [T_{\xi_{i}}]_{\Sigma^2} + [T_{\xi_{i}}]_{B(H)} = T_{\xi_{i}}^{0} [T_{\xi_{i}}]_{(\Sigma^2 \oplus sp(n))^\perp} - [T_{\xi_{i}}]_{sp(1)}.$$  

Using (3.26), Lemma 3.2 and the fact that $I_{s}(\nabla_{\xi_{i}} I_{s}) \in sp(1)$, we compute

$$(3.29) \quad 4[T_{\xi_{i}}]_{(\Sigma^2 \oplus sp(n))^\perp} - [T_{\xi_{i}}]_{sp(1)} = - \sum_{s=1}^{3} \{ \text{skew}(I_{s}(L'_{\xi_{i}} I_{s})) - [I_{s}(L'_{\xi_{i}} I_{s})]_{sp(1)} \} = \sum_{s=1}^{3} \text{skew}(2 I_{s} T_{\xi_{i}}^{0} I_{s}) + 4u = 4u$$

Plugging (3.29) into (3.28) completes the proof. \hfill \Box

The $Sp(n)$-invariant splitting of (3.24) leads to the following Corollary.

**Corollary 3.4.** The (1.1)-tensors $A_{i}$ satisfy the equalities

$$A_{3}^{+++} = 2 T_{\xi_{1}}^{0} I_{2}$$

$$A_{3}^{++-} = d\eta_{1}(\xi_{2}, \xi_{1}) I_{1}$$

$$A_{3}^{+-+} = - d\eta_{2}(\xi_{1}, \xi_{2}) I_{2}$$

$$A_{3}^{-++} = - 2 I_{3} \tilde{u} + \frac{1}{2} (d\eta_{1}(\xi_{2}, \xi_{3}) + d\eta_{2}(\xi_{3}, \xi_{1}) - d\eta_{3}(\xi_{1}, \xi_{2})) I_{3}$$

Analogous formulas for $A_{1}$ and $A_{2}$ can be obtained by a cyclic permutation of $(1, 2, 3)$.  

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Proposition 3.5. The covariant derivative of the quaternionic contact structure with respect to the Biquard connection is given by

\[ \nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \]

where the \( sp(1) \)-connection 1-forms \( \alpha_s \) are determined by

\[ \alpha_i (X) = d\eta_k (\xi_j, X) = -d\eta_j (\xi_k, X), \quad X \in H, \quad \xi_i \in V; \]

\[ \alpha_i (\xi_s) = d\eta_s (\xi_j, \xi_k) - \delta_{is} \left( \frac{\text{tr}(\tilde{u})}{2n} + \frac{1}{2} (d\eta_1 (\xi_2, \xi_3) + d\eta_2 (\xi_3, \xi_1) + d\eta_3 (\xi_1, \xi_2)) \right), \]

where the \( sp(1) \)-connection 1-forms \( \alpha_s \) are determined by

\[ \alpha_i (X) = d\eta_k (\xi_j, X) = -d\eta_j (\xi_k, X), \quad X \in H, \quad \xi_i \in V; \]

\[ \alpha_i (\xi_s) = d\eta_s (\xi_j, \xi_k) - \delta_{is} \left( \frac{\text{tr}(\tilde{u})}{2n} + \frac{1}{2} (d\eta_1 (\xi_2, \xi_3) + d\eta_2 (\xi_3, \xi_1) + d\eta_3 (\xi_1, \xi_2)) \right), \]

s = 1, 2, 3 and \((i, j, k)\) is any cyclic permutation of \((1, 2, 3)\).

Proof. The equality \((3.31)\) is proved by Biquard in [Bi1].

Using \((3.25)\), we obtain

\[ \nabla_{\xi_i} I_i = [T_{\xi_i}, I_i] + L_{\xi_i} I_i = [T_{\xi_i}, I_i] + u [I_s, I_i] + L_{\xi_i} I_i. \]

An application of Lemma \(\Box\) completes the proof.

Corollary 3.6. The covariant derivative of the distribution \( V \) is given by

\[ \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j, \]

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

3.2. The Curvature Tensor. Let \( R = [\nabla, \nabla] - \nabla [\cdot, \cdot] \) be the curvature tensor of \( \nabla \). For any \( B, C \in \Gamma(TM) \) the curvature operator \( R_{BC} \) preserves the QC structure on \( M \) since \( \nabla \) preserves it. In particular \( R_{BC} \) preserves the distributions \( H \) and \( V \), the quaternionic structure \( \mathbb{Q} \) on \( H \) and the \((2, 1)\) tensor \( \varphi \). Moreover, the action of \( R_{BC} \) on \( V \) is completely determined by its action on \( H \),

\[ R_{BC} \xi_i = \varphi^{-1} ([R_{BC}, I_i]), \quad i = 1, 2, 3. \]

Thus, we may regard \( R_{BC} \) as an endomorphism of \( H \) and we have \( R_{BC} \in sp(n) \oplus sp(1) \).

Definition 3.7. The Ricci 2-forms \( \rho_i \) are defined by

\[ \rho_i (B, C) = \frac{1}{4n} \sum_{a=1}^{4n} g(R(B, C)e_a, I_i e_a), \quad B, C \in \Gamma(TM). \]

Hereafter \( e_1, \ldots, e_{4n} \) is an orthonormal quaternionic basis of \( H \). We decompose the curvature into \( sp(n) \oplus sp(1) \)-parts. Let \( R_{BC}^0 \in sp(n) \) denote the \( sp(n) \)-component.
**Lemma 3.8.** The curvature of the Biquard connection decomposes as follows

$$R_{BC} = R_{BC}^0 + \rho_1(B,C)I_1 + \rho_2(B,C)I_2 + \rho_3(B,C)I_3.$$  

For any cyclic permutation $i, j, k$ of $(1, 2, 3)$ we also have

(3.33) $$[R_{BC}, I_i] = 2(-\rho_j(B,C)I_k + \rho_k(B,C)I_j), \quad B, C \in \Gamma(TM),$$

(3.34) $$\rho_i = \frac{1}{2}(d\alpha_i + \alpha_j \wedge \alpha_k),$$

where the connection 1-forms $\alpha_s$ are determined in (3.31), (3.32).

**Proof.** The first two identities follow directly from the definitions. Using (3.30), we calculate

$$[R_{BC}, I_i] = \nabla_B(\alpha_3(C))I_2 - \alpha_2(C)I_3 - \nabla_C(\alpha_3(B))I_2 - \alpha_2(B)I_3 - (\alpha_3([B, C])I_2 - \alpha_2([B, C])I_3)$$

$$= -(d\alpha_2 + \alpha_3 \wedge \alpha_1)(B,C)I_3 + (d\alpha_3 + \alpha_1 \wedge \alpha_2)(B,C)I_2.$$  

Now (3.33) completes the proof. □

**Definition 3.9.** The quaternionic contact Ricci tensor (qc-Ricci tensor for short) and the qc-scalar curvature $\text{Scal}$ of the Biquard connection are defined by

(3.35) $$\text{Ric}(B,C) = \sum_{a=1}^{4n} g(R(e_a, B)C,e_a), \quad \text{Scal} = \sum_{a=1}^{4n} \text{Ric}(e_a,e_a).$$

It is known, cf. [Biq1], that the Ricci tensor restricted to $H$ is symmetric. In addition, we define six Ricci-type tensors $\zeta_i, \tau_i$, $i = 1, 2, 3$ as follows

(3.36) $$\zeta_i(B,C) = \frac{1}{4n} \sum_{a=1}^{4n} g(R(e_a, B)C,e_a), \quad \tau_i(B,C) = \frac{1}{4n} \sum_{a=1}^{4n} g(R(e_a, I_i e_a)B,C).$$

We shall show that all Ricci-type contractions evaluated on the horizontal space $H$ are determined by the components of the torsion. First, define the following 2-tensors on $H$

(3.37) $$T^0(X,Y) = g((T_{\xi_1}^0 I_1 + T_{\xi_2}^0 I_2 + T_{\xi_3}^0 I_3)X,Y), \quad U(X,Y) = g(uX,Y), \quad X, Y \in H.$$

**Lemma 3.10.** The tensors $T^0$ and $U$ are $\text{Sp}(n)\text{Sp}(1)$-invariant trace-free symmetric tensors with the properties:

(3.38) $$T^0(X,Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) = 0,$$

(3.39) $$3U(X,Y) - U(I_1 X, I_1 Y) - U(I_2 X, I_2 Y) - U(I_3 X, I_3 Y) = 0.$$

**Proof.** The lemma follows directly from [214], [215] of Proposition 2.6 □
We turn to a Lemma, which shall be used later.

**Lemma 3.11.** For any cyclic permutation $i, j, k$ of $(1, 2, 3)$, $X, Y \in H$ and $B \in H \oplus V$, we have

\begin{align*}
\text{(3.40)} & \quad \text{Ric}(B, I_i Y) + 4n \zeta_i(B, Y) = 2\rho_j(B, I_k Y) - 2\rho_k(B, I_j Y), \\
\text{(3.41)} & \quad \zeta_i(X, Y) = -\frac{1}{2} \rho_i(X, Y) + \frac{1}{2n} g(I_i u X, Y) + \frac{2n - 1}{2n} g(T_{\xi_i}^0 X, Y) \\
& \quad + \frac{1}{2n} g(I_j T_{\xi_i}^0 X, Y) - \frac{1}{2n} g(I_k T_{\xi_i}^0 X, Y).
\end{align*}

The Ricci 2-forms evaluated on $H$ satisfy

\begin{align*}
\rho_1(X, Y) &= 2g(T_{\xi_2}^{0-} I_3 X, Y) - 2g(I_1 u X, Y) - \frac{\text{tr}(\tilde{u})}{n} \omega_1(X, Y), \\
\rho_2(X, Y) &= 2g(T_{\xi_3}^{0+} I_1 X, Y) - 2g(I_2 u X, Y) - \frac{\text{tr}(\tilde{u})}{n} \omega_2(X, Y), \\
\rho_3(X, Y) &= 2g(T_{\xi_1}^{0-} I_2 X, Y) - 2g(I_3 u X, Y) - \frac{\text{tr}(\tilde{u})}{n} \omega_3(X, Y).
\end{align*}

The 2-forms $\tau_s$ evaluated on $H$ satisfy

\begin{align*}
\tau_1(X, Y) &= \rho_1(X, Y) + 2g(I_1 u X, Y) + \frac{4}{n} g(T_{\xi_2}^{0-} I_3 X, Y), \\
\tau_2(X, Y) &= \rho_2(X, Y) + 2g(I_2 X, Y) + \frac{4}{n} g(T_{\xi_3}^{0+} I_1 X, Y), \\
\tau_3(X, Y) &= \rho_3(X, Y) + 2g(I_3 X, Y) + \frac{4}{n} g(T_{\xi_1}^{0-} I_2 X, Y).
\end{align*}

**Proof.** From (3.33) we have

\[
\text{Ric}(B, I_1 Y) + 4n \zeta_1(B, Y) = \sum_{a=1}^{4n} \{R(e_a, B, I_1 Y, e_a) + R(e_a, B, Y, I_1 e_a)\}
\]

\[
= \sum_{a=1}^{4n} \{-2\rho_2(e_a, B) \omega_3(Y, e_a) + 2\rho_3(e_a, B) \omega_2(Y, e_a)\} = 2\rho_2(B, I_3 Y) - 2\rho_3(B, I_2 Y),
\]

Using (3.2) and (3.34) we obtain

\[
\rho_1(X, Y) = A_1(X, Y) - \frac{1}{2} \alpha_1([X, Y]) = A_1(X, Y) + \sum_{s=1}^{3} \omega_s(X, Y) \alpha_1(\xi_s).
\]

Now, Corollary 3.4 and Corollary 3.5 imply the first equality in (3.42). The other two equalities in (3.42) can be obtained in the same manner.
Letting \( b(X, Y, Z, W) = 2\sigma_{X,Y,Z}\{\sum_{l=1}^{3} \omega_l(X, Y)g(T_{\xi_l}^0, Z, W)\} \), where \( \sigma_{X,Y,Z} \) is the cyclic sum over \( X, Y, Z \), we have

\[
\sum_{a=1}^{4n} b(X, Y, e_a, I_1 e_a) = 4g(I_1 uX, Y) + 8g(I_2 T_{\xi_3}^0 X, Y) + (8n - 4)g(T_{\xi_3}^0 X, Y) + (8n + 4)g(I_1 uX, Y)
\]

\[
+ 4g(T_{\xi_3}^0 I_3 X, Y) - 4g(T_{\xi_3}^0 I_2 X, Y).
\]  

The first Bianchi identity gives

\[
4n\sum_{a=1}^{4n} b(e_a, I_1 e_a, X, Y) = (8n - 4)g(T_{\xi_3}^0 X, Y) + (8n + 4)g(I_1 uX, Y).
\]  

Consequently, (3.44), (3.45), (3.46) and (3.47) yield the first set of equalities in (3.43) and (3.41). The other equalities in (3.43) and (3.41) can be shown similarly. This completes the proof of Lemma 3.11.

**Theorem 3.12.** Let \((M^{4n+3}, \eta, \mathbb{Q})\) be a quaternionic contact \((4n+3)\)-dimensional manifold, \(n > 1\). For any \(X, Y \in H\) the qc-Ricci tensor and the qc-scalar curvature satisfy

\[
Ric(X, Y) = (2n + 2)T^0(X, Y) + (4n + 10)U(X, Y) + (2n + 4)\frac{tr(\tilde{\eta})}{n}g(X, Y),
\]

\[
Scal = (8n + 16)tr(\tilde{\eta}).
\]

**Proof.** The proof follows from Lemma 3.11, (3.41), (3.42) and (3.40).
Corollary 3.13. The qc-scalar curvature satisfies the equalities
\[
\frac{\text{Scal}}{2(n+2)} = \sum_{a=1}^{4n} \rho_i(I_ie_a, e_a) = \sum_{a=1}^{4n} \tau_i(I_ie_a, e_a) = -2 \sum_{a=1}^{4n} \zeta_i(I_ie_a, e_a), \quad i = 1, 2, 3.
\]
The next result shows that the unknown function \( \lambda \) in (2.13) is a scalar multiple of the qc-scalar curvature.

Corollary 3.14. The torsion of the Biquard connection restricted to \( V \) satisfies the equality
\[
(3.49) \quad T(\xi_i, \xi_j) = -\frac{\text{Scal}}{8n(n+2)} \xi_k - [\xi_i, \xi_j]_H,
\]
where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

Proof. A small calculation using Corollary 3.6 and Proposition 3.5 gives
\[
T(\xi_i, \xi_j) = \nabla_{\xi_i} \xi_j - \nabla_{\xi_j} \xi_i - [\xi_i, \xi_j] = -\frac{\text{tr}(\tilde{u})}{n} \xi_k - [\xi_i, \xi_j]_H.
\]
Now, the assertion follows from the second equality in (3.48). \( \square \)

Corollary 3.15. The tensors \( T^0, U, \tilde{u} \) do not depend on the choice of the local basis.

4. QC-Einstein quaternionic contact structures

The aim of this section is to analyse the information encoded in the Bianchi identities to show that the vanishing of the torsion of the quaternionic contact structure implies that the qc-scalar curvature is constant and to prove our classification Theorem 1.3.

Definition 4.1. A quaternionic contact structure is qc-Einstein if the qc-Ricci tensor is trace-free,
\[
\text{Ric}(X,Y) = \frac{\text{Scal}}{4n} g(X,Y), \quad X,Y \in H.
\]

Proposition 4.2. A quaternionic contact manifold \((M^{4n+3}, \eta, \mathbb{Q})\) is a qc-Einstein if and only if the quaternionic contact torsion vanishes identically, \( T_\xi = 0, \xi \in V \).

Proof. If \((\eta, \mathbb{Q})\) is qc-Einstein structure then \( T^0 = U = 0 \) because of (3.48). We use the same symbol \( T^0 \) for the corresponding endomorphism of the 2-tensor \( T^0 \) on \( H \). According to (3.37), we have
\[
T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3.
\]
Using first (2.16) and then (2.17), we compute

\[(T^0)^{++} = (T^0_{\xi_2})^{++} I_2 + (T^0_{\xi_3})^{++} I_3 = 2(T^0_{\xi_2})^{++} I_2\]

Hence, \(T_{\xi_2} = T^0_{\xi_2} + I_2 u\) vanishes. Similarly \(T_{\xi_1} = T^0_{\xi_1} = 0\).

The converse follows from (3.48). \(\square\)

**Proposition 4.3.** For \(X \in V\) and any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\) we have

\[(4.2)\quad \rho_i(X, \xi_i) = -\frac{X(\text{Scal})}{32n(n + 2)} + \frac{1}{2} (\omega_i([\xi_j, \xi_k], X) - \omega_j([\xi_i, \xi_k], X) - \omega_k([\xi_i, \xi_j], X)),\]

\[(4.3)\quad \rho_i(X, \xi_j) = \omega_j([\xi_j, \xi_k], X), \quad \rho_i(X, \xi_k) = \omega_k([\xi_j, \xi_k], X),\]

\[(4.4)\quad \rho_i(I_k X, \xi_j) = -\rho_i(I_j X, \xi_k) = g(T(\xi_j, \xi_k), I_i X) = \omega_i([\xi_j, \xi_k], X).\]

**Proof.** Since \(\nabla\) preserves the splitting \(H \oplus V\), the first Bianchi identity, (3.49) and (3.33) imply

\[(4.5)\quad 2\rho_i(X, \xi_i) + 2\rho_j(X, \xi_j) = g(R(X, \xi_i)\xi_j, \xi_k) + g(R(\xi_j, X)\xi_i, \xi_k)\]

\[= \sigma_{\xi_i, \xi_j, X} \{ g(\nabla_{\xi_i} T)(\xi_j, X), \xi_k) + g(T(T(\xi_i, \xi_j), X), \xi_k) \}
\[= g((\nabla_X T)(\xi_i, \xi_j), \xi_k) + g(T(T(\xi_i, \xi_j), X), \xi_k)\]

\[= -\frac{X(\text{Scal})}{8n(n + 2)} - 2\omega_k([\xi_i, \xi_j], X).\]

Summing up the first two equalities in (4.5) and subtracting the third one, we obtain (4.2).

Similarly,

\[2\rho_k(\xi_j, X) = g(R(\xi_j, X)\xi_i, \xi_j) = \sigma_{\xi_i, \xi_j, X} \{ g(\nabla_{\xi_i} T)(\xi_j, X), \xi_j) + g(T(T(\xi_i, \xi_j), X), \xi_j) \}
\[= g(T(T(\xi_i, \xi_j), X), \xi_j) = g(T([\xi_i, \xi_j]_H, X), \xi_j) = g([\xi_i, \xi_j]_H, X), \xi_j)\]
\[= -d\eta_j([\xi_i, \xi_j]_H, X) = -2\omega_j([\xi_i, \xi_j], X).\]

Hence, the second equality in (4.5) follows. Analogous calculations show the validity of the first equality in (4.3). Then, (4.4) is a consequence of (4.3) and (3.49). \(\square\)

The vertical derivative of the qc-scalar curvature is determined in the next Proposition.
Proposition 4.4. For any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\) we have
\[
\rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j) = \frac{1}{8n(n + 2)} \xi_j(\text{Scal}).
\]

Proof. Since \(\nabla\) preserves the splitting \(H \oplus V\), the first Bianchi identity and [3,49] imply
\[
- (\rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j)) = \sigma_{\xi_i, \xi_j, \xi_k} \{g(R(\xi_i, \xi_j)\xi_k, \xi_j)\} = \frac{1}{8n(n + 2)} \xi_j(\text{Scal})
\]
\[
\sigma_{\xi_i, \xi_j, \xi_k} \{g((\nabla_{\xi_j} T)(\xi_j, \xi_k), \xi_j) + g(T(T(\xi_i, \xi_j), \xi_k), \xi_j)\} = \frac{1}{8n(n + 2)} \xi_j(\text{Scal})
\]
\[
\square
\]

4.1. The Bianchi identities. In order to derive the essential information contained in the
Bianchi identities we need the next Lemma, which is an application of a standard result in
differential geometry.

Lemma 4.5. In a neighborhood of any point \(p \in M^{4n+3}\) and an orthonormal basis
\([X_1(p), \ldots, X_{4n}(p), \xi_1(p), \xi_2(p), \xi_3(p)]\) of the tangential space at \(p\) there exists a \(Q\) - orthonormal frame field
\([X_1, \ldots, X_{4n}, \xi_1, \xi_2, \xi_3]\), \(X_{a,p} = X_a(p), \xi_i,p = \xi_i(p), \ a = 1, \ldots, 4n, i = 1, 2, 3\)
such that the connection 1-forms of the Biquard connection are all zero at the point \(p\):
\[
(\nabla_{X_a} X_b)_p = (\nabla_{X_a} X_b)_p = (\nabla_{X_a} \xi_t)_p = (\nabla_{X_a} \xi_s)_p = 0
\]
for \(a, b = 1, \ldots, 4n, s, t, r = 1, 2, 3\).

In particular,
\[
((\nabla_{X_a} I_s) X_b)_p = ((\nabla_{X_a} I_s) \xi_t)_p = ((\nabla_{X_a} I_s) X_b)_p = ((\nabla_{X_a} I_s) \xi_r)_p = 0
\]

Proof. Since \(\nabla\) preserves the splitting \(H \oplus V\) we can apply the standard arguments for the
existence of a normal frame with respect to a metric connection (see e.g. [Wu]). We sketch
the proof for completeness.

Let \([\tilde{X}_1, \ldots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3]\) be an orthonormal basis around \(p\) such that \(\tilde{X}_{a,p} = X_a(p), \tilde{\xi}_i,p = \xi_i(p)\). We want to find a modified frame
\[
X_a = \sigma_a^b \tilde{X}_b, \quad \xi_i = \sigma_i^j \tilde{\xi}_j,
\]
which satisfies the normality conditions of the lemma.

Let \(\varpi\) be the \(sp(n) \oplus sp(1)\)-valued connection 1-forms with respect to \([\tilde{X}_1, \ldots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3]\),
\[
\nabla \tilde{X}_b = \varpi^c_b \tilde{X}_c, \quad \nabla \tilde{\xi}_s = \varpi^t_s \tilde{\xi}_t, \quad B \in \{\tilde{X}_1, \ldots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}.
\]
Let \( \{x^1, \ldots, x^{4n+3}\} \) be a coordinate system around \( p \) such that
\[
\frac{\partial}{\partial x^a}(p) = X_a(p), \quad \frac{\partial}{\partial x^{4n+t}}(p) = \xi_t(p), \quad a = 1, \ldots, 4n, \quad t = 1, 2, 3.
\]
One can easily check that the matrices
\[
o^b_a = \exp \left( - \sum_{c=1}^{4n+3} \omega_a^b \left( \frac{\partial}{\partial x^c} \right) \right) \in Sp(n), \quad o^s_t = \exp \left( - \sum_{c=1}^{4n+3} \omega^s_t \left( \frac{\partial}{\partial x^c} \right) \right) \in Sp(1)
\]
are the desired matrices making the identities (4.7) true.

Now, the last identity in the lemma is a consequence of the fact that the choice of the orthonormal basis of \( V \) does not depend on the action of \( SO(3) \) on \( V \) combined with Corollary 3.6 and Proposition 3.5.

\[\square\]

**Definition 4.6.** We refer to the orthonormal frame constructed in Lemma 4.5 as a qc-normal frame.

Let us fix a qc-normal frame \( \{e_1, \ldots, e_{4n}, \xi_1, \xi_2, \xi_3\} \). We shall denote with \( X, Y, Z \) horizontal vector fields \( X, Y, Z \in H \) and keep the notation for the torsion of type (0,3) \( T(B, C, D) = g(T(B, C), D), B, C, D \in H \oplus V \).

**Proposition 4.7.** On a quaternionic contact manifold \( (M^{4n+3}, \eta, Q) \) the following identities hold
\[
\begin{align*}
2 \sum_{a=1}^{4n} (\nabla_{e_a} \text{Ric})(e_a, X) - X(\text{Scal}) & = 4 \sum_{r=1}^{3} \text{Ric}(\xi_r, I_r X) - 8n \sum_{r=1}^{3} \rho_r(\xi_r, X); \\
\text{Ric}(\xi_s, I_s X) & = 2\rho_q(\xi_t X, \xi_s) + 2\rho_t(\xi_s X, \xi_q) + 4n \sum_{a=1}^{4n} (\nabla_{e_a} T)(\xi_s, I_s X, e_a); \\
4n(\rho_s(X, \xi_s) - \xi_s(\xi_q, X)) & = 2\rho_q(\xi_t X, \xi_s) + 2\rho_t(\xi_s X, \xi_q) - 4n \sum_{a=1}^{4n} (\nabla_{e_a} T)(\xi_s, I_s X, e_a); \\
\zeta_s(\xi_s, X) & = -\frac{1}{4n} \sum_{a=1}^{4n} (\nabla_{e_a} T)(\xi_s, I_s X, e_a),
\end{align*}
\]
where \( s \in \{1, 2, 3\} \) is fixed and \( (s, t, q) \) is an even permutation of \( (1, 2, 3) \).

**Proof.** The second Bianchi identity implies
\[
2 \sum_{a=1}^{4n} (\nabla_{e_a} \text{Ric})(e_a, X) - X(\text{Scal}) + 2 \sum_{a=1}^{4n} \text{Ric}(T(e_a, X), e_a) + \sum_{a,b=1}^{4n} R(T(e_b, e_a), X, e_b, e_a) = 0.
\]
Apply (3.4) in the last equality to get (4.8).
The first Bianchi identity combined with (2.15), (3.4) and the fact that \( \nabla \) preserves the orthogonal splitting \( H \oplus V \) yield

\[
\text{Ric}(\xi_s, I_s X) = 4n \sum_{a=1}^{4n} ( \nabla e_a T)(\xi_s, I_s X, e_a) + 2 \sum_{r=1}^{3} \omega_r(I_s X, e_a) T(\xi_r, \xi_s, e_a)
\]

and (4.4) completes the proof of (4.9).

In a similar fashion, from the first Bianchi identity, (2.15), (3.4) and the fact that \( \nabla \) preserves the orthogonal splitting \( H \oplus V \) we can obtain the proof of (4.10). Finally, take (3.40) with \( B = \xi_i \) and combine the result with (4.9) to get (4.11). \( \square \)

The following Theorem gives relations between \( Sp(n)Sp(1) \)-invariant tensors and is crucial for the solution of the Yamabe problem, which we shall undertake in the last Section.

We define the horizontal divergence \( \nabla^* P \) of a (0,2)-tensor field \( P \) with respect to Biquard connection to be the (0,1)-tensor defined by

\[
\nabla^* P(.) = \sum_{a=1}^{4n} (\nabla e_a P)(e_a, .)
\]

where \( e_a, a = 1, \ldots, 4n \) is an orthonormal basis on \( H \).

**Theorem 4.8.** The horizontal divergences of the curvature and torsion tensors satisfy the system \( B b = 0 \), where

\[
B = \begin{pmatrix}
-1 & 6 & 4n - 1 & \frac{3}{16(n+2)} & 0 \\
-1 & 0 & n + 2 & \frac{3}{16(n+2)} & 0 \\
1 & -3 & 4 & 0 & -1
\end{pmatrix},
\]

\[
b = \begin{pmatrix}
\nabla^* T^0, \nabla^* U, \mathbb{A}, d(\text{Scal}), \sum_{j=1}^{3} \text{Ric}(\xi_j, I_j .) \end{pmatrix}^t,
\]

with \( T^0 \) and \( U \) defined in (3.37) and

\[
\mathbb{A}(X) = g(I_1[\xi_2, \xi_3] + I_2[\xi_3, \xi_1] + I_3[\xi_1, \xi_2], X).
\]

**Proof.** Throughout the proof of Theorem 4.8 \((s, t, q)\) will denote an even permutation of \((1, 2, 3)\). Equations (4.2) and (4.4) yield

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(4.12) \[ \sum_{r=1}^{3} \rho_r(X, \xi_r) = -\frac{3}{32n(n+2)}X(Scal) - \frac{1}{2}A(X), \]

(4.13) \[ \sum_{s=1}^{3} \rho_q(I_sX, \xi_s) = A(X). \]

Using the properties of the torsion described in Proposition 3.3 and (2.16), we obtain

(4.14) \[ \sum_{s=1}^{3} \sum_{a=1}^{4n} (\mathcal{R}(\xi_s, I_sX, e_a) = \nabla^* T^0(X) - 3\nabla^* U(X), \]

(4.15) \[ \sum_{s=1}^{3} \sum_{a=1}^{4n} (\mathcal{R}(\xi_s, X, I_s e_a) = \nabla^* T^0(X) + 3\nabla^* U(X). \]

Substitute (4.13) and (4.14) into the sum of (4.9) written for \( s = 1, 2, 3 \) to get the third row of \( B \). The second row of \( B \) can be obtained by inserting (4.11) into (4.10), taking the sum over \( s = 1, 2, 3 \) and applying (4.12), (4.13), (4.14), (4.15).

The second Bianchi identity and applications of (3.4) give

(4.16) \[ \sum_{s=1}^{3} \left( \sum_{a=1}^{4n} \left[ (\nabla e_a \mathcal{R})(I_sX, I_s e_a) + 4n(\nabla e_a \zeta_s)(I_sX, e_a) \right] - 2\mathcal{R}(\xi_s, I_sX) + 8n\zeta_s(\xi_s, X) \right) \]

\[ + 8n \sum_{s=1}^{3} \left[ \zeta_s(\xi_t, I_qX) - \zeta_s(\xi_q, I_tX) - \rho_s(\xi_t, I_qX) + \rho_s(\xi_q, I_tX) \right] = 0. \]

Using (3.40), (3.42) as well as (2.16), (2.17) and (4.1) we obtain the next sequence of equalities

(4.17) \[ \sum_{s=1}^{3} [\mathcal{R}(I_sX, I_s e_a) + 4n\zeta_s(I_sX, e_a)] = 2 \sum_{s=1}^{3} (\rho_s(I_qX, I_te_a) - \rho_s(I_tX, I_q e_a)) \]

\[ = -4T^0(X, e_a) + 24U(X, e_a) + \frac{3}{2n(n+2)}g(X, e_a), \]
\[ \sum_{s=1}^{3} (-2Ric(\xi_s, I_s X) + 8n \zeta_s(\xi_s, X)) = \sum_{s=1}^{3} (-4Ric(\xi_s, I_s X) - 4\rho_s(\xi_t, I_q X) + 4\rho_s(\xi_q, I_t X)), \]

(4.18) \[ 8n \sum_{s=1}^{3} \left[ \zeta_s(\xi_t, I_q X) - \zeta_s(\xi_q, I_t X) \right] = \sum_{s=1}^{3} (4Ric(\xi_s, I_s X) - 8\rho_s(\xi_s, X) + 4\rho_s(\xi_t, I_q X) - 4\rho_s(\xi_q, I_t X)). \]

Substitute (4.17), (4.18) and (4.19) into (4.16) and use (4.12) and (4.13) to obtain the first row of the matrix \( B \). \[ \square \]

We are ready to prove one of our main observations.

**Theorem 4.9.** The qc-scalar curvature of a qc-Einstein quaternionic contact manifold of dimension \( 4n+3 \), \( n > 1 \), is a global constant.

In addition, the vertical distribution \( V \) of a qc-Einstein structure is integrable and the Ricci tensors are given by

\[
\rho_t |_{\mathcal{H}} = \tau_t |_{\mathcal{H}} = -2\zeta_t |_{\mathcal{H}} = -\frac{8n(n+2)}{2n} \omega_t \quad s, t = 1, 2, 3, \]

\[
Ric(\xi_s, X) = \rho_s(X, \xi_t) = \zeta_s(X, \xi_t) = 0, \quad s, t = 1, 2, 3. \]

**Proof.** Suppose the quaternionic contact manifold is qc-Einstein. Then the quaternionic contact torsion vanishes, \( T_\xi = 0, \xi \in V \) according to Proposition 4.2. Then (4.11) yields \( \zeta_s(\xi_s, X) = 0 \). Substituting the latter into (4.10), using (4.2) and (4.3), we find

\[
n\rho_s(I_q X, \xi_t) - (n + 1)\rho_t(I_s X, \xi_q) - (n + 1)\rho_q(I_t X, \xi_s) = \frac{3}{16(n+2)} X(Scal)\]

for any even permutation \((s, t, q)\) of \((1, 2, 3)\) and any \( X \in \mathcal{H} \). The last identities imply

(4.20) \[ \rho_s(I_q X, \xi_t) = \rho_s(I_q X, \xi_t) = \rho_s(I_q X, \xi_t) = \frac{3}{16(n+2)^2} X(Scal). \]

On the other hand, from (4.8) and the qc-Einstein condition, (4.2) and (4.3), we have

\[
\frac{-4n^2 - 3n + 4}{4n(n+2)} X(Scal) = 4 \sum_{r=1}^{3} Ric(\xi_r, I_r X) - 4n \sum_{(s,t,q)} \rho_q(I_t X, \xi_s),
\]

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where the summation in the third term is over even permutations of (1,2,3). Apply (4.9) and (4.20) to the latter to obtain
\[ n^3 + 5n^2 - 10n - 2 \frac{n(n + 2)}{} X(\text{Scal}) = 0. \]
Hence, \( X(\text{Scal}) = 0 \), \( X \in \mathcal{H} \). This implies also \( \xi(\text{Scal}) = 0, \xi \in V \) because for any \( p \in M \) one has \([e_a, I_s e_a]_p = T(e_a, I_s e_a)_p = 2 \xi_{sip} \). Now, (4.20), (4.9), (4.3), (3.41), (3.42) and (3.43) complete the proof. \( \square \)

4.1.1. Examples of qc-Einstein structures.

Example 4.10. The flat model.

The quaternionic Heisenberg group \( G(\mathbb{H}) \) with its standard left invariant quaternionic contact structure (see Section 5.2) is the simplest example. The Biquard connection coincides with the flat left-invariant connection on \( G(\mathbb{H}) \). More precisely, we have

Proposition 4.11. Any \((4n+3)\)-dimensional quaternionic contact manifold with flat Biquard connection is locally isomorphic to \( G(\mathbb{H}) \).

Proof. Since the Biquard connection \( \nabla \) is flat, there exist local \( \mathbb{Q} \)-orthonormal frame \( \{T_a, I_1 T_a, I_2 T_a, I_3 T_a, \xi_1, \xi_2, \xi_3\}, a = 1, \ldots, n \) which is \( \nabla \)-parallel. Theorem 4.9 tells us that the quaternionic contact torsion vanishes and the vertical distribution is integrable. In addition, (3.49) and (3.3) yield \([\xi_i, \xi_j] = -2 \xi_k \), for any cyclic permutation \((i,j,k)\) of \((1,2,3)\) (conf. (5.21)). Hence, the manifold has a local Lie group structure which is locally isomorphic to \( G(\mathbb{H}) \) by the Lie theorems. \( \square \)

Example 4.12. The 3-Sasakian Case.

Suppose \((M,g)\) is a \((4n+3)\)-dimensional Riemannian manifold with a given 3-Sasakian structure, i.e., the cone metric on \( M \times \mathbb{R} \) is a hyper Kähler metric or equivalently it has holonomy contained in \( \text{Sp}(n+1) \) [BGN]. In other words there are three Killing vector fields \( \{\xi_1, \xi_2, \xi_3\} \), which satisfy

(i) \( g(\xi_i, \xi_j) = \delta_{ij}, \ i, j = 1, 2, 3 \)
(ii) \( [\xi_i, \xi_j] = -2 \xi_k \), for any cyclic permutation \((i,j,k)\) of \((1,2,3)\)
(iii) \( (D_B \tilde{I}_i)C = g(\xi_i, C)B - g(B, C)\xi_i, \ i = 1, 2, 3, \ B, C \in \Gamma(TM) \), where \( \tilde{I}_i(B) = D_B \xi_i \) and \( D \) denotes the Levi-Civita connection.
A 3-Sasakian manifold is Einstein with positive Riemannian scalar curvature \((4n + 2)\) and if complete it is compact with finite fundamental group due to Mayer’s theorem (see [BG] for a nice overview of 3-Sasakian spaces).

Let \(H = \{\xi_1, \xi_2, \xi_3\}^\perp\). Then
\[
\tilde{I}_i(\xi_j) = \xi_k, \quad \tilde{I}_i \circ \tilde{I}_j(X) = \tilde{I}_k X, \quad \tilde{I}_i \circ \tilde{I}_i(X) = -X, \ X \in H,
\]
\[
d\eta_i(X,Y) = 2g(\tilde{I}_i X, Y), \ X, Y \in H.
\]

Defining \(V = \text{span}\{\xi_1, \xi_2, \xi_3\}\), \(I_i|_H = \tilde{I}_i|_H, I_i|_H = 0\) we obtain a quaternionic contact structure on \(M\) [Biq1]. It is easy to calculate that
\[
\xi_i \cdot d\eta_j|_H = 0, \ d\eta_i(\xi_j, \xi_k) = 2, \ d\eta_i(\xi_i, \xi_k) = d\eta_i(\xi_i, \xi_j) = 0,
\]
\[
A_1 = A_2 = A_3 = 0 \text{ cf. (3.3), (3.2)}, \quad \tilde{u} = \frac{1}{2}Id_H \text{ cf. (3.8)}.
\]

This quaternionic contact structure satisfies the conditions (2.11) and therefore it admits the Biquard connection \(\nabla\). More precisely, we have

(i) \(\nabla X I_i = 0, X \in H\), \(\nabla_{\xi_i} I_j = 0\), \(\nabla_{\xi_j} I_j = -2I_k, \ \nabla_{\xi_i} I_i = 2I_k\),

(ii) \(T(\xi_i, \xi_j) = -2\xi_k\)

(iii) \(T(\xi_i, X) = 0, \ X \in H\).

From Proposition 4.2, Theorem 4.9, (3.32) and (3.34), we obtain

**Corollary 4.13.** Any 3-Sasakian manifold of dimension \(4n + 3 > 7\) is a qc-Einstein with positive qc-scalar curvature
\[
\text{Scal} = 16n(n + 2).
\]

For any \(s, t, r = 1, 2, 3\), the Ricci-type tensors are given by
\[
\rho_{(\xi_i|_H} = \tau_{(i|H} = -2\zeta_{(i|H} = -2\omega_{i},
\]
\[
Ric(\xi_s, X) = \rho_s(X, \xi_t) = \zeta_s(X, \xi_t) = 0,
\]
\[
\rho_s(\xi_{s(t}}, \xi_r) = 0.
\]

In fact, 3-Sasakian spaces are locally the only qc-Einstein manifolds in dimension grater than 7 (cf. Theorem 1.3). Before we turn to the proof of this fact we shall consider some special cases of QC-structures suggested by above example. We recall that the Nijenhuis tensor \(N_i\) corresponding to \(I_i\) on \(H\) is defined as usual by
\[
N_i(X,Y) = [I_iX, I_iY] - [X, Y] - I_i[I_iX, Y] - I_i[X, I_iY], \ X, Y \in H.
\]
Definition 4.14. A quaternionic contact structure \((\eta, Q)\) is said to be hyperhermitian contact (HC structure for short) if the horizontal bundle \(H\) is formally integrable with respect to \(I_1, I_2, I_3\) simultaneously, i.e. for \(i = 1, 2, 3\) and any \(X, Y \in H\), we have

\[
N_{I_i}(X, Y) = 0 \mod V.
\]

In fact a QC structure is locally a HC structure exactly when two of the almost complex structures on \(H\) are formally integrable due to the following identity essentially established in [AM, (3.4.4)]

\[
2N_{I_3}(X, Y) - N_{I_1}(X, Y) + I_2N_{I_1}(I_2X, Y) + I_2N_{I_1}(X, I_2Y) - N_{I_1}(I_2X, I_2Y) - N_{I_2}(X, I_1Y) = 0 \mod V.
\]

On the other hand, the Nijenhuis tensor has the following expression in terms of a connection \(\nabla\) with torsion \(T\) satisfying (3.30) (see e.g. [IV])

\[
N_{I_i}(X, Y) = T^{0,2}_{I_i}(X, Y) + \beta_i(Y)I_jX - \beta_i(X)I_jY - I_i\beta_i(Y)I_kX + I_i\beta_i(X)I_kY,
\]

where \((i, j, k)\) is an even permutation of \((1, 2, 3)\), the \(1\)-forms \(\beta_i\) and the \((0,2)\)-part of the torsion \(T^{0,2}_{I_i}\) with respect to the almost complex structure \(I_i\) are defined on \(H\), correspondingly, by

\[
\beta_i = \alpha_j + I_i\alpha_k,
\]

\[
T^{0,2}_{I_i}(X, Y) = T(X, Y) - T(I_iX, I_iY) + I_iT(I_iX, Y) + I_iT(X, I_iY).
\]

Applying the above formulas to the Biquard connection and taking into account (3.21) one sees that \((\beta_i)|_H = 0\). Hence we have the following proposition.

Proposition 4.15. A quaternionic contact structure (abbr. QC) \((\eta, Q)\) is a hyperhermitian contact structure if and only if the connection 1-forms satisfy the relations

\[
\alpha_j(X) = \alpha_k(I_iX), \quad X \in H
\]

for any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\). The Nijenhuis tensors of a HC structure satisfy \(N_{I_i}(X, Y) = T^{0,2}_{I_i}(X, Y)\), \(X, Y \in H\).

Given a QC structure \((\eta, Q)\) let us consider for any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\) the three almost complex structures \((\eta_i, \tilde{I}_i)\) defined by
\[ \tilde{I}_i X = I_i X, \quad X \in H, \quad \tilde{I}_i(\xi_j) = \xi_k, \quad \tilde{I}_i(\xi_i) = 0. \]

With these definitions \((\eta_i, \tilde{I}_i)\) are almost CR structures (i.e. possibly non-integrable) exactly when the QC structure is HC since the condition

\[ d\eta(\tilde{I}_i X, \tilde{I}_i \xi_j) = d\eta(X, \xi_j) \]

is equivalent to \(\alpha_k(X) = \alpha_j(I_1 X)\) in view of (3.31). Hence, \(d\eta_i\) is an (1,1)-form with respect to \(\tilde{I}_i\) on \(\xi_i^\perp = H \oplus \{ \xi_j, \xi_k \}\) and a HC structure supports a non integrable hyper CR-structure \((\eta_i, \tilde{I}_i)\).

A natural question is to examine when \(\tilde{I}_i\) is formally integrable, i.e \(N_{\tilde{I}_i} = 0 \mod \xi_i\).

**Proposition 4.16.** Let \((\eta, Q)\) be a hyperhermitian contact structure. Then the CR structures \((\eta_i, \tilde{I}_i)\) are integrable if and only if the next two equalities hold

\[ d\eta_j(\xi_k, \xi_i) = d\eta_k(\xi_i, \xi_j), \quad d\eta_j(\xi_j, \xi_i) - d\eta_k(\xi_k, \xi_i) = 0. \]  

**Proof.** From (3.4) it follows \(T^{0,2}_{\tilde{I}_1}(X, Y) = 0\) using also (4.24). Substituting the latter into (4.22) taken with respect to \(\tilde{I}_i\) shows \(N_{\tilde{I}_i}|_H = 0 \mod \xi_1\) is equivalent to (4.25). Corollary 3.6 implies

\[ N_{\tilde{I}_i}(X, \xi_j) = (\alpha_j(I_1 X) + \alpha_k(X))\xi_1 + (\alpha_j(\xi_k) + \alpha_k(\xi_j))I_kX + (\alpha_j(\xi_j) - \alpha_k(\xi_k))I_jX + T(\xi_k, I_1 X) - I_1 T(\xi_k, X) - T(\xi_j, X) - I_1 T(\xi_j, I_1 X). \]

Take the trace part and the trace-free part in the right-hand side to conclude \(N_{\tilde{I}_i}(X, \xi_j) = 0 \mod \xi_i\) is equivalent to the system

\[ T(\xi_k, I_1 X) - I_1 T(\xi_k, X) - T(\xi_j, X) - I_1 T(\xi_j, I_1 X) = 0, \]

\[ \alpha_j(\xi_k) + \alpha_k(\xi_j) = 0, \quad \alpha_j(\xi_j) - \alpha_k(\xi_k) = 0. \]

An application of Proposition 3.3, (2.16) and (2.17) shows the first equality is trivially satisfied while (3.32) tells us that the other equalities are equivalent to (4.27). Formula (4.6) together with (4.27) yields the constancy assertion. \(\square\)
4.2. Proof of Theorem 1.3

Proof of Theorem 1.3. Let \((M^{4n+3}, \tilde{\eta}, Q)\) be a qc-Einstein manifold with qc-scalar curvature \(\text{Scal}\). Accordingly to the Theorem 1.9, \(\text{Scal}\) is a global constant on \(M\). We define \(\eta = \frac{\text{Scal}}{16n(n+2)}\tilde{\eta}\). Then \((M^{4n+3}, \eta, Q)\) is a qc-Einstein manifold with qc-scalar curvature \(\text{Scal} = 16n(n+2)\), horizontal distribution \(H = \text{Ker}(\eta)\) and involutive vertical distribution \(V = \text{span}\{\xi_1, \xi_2, \xi_3\}\) (see (5.11), (5.13) and (5.14)).

We shall show that the Riemannian cone is a hyper-Kähler manifold. Consider the structures defined by (4.26). It is easy to check the relations

\[
\eta_i(\xi_j) = \delta_{ij}, \quad \eta_i \tilde{I}_j = -\eta_j \tilde{I}_i = \eta_k, \quad \tilde{I}_i \xi_j = -\tilde{I}_j \xi_i = \xi_k
\]

(4.28)

\[
\tilde{I}_i \tilde{I}_j - \eta_i \otimes \xi_i = -\tilde{I}_j \tilde{I}_i + \eta_i \otimes \xi_j = \tilde{I}_k
\]

\[
\tilde{I}_j^2 = -\text{Id} + \eta_i \otimes \xi_i, \quad \eta_i \tilde{I}_j = 0, \quad \tilde{I}_i \xi_i = 0, \quad g(\tilde{I}_i, \tilde{I}_j) = g(\ldots) - \eta_i(\ldots)\eta_j(\ldots),
\]

where \((i, j, k)\) is any cyclic permutation of \((1,2,3)\).

Let \(D\) be the Levi-Civita connection of the metric \(g\) on \(M\) determined by the structure \((\eta, Q)\). The next step is to show

\[
D\tilde{I}_i = \text{Id} \otimes \eta_i - g \otimes \xi_i - \sigma_j \otimes \tilde{I}_k + \sigma_k \otimes \tilde{I}_j,
\]

(4.29)

for some appropriate 1-forms \(\sigma_s\) on \(M\). We consider all possible cases:

Case 1\([X, Y, Z \in H]\) The well known formula

\[
2g(D_AB, C) = Ag(B, C) + Bg(A, C) - Cg(A, B)
\]

\[
+ g([A, B], C) - g([B, C], A) - g([A, C], B), \quad A, B, C \in \Gamma(TM)
\]

yields

\[
2g((D_X \tilde{I}_i)Y, Z) = d\omega_i(X, Y, Z) - d\omega_i(X, I_i Y, I_i Z) + g(N_i(Y, Z), I_i X).
\]

We compute \(d\omega_i\) in terms of Biquard connection. Using (3.34), (3.30) and (1.23), we calculate

\[
d\omega_i(X, Y, Z) - d\omega_i(X, I_i Y, I_i Z) = -2\alpha_j(X)\omega_k(Y, Z) + 2\alpha_k(X)\omega_j(Y, Z)
\]

\[
- \beta_j(Y)\omega_k(Z, X) - I_i \beta_j(Y)\omega_j(Z, X) - \beta_i(Z)\omega_j(X, Y) - I_i \beta_i(Z)\omega_j(X, Y).
\]

(4.32)

Substitute (1.22) and (4.32) into (4.31) to derive

\[
g((D_X \tilde{I}_i)Y, Z) = -\alpha_j(X)\omega_k(Y, Z) + \alpha_k(X)\omega_j(Y, Z).
\]

Set \(\sigma_i(X) = \alpha_i(X)\) to get (1.22).
Case 2 [A, B ∈ V and Z ∈ H] Using the integrability of the vertical distribution V and (4.30), we compute

\begin{equation}
2g((D_A \tilde{I})B, Z) = 2g(D_A \tilde{I}B, Z) + 2g(D_A B, IZ) = -Zg(\tilde{I}B, A) - IZg(A, B) - g([I, B, Z], A) - g([A, Z], \tilde{I}B) - g([A, I, Z], B) - g([B, I, Z], A)
\end{equation}

\begin{equation}
\eta_s(A)\eta_2(B)d\eta_s(\xi_3, Z) - \eta_s(A)\eta_2(B)d\eta_s(\xi_2, Z) + \eta_s(A)\eta_2(B)d\eta_3(\xi_s, Z)
\end{equation}

Apply (2.11) to conclude \( g((D_A \tilde{I})B, Z) = 0 \).

Case 3 [X, Y ∈ H and C ∈ V] We have

\begin{align*}
2g((D_X \tilde{I}I)Y, C) &= 2g(D_X \tilde{I}I, Y) + 2g(D_X Y, \tilde{I}C) \\
&= -Cg(X, IY) - \tilde{I}Cg(X, Y) + g([X, IY], C) - g([X, C], IY) \\
&- g([IY, C], Y) + g([X, Y], \tilde{I}C) - g([X, I, Y], C) - g([Y, I, C], X)
\end{align*}

\begin{align*}
&= -\eta_s(C)(L_{\xi_s}g)(X, IY) - \eta_2(C)(L_{\xi_s}g)(X, Y) + \eta_3(C)(L_{\xi_s}g)(X, Y) - 2\eta(C)g(X, Y) \\
&= -2\eta(C)g(X, Y).
\end{align*}

where we have used (3.20) and \( T_{\xi_s} = 0 \), \( s = 1, 2, 3 \).

Case 4 [A, B, C ∈ V] We extend the definition of the three 1-forms \( \sigma_s \) on V as follows

\begin{equation}
\sigma_i(\xi_i) = 1 + \frac{1}{2}(d\eta_j(\xi_j, \xi_k) - d\eta_j(\xi_k, \xi_j) - d\eta_k(\xi_i, \xi_j))
\end{equation}

\begin{equation}
\sigma_i(\xi_j) = d\eta_j(\xi_j, \xi_k), \quad \sigma_i(\xi_k) = d\eta_k(\xi_j, \xi_k).
\end{equation}

An easy calculation leads to the formula

\begin{equation}
g(\tilde{I}A, B) = (\eta_j \wedge \eta_k)(A, B).
\end{equation}

On the other hand we have

\begin{equation}
2(D_A \eta_i)(B) = 2g((D_A \xi_i, B)
\end{equation}

\begin{align*}
&= A\eta_i(B) + \xi_i g(A, B) - B\eta_i(A) + g([A, \xi_i], B) - \eta([A, B], \xi_i) - 2\eta_i(A)\eta_j(\xi_j, \xi_k) - \eta_s(A)\eta_k(\xi_s, \xi_i) - \eta_s(A)\eta_k(\xi_s, \xi_i) + c\eta_s(B)\eta_k(A)d\eta_k(\xi_s, \xi_i) \\
&= 2\eta_j(\eta_k(A, B) - 2\sigma_j(A)\eta_k(B) + 2\sigma_k(A)\eta_j(B).
\end{align*}
Using (4.36) and (4.37) we compute

\[
\Psi(D\tilde{A}I)B,C) = D\tilde{A}(\eta_j \wedge \eta_k)(B,C) = [D\tilde{A}(\eta_j) \wedge \eta_k + \eta_j \wedge D\tilde{A}(\eta_k)](B,C)
\]

\[
= ((A,\eta_k \wedge \eta_j + \eta_j \wedge \eta_k)(B,C)
\]

\[
= (\eta_k(B)\eta_k, -\eta_j(B)\eta_j)(B,C) - \eta_j(A)\sigma_j(A)\eta_j - \eta_k(A)\sigma_k(A)\eta_k)
\]

\[
= \eta_i(B)g(A,C) - g(A,B)\eta_i(C) - \sigma_j(A)g(\tilde{I}_jB,C) + \sigma_k(A)g(\tilde{I}_kB,C)
\]

which completes the proof of (4.29). Now consider the Riemannian cone \( N = M \times \mathbb{R}^+ \) with the cone metric \( g_N = t^2g + dt \) and the almost complex structures

\[
\phi_i(E,f\frac{dt}{dt}) = (\tilde{I}_iE + \frac{f}{t}\xi_i, -t\eta_i(E)\frac{dt}{dt}), \quad i = 1, 2, 3, \quad E \in \Gamma(TM).
\]

Using the O'Neill formulas for warped product, [On], p.206, (4.28) and the just proved (4.29), one gets (see also [MO]) that the Riemannian cone \((N, g_N, \phi_i, i = 1, 2, 3)\) is a quaternionic Kähler manifold with connection 1-forms defined by (4.33) and (4.35). It is classical result (see e.g. [Bes]) that a quaternionic Kähler manifolds of dimension bigger than 4 are Einstein with non-negative scalar curvature. This fact implies that the cone \( N = M \times \mathbb{R}^+ \) with the warped product metric \( g_N \) must be Ricci flat (see e.g. [Bes], p.267) and therefore it is locally hyper-Kähler (see e.g. [Bes], p.397). This means that there locally exists a \( SO(3) \)-matrix \( \Psi \) with smooth entries such that the triple \( (\phi_1^*, \phi_2^*, \phi_3^*) = \Psi \cdot (\phi_1, \phi_2, \phi_3)^t \) is D-parallel. Consequently \((M, \Psi \cdot \eta, \mathbb{Q})\) is locally 3-Sasakian. Example 4.12 and Proposition 4.2 complete the proof.

\[\square\]

**Corollary 4.17.** Let \((M, \eta, \mathbb{Q})\) be a QC structure on a \((4n+3)\)-dimensional \((n > 1)\) manifold with non-zero qc-scalar curvature \( Scal \). The next conditions are equivalent

i) The structure \((\frac{16n(n+2)}{Scal}, \eta, \mathbb{Q})\) is locally 3-sasakian;

ii) The connection 1-forms of the Biquard connection vanish on \( H \),

\[
\alpha_i(X) = -d\eta_j(\xi_k, X) = 0, X \in H, \quad i,j,k = 1, 2, 3.
\]

**Proof.** In view of Theorem 4.3 and Example 4.12 it is sufficient to prove
Lemma 4.18. If a QC structure has zero connection one forms restricted to the horizontal space $H$ that it is qc-Einstein, or equivalently, it has zero torsion.

If $\alpha_i(X) = 0$ for $i = 1, 2, 3$ and $X \in H$ then (3.34) together with (3.4) yield

$$(4.39) \quad \rho_i(X,Y) = -\frac{1}{2}\alpha_i([X,Y]) = \frac{1}{2}\alpha_i(T(X,Y)) = \sum_{s=1}^{3} \alpha_i(\xi_s)\omega_s(X,Y).$$

Substitute (4.39) into (3.42) to conclude considering the $Sp(n)Sp(1)$-invariant parts of the obtained equalities that

$$T_0(X,Y) = U(X,Y) = \alpha_i(\xi_j) = 0, \quad \alpha_i(\xi_i) = -\frac{\text{Scal}}{8n(n+2)}. \quad \square$$

5. Conformal transformations of a QC-structure

Let $h$ be a positive smooth function on a QC manifold $(M,\eta,\mathbb{Q})$. Let $\tilde{\eta} = \frac{1}{2h}\eta$ be a conformal deformation of the QC structure $\eta$. We denote the objects related to $\tilde{\eta}$ by overlining the same object corresponding to $\eta$. Thus, $\overline{d\tilde{\eta}} = -\frac{1}{2h}dh \wedge \eta + \frac{1}{2h}d\eta$ and $\overline{g} = \frac{1}{2h}g$.

The new triple $\{\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\}$ is determined by (2.11). We have

$$(5.1) \quad \overline{\xi}_s = 2h\xi_s + I_s\overline{\nabla}h, \quad s = 1, 2, 3,$$

where $\overline{\nabla}h$ is the horizontal gradient defined by $g(\overline{\nabla}h, X) = dh(X), \quad X \in H$. The Biquard connections $\nabla$ and $\overline{\nabla}$ are connected by a $(1,2)$ tensor $S$,

$$(5.2) \quad \overline{\nabla}A \overline{B} = \nabla A \overline{B} + S_A \overline{B}, \quad A, B \in \Gamma(TM).$$

From $\overline{\nabla} \overline{g} = 0$ we get

$$(5.3) \quad g(S_X Y, Z) + g(S_X Z, Y) = -\frac{1}{h}dh(X)g(Y, Z), \quad X, Y, Z \in H.$$ 

The condition (5.4) yields

$$(5.4) \quad g(S_X Y, Z) - g(S_Y X, Z) = -\frac{1}{h}\sum_{s} \omega_s(X,Y)dh(I_s Z), \quad X, Y, Z \in H.$$ 

The equations (5.3) and (5.4) determine $g(S_X Y, Z)$ for $X, Y, Z \in H$ due to

$$(5.5) \quad g(S_X Y, Z) = -\frac{1}{2h}(dh(X)g(Y, Z) - \sum_{j=1}^{3} dh(I_sX)\omega_s(Y, Z) + dh(Y)g(Z, X) + \sum_{j=1}^{3} dh(I_sY)\omega_s(Z, X) - dh(Z)g(X, Y) + \sum_{j=1}^{3} dh(I_sZ)\omega_s(X, Y))$$
Using the Biquard’s Theorem 2.5, we obtain after some calculations that

\( g(\tilde{T}_{\xi_1} X, Y) - 2h g(T_{\xi_1} X, Y) - g(S_{\xi_1} X, Y) = \)

\( - \nabla dh(X, I_1 Y) + \frac{1}{h} (dh(I_3 X) dh(I_2 Y) - dh(I_2 X) dh(I_3 Y)) \).

Decomposing (5.6) into [3] and [-1] parts according to (2.8), and using the properties of the torsion tensor \( T_{\xi_1} \) we come to the formulas

\[
(5.7) \quad g(\sum_{s=1}^{3} T_0^{0_{s}} I_s X, Y) = \bar{g}(\sum_{s=1}^{3} \bar{T}_0^{0_{s}} I_s X, Y) - \frac{1}{h} [\nabla dh]_{[sym][-1]}(X, Y),
\]

\[
(5.8) \quad g(u X, Y) = \bar{g}(\bar{u} X, Y) - \frac{1}{2h} [\nabla dh - \frac{2}{h} \nabla \otimes dh]_{[3][0]}(X, Y),
\]

where \([.]_{[sym][-1]}\) and \([.]_{[3][0]}\) denote the symmetric \([-1]\)-component and the traceless \([3]\) part of the corresponding \((0,2)\) tensors on \( H \), respectively.

The identity \( d^2 = 0 \) yields

\[
(5.9) \quad \nabla dh(X, Y) - \nabla dh(Y, X) = -dh(T(X, Y)).
\]

Applying (5.9), we can write

\[
(5.10) \quad \nabla dh(X, Y) = [\nabla dh]_{[sym]}(X, Y) - \sum_{s=1}^{3} dh(\xi_s) \omega_s(X, Y),
\]

where \([.]_{[sym]}\) denotes the symmetric part of the corresponding \((0,2)\)-tensor.

Equations (5.7), (5.8) and (5.10) imply the next transformation formulas:

\[
(5.11) \quad \tilde{T}^{0}(X, Y) = T^{0}(X, Y) + \frac{1}{h} [\nabla dh]_{[sym][-1]},
\]

\[
(5.12) \quad \tilde{U}(X, Y) = U(X, Y) + \frac{1}{2h} [\nabla dh - \frac{2}{h} \nabla \otimes dh]_{[3][0]},
\]

\[
g(S_{\xi_1} X, Y) = -\frac{1}{4} (-\nabla dh(X, I_1 Y) + \nabla dh(I_1 X, Y) - \nabla dh(I_2 X, I_3 Y) + \nabla dh(I_3 X, I_2 Y))
\]

\[
- \frac{1}{2h} (dh(I_3 X) dh(I_2 Y) - dh(I_2 X) dh(I_3 Y) + dh(I_1 X) dh(Y) - dh(X) dh(I_1 Y))
\]

\[
+ \frac{1}{4h} \left( -\triangle h + \frac{2}{h} |\nabla h|^2 \right) g(I_1 X, Y) - dh(\xi_3) g(I_2 X, Y) + dh(\xi_2) g(I_3 X, Y),
\]

where \( \triangle h = tr^g_H(\nabla dh) = \sum_{a=1}^{4n} \nabla dh(e_a, e_a) \) is the sub Laplacian and \( |\nabla h|^2 = \sum_{a=1}^{4n} dh(a_a)^2 \) is the horizontal norm of \( dh \).

Thus, we proved the following Proposition.
Proposition 5.1. Let $\bar{\eta} = \frac{1}{2h}\eta$ be a conformal transformation of a given QC structure $\eta$. Then the trace-free parts of the corresponding qc-Ricci tensors are related by the equation

$$Ric_0(X,Y) - \bar{Ric}_0(X,Y) = -\frac{2n + 2}{h}[\nabla dh]_{\text{sym}}[-1](X,Y) - \frac{2n + 5}{h}[\nabla dh - \frac{2}{h}dh \otimes dh]_{[3][0]}(X,Y).$$

In addition, the qc-scalar curvature transforms by the formula

$$\bar{Scal} = 2h(Scal) - 8(n + 2)^2\frac{\|\nabla h\|^2}{h} + 8(n + 2)\Delta h.$$

5.1. Conformal transformations preserving the qc-Einstein condition. In this section we investigate the question of conformal transformations, which preserve the qc-Einstein condition. A straightforward consequence of (5.13) is the following

Proposition 5.2. Let $\tilde{\eta} = \frac{1}{2h}\eta$ be a conformal deformation of a given qc-structure $\eta$. Then the trace-free part of the qc-Ricci tensor does not change if and only if the function $h$ satisfies the differential equations

$$3(\nabla_X dh)Y - \sum_{s=1}^{3}(\nabla_{I_sX}dh)I_sY = -4\sum_{s=1}^{3}dh(\xi_s)\omega_s(X,Y),$$

$$3(\nabla_X dh)Y - \frac{2}{h}dh(X)dh(Y) + \sum_{s=1}^{3}\left[(\nabla_{I_sX}dh)I_sY - \frac{2}{h}dh(I_sX)dh(I_sY)\right] = \lambda g(X,Y),$$

for some smooth function $\lambda$ and any $X,Y \in H$.

Let us fix a normal frame $\{T_\alpha, X_\alpha = I_1T_\alpha, Y_\alpha = I_2T_\alpha, Z_\alpha = I_3T_\alpha, \xi_1, \xi_2, \xi_3\}, \alpha = 1, \ldots, n$ at a point $p \in M$.

Lemma 5.3. If $h$ satisfies (5.15) then we have at $p \in M$ the relations

$$T_\alpha X_\alpha(h) = Y_\alpha Z_\alpha(h) = -X_\alpha T_\alpha(h) = -Z_\alpha Y_\alpha(h) = -\xi_1(h),$$

$$T_\alpha Y_\alpha(h) = Z_\alpha X_\alpha(h) = -Y_\alpha T_\alpha(h) = -X_\alpha Z_\alpha(h) = -\xi_2(h),$$

$$T_\alpha Z_\alpha(h) = X_\alpha Y_\alpha(h) = -Z_\alpha T_\alpha(h) = -Y_\alpha X_\alpha(h) = -\xi_3(h).$$

Equivalently, we have

$$\epsilon_{ijk}T_\alpha(h) = -\epsilon_{ijk}(I_j T_\alpha h) = \xi_j h$$

$$\epsilon_{ijk}(I_j T_\alpha) = -\epsilon_{ijk}(I_i T_\alpha) = \xi_k h,$$

where $(i,j,k)$ is a cyclic permutation of $(1,2,3)$. 

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Proof. We derive from (5.15) that
\[4T_\alpha X_\alpha h - [T_\alpha, X_\alpha]h + [Y_\alpha Z_\alpha]h = -4\xi_1(h) .\]
Apply (3.4) and use Lemma 4.5 to get \([T_\alpha, X_\alpha] - [Y_\alpha Z_\alpha] = 0\). Hence, (5.17) follow. \(\square\)

5.2. Quaternionic Heisenberg group. Proof of Theorem 1.1. The proof of Theorem 1.1 will be presented as separate Propositions and Lemmas in the rest of the Section, see (5.49) for the final formula. The definition of the standard quaternionic contact structure on \(G(\mathbb{H})\) is in the next paragraph.

We will use the following model of the quaternionic Heisenberg group \(G(\mathbb{H})\). Define
\[G(\mathbb{H}) = \mathbb{H}^n \times \text{Im} \mathbb{H}\]
with the group law given by
\[(q', \omega') = (q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2 \text{ Im } q o \bar{q}) ,\]
where \(q, q_o \in \mathbb{H}^n\) and \(\omega, \omega_o \in \text{Im} \mathbb{H}\). In coordinates, with the obvious notation, the multiplication formula is
\[
t^\alpha = t^\alpha + t^\alpha_o , \quad x^\alpha = x^\alpha + x^\alpha_o , \quad y^\alpha = y^\alpha + y^\alpha_o , \quad z^\alpha = z^\alpha + z^\alpha_o ,
\]
\[x' = x + x_o + 2(z^\alpha_o t^\alpha_o - t^\alpha_o x^\alpha_o + z^\alpha_o y^\alpha_o - y^\alpha_o z^\alpha_o) ,\]
\[y' = y + y_o + 2(y^\alpha_o t^\alpha_o - z^\alpha_o x^\alpha_o - t^\alpha_o y^\alpha_o + x^\alpha_o z^\alpha_o) ,\]
\[z' = z + z_o + 2(z^\alpha_o t^\alpha_o + y^\alpha_o x^\alpha_o - x^\alpha_o y^\alpha_o - t^\alpha_o z^\alpha_o) .\]

A basis of left invariant horizontal vector fields \(T_\alpha, X_\alpha = I_1T_\alpha, Y_\alpha = I_2T_\alpha, Z_\alpha = I_3T_\alpha, \alpha = 1 \ldots n\) is given by
\[
T_\alpha = \frac{\partial}{\partial t^\alpha} + 2x^\alpha \frac{\partial}{\partial x} + 2y^\alpha \frac{\partial}{\partial y} + 2z^\alpha \frac{\partial}{\partial z} ,
\]
\[
X_\alpha = \frac{\partial}{\partial x} - 2t^\alpha \frac{\partial}{\partial z} - 2x^\alpha \frac{\partial}{\partial x} + 2y^\alpha \frac{\partial}{\partial y} ,
\]
\[
Y_\alpha = \frac{\partial}{\partial y} - 2z^\alpha \frac{\partial}{\partial x} - 2t^\alpha \frac{\partial}{\partial y} + 2x^\alpha \frac{\partial}{\partial z} ,
\]
\[
Z_\alpha = \frac{\partial}{\partial z} - 2y^\alpha \frac{\partial}{\partial x} - 2x^\alpha \frac{\partial}{\partial y} + 2t^\alpha \frac{\partial}{\partial z} .
\]

The central (vertical) vector fields \(\xi_1, \xi_2, \xi_3\) are described as follows
\[
\xi_1 = 2 \frac{\partial}{\partial x} , \quad \xi_2 = 2 \frac{\partial}{\partial y} , \quad \xi_3 = 2 \frac{\partial}{\partial z} .
\]

A small calculation shows the following commutator relations
\[
[I_j T_\alpha, T_\alpha] = 2\xi_j , \quad [I_j T_\alpha, I_k T_\alpha] = 2\xi_k .
\]
where \(i, j, k\) is a cyclic permutation of 1, 2, 3. The contact forms shall be denoted with \(\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2} (d\omega - q^\prime \cdot dq^\prime + dq^\prime \cdot \bar{q}^\prime),\) i.e.,

\[
\begin{align*}
\tilde{\Theta}_1 &= \frac{1}{2} dx - x^\alpha dt^\alpha + t^\alpha dx^\alpha - z^\alpha dy^\alpha + y^\alpha dz^\alpha, \\
\tilde{\Theta}_2 &= \frac{1}{2} dy - y^\alpha dt^\alpha + z^\alpha dx^\alpha + t^\alpha dy^\alpha - x^\alpha dz^\alpha, \\
\tilde{\Theta}_2 &= \frac{1}{2} dz - z^\alpha dt^\alpha - y^\alpha dx^\alpha + x^\alpha dy^\alpha + t^\alpha dz^\alpha.
\end{align*}
\]

The described horizontal and vertical vector fields are parallel for the Biquard connection and are an orthonormal basis of the tangent space.

We turn to the proof of Theorem 1.1. We start with a Proposition in which we shall determine the vertical Hessian of \(h\).

**Proposition 5.4.** If \(h\) satisfies \((5.15)\) on \(G(\mathbb{H})\) then we have the relations

\[
\xi_i \xi_j h = \xi_j \xi_i h = \xi_k \xi_i h = 0, \quad i \neq j = 1, 2, 3,
\]

where \(\mu_o > 0\) is a constant. In particular,

\[
\begin{align*}
\xi_i \xi_i h &= 8 \mu_o, \\
\xi_i \xi_j h &= 0, \quad i \neq j = 1, 2, 3,
\end{align*}
\]

for some real valued functions \(x_o, y_o, z_o\) on \(\mathbb{H}\). Furthermore we have

\[
T_\alpha Z_\alpha X_\alpha^2(h) = T_\alpha Z_\alpha Y_\alpha^2(h) = 0, \quad T_\alpha^2 \xi_j(h) = 0.
\]

**Proof.** With \(i, j, k\) denoting a cyclic permutation of 1, 2, 3 equations \((5.18)\) and \((5.21)\) yield the next sequence of equalities

\[
\begin{align*}
\xi_i \xi_j h &= -T_\alpha (I_i T_\alpha) \xi_j h = -T_\alpha \xi_j (I_i T_\alpha) h \\
&= \frac{1}{2} T_\alpha [T_\alpha, I_j T_\alpha] (I_i T_\alpha) h = \frac{1}{2} T_\alpha^2 (I_j T_\alpha) (I_i T_\alpha) h - T_\alpha (I_j T_\alpha) T_\alpha (I_i T_\alpha) h \\
&= \frac{1}{2} T_\alpha^2 \xi_k h - \frac{1}{2} \xi_i \xi_j h.
\end{align*}
\]

Hence, \(3 \xi_i \xi_j h = T_\alpha^2 \xi_k h\). Similarly, interchanging the roles of \(i\) and \(j\) together with \(\{I_i T_\alpha, I_j T_\alpha\} = 0\) we find \(3 \xi_j \xi_i h = -T_\alpha^2 \xi_k(h)\). Consequently

\[
\xi_i \xi_j h = T_\alpha^2 \xi_k h = 0.
\]

An analogous calculation shows that

\[
\xi_i \xi_k h = 0.
\]
Furthermore, we have
\[ \xi_1^2(h) = X_a T_a \xi_1(h) = X_a \xi_1 T_a(h) = \frac{1}{2} X_a [Y_a Z_a] T_a(h) \]
\[ = \frac{1}{2} (X_a Z_a Y_a T_a(h) - X_a Y_a Z_a T_a(h)) = \frac{1}{2} (\xi_3^2(h) + \xi_3^2(h)). \]
We derive similarly
\[ \xi_2^2(h) = \frac{1}{2} (\xi_1^2(h) + \xi_3^2(h)), \quad \xi_3^2(h) = \frac{1}{2} (\xi_2^2(h) + \xi_1^2(h)). \]
Therefore
\[ \xi_1^2(h) = \xi_2^2(h) = \xi_3^2(h), \quad \xi_1^3(h) = \xi_2^3(h) = \xi_3^3(h) = 0, \quad i \neq j = 1, 2, 3 \]
which proves part of (5.26). Next we prove that the common value of the second derivatives is a constant. For this we differentiate the equation \( T_a^2 \xi_k h = 0 \) with respect to \( I_k T_a \) from where and (5.18) and (5.21) it follows
\[ (5.26) \quad 0 = \xi_k (I_k T_a) T_a^2 h = \xi_k T_a (I_k T_a) T_a h + \xi_k [I_k T_a, T_a] h \]
\[ = T_a \xi_k^2 h + 2 T_a \xi_k^2 h = 3 T_a \xi_k^2 h. \]
In order to see the vanishing of \( (I_i T_a) \xi_k^2 h \) we shall need
\[ (5.27) \quad (I_i T_a)^2 \xi_k h = 0. \]
The latter can be seen by the following calculation.
\[ \xi_i \xi_j h = \xi_i (I_i T_a) (I_k T_a) h = (I_i T_a) \xi_i (I_k T_a) h \]
\[ = \frac{1}{2} T_a [T_a, I_j T_a] (I_i T_a) h = \frac{1}{2} (I_i T_a)^2 T_a (I_k T_a) h - \frac{1}{2} (I_i T_a) T_a (I_i T_a) (I_k T_a) h \]
\[ = - \frac{1}{2} (I_i T_a)^2 \xi_k h - \frac{1}{2} \xi_i \xi_j h, \]
from where
\[ 0 = 3 \xi_i \xi_j h = - (I_i T_a)^2 \xi_k h. \]
Hence, differentiating \( (I_i T_a)^2 \xi_k h = 0 \) with respect to \( I_j T_a \) gives
\[ (5.28) \quad 0 = \xi_k (I_j T_a) (I_i T_a)^2 \xi_k h = \xi_k (I_i T_a) (I_k T_a) [I_i, I_k] (I_i T_a) h \]
\[ = (I_i T_a) \xi_k^2 h + 2 (I_i T_a) \xi_k^2 h = 3 (I_i T_a) \xi_k^2 h. \]
We proved the vanishing of all derivatives of the common value of \( \xi_j^2 h \), i.e., this common value is a constant, which we denote by \( 8 \mu_o \). Let us note that \( \mu_o > 0 \) follows easily from the fact that \( h > 0 \) since \( g \) is independent of \( x, y \) and \( z \).
The rest equalities of the proposition follow easily from (5.17) and (5.23). □

In view of the above Proposition we define $h = g + \mu_0 f$, where

$$f = (x + x_0(q))^2 + (y + y_0(q))^2 + (z + z_0(q))^2.$$  

The following simple Lemma is one of the keys to integrating our system.

**Lemma 5.5.** Let $X$ and $Y$ be two parallel horizontal vectors

a) If $\omega_s(X, Y) = 0$ for $s = 1, 2, 3$ then

$$4XYh - \frac{2}{h} \{ dh(X) \, dh(Y) + \sum_{s=1}^{3}dh(I_sX) \, dh(I_sY) \} = \lambda g(X, Y).$$

b) If $g(X, Y) = 0$ then

$$2XYh - \frac{1}{h} \{ dh(X) \, dh(Y) + \sum_{s=1}^{3}dh(I_sX) \, dh(I_sY) \} = 2 \sum_{s=1}^{3} \{ (\xi_s)h \, \omega_s(X, Y) \}.$$

c) If $g(X, Y) = \omega_s(X, Y) = 0$ for $s = 1, 2, 3$ we have for any $j \in \{1, 2, 3\}$

$$(5.32) \quad XY(\xi_jh) = 0$$

$$(5.33) \quad 8XYh = \mu_0 \{ (X\xi_jf)(Y\xi_jf) + \sum_{s=1}^{3}(I_sX\xi_jf)(I_sY\xi_jf) \}.$$

**Proof.** The equation of a) and b) are obtained by adding (5.15) and (5.16). Let us prove part c). From (5.15) and (5.16) taking any two horizontal vectors $X$ and $Y$, such that, $g(X, Y) = \omega_s(X, Y) = 0$ we have

$$2h\nabla dh(X, Y) = dh(X) \, dh(Y) + \sum_{s=1}^{3}dh(I_sX) \, dh(I_sY).$$

If $X, Y$ are also parallel, differentiating along $\xi_j$ gives

$$2\xi_jh XYh + 2h XY \xi_jh$$

$$= (X\xi_jh)(Yh) + \sum_{s=1}^{3}[(I_sX\xi_jh)(I_sYh) + (I_sX\xi_jh)(I_sYh)].$$

Taking one more derivative we come to

$$2\xi_j^2h XYh + 4h XY \xi_jh = 2\{ (X\xi_jh)(Y\xi_jh) + \sum_{s=1}^{3}(I_sX\xi_jh)(I_sY\xi_jh) \}.$$ 

Hence, differentiating three times along $\xi_j$ and using $\xi_j^2h = \text{const}$, cf. (5.23), we obtain

$$2 (\xi_j^2h) XY(\xi_jh) = 0,$$
from where \[ (5.32). \] With this information \( (5.36) \) reduces to \( (5.33). \)

In order to see that after a suitable translation the functions \( x_o, y_o \) and \( z_o \) can be made equal to zero we prove the following proposition.

**Proposition 5.6.** If \( h \) satisfies \( (5.15) \) and \( (5.16) \) on \( G (\mathbb{H}) \) then we have

a) For \( s \in \{1, 2, 3\} \) and \( i, j, k \) a cyclic permutation of \( 1, 2, 3 \)

\[
\begin{align*}
T_\alpha T_\beta (\xi_s h) &= (I_i T_\alpha) (I_j T_\beta) (\xi_s h) = 0 \quad \forall \alpha, \beta \\
(I_i T_\alpha) T_\beta (\xi_s h) &= (I_i T_\alpha) (I_j T_\beta) (\xi_s h) = 0, \alpha \neq \beta \\
(I_j T_\alpha) T_\alpha (\xi_s h) &= - T_\alpha (I_j T_\alpha) (\xi_s h) = 8 \delta_{kj} \mu_o \\
(I_j T_\alpha) (I_i T_\alpha) (\xi_s h) &= - (I_i T_\alpha) (I_j T_\alpha) (\xi_s h) = 8 \delta_{ik} \mu_o , \\
\end{align*}
\]

i.e., the horizontal Hessian of a vertical derivative of \( h \) is determined completely.

b) There is a point \( (q_o, \omega_o) \in G (\mathbb{H}) \), \( q_o = (q_o^1, q_o^2, \ldots, q_o^n) \in \mathbb{H}^n \) and \( \omega = ix_o + jy_o + kz_o \in \text{Im}(\mathbb{H}) \), such that,

\[
\begin{align*}
x_o(q) &= x_o + 2(x_o^\alpha t_\alpha - t_o^\alpha x^\alpha + z_o^\alpha y^\alpha - y_o^\alpha z^\alpha) \\
y_o(q) &= y_o + 2(y_o^\alpha t_\alpha - z_o^\alpha x^\alpha - t_o^\alpha y^\alpha + x_o^\alpha z^\alpha) \\
z_o(q) &= z_o + 2(z_o^\alpha t_\alpha + y_o^\alpha x^\alpha - x_o^\alpha y^\alpha - t_o^\alpha z^\alpha) .
\end{align*}
\]

In other words we have

\[
ix_o(q) + jy_o(q) + kz_o(q) = w_o + 2 \text{ Im } q_o \bar{q}.
\]

**Proof.** a) Taking \( \alpha \neq \beta \) and using \( X = T_\alpha \) and \( Y = T_\beta \) in \( (5.32) \) we obtain

\[
T_\alpha T_\beta \xi_s h = 0, \quad \alpha \neq \beta.
\]

When \( \alpha = \beta \) the same equality holds by \( (5.27) \).

The vanishing of the other derivatives can be obtained similarly. Finally, the rest of the second derivatives can be determined from \( (5.15) \).

b) From the identities in \( (5.37) \) all second derivatives of \( x_o, y_o \) and \( z_o \) vanish. Thus \( x_o, y_o \) and \( z_o \) are linear function. The fact that the coefficients are related as required amounts to the following system

\[
\begin{align*}
T_\alpha x_o &= Z_\alpha y_o = -Y_\alpha z_o \\
X_\alpha x_o &= Y_\alpha y_o = Z_\alpha z_o \\
Y_\alpha x_o &= X_\alpha y_o = -T_\alpha z_o \\
Z_\alpha x_o &= -T_\alpha y_o = -X_\alpha z_o.
\end{align*}
\]
From (5.24) we have
\[\xi_1 h = 4\mu_o (x + x_o(q)), \quad \xi_2 h = 4\mu_o (y + y_o(q)), \quad \xi_3 h = 4\mu_o (z + z_o(q)).\]

Therefore the above system is equivalent to
\[
\begin{align*}
    T_\alpha \xi_1 h &= Z_\alpha \xi_2 h = -Y_\alpha \xi_3 h \\
    X_\alpha \xi_1 h &= Y_\alpha \xi_2 h = Z_\alpha \xi_3 h \\
    Y_\alpha \xi_1 h &= X_\alpha \xi_2 h = -T_\alpha \xi_3 h \\
    Z_\alpha \xi_1 h &= -T_\alpha \xi_2 h = -X_\alpha \xi_3 h
\end{align*}
\]

Let us prove the first line. Denote
\[a = T_\alpha \xi_1 h \quad b = Z_\alpha \xi_2 h \quad c = -Y_\alpha \xi_3 h.\]

From (5.17) and (5.21) it follows
\[
\begin{align*}
    a &= T_\alpha Z_\alpha Y_\alpha h = Z_\alpha T_\alpha Y_\alpha h + [T_\alpha, Z_\alpha] Y_\alpha h = -b + 2c \\
    b &= Z_\alpha Y_\alpha T_\alpha h = Y_\alpha Z_\alpha T_\alpha h + [Z_\alpha, Y_\alpha] T_\alpha h = -c + 2a \\
    c &= Y_\alpha T_\alpha Z_\alpha h = T_\alpha Y_\alpha Z_\alpha h + [Y_\alpha, T_\alpha] Z_\alpha h = -a + 2b,
\end{align*}
\]

which implies \(a = b = c\). The rest of the identities of the system can be obtained analogously.

So far we have proved that if \(h\) satisfies the system (5.15) and (5.16) on \(G(\mathbb{H})\) then, in view of the translation invariance of the system, after a suitable translation we have
\[h(q, \omega) = g(q) + \mu_o (x^2 + y^2 + z^2).\]

Our goal is to show that
\[g(q) = (b + 1 + \sqrt{\mu_o} |q|^2)^2.\]

**Proposition 5.7.** If \(h\) satisfies the system (5.15) and (5.16) on \(G(\mathbb{H})\) then after a suitable translation we have
\[g(q) = (b + 1 + \sqrt{\mu_o} |q|^2)^2, \quad b + 1 > 0.\]

**Proof.** As we already noted above the statement of the Proposition, we are left with finding the function \(g\). Notice that
\[
\begin{align*}
    \xi_1 h &= 4\mu_o x \\
    \xi_2 h &= 4\mu_o y \\
    \xi_3 h &= 4\mu_o z.
\end{align*}
\]
With this equations (5.17) become
\[ T_\alpha X_\alpha(h) = Y_\alpha Z_\alpha(h) = -X_\alpha T_\alpha(h) = -Z_\alpha Y_\alpha(h) = -4\mu_o x, \]
\[ T_\alpha Y_\alpha(h) = Z_\alpha X_\alpha(h) = -Y_\alpha T_\alpha(h) = -X_\alpha Z_\alpha(h) = -4\mu_o y, \]
\[ T_\alpha Z_\alpha(h) = X_\alpha Y_\alpha(h) = -Z_\alpha T_\alpha(h) = -Y_\alpha X_\alpha(h) = -4\mu_o z. \]

Let us also write explicitly some of the derivatives of \( f \), which shall be used to express the derivatives of \( g \) by the derivatives of \( h \). For all \( \alpha \) and \( \beta \) we have

\[
\begin{align*}
T_\beta f &= 4(x^\beta x + y^\beta y + z^\beta z) & X_\beta f &= 4(-t^\beta x - z^\beta y + y^\beta z) \\
Y_\beta f &= 4(z^\beta x - t^\beta y - x^\beta z) & Z_\beta f &= 4(-y^\beta x + x^\beta y - t^\beta z) \\
T_\alpha T_\beta f &= 8(x^\alpha x^\beta + y^\alpha y^\beta + z^\alpha z^\beta) & X_\alpha X_\beta f &= 8(t^\alpha t^\beta + z^\alpha z^\beta + y^\alpha y^\beta) \\
Y_\alpha Y_\beta f &= 8(z^\alpha z^\beta + t^\alpha t^\beta + x^\alpha x^\beta) & Z_\alpha Z_\beta f &= 8(y^\alpha y^\beta + x^\alpha x^\beta + t^\alpha t^\beta) \\
T_\alpha X_\beta f &= -4\delta_\alpha\beta x + 8(-x^\alpha t^\beta - y^\alpha z^\beta + z^\alpha y^\beta) \\
T_\alpha Y_\beta f &= -4\delta_\alpha\beta y + 8(x^\alpha z^\beta - y^\alpha t^\beta - z^\alpha x^\beta) \\
T_\alpha Z_\beta f &= -4\delta_\alpha\beta z + 8(-x^\alpha y^\beta + y^\alpha x^\beta - z^\alpha t^\beta) \\
X_\alpha T_\beta f &= 4\delta_\alpha\beta x + 8(-t^\alpha x^\beta - z^\alpha y^\beta + y^\alpha z^\beta) \\
X_\alpha Y_\beta f &= -4\delta_\alpha\beta y + 8(-t^\alpha z^\beta + z^\alpha t^\beta - y^\alpha x^\beta) \\
X_\alpha Z_\beta f &= 4\delta_\alpha\beta y + 8(t^\alpha y^\beta - z^\alpha x^\beta + y^\alpha t^\beta) & \forall \alpha, \beta, \text{ etc.}
\end{align*}
\]

From the above formulas we see that the fifth order horizontal derivatives of \( f \) vanish. In particular the fifth order derivatives of \( h \) and \( g \) coincide.

Taking \( X = Y = T_\alpha \) in (5.30) we obtain
\[ 4T_\alpha^2 h - \frac{2}{h} \{ (T_\alpha h)^2 + (X_\alpha h)^2 + (Y_\alpha h)^2 + (Z_\alpha h)^2 \} = \lambda. \]

Using in the same manner \( X_\alpha, Y_\alpha \) and \( Z_\alpha \) we see the equality of the second derivatives
\[ T_\alpha^2 h = X_\alpha^2 h = Y_\alpha^2 h = Z_\alpha^2 h. \]

Therefore, using (5.17) and (5.23), we have
\[ T_\alpha^3 h = T_\alpha X_\alpha^2 h = X_\alpha T_\alpha X_\alpha h + [T_\alpha, X_\alpha]X_\alpha h = 3X_\alpha \xi_1 h = 24\mu_o t^\alpha \]
and thus
\[ T_\alpha^4 h = 24\mu_o. \]

In the same fashion we conclude
\[ T_\alpha^3 h = 24\mu_o t^\alpha \quad X_\alpha^3 h = 24\mu_o x^\alpha \quad Y_\alpha^3 h = 24\mu_o y^\alpha \quad Z_\alpha^3 h = 24\mu_o z^\alpha. \]
Similarly, taking $X = T_\alpha$, $Y = X_\beta$ and $j = 1$ in (5.33) we find
\[ T_\alpha X_\beta h = 8\mu_0 (-x^\alpha t^\beta + t^\alpha x^\beta - y^\alpha z^\beta + z^\alpha y^\beta), \quad \alpha \neq \beta. \]

Plugging $X = T_\alpha$, $Y = T_\beta$ with $\alpha \neq \beta$ in (5.33) we obtain
\[ T_\alpha T_\beta h = X_\alpha h X_\beta h = Y_\alpha h Y_\beta h = Z_\alpha h Z_\beta h = 8\mu_0(t^\alpha t^\beta + x^\alpha x^\beta + y^\alpha y^\beta + z^\alpha z^\beta). \]

The other mixed second order derivatives when $\alpha \neq \beta$ can be obtained by taking suitable $X$ and $Y$. In view of the formulas for the derivatives of $f$ we conclude
\[ (5.44) \]
\[ T_\alpha X_\beta g = 8\mu_0 t^\alpha x^\beta, \quad T_\alpha Y_\beta g = 8\mu_0 t^\alpha y^\beta, \quad T_\alpha Z_\beta g = 8\mu_0 t^\alpha z^\beta \]
\[ X_\alpha Y_\beta g = 8\mu_0 x^\alpha y^\beta, \quad X_\alpha Z_\beta g = 8\mu_0 x^\alpha z^\beta, \quad Y_\alpha Z_\beta g = 8\mu_0 y^\alpha z^\beta \]
and also, again for $\alpha \neq \beta$,
\[ (5.45) \]
\[ T_\alpha T_\beta g = 8\mu_0 t^\alpha t^\beta, \quad X_\alpha X_\beta g = 8\mu_0 x^\alpha x^\beta \]
\[ Y_\alpha Y_\beta g = 8\mu_0 y^\alpha y^\beta, \quad Z_\alpha Z_\beta g = 8\mu_0 z^\alpha z^\beta \]

Finally, from (5.38) we see
\[ (5.46) \]
\[ T_\alpha X_\alpha g = 8\mu_0 t^\alpha x^\alpha, \quad T_\alpha Y_\alpha g = 8\mu_0 t^\alpha y^\alpha, \quad T_\alpha Z_\alpha g = 8\mu_0 t^\alpha z^\alpha \]
\[ X_\alpha Y_\alpha g = 8\mu_0 x^\alpha y^\alpha, \quad X_\alpha Z_\alpha g = 8\mu_0 x^\alpha z^\alpha, \quad Y_\alpha Z_\alpha g = 8\mu_0 y^\alpha z^\alpha. \]

A consequence of the considerations so far is the fact that all second order derivative are quadratic functions of the variables from the first layer, except the pure second derivatives, in which case we know (5.40) and (5.43). It is easy to see then that the fifth order horizontal derivatives of $h$ vanish. With the information so far after a small argument we can assert that $g$ is a polynomial of degree 4 without terms of degree 3, and of the form
\[ g = \mu_0 \sum_{\alpha=1}^n (t_\alpha^4 + x_\alpha^4 + y_\alpha^4 + z_\alpha^4) + p_2, \]
where $p_2$ is a polynomial of degree two. Furthermore, the mixed second order derivatives of $g$ are determined, while the same second order derivatives are equal. The latter follows from (5.39) taking $q = 0$, $\omega = 0$. Let us see that there are no terms of degree one on $p_2$. 


Taking $X = T_\alpha$, $Y = T_\beta$, $\alpha \neq \beta$ and $j = 1$ in (5.35) we find

$$(4 \mu_o x) \left\{ 4T_\alpha T_\beta g + 32 \mu_o \left( x^\alpha x^\beta + y^\alpha y^\beta + z^\alpha z^\beta \right) \right\}$$

$$= 2 \left\{ (8x^\alpha)(T_\beta g + 4 \mu_o (x^\beta x + y^\beta y + z^\beta z)) + (-8t^\alpha)(x_\beta g + 4 \mu_o (-t^\beta x - 3y^\beta y + z^\beta z)) \right.$$ 

$$+ (8z^\alpha)(Y_\beta g + 4 \mu_o (z^\beta x - t^\beta y - x^\beta z)) + (-8y^\alpha)(Z_\beta g + 4 \mu_o (-y^\beta x + x^\beta y - t^\beta z)) \right.$$ 

$$+ (8x^\beta)(T_\alpha g + 4 \mu_o (x^\alpha x + y^\alpha y + z^\alpha z)) + (-8t^\beta)(X_\alpha g + 4 \mu_o (-t^\alpha x - 3y^\alpha y + z^\alpha z)) \right.$$ 

$$+ (8z^\beta)(Y_\alpha g + 4 \mu_o (z^\alpha x - t^\alpha y - x^\alpha z)) + (-8y^\beta)(Z_\alpha g + 4 \mu_o (-y^\alpha x + x^\alpha y - t^\alpha z)) \left\} \right.$$ 

$$= 16 \left( x^\alpha T_\beta g + x^\beta T_\alpha g - t^\alpha X_\beta g - t^\beta X_\alpha g + z^\alpha Y_\beta g + z^\beta Y_\alpha g - y^\alpha Z_\beta g - y^\beta Z_\alpha g \right)$$

$$+ 128 \mu_o \ (t^\alpha t^\beta + x^\alpha x^\beta + t^\alpha t^\beta + x^\alpha x^\beta)$$

Taking into account (5.45) we proved

$$(5.47) \quad x^\alpha T_\beta g + x^\beta T_\alpha g - t^\alpha X_\beta g - t^\beta X_\alpha g$$

$$+ z^\alpha Y_\beta g + z^\beta Y_\alpha g - y^\alpha Z_\beta g - y^\beta Z_\alpha g = 0 \quad \alpha \neq \beta.$$

Comparing coefficients in front of the linear terms implies that $g$ has no first order terms.

Thus, we can assert that $g$ can be written in the following form

$$(5.48) \quad g = \left( 1 + \sqrt{\mu_o} \ |q|^2 \right)^2 + 2a \ |q|^2 + b$$

hence

$$h = \left( 1 + \sqrt{\mu_o} \ |q|^2 \right)^2 + a \ |q|^2 + b + \mu_o (x^2 + y^2 + z^2).$$

Taking $X = T_\alpha$, $Y = T_\beta$ in (5.31) we obtain

$$16 \mu_o (1 + b) = 4 (a + 2 \sqrt{\mu_o})^2.$$
Therefore,

\[ g = \mu_o |q|^4 + (a + 2\sqrt{\mu_o}) \sqrt{\mu_o} |q|^2 + b + 1 = 2\sqrt{b + 1} |q|^2 + b + 1 = (b + 1 + \sqrt{\mu_o} |q|^2)^2. \]

In turn the formula for \( h \) becomes

\[ (5.49) \quad h = (b + 1 + \sqrt{\mu_o} |q|^2)^2 + \mu_o (x^2 + y^2 + z^2). \]

Setting

\[ c = (b + 1)^2 \quad \text{and} \quad \nu = \frac{\sqrt{\mu_o}}{1 + b} > 0 \]

the solution takes the following form

\[ h = c \left[ (1 + \nu |q|^2)^2 + \nu^2 (x^2 + y^2 + z^2) \right], \]

which completes the proof. Let us note that the final conclusion can be reached also using the fact that a qc-Einstein structure has necessarily constant scalar curvature by Theorem 4.9 together with the result of [GV1] identifying all partially symmetric solutions of the Yamabe equation on \( G(\mathbb{H}) \), i.e., of the equation

\[ \sum_{\alpha=1}^{n} (T_{\alpha}^2 u + X_{\alpha}^2 u + Y_{\alpha}^2 u + Z_{\alpha}^2 u) = -u^{\frac{n+2}{2}}. \]

The fact that we are dealing with such a solution follows from (5.48). The current solution depends on one more parameter as the scalar curvature can be an arbitrary constant. This constant will appear in the argument of [GV1] by first using scalings to reduce to a fixed scalar curvature one for example.

\[ \square \]

6. Special functions and pseudo-Einstein quaternionic contact structures

Considering only the \([3]\)-component of the Einstein tensor of the Biquard connection due to Theorem 3.12 and by analogy with the CR-case [L1], it seems useful to give the following

**Definition 6.1.** Let \((M, \eta, Q)\) be a quaternionic contact manifold of dimension bigger than 7. We call \((M, \eta, Q)\) qc-pseudo-Einstein if the trace-free part of the \([3]\)-component of the qc-Einstein tensor vanishes.
According to Theorem 3.12 \((M, \eta, Q)\) is quaternionic qc-pseudo-Einstein exactly when the trace-free part of the \([3]\)-component of the torsion vanishes, \(U = 0\). Proposition 5.1 yields the following claim.

**Proposition 6.2.** Let \(\tilde{\eta} = u\eta\) be a conformal deformation of a given qc-structure. Then the trace-free part of the qc-Ricci tensor is preserved if and only if the function \(u\) satisfies the differential equations

\[
(\nabla_X du)Y + (\nabla_{I_1X} du)I_1 Y + (\nabla_{I_2X} du)I_2 Y + (\nabla_{I_3X} du)I_3 Y = \frac{1}{n} \Delta u g(X,Y).
\]

In particular, the qc-pseudo-Einstein condition persists under conformal deformation \(\tilde{\eta} = u\eta\) exactly when the function \(u\) satisfies \((6.1)\).

**Proof.** Defining \(h = \frac{1}{u}\) a small calculation shows

\[
(\nabla dh - \frac{2}{h} dh \otimes dh) = \frac{1}{u} \nabla du.
\]

Insert \((6.2)\) into \((5.16)\) to get \((6.1)\). \(\square\)

Our next goal is to investigate solutions to \((6.1)\). In this section we find geometrically defined functions solving \((6.1)\).

### 6.1. Quaternionic pluriharmonic functions.**

**We start with some analysis on the quaternion space \(\mathbb{H}^n\).**

#### 6.1.1. Pluriharmonic functions in \(\mathbb{H}^n\).

Let \(\mathbb{H}\) be the four-dimensional real associative algebra of the quaternions. The elements of \(\mathbb{H}\) are of the form \(q = t + ix + jy + kz\), where \(t, x, y, z \in \mathbb{R}\) and \(i, j, k\) are the basic quaternions satisfying the multiplication rules

\[ i^2 = j^2 = k^2 = -1 \text{ and } ijk = -1. \]

For a quaternion \(q\) we define its conjugate \(\bar{q} = t - ix - jy - kz\), and real and imaginary parts, correspondingly, by

\[
\Re q = t \quad \text{and} \quad \Im q = xi + yj + zk.
\]

The most important operator for us will be the Dirac-Feuter operator \(\nabla F = \frac{\partial}{\partial t} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\), i.e.,

\[
\nabla F = \frac{\partial F}{\partial t} + i \frac{\partial F}{\partial x} + j \frac{\partial F}{\partial y} + k \frac{\partial F}{\partial z}.
\]
and in addition

\[ \mathcal{D} F = \frac{\partial F}{\partial t} - i\frac{\partial F}{\partial x} - j\frac{\partial F}{\partial y} - k\frac{\partial F}{\partial z}. \]

Note that if \( F \) is a quaternionic valued function due to the non-commutativity of the multiplication the above expression is not the same as \( F \mathcal{D} \) defined as

\[ F \mathcal{D} \overset{\text{def}}{=} \frac{\partial F}{\partial t} + i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}. \]

Also, when conjugating \( \mathcal{D} F \neq \mathcal{D} \bar{F} \).

**Definition 6.3.** A function \( F : \mathbb{H} \to \mathbb{H} \), which is continuously differentiable when regarded as a function of \( \mathbb{R}^4 \) into \( \mathbb{R}^4 \) is called quaternionic anti-regular (quaternionic regular), or just anti-regular (regular) for short, if \( \mathcal{D} F = 0 \) (\( \mathcal{D} \bar{F} = 0 \)).

These functions were introduced by Fueter [F]. The reader can consult the paper of A. Sudbery [S] for the basics of the quaternionic analysis on \( \mathbb{H} \). Let us note explicitly one of the most striking differences between complex and quaternionic analysis. As it is well known the theory of functions of a complex variable \( z \) is equivalent to the theory of power series of \( z \). In the quaternionic case, each of the coordinates \( t, x, y \) and \( z \) can be written as a polynomial in \( q \), see eq. (3.1) of [S], and hence the theory of power series of \( q \) is just the theory of real analytic functions. Our goal here is to consider functions of several quaternionic variables in \( \mathbb{H}^n \) and on manifolds with quaternionic structure and present some applications in geometry.

For a point \( q \in \mathbb{H}^n \) we shall write \( q = (q^1, \ldots, q^n) \) with \( q^\alpha \in \mathbb{H} \), \( q^\alpha = t^\alpha + ix^\alpha + jy^\alpha + kz^\alpha \) for \( \alpha = 1, \ldots, n \). Furthermore, \( q^\bar{\alpha} = q^{\bar{\alpha}} \), i.e., \( q^\bar{\alpha} = t^{\alpha} - ix^{\alpha} - jy^{\alpha} - kz^{\alpha} \).

We recall that a function \( F : \mathbb{H}^n \to \mathbb{H} \), which is continuously differentiable when regarded as a function of \( \mathbb{R}^{4n} \) into \( \mathbb{R}^4 \) is called quaternionic regular, or just regular for short, if

\[ \mathcal{D}_\alpha F = \frac{\partial F}{\partial t_\alpha} + i\frac{\partial F}{\partial x_\alpha} + j\frac{\partial F}{\partial y_\alpha} + k\frac{\partial F}{\partial z_\alpha} = 0, \]

for every \( \alpha = 1, \ldots, n \).

In other words, a real-differentiable function of several quaternionic variables is regular if it is regular in each of the variables (see [Per1], [Per2], [Joy]). The condition that \( F = \)
\( f + iw + ju + kv \) is regular is equivalent to the following Cauchy-Riemann-Feuter equations

\[
\begin{align*}
\frac{\partial f}{\partial t} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} &= 0 \\
\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial u}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} - \frac{\partial f}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{align*}
\]

**Definition 6.4.** A real-differentiable function \( f : \mathbb{H}^n \to \mathbb{R} \) is called \( Q \)-pluriharmonic if it is the real part of a regular function.

**Proposition 6.5.** Let \( f \) be a real-differentiable function \( f : \mathbb{H}^n \to \mathbb{R} \). The following conditions are equivalent

i) \( f \) is \( Q \)-pluriharmonic;

ii) \( \overline{D}_\beta D_\alpha f = 0 \) for every \( \alpha, \beta \in \{1, \ldots, n\} \), where

\[ D_\beta = \frac{\partial}{\partial t} - i \frac{\partial}{\partial x} - j \frac{\partial}{\partial y} - k \frac{\partial}{\partial z} \]

iii) \( f \) satisfies the following system of PDEs

\[
\begin{align*}
\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= 0 \\
\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= 0 \\
\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} &= 0 \\
\frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2} &= 0.
\end{align*}
\]

**Proof.** It is easy to check that \( \overline{D}_\beta D_\alpha f = 0 \) is equivalent to (6.5).

We turn to the proof of ii) implies i). Let \( f \) be real valued function on \( \mathbb{H}^n \), such that, \( \overline{D}_\beta D_\alpha f = 0 \). We shall construct a real-differentiable regular function \( F : \mathbb{H}^n \to \mathbb{H} \). In fact, for \( q \in \mathbb{H}^n \) we define

\[
F(q) = f(q) + \Im \int_0^1 s^2 (D_\alpha f)(sq) q^\alpha \, ds.
\]
In order to rewrite the imaginary part in a different way we compute

\[ \Re \int_0^1 s^2 (D_\alpha f)(sq) q_\alpha ds \]

\[ = \Re \int_0^1 s^2 \left( \frac{\partial f}{\partial t_\alpha} - i \frac{\partial f}{\partial x_\alpha} - j \frac{\partial f}{\partial y_\alpha} - k \frac{\partial f}{\partial z_\alpha} \right) (sq) \left( t_\alpha + ix_\alpha + jy_\alpha + kz_\alpha \right) ds \]

\[ = \int_0^1 s^2 \left( \frac{\partial f}{\partial t_\alpha} (sq) t_\alpha + \frac{\partial f}{\partial x_\alpha} (sq) x_\alpha + \frac{\partial f}{\partial y_\alpha} (sq) y_\alpha + \frac{\partial f}{\partial z_\alpha} (sq) z_\alpha \right) ds \]

\[ = \int_0^1 s^2 \frac{d}{ds} \left( f(sq) \right) ds = s^2 f(sq) \big|_0^1 - 2 \int_0^1 s f(sq) ds \]

\[ = f(q) - 2 \int_0^1 s f(sq) ds. \]

Therefore we have

\[ \Im \int_0^1 s^2 (D_\alpha f)(sq) q_\alpha ds = \int_0^1 s^2 (D_\alpha f)(sq) q_\alpha ds - f(q) + 2 \int_0^1 s f(sq) ds. \]

In turn, the formula for \( F(q) \) becomes

\[ F(q) = \int_0^1 s^2 (D_\alpha f)(sq) q_\alpha ds + 2 \int_0^1 s f(sq) ds. \]

Hence

\[ \overline{D}_\beta F(q) = \int_0^1 s^2 \overline{D}_\beta \left[ (D_\alpha f)(sq) q_\alpha \right] ds + 2 \int_0^1 s \overline{D}_\beta \left[ f(sq) \right] ds. \]

Next we compute the term in the first integral above,

\[ \overline{D}_\beta \left[ (D_\alpha f)(sq) q_\alpha \right] = \left( \frac{\partial}{\partial t_\beta} + i \frac{\partial}{\partial x_\beta} + j \frac{\partial}{\partial y_\beta} + k \frac{\partial}{\partial z_\beta} \right) \left[ (D_\alpha f)(sq) q_\alpha \right] \]

\[ = \overline{D}_\beta \left[ (D_\alpha f)(sq) \right] q_\alpha + D_\alpha f(sq) \frac{\partial q_\alpha}{\partial t_\beta} + i D_\alpha f(sq) \frac{\partial q_\alpha}{\partial x_\beta} + j D_\alpha f(sq) \frac{\partial q_\alpha}{\partial y_\beta} + k D_\alpha f(sq) \frac{\partial q_\alpha}{\partial z_\beta} \]

\[ = \overline{D}_\beta \left[ (D_\alpha f)(sq) \right] q_\alpha + \delta_\alpha_\beta \{ D_\alpha f(sq) + i D_\alpha f(sq)i + j D_\alpha f(sq)j + k D_\alpha f(sq)k \}. \]
The last term can be simplified, using the fundamental property that the coordinates of a quaternion can be expressed by the quaternion only, as follows

\[ D_\beta f(sq) + iD_\beta f(sq)i + jD_\beta f(sq)j + kD_\beta f(sq)k \]

\[ = \left( \frac{\partial f}{\partial t_\beta} - i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} \right) + i\left( \frac{\partial f}{\partial t_\beta} - i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} \right)i + j\left( \frac{\partial f}{\partial t_\beta} - i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} \right)j + k\left( \frac{\partial f}{\partial t_\beta} - i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} \right)k \]

\[ = -2 \frac{\partial f}{\partial t_\beta} - i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} + i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} \]

\[ - i \frac{\partial f}{\partial x_\beta} + j \frac{\partial f}{\partial y_\beta} - k \frac{\partial f}{\partial z_\beta} - i \frac{\partial f}{\partial x_\beta} - j \frac{\partial f}{\partial y_\beta} + k \frac{\partial f}{\partial z_\beta} \]

\[ = -2 \frac{\partial f}{\partial t_\beta} - 2i \frac{\partial f}{\partial x_\beta} - 2j \frac{\partial f}{\partial y_\beta} - 2k \frac{\partial f}{\partial z_\beta} = -2D_\beta f(sq). \]

Going back to the computation of \( \overline{D}_\beta F(q) \) we find

\[ \overline{D}_\beta F(q) = \int_0^1 \overline{D}_\beta \left[ (D_\alpha f)(sq) \right] q^\alpha ds - 2 \int_0^1 s^2\overline{D}_\beta f(sq) ds + 2 \int_0^1 s^2\overline{D}_\beta f(sq) ds \]

\[ = \int_0^1 \overline{D}_\beta \left[ (D_\alpha f)(sq) \right] q^\alpha ds. \]

Hence, if \( \overline{D}_\beta D_\alpha f = 0 \) for every \( \alpha \) and \( \beta \) we have \( \overline{D}_\beta F(q) = 0 \).

Next we show that i) implies ii). We calculate using (6.3) that

\[ \frac{\partial^2 f}{\partial x_\beta \partial t_\alpha} - \frac{\partial^2 f}{\partial t_\beta \partial x_\alpha} + \frac{\partial^2 f}{\partial z_\beta \partial y_\alpha} - \frac{\partial^2 f}{\partial y_\beta \partial z_\alpha} = \]

\[ \frac{\partial^2 w}{\partial x_\beta \partial x_\alpha} + \frac{\partial^2 u}{\partial x_\beta \partial y_\alpha} + \frac{\partial^2 v}{\partial x_\beta \partial z_\alpha} - \frac{\partial^2 w}{\partial t_\beta \partial t_\alpha} - \frac{\partial^2 u}{\partial t_\beta \partial y_\alpha} - \frac{\partial^2 v}{\partial t_\beta \partial z_\alpha} \]

\[ - \frac{\partial^2 u}{\partial z_\beta \partial t_\alpha} - \frac{\partial^2 v}{\partial z_\beta \partial y_\alpha} - \frac{\partial^2 w}{\partial z_\beta \partial z_\alpha} - \frac{\partial^2 u}{\partial y_\beta \partial t_\alpha} - \frac{\partial^2 v}{\partial y_\beta \partial y_\alpha} - \frac{\partial^2 w}{\partial y_\beta \partial z_\alpha}. \]

Both sides must be equal to zero by noticing that the left hand side is antisymmetric while on the right we have an expression symmetric with respect to exchanging \( \alpha \) with \( \beta \). The other identities can be obtained similarly. \( \square \)

According to [Sti] there are exactly two kinds of Cauchy-Riemann equations for functions of several quaternionic variables. The second one turns out to be most suitable for the geometric purposes considered in this paper.
Definition 6.6. A function \( F : \mathbb{H}^n \to \mathbb{H} \), which is continuously differentiable when regarded as a function of \( \mathbb{R}^{4n} \) into \( \mathbb{R}^4 \) is called quaternionic anti-regular, or just anti-regular for short, if
\[
\mathcal{D}F = \frac{\partial F}{\partial t_\alpha} - i\frac{\partial F}{\partial x_\alpha} - j\frac{\partial F}{\partial y_\alpha} - k\frac{\partial F}{\partial z_\alpha} = 0,
\]
for every \( \alpha = 1, \ldots, n \).

The condition that \( F = f + iw + ju + kv \) is anti-regular function is equivalent to the following Cauchy-Riemann-Feuter equations
\begin{align*}
\frac{\partial f}{\partial t_\alpha} + \frac{\partial u}{\partial x_\alpha} + \frac{\partial v}{\partial y_\alpha} + \frac{\partial w}{\partial z_\alpha} &= 0, \\
\frac{\partial w}{\partial t_\alpha} - \frac{\partial f}{\partial x_\alpha} - \frac{\partial v}{\partial y_\alpha} + \frac{\partial u}{\partial z_\alpha} &= 0, \\
\frac{\partial u}{\partial t_\alpha} + \frac{\partial v}{\partial x_\alpha} - \frac{\partial f}{\partial y_\alpha} - \frac{\partial w}{\partial z_\alpha} &= 0, \\
\frac{\partial v}{\partial t_\alpha} - \frac{\partial u}{\partial x_\alpha} + \frac{\partial w}{\partial y_\alpha} - \frac{\partial f}{\partial z_\alpha} &= 0.
\end{align*}
(6.6)

See also (6.9) for an equivalent form of the above system.

Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in \cite{CL1,CL2,LZ} in connection with minimal surfaces and quaternionic maps between quaternionic Kähler manifolds.

Definition 6.7. A real-differentiable function \( f : \mathbb{H}^n \to \mathbb{R} \) is called pluriharmonic if it is the real part of an anti-regular function.

The anti-regular functions and their real part play a significant rôle in the theory of hypercomplex manifold as well as in the theory of quaternionic contact (hypercomplex contact) manifolds as we shall see further in the paper. We need a real expression of the second order differential operator \( \mathcal{D}_\alpha \mathcal{D}_\beta f \) acting on a real function \( f \).

We use the standard hypercomplex structure on \( \mathbb{H}^n \) determined by the action of the imaginary quaternions
\begin{align*}
J_1 dt^\alpha &= dx^\alpha & J_1 dy^\alpha &= dz^\alpha \\
J_2 dt^\alpha &= dy^\alpha & J_2 dx^\alpha &= -dz^\alpha
\end{align*}
We recall a convention. For any p-form \( \psi \) we consider the p-form \( I_s \psi \) and three \((p+1)\)-forms \( d_s \psi \), \( s = 1, 2, 3 \) defined by

\[
J_s \psi (X_1, \ldots, X_p) := (-1)^p \psi (J_s X_1, \ldots, J_s X_p), \quad d_s \psi := (-1)^p J_s d J_s \psi.
\]

Consider the second order differential operators \( DD_{J_s} \) acting on the exterior algebra defined in [HP] by

\[
(6.7) \quad DD_{J_s} := \frac{\partial^2}{\partial t_\alpha \partial t_\alpha} + \frac{\partial^2}{\partial x_\beta \partial x_\alpha} + \frac{\partial^2}{\partial y_\beta \partial y_\alpha} + \frac{\partial^2}{\partial z_\beta \partial z_\alpha} = 0,
\]

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

**Proposition 6.8.** Let \( f \) be a real-differentiable function \( f : \mathbb{H}^n \to \mathbb{R} \). The following conditions are equivalent

i) \( f \) is pluriharmonic, i.e. it is the real part of an anti-regular function;

ii) \( DD_{I_s} f = 0, \quad s = 1, 2, 3; \)

iii) \( \mathcal{D}_\alpha \overline{\mathcal{D}_\beta} f = 0 \) for every \( \alpha, \beta \in \{1, \ldots, n\} \), where

\[
\mathcal{D}_\beta = \frac{\partial}{\partial t_\alpha} - i \frac{\partial}{\partial x_\alpha} - j \frac{\partial}{\partial y_\alpha} - k \frac{\partial}{\partial z_\alpha},
\]

iv) \( f \) satisfies the following system of PDEs

\[
\frac{\partial^2 f}{\partial t_\beta \partial t_\alpha} + \frac{\partial^2 f}{\partial x_\beta \partial x_\alpha} + \frac{\partial^2 f}{\partial y_\beta \partial y_\alpha} + \frac{\partial^2 f}{\partial z_\beta \partial z_\alpha} = 0,
\]

\[
-\frac{\partial^2 f}{\partial x_\beta \partial t_\alpha} + \frac{\partial^2 f}{\partial t_\beta \partial t_\alpha} - \frac{\partial^2 f}{\partial y_\beta \partial y_\alpha} + \frac{\partial^2 f}{\partial z_\beta \partial z_\alpha} = 0,
\]

\[
\frac{\partial^2 f}{\partial t_\beta \partial y_\alpha} + \frac{\partial^2 f}{\partial t_\beta \partial x_\alpha} - \frac{\partial^2 f}{\partial y_\beta \partial t_\alpha} - \frac{\partial^2 f}{\partial z_\beta \partial x_\alpha} = 0,
\]

\[
\frac{\partial^2 f}{\partial t_\beta \partial z_\alpha} - \frac{\partial^2 f}{\partial x_\beta \partial y_\alpha} + \frac{\partial^2 f}{\partial y_\beta \partial x_\alpha} - \frac{\partial^2 f}{\partial z_\beta \partial t_\alpha} = 0.
\]

**Proof.** A simple calculation of \( \mathcal{D}_\beta \overline{\mathcal{D}_\alpha} f \) gives the equivalence between iii) and iv).

Next we shall show that ii) is equivalent to iii). As \( df = \frac{\partial f}{\partial t_\alpha} dt^{\alpha} + \frac{\partial f}{\partial x_\alpha} dx^{\alpha} + \frac{\partial f}{\partial y_\alpha} dy^{\alpha} + \frac{\partial f}{\partial z_\alpha} dz^{\alpha} \) we have

\[
I_1 df = \frac{\partial f}{\partial t_\alpha} dx^{\alpha} - \frac{\partial f}{\partial x_\alpha} dt^{\alpha} + \frac{\partial f}{\partial y_\alpha} dz^{\alpha} - \frac{\partial f}{\partial z_\alpha} dy^{\alpha}.
\]
Now we calculate using (6.21) that

\[
DD_{I_1} f = \Re(D_\beta \overline{D}_\alpha f) \, dt^\alpha \wedge dx^\beta - \Re(kD_\beta \overline{D}_\alpha f) \, dt^\alpha \wedge dy^\beta + \Re(jD_\beta \overline{D}_\alpha f) \, dt^\alpha \wedge dz^\beta \]

and similar formulas hold for \(DD_{I_2}\) and \(DD_{I_3}\).

Hence the equivalence of ii) and iii) follows.

The proof of the implication iii) implies i) is analogous to the proof of the corresponding implication in the Proposition 6.5. We define

\[
F(q) = f(q) + \Im \int_0^1 s^2 (D_\beta f)(sq) \overline{q} \, ds,
\]

and a small calculation shows that this defines an anti-regular function, i.e., \(D_\alpha F = 0\) for every \(\alpha\).

In order to see that iii) follows from i) we can proceed as in Proposition 6.5 and hence we skip the details. See also another proof in Proposition 6.11

Remark 6.9. We note that Proposition 6.5 and Proposition 6.8 imply that the real part of a regular function is not in the kernel of the operators \(DD_{I_\alpha}\) which is one of the main difference between regular and anti-regular function.

6.2. Quaternionic pluriharmonic functions on hypercomplex manifold. We recall that a hypercomplex manifold is a smooth 4n-dimensional manifold \(M\) together with a triple \((J_1, J_2, J_3)\) of integrable almost complex structures satisfying the quaternionic relations \(J_1J_2 = -J_2J_1 = J_3\). The second order differential operators \(DD_{J_\alpha}\) defined in [HP] by (6.21) having the origin in the papers [Sal1, Sal2, CSal] play an important rôle in the theory of quaternionic plurisubharmonic functions (i.e. a real function for which \(DD_{J_\alpha}(., ., J_\alpha)\) is positive definite) on hypercomplex manifold [A1, A2] as well as the potential theory of HKT-manifolds. We recall that Riemannian metric \(g\) on a hypercomplex manifold compatible with the three complex structures is said to be HKT-metric [HP] if the three
corresponding Kähler forms $\Omega_s = g(J_s,.)$ satisfy $d_1\Omega_1 = d_2\Omega_2 = d_3\Omega_3$. A smooth real function is a HKT-potential if locally it generates the three Kähler forms, $\Omega_s = DD_J_s f$ [MS, GP], in particular such a function is quaternionic plurisubharmonic. The existence of a HKT potential on any HKT metric on $\mathbb{H}^n$ is proved in [MS] and for any HKT metric in [BS].

Regular functions on hypercomplex manifold are studied from analytical point [Per1, Per2], from algebraic point [Joy, Q]. However, as we have already mentioned, regular functions are not the appropriate functions for our purposes mainly because they have no direct connection with the second order differential operator $DD_J_s$.

Here we consider anti-regular functions and their real parts on hypercomplex manifold.

**Definition 6.10.** Let $(M, J_1, J_2, J_3)$ be a hypercomplex manifold. A quaternionic valued function $F : M \rightarrow f + iw + ju + kv \in \mathbb{H}$ is said to be anti-regular if any one of the following relations between the differentials of the coordinates hold

\[
\begin{align*}
df - d_1w - d_2u - d_3v &= 0 \\
d_1f + dw - d_3u + d_2v &= 0 \\
d_2f + d_3w + du - d_1v &= 0 \\
d_3f - d_2w + d_1u + dv &= 0.
\end{align*}
\]

(6.9)

A real valued function $f : M \rightarrow \mathbb{R}$ is said to be pluriharmonic if it is the real part of anti-regular function.

Observe that the system (6.6) is equivalent to (6.9). We have the hypercomplex manifold analogue of Proposition 6.8.

**Proposition 6.11.** Let $(M, J_1, J_2, J_3)$ be a hypercomplex manifold and let $f$ be a real-differentiable function on $M$, $f : M \rightarrow \mathbb{R}$. The following conditions are equivalent

i) $f$ is pluriharmonic, i.e. it is the real part of an anti-regular function;

ii) $DD_J_s f = 0, \ s = 1,2,3,.$

**Proof.** It is easy to verify that if each $I_s$ is integrable almost complex structure then we have the identities [HP]

\[
\begin{align*}
dd_s + d_s d &= 0, \\
d_s d_r + d_r d_s &= 0, s, r = 1,2,3.
\end{align*}
\]

(6.10)
Using the commutation relations (6.10), we easily get i) implies ii). For example, (6.9) yields

\[ dd_1 f + d_2 d_3 f + d_1^2 w + d_2^2 w + dd_3 u - d_2 d_1 u - dd_2 v - d_2 dv = 0, \]
\[ d_1 df + d_3 d_2 f + d_2^2 w - d_3 w + d_1 d_2 u - d_3 d_1 u + d_1 d_3 v + d_3 d_1 v = 0. \]

Subtracting the two equations and using the commutation relations (6.10) we get \( DD J_1 = 0. \)

For the converse, observe that \( DD I_1 f = 0 \) if and only if \( dd_1 f = I_2 dd_1 f. \) The \( \partial\bar{\partial} \)-lemma for \( I_2 \) gives the existence of a smooth function \( A_1 \) such that \( dd_1 f = dd_2 A_1. \) Similarly, using the Poincare lemma, we obtain

\[ d_1 f - d_2 A_1 - dB_1 = 0, \quad d_2 f - d_3 A_2 - dB_2 = 0, \quad d_3 f - d_1 A_3 - dB_3 = 0 \]

for a smooth functions \( A_1, A_2, A_3, B_1, B_2, B_3 \) which yields

\[ df + d_3 A_1 + d_1 B_1 = 0, \]
\[ df + d_1 A_2 + d_2 B_2 = 0, \]
\[ df + d_2 A_3 + d_3 B_3 = 0. \]

The latter implies

\[ df + d_1 (A_2 + B_1) + d_2 (B_2 - A_3) + d_3 (A_1 - B_3) = 0. \]

Set \( w = -A_2 - B_1, u = A_3 - B_2, v = B_3 - A_1 \) to get the equivalence between i) and ii). □

6.2.1. \textit{Restriction on hyper-surfaces.} In this section we shall denote with \( \langle ., . \rangle \) the Euclidean scalar product in \( \mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1} \) and with \( \tilde{I}_j, j = 1, 2, 3 \) or, respectively, \( \tilde{I}, \tilde{J} \) and \( \tilde{K} \) the standard almost complex structures on \( \mathbb{H}^{n+1} \). Let \( M \) be a smooth hyper-surface in \( \mathbb{H}^{n+1} \) with a defining function \( \rho, M = \{ \rho = 0 \}, d\rho \neq 0, \) and \( i : M \hookrightarrow \mathbb{H}^{n+1} \) be the embedding. It is not hard to see that at every point \( p \in M \) the subspace \( H_p = \bigcap_{j=1}^3 \tilde{I}_j (T_p M) \) of the tangent space \( T_p M \) of \( M \) at \( p \) is the largest subspace invariant under the almost complex structures and \( \dim H_p = 4n. \) We shall call \( H_p \) the horizontal space at \( p. \) Thus on the horizontal space \( H \) the almost complex structures \( I_j, j = 1, 2, 3, \) are the restrictions of the standard almost complex structures on \( \mathbb{H}^{n+1}. \) In particular, for a horizontal vector \( X \) we have

\[ (6.11) \quad \tilde{I}_j i_* X = i_* (I_j X). \]
We shall also use the notation \( I, J \) and \( K \), correspondingly, for \( I_1, I_2 \) and \( I_3 \). Let \( \tilde{\theta}^j = \tilde{I}_j \frac{d\rho}{|d\rho|} \). Finally, we shall drop the tilda in the notation of the almost complex structures when there is no ambiguity.

We define three one-forms on \( M \) by setting
\[
\theta_j (\cdot) = -\frac{d\rho (\tilde{I}_j \cdot)}{|d\rho|} = \langle \cdot, \tilde{I}_j N \rangle,
\]
where \( N = \frac{D\rho}{|D\rho|} \) is the unit normal vector to \( M \). In the next proposition we describe the hypersurfaces which inherit a natural quaternionic contact structure from the standard structures on \( \mathbb{H}^{n+1} \) (see also [D1]).

**Proposition 6.12.** If \( M \) is a smooth hypersurface of \( \mathbb{H}^{n+1} \) then we have
\[
\theta_1 (I_1 X, Y) = \theta_2 (I_2 X, Y) = \theta_3 (I_3 X, Y) \quad (X, Y \in H)
\]
if and only if the restriction of the second fundamental form of \( M \) to the horizontal space is invariant with respect to the almost complex structures, i.e. if \( X \) and \( Y \) are two horizontal vectors we have \( II(I_1 X, I_j Y) = II(X, Y) \). Furthermore, if the restriction of the second fundamental form of \( M \) to the horizontal space is positive definite, \( II(X, X) > 0 \) for any non-zero horizontal vector \( X \), then \((M, \theta, I, J)\) is a quaternionic contact manifold.

**Proof.** Let \( D \) be the Levi-Civita connection on \( \mathbb{R}^{4n+4} \), \( N = \frac{D\rho}{|D\rho|} \) be the unit normal vector to \( M \), and \( X, Y \) be two horizontal vectors. As the horizontal space is the intersection of the kernels of the one forms \( \theta_j \) we have
\[
\theta_1 (I_1 X, Y) = -\theta_1 ([\tilde{I}_1 X, Y]) = -\langle [\tilde{I}_1 X, Y], \tilde{I}_1 N \rangle
\]
\[
= -\langle D_{\tilde{I}_1 Y} X - D_Y (\tilde{I}_1 X), \tilde{I}_1 N \rangle = -\langle D_{\tilde{I}_1 X} Y, \tilde{I}_1 N \rangle + \langle D_Y (\tilde{I}_1 X), \tilde{I}_1 N \rangle
\]
\[
= \langle D_{\tilde{I}_1 X} (\tilde{I}_1 Y), N \rangle + \langle D_Y X, N \rangle \quad \text{(as \( D\tilde{I}_j = 0 \))}
\]
\[
= II(\tilde{I}_1 X, \tilde{I}_1 Y) + II(X, Y).
\]
Therefore \( \theta_1 (I_1 X, Y) = \theta_2 (I_2 X, Y) \) iff \( II(I_j X, I_j Y) = II(X, Y) \).

The last claim of the proposition is obvious from the above formula. In particular, \( g_H(X, Y) = II(X, Y) \) is a metric on the horizontal space when the second fundamental form is positive definite on the horizontal space and we have
\[
d\theta_1 (I_1 X, Y) = 2g_H(X, Y).
\]
and hence \((M, \eta, I, J)\) becomes a quaternionic contact structure. We shall denote the corresponding horizontal forms with \(\omega_j\), i.e., \(\omega_j(X, Y)g_H(I_jX, Y)\).

Let us note also that in the situation as above \(g = g_H + \eta_j \otimes \eta_j\) is a Riemannian metric on \(M\). In view of the above observations we define a QC-hypersurface of \(\mathbb{H}^{n+1}\) as follows.

**Definition 6.13.** We say that a smooth embedded hypersurface of \(\mathbb{H}^{n+1}\) is a QC-hypersurface if the restriction of the second fundamental form of \(M\) to the horizontal space is a definite symmetric form, which is invariant with respect to the almost complex structures.

Clearly every sphere in \(\mathbb{H}^{n+1}\) is a QC-hypersurface and this is true also for many ellipsoids, for example \(\sum a|q_a|^2 b_a = 1\). In fact, a hypersurface of \(\mathbb{H}^{n+1}\) is a QC-hypersurface if and only if the (Euclidean) Hessian of the defining function \(\rho\) is a symmetric definite matrix from \(GL(n + 1, \mathbb{H})\), the latter being the linear group of invertible matrices which commute with the standard complex structures on \(\mathbb{H}^{n+1}\).

**Proposition 6.14.** Let \(i: M \to \mathbb{H}^n\) be a QC hypersurface in \(\mathbb{H}^n\), \(f\) a real-valued function on \(M\). If \(f = i^*F\) is the restriction to \(M\) of a real-valued function \(F\) defined on \(\mathbb{H}^n\), which is the real part of an anti-regular function \(F + iW + jU + kV\), then:

\[
\begin{align*}
\text{(6.15)} & \quad df = d(i^*F) = d_1(i^*W) + d_2(i^*U) + d(i^*V),
\text{(6.16)} & \quad DD_1f(X, I_1Y) = -4dF(D\rho)g_H(X, Y) - 4(\xi_2 f)\omega_2(X, Y)
\end{align*}
\]

for any horizontal vector fields \(X, Y \in H\).

**Proof.** Let us prove first (6.15). Denote with small letters the restrictions of the functions defined on \(\mathbb{H}^n\). For \(X \in H\) from (6.11) we have

\[
(i^* \tilde{I}_1 dW)(X) = (\tilde{I}_1 dW)(i_+ X) = -dW(\tilde{I}_1 i_+ X) = -dW(i_+ (I_1 X)) = -dw(I_1 X) = d_1 w(X).
\]

Applying the same argument to the functions \(U\) and \(V\) we see the validity of (6.16).

Our goal is to write the equation for \(f\) on \(M\), using the fact that \(f = i^*F\).

We define the function \(\lambda\) by

\[
\lambda = \frac{dF(D\rho)}{|D\rho|^2},
\]

and the one-form \(d_M F\) equal to

\[
d_M F = df - \lambda dp.
\]
Thus the one-form $df$ satisfies the equation

\[(6.17) \quad df = i^*(d_M F + \lambda d\rho) = i^*(d_M F),\]

taking into account that $(\lambda \circ i) d(\rho \circ i) = 0$ as $\rho$ is constant on $M$. From Proposition 6.8, the assumption on $F$ is equivalent to $DD_{\tilde{I}} F = 0$. Therefore, we have

\[
0 = DD_{\tilde{I}} F = d\tilde{I}dF - \tilde{J}d\tilde{I}dF
\]

\[
= d(\tilde{I}d_M F + \lambda \tilde{I}d\rho) - \tilde{J}d(\tilde{I}d_M F + \lambda \tilde{I}d\rho)
\]

\[
= d\tilde{I}d_M F + d\lambda \wedge \tilde{I}d\rho - \tilde{J}d\tilde{I}d_M F - \tilde{J}(d\lambda \wedge \tilde{I}d\rho) - \lambda \tilde{J}\tilde{I}d\rho.
\]

Restricting to $M$, and in fact, to the horizontal space $H$ we find

\[(6.18) \quad 0 = i^*(DD_{\tilde{I}} F)|_H
\]

\[
= i^*d(\tilde{I}d_M F)|_H + d(\lambda \circ i) \wedge i^*(\tilde{I}d\rho)|_H + (\lambda \circ i)di^*(\tilde{I}d\rho)|_H
\]

\[
- i^*(\tilde{J}d\tilde{I}d_M F)|_H - i^*\tilde{J}(d\lambda \wedge \tilde{I}d\rho)|_H - (\lambda \circ i) i^*(\tilde{J}d\tilde{I}d\rho)|_H.
\]

Since the horizontal space is in the kernel of the one-forms $\theta_j = \tilde{I}_j d\rho|_H$ it follows that

\[(6.19) \quad i^* (\tilde{I}_j d\rho)|_H = 0.
\]

Hence, two of the terms in (6.18) are equal to zero, and we have

\[(6.20) \quad 0 = i^*(DD_{\tilde{I}} F)|_H = i^*(d\tilde{I}d_M F - \tilde{J}d\tilde{I}d_M F)|_H
\]

\[
+ (\lambda \circ i) i^*(d\tilde{I}d\rho - \tilde{J}d\tilde{I}d\rho)|_H.
\]

In other words for horizontal $X$ and $Y$ we have

\[(6.21) \quad i^*(d\tilde{I}d_M F - \tilde{J}d\tilde{I}d_M F)(X, IY) = -(\lambda \circ i) i^*(d\tilde{I}d\rho - \tilde{J}d\tilde{I}d\rho)(X, IY)
\]

The right-hand side is proportional to the metric. Indeed, recall

\[(6.22) \quad i^*(\tilde{I}_j d\rho)(X) = |d\rho| \theta_j(X) \quad d\theta_j(X, Y) = 2g(I_j X, Y).
\]

Hence the identity

\[(6.23) \quad i^*(d\tilde{I}d\rho - \tilde{J}d\tilde{I}d\rho)(X, Y) = 2|d\rho| g(I X, Y) - 2|d\rho| g(I J X, J Y)
\]

\[
= 2|d\rho| g(I X, Y) - 2g(K X, J Y) = 4|d\rho| g(I X, Y).
\]
Let us consider now the term in the left-hand side of (6.21). Decomposing \(d_M F\) into horizontal and vertical parts we write

\[ d_M F = d_H f + F_j \tilde{\theta}^j. \]

From the definitions of the forms \(\tilde{\theta}^j\) we have

\[ \tilde{I}_1 \tilde{\theta}^1 = \frac{dp}{|dp|}, \quad \tilde{I}_1 \tilde{\theta}^2 = \tilde{\theta}^3, \quad \tilde{I}_1 \tilde{\theta}^1 \tilde{\theta}^3 = -\tilde{\theta}^2. \]

Therefore

\[(6.24) \quad d\tilde{I}d_M F = d\tilde{I}d_H F + dF_j \wedge \tilde{I} \theta^j + F_1 d\left(\frac{dp}{|dp|}\right) + F_2 d\tilde{\theta}^3 - F_3 d\tilde{\theta}^2 \]

and also

\[(6.25) \quad \tilde{J}d\tilde{I}d_M F = \tilde{J}d\tilde{I}d_H F + \tilde{J}dF_j \wedge \tilde{I} \theta^j - |dp|^{-2}\tilde{J}d|dp| \wedge d\rho + F_2 \tilde{J}d\tilde{\theta}^3 - F_3 \tilde{J}d\tilde{\theta}^2. \]

From \(J \theta^3 = -d\theta^3, \quad Jd\theta^2 = d\theta^2\) and the above it follows

\[(6.26) \quad i^*(d\tilde{I}d_M F - \tilde{J}d\tilde{I}d_M F)|_H = DD_1 f + F_2 d\theta^3 - F_3 d\theta^2 + F_2 d\theta^3 + F_2 d\theta^2 \]

In conclusion, we proved

\[ DD_1 f(X,Y) = -4(\lambda \circ i) |\nabla \rho| g(IX,Y) - 4F_2 \omega_3(X,Y) \]

from where the claim of the Proposition.

\[ \square \]

6.3. **Anti-CRF functions on Quaternionic contact manifold.** Let \((M, \eta, \mathbb{Q})\) be a \((4n+3)\)-dimensional quaternionic contact manifold and \(\nabla\) denote the Biquard connection on \(M\). The equation (6.15) suggests the following

**Definition 6.15.** A smooth \(\mathbb{H}\)-valued function \(F : M \rightarrow \mathbb{H}\),

\[ F = f + iw + ju + kv, \]

is said to be an anti-CRF function if the smooth real valued functions \(f, w, u, v\) satisfy

\[(6.27) \quad df = d_1 w + d_2 u + d_3 v \mod \eta, \]

where \(d_i = I_i \circ d\).

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Choosing a local frame \( \{ T_a, X_a = I_1 T_a, Y_a = I_2 T_a, Z_a = I_3 T_a, \xi_1, \xi_2, \xi_3 \} \), \( a = 1, \ldots, n \) it is easy to check that a \( \mathbb{H} \)-valued function \( F = f + iw + ju + kv \) is an anti-CRF function if it belongs to the kernel of the operators

\[
D_{T_a} = T_a - iX_a - jY_a - kZ_a, \quad D_{T_a} F = 0, \quad \alpha = 1, \ldots n.
\]

**Remark 6.16.** We note that anti-CRF functions have different properties than the CRF functions \([Per1, Per2]\) which are defined to be in the kernel of the operator

\[
\overline{D}_{T_a} = T_a + iX_a + jY_a + kZ_a, \quad \overline{D}_{T_a} F = 0, \quad \alpha = 1, \ldots n.
\]

Equation (6.27) and a small calculation give the following Proposition.

**Proposition 6.17.** A \( \mathbb{H} \)-valued function

\[
F = f + iw + ju + kv
\]

is an anti-CRF function if and only if the smooth functions \( f, w, u, v \) satisfy the horizontal Cauchy-Riemann-Fueter equations

\[
(6.29)
\]

Having the quaternionic contact form \( \eta \) fixed, we may extend the definitions \([6.7]\) of \( DD_{I_i} \) to the second order differential operator \( DD_{I_i} \) acting on the real-differentiable functions \( f : M \to \mathbb{R} \) by

\[
(6.30)
\]

The following proposition provides some formulas, which shall be used later.

**Proposition 6.18.** On a QC-manifold we have the following commutation relations

\[
(6.31)
\]

\[
(6.32)
\]

where \( i, j, k \) is a cyclic permutation of \( \{1, 2, 3\} \) and \( X, Y \in H \).
In particular, if $H$ is formally integrable then

$$DD_{I_i}f(X, I_iY) - DD_{I_k}(X, I_kY) = 4\xi_i(f)\omega_i(X, Y) - 4\xi_j(f)\omega_j(X, Y),$$

$(6.33)$

$$d_i d_j f(X, Y) + d_j d_i f(X, Y) = -4(\xi_i(f)\omega_j + \xi_j(f)\omega_i),$$

$$dd_i f + d_i df = 4(\xi_k(f)\omega_j - \xi_j(f)\omega_k).$$

**Proof.** By the definition $(6.30)$ we obtain the second and the third formulas in $(6.31)$ as well as $DD_{I_i}(X, Y) + (dd_k - d_j d_i)(X, I_j Y) = -I_i N_{I_j}(X, Y)$. The first equality in $(6.31)$ is a consequence of the latter and the second equality in $(6.31)$. We have

$$d^2 f(X, Y) = -I_i d(I_i^2 df)(X, Y) = d(df - \sum_{s=1}^3 \xi_s(f)\eta_s)(I_i X, I_i Y)$$

which is exactly $(6.32)$. If $H$ is formally integrable then the formula $(4.22)$ reduces to $N_{I_i}(X, Y) = T_{i,2}^0(X, Y)$. The equation $(6.33)$ is an easy consequence of the latter equality, $(6.31)$ and $(3.4)$. □

Let us make the conformal change $\tilde{\eta} = \frac{1}{2h}\eta$. The endomorphisms $I_i$ will coincide with $I_i$ on the horizontal distribution $H$ but they will have a different kernel - the new vertical space $\text{span}\{\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}$, where $\tilde{\xi}_s = 2h\xi_s + I_s(\nabla h)$ (see $(5.1)$). Hence, for any cyclic permutation $(i, j, k)$ of $(1,2,3)$ and any $P \in \Gamma(TM)$ we have

$$\tilde{I}_i(P) = I_i(P - \sum_{s=1}^3 \eta_s(P)\tilde{\xi}_s) = I_i(P - \frac{1}{2h_i} \sum_{s=1}^3 \eta_s(P)(2h\xi_s + I_s(\nabla h)))$$

$$= I_i(P) + \frac{1}{2h} \{\eta_i(P)\nabla h - \eta_j(P)I_k\nabla h + \eta_k(P)I_j\nabla h\}.$$  

$(6.34)$

**Proposition 6.19.** Suppose $\tilde{\eta} = \frac{1}{2h}\eta$ are two conformally equivalent structures.

$a)$ The second order differential operator $DD_{I_i}$ (restricted on functions) transforms by the following formula

$$DD_{I_i} f - DD_{I_k} f = -\frac{2df(\nabla h)}{h} \omega_i - \frac{2df(I_j \nabla h)}{h} \omega_k$$

$(6.35)$

$b)$ If $f$ is the real part of the anti-CRF function $f + iw + ju + kv$ then the two forms

$$\Omega_i = DD_{I_i} f - \lambda \omega_i + 4(\xi_j(f)\omega_k$$

are conformally invariant, where $\lambda = 4(\xi_1w + \xi_2u + \xi_3v)$ and $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$. 

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Proof. a) For any \(X, Y \in H\), we compute

\[
d(\bar{I}_i df)(X, Y) = X(I_i df(Y)) - Y(I_i df(X)) - \bar{I}_i df[X, Y] = d(I_i df)(X, Y) + df(\bar{I}_i[X, Y] - I_i[X, Y])
\]

Here, we apply (6.34) to get

\[
d(\bar{I}_i df)(X, Y) = d(I_i df)(X, Y) + \frac{1}{h}\{-df(\nabla h)\omega_i(X, Y) + df(I_k \nabla h)\omega_j(X, Y) - df(I_j \nabla h)\omega_k(X, Y)\}.
\]

Now, apply the defining equation (6.30) to get the statement of the Lemma.

b) Assuming that \(f\) is the real part of an anti-CRF function, from part a) and Theorem 6.20 we have

\[
\bar{\Omega}_i - \Omega_i = DD_I f - DD_I f - \bar{\lambda}\omega_i + \lambda\omega_i + 4\xi_j f \omega_k = 4\xi_j f \omega_k
\]

\[
= \frac{2}{h} g_H(df, dh) \omega_i - \frac{2}{h} g_H(df, d_j h) \omega_k
\]

\[
- 4(\xi_1 w + \xi_2 u + \xi_3 v) \omega_i - 4((I_1 dh)w + (I_2 dh)u + (I_3 dh)v) \omega_i \frac{\omega_k}{2h}
\]

\[
+ 4(\xi_1 w + \xi_2 u + \xi_3 v) \omega_i + 4g_H(f, d_j h) \omega_k \frac{\omega_k}{2h} = 0,
\]

taking into account (6.27). \(\square\)

We restrict our consideration on hyperhermitian contact manifolds.

**Theorem 6.20.** Let \(f : M \rightarrow \mathbb{R}\) be a smooth real function on a \((4n+3)\)-dimensional \((n > 1)\) hyperhermitian contact manifold \((M, \eta, Q)\).

a) If \(f\) is the real part of an anti-CRF function \(f + iw + ju + kv\) then the following equivalent conditions hold true:

i) For any cyclic permutation \(\{i, j, k\}\) of \(\{1, 2, 3\}\) the following equalities hold

\[
DD_I f = \lambda \omega_i - 4\xi_j(f)\omega_k.
\]
ii) For any $X, Y \in H$ the next equality holds

\[(\nabla_X df)(Y) + (\nabla_{I_1 X} df)(I_1 Y) + (\nabla_{I_2 X} df)(I_2 Y) + (\nabla_{I_3 X} df)(I_3 Y) \]
\[= 4\lambda g(X, Y)\]

(6.39)

\[+ df(X) \alpha_3(I_3 Y) + df(I_1 X) \alpha_3(I_2 Y) - df(I_2 X) \alpha_3(I_1 Y) - df(I_3 X) \alpha_3(Y)\]
\[+ df(Y) \alpha_3(I_3 X) + df(I_1 Y) \alpha_3(I_2 X) - df(I_2 Y) \alpha_3(I_1 X) - df(I_3 Y) \alpha_3(X).\]

iii) The function $f$ satisfies the following second order differential equations

\[\Re(D_{T_3} \overline{D}_{T_3} f)\]
\[= \lambda g(T_3, T_\alpha)\]

(6.40)

\[+ df(\nabla_{T_\beta} T_\alpha) + df(\nabla_{I_1 T_\beta} I_1 T_\alpha) + df(\nabla_{I_2 T_\beta} I_2 T_\alpha) + df(\nabla_{I_3 T_\beta} I_3 T_\alpha)\]
\[+ df(T_\beta) \alpha_3(I_3 T_\alpha) + df(I_1 T_\beta) \alpha_3(I_2 T_\alpha) - df(I_2 T_\beta) \alpha_3(I_1 T_\alpha) - df(I_3 T_\beta) \alpha_3(T_\alpha)\]
\[+ df(T_\alpha) \alpha_3(I_3 T_\beta) + df(I_1 T_\alpha) \alpha_3(I_2 T_\beta) - df(I_2 T_\alpha) \alpha_3(I_1 T_\beta) - df(I_3 T_\alpha) \alpha_3(T_\beta)\]

(6.41)

\[\Re(iD_{T_3} \overline{D}_{T_3} f) = \Re(D_{I_1 T_3} \overline{D}_{I_1 T_3} f), \quad \Re(jD_{T_3} \overline{D}_{T_3} f) = \Re(D_{I_2 T_3} \overline{D}_{I_2 T_3} f),\]
\[\Re(jD_{I_3 T_3} \overline{D}_{I_3 T_3} f) = \Re(D_{I_2 T_3} \overline{D}_{I_2 T_3} f).\]

The function $\lambda$ is determined by

(6.42)

\[\lambda = 4(\xi_1(w) + \xi_2(u) + \xi_3(v)).\]

**Proof.** The proof includes a number of steps and occupies the rest of the section.

a) $\Rightarrow$ i) Suppose that there exists a smooth functions $w, u, v$ such that $F = f + iw + ju + kv$ is an anti-CRF function. The defining equation (6.30) yields

\[df = d_1 w + d_2 u + d_3 v + \sum_{s=1}^{3} \xi_s(f) \eta_s,\]
\[d_1 f = -dw + d_3 u - d_2 v + \sum_{s=1}^{3} \xi_s(w) \eta_s,\]
\[d_2 f = -d_3 w - du + d_1 v + \sum_{s=1}^{3} \xi_s(u) \eta_s,\]
\[d_3 f = d_2 w - d_1 u + dv + \sum_{s=1}^{3} \xi_s(v) \eta_s.\]

Observing $d_i \eta_j(X, Y) = 0$, $i, j \in \{1, 2, 3\}$, $X, Y \in H$, applying (6.32) and (2.1), we obtain from (6.43)

\[(dd_1 f - dd_3 u + dd_2 v - 2\xi_1(w)\omega_1 - 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X, Y) = 0,\]
\[(d_1 df - d_1 d_2 u - d_1 d_3 v + 2\xi_1(w)\omega_1 - 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X, Y) = 0,\]
\[(d_2 df + d_2 d_1 u + d_2 dv - 2\xi_1(w)\omega_1 + 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X, Y) = 0,\]
\[(d_3 df + d_3 d_1 u + d_3 dv + 2\xi_1(w)\omega_1 + 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X, Y) = 0.\]
Summing the first and the third equations, subtracting the second and the fourth and using the commutation relations (6.33) we obtain (6.48) with the condition (6.42) which proves a) \(\Rightarrow\) i).

a) \(\Rightarrow\) iii). Equations (6.28) and (6.29) yield

\[
2\Re(D_{T\beta}D_{T\alpha}f) = 2(T_{\beta}T_{\alpha}f + X_{\beta}X_{\alpha}f + Y_{\beta}Y_{\alpha}f + Z_{\beta}Z_{\alpha}f)
= (\Re(D_{T\beta}D_{T\alpha}f) + \Re(D_{T\alpha}D_{T\beta}f)) + (\Re(D_{T\beta}D_{T\alpha}f) - \Re(D_{T\alpha}D_{T\beta}f))
= ([T_{\beta}, T_{\alpha}] + [X_{\beta}, X_{\alpha}] + [Y_{\beta}, Y_{\alpha}] + [Z_{\beta}, Z_{\alpha}])f
\]

(6.44)

Similarly, one can check the validity of (6.27), (6.31), (6.30) and (6.26) gives (6.40).

\(i) \Leftrightarrow ii) \Leftrightarrow iii)\) The next lemma establishes the equivalence between i), ii) and iii).

Lemma 6.21. For any \(X, Y \in H\) on a quaternionic contact manifold we have the identity

\[
DD_{I_1}f(X, I_1Y) = (\nabla_X df)Y + (\nabla_{I_1X} df)I_1Y + (\nabla_{I_2X} df)I_2Y + (\nabla_{I_2X} df)I_3X - 4\xi_2(f)\omega_2(X, Y)
- df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_2(I_3Y) + df(I_2X)\alpha_3(I_3Y) - df(I_3X)\alpha_2(I_1Y)
- df(Y)\alpha_2(I_2X) - df(I_1Y)\alpha_3(I_2X) + df(I_2Y)\alpha_2(I_3X) + df(I_3Y)\alpha_3(I_3X).
\]

Proof. (Lemma 6.21) Using the definition and also (6.30), (6.31) and (6.32) we derive the next sequence of equalities

\[
(dd_{I_1}f)(X, Y) = d(I_1df)(X, Y)
= -X df(I_1Y) + Y df(I_1X) + df(I_1[X, Y])
= -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X) - df(\nabla_X(I_1Y) - \nabla_Y(I_1X) - I_1[X, Y])
= -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X)
+ \alpha_2(X)df(I_3Y) - \alpha_3(X)df(I_2Y) - \alpha_2(Y)df(I_3X) + \alpha_3(Y)df(I_2X)
= -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X)
+ \alpha_2(X)df(I_3Y) - \alpha_3(X)df(I_2Y) - \alpha_2(Y)df(I_3X) + \alpha_3(Y)df(I_2X).
\]
Thus

\[
DD_1 f(X, I_1 Y) = (dd_1 - I_2 dd_1) f(X, I_1 Y)
\]

\[
= (\nabla_X df) Y + (\nabla_{I_1 X} df) I_1 Y + (\nabla_{I_2 X} df) I_2 Y + (\nabla_{I_3 X} df) I_3 X
\]

\[
- df(T(I_1 Y, I_1 X)) - df(T(I_3 Y, I_3 X))
\]

\[
- df(X)\alpha_3(I_3 Y) + df(I_1 X)\alpha_2(I_3 Y) + df(I_2 X)\alpha_3(I_1 Y) - df(I_3 X)\alpha_2(I_1 Y)
\]

\[
- df(Y)\alpha_2(I_2 X) - df(I_1 Y)\alpha_3(I_2 X) + df(I_2 Y)\alpha_2(X) + df(I_3 Y)\alpha_3(X).
\]

A short calculation using (6.46) gives

\[
df(T(I_1 Y, I_1 X)) + df(T(I_3 Y, I_3 X)) = 4\xi_2(f)\omega_2(X, Y).
\]

Insert the last equality into (6.46) to get the proof of Lemma 6.21.

Since the structure is hyperhermitian contact, an application of (4.25) to Lemma 6.21 makes at this point the proof of Theorem 6.20 complete.

We conjecture that i) implies a). At this point we can prove Lemma 6.23, which supports the conjecture. First we prove a useful technical result.

**Lemma 6.22.** Suppose M is a quaternionic contact manifold. If \( \psi \) is a smooth closed two-form whose restriction to \( H \) vanishes, then \( \psi \) vanishes identically.

**Proof.** (Lemma 6.22) The hypothesis on \( \psi \) show that \( \psi \) is of the form

\[
\psi = \sum_{s=1}^{3} \sigma_s \wedge \eta_s + \sum_{1 \leq i < j \leq 3} A_{ij} \eta_i \wedge \eta_j,
\]

where \( A_{ij} \) are smooth functions and \( \sigma_s \) are horizontal 1-forms in the sense that \( \sigma_i(\xi_j) = 0, \ i, j = 1, 2, 3 \). Using (2.1), we obtain from (6.47)

\[
d\psi = \sum_{s=1}^{3} (d\sigma_s \wedge \eta_s - 2\eta_s \wedge \omega_s)
\]

\[
+ \sum_{1 \leq i < j \leq 3} [2A_{ij}(\omega_i \wedge \eta_j - \eta_i \wedge \eta_j) + dA_{ij} \wedge \eta_i \wedge \eta_j].
\]

Consequently, (6.48) yields
\begin{align}
0 = \sum_{a=1}^{4n} d\Psi(e_a, I_t e_a, X) &= -2 \sum_{a=1}^{4n} \sum_{s=1}^3 \sigma_s \wedge \omega_s(e_a, I_t e_a, X) = (4n - 2)\sigma_t(X).
0 = d\Psi(e_a, I_t e_a, \xi_j) = 2A_{ij}.
\end{align}

The assumption in the next Lemma is a kind of \( \partial \bar{\partial}_H \)-lemma result, which we do not know how to prove at the moment, but we believe that it is true. We show how it implies \( d \Rightarrow a \) of Theorem 6.20.

\textbf{Lemma 6.23.} Suppose \( D_b D_t f = d_b d_t f + d_j d_k f = 0 \), i.e., \( dd_1 f + d_2 d_3 f = p_s \omega_s \) implies
\begin{align}
(6.50) \quad dd_1 f - dd_2 A_1 &= 2r_s \omega_s
\end{align}
for some function \( A_1 \). Then \( f \) is a real part of an anti-CRF-function.

\textbf{Proof.} (Lemma 6.23) Consider the closed 2-form
\[
\Omega = d(d_1 f - d_2 A_1 - r_s \eta_s).
\]
We have \( d\Omega = 0 \) and \( \Omega|_H = 0 \) because of (6.50) and (3.1). Apply our Lemma 6.22, we conclude \( \Omega = 0 \) and the Poincare lemma yields
\[
d_1 f - d_2 A_1 - dB_1 = 0 \mod \eta,
d_2 f - d_3 A_2 - dB_2 = 0 \mod \eta,
d_3 f - d_1 A_3 - dB_3 = 0 \mod \eta.
\]
for a smooth functions \( A_1, A_2, A_3, B_1, B_2, B_3 \) which yields
\[
df + d_3 A_1 + d_1 B_1 = 0 \mod \eta,
df + d_1 A_2 + d_2 B_2 = 0 \mod \eta,
df + d_2 A_3 + d_3 B_3 = 0 \mod \eta.
\]
The latter implies
\[
df + d_1(A_2 + B_1) + d_2(B_2 - A_3) + d_3(A_1 - B_3) = 0 \mod \eta.
\]
Set \( w = -A_2 - B_1, \ u = A_3 - B_2, \ v = B_3 - A_1 \) to get \( d \Rightarrow a \). \( \square \)

\textbf{Corollary 6.24.} Let \( f : M \to \mathbb{R} \) be a smooth real function on a \((4n+3)\)-dimensional \((n > 1)\) \( 3 \)-Sasakian manifold \((M, \eta, \mathcal{Q})\).
a) If \( f \) is the real part of an anti-CRF function \( f + iw + ju + kv \) then the following equivalent conditions hold true:

i) The equation (6.38) holds.

ii) For any \( X,Y \in H \) the next equality holds

\[
(\nabla_X df)(Y) + (\nabla_{I_1 X} df)(I_1 Y) + (\nabla_{I_2 X} df)(I_2 Y) + (\nabla_{I_3 X} df)(I_3 Y) = 4\lambda g(X,Y) \tag{6.51}
\]

The function \( \lambda \) is determined in (6.42).

**Corollary 6.25.** Let \( f : G(\mathbb{H}) \to \mathbb{R} \) be a smooth real function on the \((4n+3)\)-dimensional \((n > 1)\) quaternionic Heisenberg group endowed with the standard flat quaternionic contact structure and \( \{T_a, X_a, Y_a, Z_a, \quad a = 1, \ldots, 4n\} \) be \( \nabla \)-parallel basis on \( G(\mathbb{H}) \).

a) If \( f \) is the real part of an anti-CRF function \( f + iw + ju + kv \) then the following equivalent conditions hold true:

i) The equation (6.38) holds.

ii) The horizontal Hessian of \( f \) is given by

\[
T_b T_a f + X_b X_a f + Y_b Y_a f + Z_b Z_a f = 4\lambda g(T_b, T_a); \tag{6.52}
\]

iii) The function \( f \) satisfies the following second order differential equation

\[
D_{T_b} D_{T_a} f = \lambda (g - i\omega_1 - j\omega_2 - k\omega_3)(T_b, T_a); \tag{6.53}
\]

The function \( \lambda \) is given by (6.42).

Proposition 6.2, Corollary 6.24 and Example 4.12 imply the next Corollary.

**Corollary 6.26.** Let \( (M, \eta, Q) \) be a \((4n+3)\)-dimensional \((n > 1)\) 3-Sasakian manifold, \( f : M \to \mathbb{R} \) a positive smooth real function. Then the conformally 3-Sasakian QC structure \( \tilde{\eta} = f\eta \) is qc-pseudo Einstein if and only if the operators \( DD_{I_s} f, s = 1, 2, 3 \) satisfy (6.42). In particular, if \( f \) is real part of anti CRF function then the conformally 3-Sasakian qc structure \( \tilde{\eta} = f\eta \) is qc-pseudo Einstein.
7. Infinitesimal Automorphisms

7.1. 3-contact manifolds. We start with the more general notion of 3-contact manifold \((M, H)\), where \(H\) is orientable codimension three distribution on \(M\). Let \(E \subset T M^*\) be the canonical bundle determined by \(H\), i.e. the bundle of 1-forms with kernel \(H\). Hence, \(M\) is orientable if and only if \(E\) is also orientable, i.e. \(E\) has a global nonvanishing section \(\eta_1 \wedge \eta_2 \wedge \eta_3\). Denote \(\eta = (\eta_1, \eta_2, \eta_3)\) the 1-form with values in \(\mathbb{R}^3\). Clearly \(H = \ker \eta\).

**Definition 7.1.** A \((4n+3)\)-dimensional orientable smooth manifold \((M, \eta, H = \ker \eta)\) is said to be 3-contact manifold if the restriction of each 2-form \(d \eta_i, i = 1, 2, 3\) to \(H\) is non-degenerate, i.e.

\[
d \eta_i^{2n} \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 = u_i \cdot \text{vol}_M, \quad u_i > 0, \quad i = 1, 2, 3,
\]

and the following compatibility conditions hold

\[
d \eta_i^p \wedge d \eta_i^q \wedge d \eta_i^r \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 = 0, \quad p + q + r = 2n, \quad 0 < p, q, r < 2n.
\]

Denote the restriction of \(d \eta_i\) on \(H\) by \(\Omega_i\), \(\Omega_i = (d \eta)|_H, i = 1, 2, 3\). The condition (7.1) is equivalent to

\[
\Omega_i^{2n} \neq 0, \quad i = 1, 2, 3, \quad \Omega_i^p \wedge \Omega_i^q \wedge \Omega_i^r = 0, \quad p + q + r = 2n, \quad 0 < p, q, r < 0.
\]

We remark that the notion of 3-contact structure is slightly more general than the notion of QC structure. For example, any real hypersurface \(M\) in \(\mathbb{H}^{n+1}\) with non-degenerate second fundamental form carries 3-contact structure defined in the beginning of Section 6.2.1 by (6.11) and (6.12) (conf. Proposition 6.12 and Definition 6.13 where this structure is a-QC if and only if (6.13) holds, or equivalently, the second fundamental form is, in addition, invariant with respect to the hypercomplex structure on \(\mathbb{H}^{(n+1)}\)). Another examples of 3-contact structure is the so called quaternionic CR structure introduced in [KN] and the so called weak QC structures considered in [D1].

On any 3-contact manifold \((M, \eta, H)\) there exist unique triple \((\xi_1, \xi_2, \xi_3)\) of vector fields transversal to \(H\) determined by the conditions

\[
\eta_i(\xi_j) = \delta_{ij}, \quad (\xi_i, d \eta_i)|_H = 0.
\]

We refer to such a triple as fundamental vector fields or Reeb vector fields and denote \(V = \text{span}\{\xi_1, \xi_2, \xi_3\}\). Hence, we have the splitting \(TM = H \oplus V\).

The 3-contact structure \((\eta, H)\) and the vertical space \(V\) are determined up to an action of \(GL(3, \mathbb{R})\), namely for any \(GL(3, \mathbb{R})\) matrix \(\Phi\) with smooth entries the structure \(\Phi \cdot \eta\) is

\[
\Phi \cdot (\eta_1, \eta_2, \eta_3) = (\Phi \cdot \eta_1, \Phi \cdot \eta_2, \Phi \cdot \eta_3).
\]
again a 3-contact structure. Indeed, it is an easy algebraic fact that the conditions (7.1) and (7.2) also hold for $\Phi \cdot \eta$. The Reeb vector field are transformed with the matrix with entries the adjunction quantities of $\Phi$, i.e. with the inverse matrix $\Phi^{-1}$. This leads to the next

**Definition 7.2.** A diffeomorphism $\phi$ of a 3-contact manifold $(M, \eta, H)$ is called a 3-contact automorphism if $\phi$ preserves the 3-contact structure $\eta$, i.e.

$$\phi^* \eta = \Phi \cdot \eta,$$

for some matrix $\Phi \in GL(3, \mathbb{R})$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$.

The infinitesimal versions of these notions lead to the following definition.

**Definition 7.3.** A vector field $Q$ on a 3-contact manifold $(M, \eta, H)$ is an infinitesimal generator of a 3-contact automorphism (3-contact vector field for short) if its flow preserves the 3-contact structure, i.e.

$$\mathcal{L}_Q \eta = \phi \cdot \eta,$$

where $\phi \in gl(3, \mathbb{R})$.

We show that any 3-contact vector field on a 3-contact manifold depend on 3-functions which satisfy certain differential relations. We begin with describing infinitesimal automorphisms of the 3-contact structure $\eta$ i.e. vector field $Q$ whose flow satisfies (7.4). Our main observation is that 3-contact vector fields on a 3-contact manifold are completely determined by their vertical components in the sense of the following

**Proposition 7.4.** Let $(M, \eta, H)$ be a 3-contact manifold. A smooth vector field $Q$ on $M$ is 3-contact vector field if and only if the functions $f_i = \eta_i(Q), i = 1, 2, 3$ satisfy the next compatibility conditions on $H$

$$u_i(df_i + f_j(\xi_j \cdot d\eta_i) + f_k(\xi_k \cdot d\eta_i))|_H \wedge \Omega_i^{(2n-1)} =$$

$$u_j(df_j + f_k(\xi_k \cdot d\eta_j) + f_i(\xi_i \cdot d\eta_j))|_H \wedge \Omega_j^{(2n-1)}$$

on $H$, where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$. The 3-contact vector field $Q$ has the form

$$Q = Q_h + f_1 \xi_1 + f_2 \xi_2 + f_3 \xi_3,$$
where $Q_h$ is the horizontal 3-contact hamiltonian field of $(f_1, f_2, f_3)$ defined on $H$ by

\[(7.8) \quad Q_h \cdot \eta_i = 0, \quad Q_h \cdot (\Omega_i) = -df_i - f_j(\xi_j \cdot d\eta_i) - f_k(\xi_k \cdot d\eta_i), \quad i = 1, 2, 3, \quad \text{on} \quad H.\]

**Proof.** For a vector field $Q \in \Gamma(TM)$ we write

\[Q = Q_H + \sum_{s=1}^{3} \eta_s(Q) \xi_s \quad \text{where} \quad Q_H \in H \quad \text{is the horizontal part of} \quad Q.\]

Applying (3.11), we calculate

\[(7.9) \quad \mathcal{L}_Q \eta_i = Q \cdot d\eta_i + d(Q \cdot \eta_i) = Q_H \cdot \Omega_i + [d(\eta_i(Q)) + \eta_i(Q) \xi_i \cdot d\eta_i + \eta_j(Q) \xi_j \cdot d\eta_i + \eta_k(Q) \xi_k \cdot d\eta_i]_H + [\xi_i(\eta_i(Q)) - \eta_j(Q) \xi_j \cdot d\eta_i + \eta_k(Q) \xi_k \cdot d\eta_i]_H + [\xi_j(\eta_i(Q)) + d\eta_i(Q, \xi_j)] \eta_j + [\xi_k(\eta_i(Q)) + d\eta_i(Q, \xi_k)] \eta_k\]

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$ and the symbol $\cdot H$ means the $H$-component of the corresponding object.

Suppose $Q$ is a 3-contact vector field. Then (7.9) and the compatibility conditions (7.2) imply that $f_i$ and $Q_H$ necessarily satisfy (7.6) and (7.8), respectively. Therefore $Q_H = Q_h$.

The converse follows from (7.9) and the conditions of the proposition. \[\Box\]

The last Proposition implies that the space of 3-contact vector fields is isomorphic to the space of triples consisting of smooth function $f_1, f_2, f_3$ satisfying the compatibility conditions (7.6).

**Corollary 7.5.** Let $(M, \eta)$ be a 3-contact manifold. Then

a) If $Q$ is a horizontal 3-contact vector field on $M$ then $Q$ vanishes identically.

b) The vector fields $\xi_i, i = 1, 2, 3$ are 3-contact vector fields if and only if

\[\xi_i \cdot d\eta_j |_H = 0, \quad i, j = 1, 2, 3.\]

### 7.2. QC-vector fields

Suppose $(M, \eta, Q)$ is a quaternionic contact manifold i.e. it is a 3-contact manifold with fixed conformal class $[g]$ of Riemannian metrics on $H$, or equivalently, fixed quaternionic structure $Q$ on $H$. This leads to the following

**Definition 7.6.** A diffeomorphism $\phi$ of a QC manifold $(M, \eta, Q)$ is called a quaternionic contact automorphism if $\phi$ preserves the QC structure $\eta, Q$, i.e.

\[(7.10) \quad \phi^* \eta = \mu \Psi \cdot \eta,\]

where $\mu \Psi$ is a unit quaternion.
for some positive smooth function $\mu$ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$.

In view of the uniqueness of the possible associated almost complex structures, a quaternionic contact automorphism will preserve also the associated (if any) almost complex structures, $\phi^*\xi = \xi$ and consequently, it will preserve the conformal class $[g]$ on $H$. Therefore, in the case we are dealing with an automorphism of a quaternionic contact manifold we shall refer to the quaternionic contact automorphisms as conformal quaternionic contact automorphism (QC-automorphism for short). We note that QC diffeomorphisms on $S^{4n+3}$ are considered in [Kam].

The infinitesimal versions of these notions lead to the following definition.

**Definition 7.7.** A vector field $Q$ on a QC manifold $(M, \eta, Q)$ is an infinitesimal generator of a conformal quaternionic contact automorphism (QC vector field for short) if its flow preserves the QC structure, i.e.

\[ L_Q \eta = (\nu I + O) \cdot \eta, \]

where $\nu$ is a smooth function and $O \in so(3)$.

In view of the discussion above a QC vector field on a QC manifold $(M, \eta, Q)$ satisfies the conditions.

\[ L_Q g = \nu g, \]
\[ L_Q I = O \cdot I, \quad O \in so(3), \quad I = (I_1, I_2, I_3)^t, \]

If the flow of a vector field $Q$ is a conformal diffeomorphism of the horizontal metric $g$, i.e. (7.12) holds, we shall call it *infinitesimal conformal isometry*. If the function $\mu = 0$ then $Q$ is said to be *infinitesimal isometry*.

A QC vector field on a QC manifold is a 3-contact vector field of special type. Indeed, let $\flat$ be the musical isomorphism between $T^*M$ and $TM$ with respect to the fixed Riemannian metric $g$ on $TM$ and recall that the forms $\alpha_j$ were defined in (3.31). We have

\[ Q = \frac{1}{2} \left( f_j I_i \alpha_k^\flat - f_k I_i \alpha_j^\flat - I_i (df_i)^\flat \right) + \sum_{s=1}^{3} f_s \xi_s, \]

\[ (7.14) \]

**Proposition 7.8.** Let $(M, \eta, \xi)$ be a quaternionic contact manifold. The vector field $Q$ is an infinitesimal conformal isometry as in (7.11) if and only if

\[ (7.14) \]
for some functions \( f_1, f_2 \) and \( f_3 \) such that for any positive permutation \((i, j, k)\) of \((1, 2, 3)\) we have

\[
(7.15) \quad f_j \, d\eta_i(\xi_j, \xi_i) + f_k \, d\eta_i(\xi_k, \xi_i) + \xi_i \, f_i = f_k \, d\eta_j(\xi_k, \xi_j) + f_i \, d\eta_j(\xi_i, \xi_j) + \xi_j \, f_j
\]

and

\[
(7.16) \quad f_i \, d\eta_i(\xi_i, \xi_j) + f_k \, d\eta_i(\xi_k, \xi_j) + \xi_j \, f_i = -f_j \, d\eta_j(\xi_j, \xi_i) - f_k \, d\eta_j(\xi_k, \xi_i) - \xi_i \, f_j,
\]

and

\[
(7.17) \quad f_j I_i(\alpha_k)^\flat - f_k I_i(\alpha_j)^\flat - I_i(df_i)^\flat = f_i I_k(\alpha_j)^\flat - f_j I_k(\alpha_i)^\flat - I_k(df_k)^\flat
\]

Proof. Notice that (7.14) implies \( f_i = \eta_i(Q) \). By Cartan’s formula (7.11) is equivalent to

\[ Q \omega d\eta_i + df_i = \nu \eta_i + o_{is} \eta_s. \]

In other words, both sides must be the same when evaluated on \( \xi_t \), \( t = 1, 2, 3 \) and also when restricted to the horizontal bundle. Let \( Q = Q_H + f_s \xi_s \). Consider first the action on the vertical vector fields. Pairing with \( \xi_i \) and taking successively \( t = i, j, k \) gives

\[
(7.18) \quad f_j \, d\eta_i(\xi_j, \xi_i) + f_k \, d\eta_i(\xi_k, \xi_i) + \xi_i \, f_i = \nu + o_{ii}
\]

Equating the restrictions to the horizontal bundle, i.e., \( d\eta_i(Q, .)|_H + df_i|_H = 0 \), gives

\[ \left( f_j \, d\eta_i(\xi_j, .) + f_k \, d\eta_i(\xi_k, .) + d\eta_i(Q H, .) + df_i \right)|_H = 0. \]

Since \( g(A, .)|_H = 0 \iff A = \eta_s(A)\xi_s \), the last equation is equivalent to

\[
(7.19) \quad -f_j \alpha_k^\flat + f_k \alpha_j^\flat - 2I_i Q_H + (df_i)^\flat = \left( -f_j \alpha_k(\xi_s) + f_k \alpha_j(\xi_s) + \xi_s f_i \right) \xi_s.
\]

Acting with \( I_i \) determines \( 2Q_H = f_j I_i(\alpha_k)^\flat - f_k I_i(\alpha_j)^\flat - I_i(df_i)^\flat \), which implies (7.14). In addition we have

\[ \alpha_j(Q_H) = \frac{1}{2} \left( f_j \alpha_j(I_i(\alpha_k)^\flat) - f_k \alpha_j(I_i(\alpha_j)^\flat) - \alpha_j(I_i(df_i)^\flat) \right). \]

On the other hand, \( o \in so(3) \) is equivalent to \( o \) being a skew symmetric which is equivalent to (7.14) and (7.16), by the above computations. Therefore, if we are given three functions \( f_1, f_2, f_3 \) satisfying (7.15), (7.16) and (7.17), then we define \( Q \) by (7.14). Using (7.14) we define \( \nu \) and \( o \) with \( o \in so(3) \). With these definitions \( Q \) is a QC vector field. \( \square \)
Using the formulas in Example 4.18 we obtain from Propositions 7.8 the following '3-hamiltonian' form of a QC vector field on 3-Sasakian manifold.

**Corollary 7.9.** Let \((M, \eta, Q)\) be a 3-Sasakian manifold. Then any QC vector field \(Q\) has the form

\[
Q = Q_h + f_1\eta_1 + f_2\eta_2 + f_3\eta_3,
\]

where the smooth functions \(f_1, f_2, f_3\) satisfy the conditions

\[
d_i f_i = d_j f_j, \quad \eta_i(f_i) = \eta_j(f_j), \quad \eta_i(f_j) = -\eta_j(f_i), \quad i, j = 1, 2, 3,
\]

and the horizontal part \(Q_h \in H\) is determined by

\[
Q_h \circ d\eta_i = d_i f_i, \quad i \in \{1, 2, 3\}.
\]

The matrix in (7.11) has the form

\[
\nu I_{d\eta} + O = \begin{pmatrix}
\xi_1(\eta_1(Q)) & -\xi_1(\eta_2(Q)) - 2\eta_3(Q) & -\xi_1(\eta_3(Q)) + 2\eta_2(Q) \\
\xi_1(\eta_2(Q)) + 2\eta_3(Q) & \xi_1(\eta_1(Q)) & -\xi_2(\eta_3(Q)) - 2\eta_1(Q) \\
\xi_1(\eta_3(Q)) - 2\eta_2(Q) & \xi_2(\eta_3(Q)) + 2\eta_1(Q) & \xi_1(\eta_1(Q))
\end{pmatrix}.
\]

In particular, the Reeb vector fields \(\xi_1, \xi_2, \xi_3\) are 3-contact vector fields.

Corollary 7.5 tells us that on a QC manifold the Reeb vector fields \(\xi_1, \xi_2, \xi_3\) are 3-contact exactly when the connection 1-forms vanish on \(H\). This combined with Corollary 7.14 gives 3-Sasakian structure compatible with the given 3-contact structure \(H\), if the qc-scalar curvature is not zero (see Corollary 7.14 below).

Next we shall investigate some useful properties of a QC vector field.

**Proposition 7.10.** Let \((M, \eta, Q)\) be QC manifold and \(Q\) be a QC vector field determined by \(\nu I_{d\eta}\) and \(O\). The next equality hold

\[
d\eta([Q, I X]^\perp, Y) + d\eta(I X, [Q, Y]^\perp) = 0
\]

**Proof.** We have using (7.11) that

\[
(7.20) \quad \mathcal{L}_Q d\eta_i(I X, Y) = 2(\mathcal{L}_Q \omega_i)(I X, Y) - d\eta([Q, I X]^\perp, Y) - d\eta(I X, [Q, Y]^\perp)
\]

\[
= -2(\mathcal{L}_Q g)(X, Y) + 2g(\mathcal{L}_Q I I X, Y) - d\eta([Q, I X]^\perp, Y) - d\eta(I X, [Q, Y]^\perp)
\]

\[
= (d\mathcal{L}_Q \eta_i)(I X, Y) = (d\nu \wedge \eta_i + \nu d\eta_i + d\nu_i \wedge \eta + o_{ij} \eta j + o_{ik} \eta k)(I X, Y)
\]

\[
= -2\nu g(X, Y) - 2o_{ij} \omega_k(X, Y) + 2o_{ik} \omega_j(X, Y),
\]

where \(o_{ij}\) are the entries of the matrix \(O\) given by (7.18) and \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\). Apply (7.12) and (7.13) to (7.20) to get the assertion. \(\square\)
We are going to characterize the vanishing of the torsion of the Biquard connection in terms of the existence of some special vertical vector fields. More precisely, we have the following Theorem.

**Theorem 7.11.** Let \((M, \eta, Q)\) be a QC manifold with non zero qc-scalar curvature. The following conditions are equivalent

i) Each Reeb vector field is a QC vector field;

ii) The torsion of the Biquard connection is identically zero;

iii) Each Reeb vector field preserves the horizontal metric and the quaternionic structure simultaneously, i.e. (7.12) with \(\nu = 0\) and (7.13) hold for \(Q = \xi_i, i = 1, 2, 3\);

iv) There exists a local 3-Sasakian structure in the sense of Theorem 1.3

**Proof.** In the course of the proof we shall prove two Lemmas of independent interest. Given a vector field \(Q\), we define the symmetric tensor \(T^0_Q\) and the skew-symmetric tensor \(u_Q\)

\[
(7.21) \quad T^0_Q = \sum_{s=1}^{3} \eta_s(Q) T^0_{\xi_s}; \quad u_Q = \sum_{s=1}^{3} \eta_s(Q) I_s u,
\]

respectively, such that,

\[
T(Q, X, Y) = g(T^0_Q X, Y) + g(u_Q X, Y),
\]

**Lemma 7.12.** The tensors \(T^0_Q\) and \(u_Q\) lie in the \([-1]\) component associated to the operator \(\Upsilon\) cf. (2.9) and (2.8).

**Proof of Lemma 7.12.** Let us consider first \(u_Q\). By its definition we have

\[
g(u_Q I_1 X, I_1 Y) = \sum_{s=1}^{3} \eta_s(Q) g(I_s u X, Y)
\]

and thus after summing we find

\[
\sum_{j=1}^{3} g(u_Q I_j X, I_j Y) = \sum_{j=1}^{3} \eta_j(Q) g(I_j u X, Y) = -g(u_Q X, Y).
\]

This proves the claim for \(u_Q\).

We turn to the second claim. Recall that \(T^0_{\xi_j}\) anti-commutes with \(I_j\), see (2.10). Hence,

\[
g(T^0_Q I_1 X, I_1 Y) = \eta_1(Q) g(T^0_{\xi_1} X, Y) - \eta_2(Q) [g(T^0_{\xi_2}^{++} X, Y) - g(T^0_{\xi_2}^{+-} X, Y)]
\]

\[
- \eta_3(Q) [g(T^0_{\xi_3}^{--} X, Y) - g(T^0_{\xi_3}^{++} X, Y)],
\]
also
\[ g(T^0_Q I_2 X, I_2 Y) = -\eta_2(Q) g(T^0_Q X, Y) - \eta_1(Q) [g(T^0_Q X, Y) - g(T^0_Q X, Y)] \]
\[ - \eta_3(Q) [g(T^0_Q X, Y) - g(T^0_Q X, Y)] \]
and finally
\[ g(T^0_Q I_3 X, I_3 Y) = -\eta_3(Q) g(T^0_Q X, Y) - \eta_1(Q) [g(T^0_Q X, Y) - g(T^0_Q X, Y)] \]
\[ - \eta_2(Q) [g(T^0_Q X, Y) - g(T^0_Q X, Y)] \]

Summing the above three equations we come to
\[ \sum_{j=1}^{3} g(T^0_Q I_j X, I_j Y) = - \sum_{j=1}^{3} g(Q, \xi_j) g(T^0_Q X, Y) = -g(T^0_Q X, Y), \]
which finishes the proof of Lemma 7.12.

\[ \square \]

**Lemma 7.13.** If \( Q \) is an infinitesimal conformal isometry whose flow preserves the quaternionic structure then the next two equalities hold
\[
(7.22) \quad g(\nabla_X Q, Y) + g(\nabla_Y Q, X) + 2g(T^0_Q X, Y) = \nu g(X, Y),
\]
\[
(7.23) \quad 3g(\nabla_X Q, Y) - \sum_{s=1}^{3} g(\nabla_{I_s} X, I_s Y) + 4g(T^0_Q X, Y) + 4g(u_Q X, Y)
\]
\[ + 2 \sum_{(ijk)} L_{ij} (Q) \omega_k (X, Y) = 0, \]
where the sum is over all even permutation of \((1,2,3)\) and
\[
(7.24) \quad L_{ij} (Q) = -L_{ji} (Q) = \xi_j (\eta_i (Q)) - \eta_i (Q) d\eta_j (\xi_i, \xi_j)
\]
\[ + \frac{1}{2} \eta_k (Q) \left( \frac{\text{Scal}}{8n(n+2)} + d\eta_j (\xi_k, \xi_i) - d\eta_i (\xi_j, \xi_k) - d\eta_k (\xi_i, \xi_j) \right). \]

**Proof of Lemma 7.13** In terms of the Biquard connection \((7.12)\) reads exactly as \((7.22)\).
Furthermore, from \((7.13)\), \((7.18)\) and \((3.30)\) it follows
\[
(7.25) \quad o_{ij} I_j X + o_{ik} I_k X = (\mathcal{L}_Q I_i) (X) =
\]
\[-\nabla_{I_i} Q + I_i \nabla_X Q - \alpha_j (Q) I_k X + \alpha_k (Q) I_j X - T(Q, I_i X) + I_i T(Q, X).\]
A use of (7.18), (3.32) and (3.48) allows us write the last equation in the form
\[ g(\nabla X Q, Y) - g(\nabla_{I_i} X, I_i Y) + T(Q, X, I_i Y) - T(Q, I_i X, I_i Y) \]
\[ = (o_{ij} - \alpha_k(Q))\omega_k(X, Y) - (o_{ik} + \alpha_j(Q))\omega_j(X, Y) \]
\[ = -L_{ij}(Q)\omega_k(X, Y) + L_{ik}(Q)\omega_j(X, Y), \]
where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\) and \(L_{ij}(Q)\) satisfy (7.24). Summing the above identities for the three almost complex structures and applying Lemma 7.12 we obtain (7.23) which completes the proof of Lemma 7.13.

We are ready to finish the proof of Theorem 7.11. Let \(\xi_i, i = 1, 2, 3\) be QC vector fields. Then (7.22) for \(Q = \xi_i\) yields \(T_{\xi} = 0, i = 1, 2, 3, \nu = 0\) since \(T_{\xi}\) is trace-free. Consequently, for any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\), (7.23) and (7.24) imply
\[ u_{\xi_i} = 0, \quad d\eta_j(\xi_i, \xi_j) = 0, \quad d\eta_i(\xi_j, \xi_k) = \frac{\text{Scal}}{8(n+2)} \]
by comparing the trace and the trace-free part. Hence ii) follow.

Conversely, if the torsion of the Biquard connection vanishes, then (7.22) is trivially satisfied for \(\nu = 0\) and (7.25) yields (7.13) with \(o_{ij} = \alpha_k\). This establishes the equivalence between ii) and iii).

The other equivalences in the theorem follow from Theorem 1.3, Example 4.12, Corollary 7.5 and Corollary 4.17.

Corollary 7.14. Let \((M, \eta, Q)\) be a QC manifold with non zero qc-scalar curvature. The following conditions are equivalent

i) There exists a local 3-Sasakian structure compatible with \(H = \text{Ker} \eta\);

ii) There are three linearly independent transversal QC-vector fields.

Proof. Let \(\gamma_1, \gamma_2, \gamma_3\) be linearly independent transversal QC-vector fields. Then there exist 1-forms \(\eta_{\gamma_1}, \eta_{\gamma_2}, \eta_{\gamma_3}\) satisfying \(\eta_{\gamma_i}(\gamma_j) = \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker symbol. In view of the proof of Theorem 7.11 it is sufficient to show \(\gamma_1, \gamma_2, \gamma_3\) are the Reeb vector field for \(\eta_{\gamma}\), i.e. we have to show that the compatibility conditions (2.11) are satisfied. Indeed, the fact that \(\gamma_i, i = 1, 2, 3\) are QC vector fields means that (7.11) hold with respect to \(\eta_{\gamma}\). Then (7.13) gives \(\nu = 0\) and the second line of (7.10), for \(\eta_{\gamma}\) and \(Q = \gamma_i, i = 1, 2, 3, \) imply (2.11) for the structure \(\eta_{\gamma}\). Theorem 7.11 completes the proof.
In the particular case when the vector field \( Q \) is the gradient of a function defined on the manifold \( M \), we have

**Corollary 7.15.** If \( h \) is a smooth real valued function on \( M \) and \( Q = \nabla h \) is a QC vector field, then for any horizontal vector fields \( X \) and \( Y \) we have

\[
\text{a) } \nabla h(X,Y) = 0 \quad \text{b) } \nabla h(X,Y) = -T_Q^0(X,Y) \quad \text{c) } u_Q(X,Y) = 0
\]

where \( i, j, k \) is a cyclic permutation of \( 1, 2, 3 \).

**Proof.** From (7.22) and (5.10), we find

\[
2\nabla dh(X,Y) + 2dh(\xi_j)\omega_j(X,Y) + 2g(T_Q^0, X,Y) = \nu g(X,Y).
\]

Decomposing in the \([-1]\) and \([3]\) components completes the proof of a) and b), taking also into account (7.23) and Lemma 7.12. The skew-symmetric part of (7.23) gives

\[
2u_Q + \sum_{(ijk)} L_{ij}(\nabla h)\omega_k = 0,
\]

where the sum is over even permutations of \( 1, 2, 3 \). Hence, c) follows by comparing the trace and trace-free parts of the latter equality. □

8. **Quaternionic contact Yamabe problem**

8.1. **The Divergence Formula.** Let \((M, \eta, g)\) be a contact quaternionic manifold. For a fixed \( j \in \{1, 2, 3\} \) the form \( \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_j^{2n-1} \) is a volume form and is independent of \( j \).

Fixing an orthonormal basis \( \{e_\alpha\} \), \( \alpha = 1, 2, \ldots, 4n \) of the horizontal bundle \( H \) we define the (horizontal) divergence of a one-form \( \sigma \in \Lambda^1(H) \) by the formula

\[
\nabla^* \sigma = -\nabla \sigma \,(e_\alpha, e_\alpha).
\]

It is justified to call the function \( \nabla^* \sigma \) divergence of \( \sigma \) in view of the following Proposition.

**Proposition 8.1.** Let \( \sigma \in \Lambda^1(H) \) and \( \eta \wedge \omega^{2n-1} \) is a volume form. We have
a) \( d (\sigma \wedge \eta \wedge \omega^{2n-1}) = d\sigma \wedge \eta \wedge \omega^{2n-1}; \)

b) \( d\sigma \wedge \eta \wedge \omega^{2n-1} = (\nabla^* \sigma) \eta \wedge \omega^{2n}. \)

Therefore, if \( M \) is compact,

\[ \int_M (\nabla^* \sigma) \eta \wedge \omega^{2n} = 0. \]

**Proof.** a) Taking the exterior derivative

\[
d (\sigma \wedge \eta \wedge \omega^{2n-1}) = d\sigma \wedge \eta \wedge \omega^{2n-1} - \sigma \wedge d\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega^{2n-1} \\
+ \sigma \wedge \eta_1 \wedge d\eta_2 \wedge \eta_3 \wedge \omega^{2n-1} - \sigma \wedge \eta_1 \wedge \eta_2 \wedge d\eta_3 \wedge \omega^{2n-1} + \sigma \wedge \eta \wedge d\omega^{2n-1},
\]

so we have to show that all except the first term in the right-hand side are equal to zero. Denote, as we have been doing so far, by \( \xi_j \) the quaternionic Reeb vector fields of the contact forms \( \eta_j \), i.e., \( \eta_j (\xi_i) = \delta_{ij} \) and \( \eta_j (d\eta_j) \big|_H = 0 \). Since we have also, by definition, \( \iota_{\omega} \xi_j = 0 \) and \( \iota_{\omega} \sigma = 0 \) for \( \xi \) in the vertical space, \( \xi \in V \), it follows \( \sigma \wedge d\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega^{2n-1} = \sigma \wedge \eta_1 \wedge d\eta_2 \wedge \eta_3 \wedge \omega^{2n-1} = \sigma \wedge \eta_1 \wedge \eta_2 \wedge d\eta_3 \wedge \omega^{2n-1} = 0 \). Turning to the last term, we observe that

\[
d\omega_1 = -d\eta_2 \wedge \iota_{\omega_2} d\eta_1 - d\eta_3 \wedge \iota_{\omega_3} d\eta_1 \mod (\eta_1, \eta_2, \eta_3),
\]

hence

\[
2d\omega_1|_H = -\omega_2 \wedge \iota_{\omega_2} d\eta_1 - \omega_3 \wedge \iota_{\omega_3} d\eta_1.
\]

Therefore

\[
-2\sigma \wedge \eta \wedge d\omega_1 \wedge \omega_1^{2n-2} = \sigma \wedge \eta \wedge \omega_2 \wedge \iota_{\omega_2} d\eta_1 \wedge \omega_1^{2n-2} + \sigma \wedge \eta \wedge \omega_3 \wedge \iota_{\omega_3} d\eta_1 \wedge \omega_1^{2n-2}.
\]

From \( \omega_2 (I_1 X, I_1 Y) = -\omega_2 (X, Y) \) and \( \omega_3 (I_1 X, I_1 Y) = -\omega_3 (X, Y) \), i.e., \( \omega_2, \omega_3 \in \Lambda^{2,0}_1 + \Lambda^{0,2}_1 \), we conclude \( \sigma \wedge \omega_2 \wedge \omega_3 \wedge \omega^{2n-2} = \sigma \wedge \omega_3 \wedge \omega^{2n-2} = 0 \) on \( H \).

b) The exterior derivative of \( \sigma \) is expressed in terms of the Biquard connection as

\[
d\sigma(e_\alpha, e_\beta) = \nabla^g \sigma(e_\alpha, e_\beta) - \nabla^g \sigma(e_\beta, e_\alpha) + \sigma(T_{e_\alpha, e_\beta}).
\]

Recalling that for the Biquard connection the torsion of two horizontal vectors is a vertical vector, \( T_{e_\alpha, e_\beta} \in V \), we see that

\[
d\sigma(e_\alpha, e_\beta) = \nabla^g \sigma(e_\alpha, e_\beta) - \nabla^g \sigma(e_\beta, e_\alpha).
\]

Let us fix an arbitrary point \( p \) of \( M \). Taking normal coordinates at \( p \), hence \( \nabla^g \sigma(e_\alpha, e_\beta) = \nabla^g \sigma(e_\alpha, e_\beta) = \sigma_{\alpha, \beta} \), the above equations shows the validity of b), which completes the proof. \( \Box \)
8.2. Partial solutions of the QC-Yamabe problem. In this Section we shall present a partial solution of the Yamabe problem on the quaternionic sphere. Equivalently, using the Cayley transform this provides a partial solution of the Yamabe problem on the quaternionic Heisenberg group. The extra assumption under which we classify the solutions of the Yamabe equation consists of assuming that the “new” quaternionic structure has an integrable vertical space. The change of the vertical space is given by (5.1). Of course, the standard quaternionic contact structure has an integrable vertical distribution. A note about the Cayley transform is in order. We shall define below the explicit Cayley transform for the considered case, but one should keep in mind the more general setting of groups of Heisenberg type \[\text{CDKR1}\]. In that respect, the solutions of the Yamabe equation on the quaternionic Heisenberg group, which we describe, coincide with the solutions on the groups of Heisenberg type \[\text{GV1}\].

As in Section 5 we are considering a conformal transformation \(\tilde{\eta} = \frac{1}{2\eta}\eta\), where \(\tilde{\eta}\) represents a fixed quaternionic contact structure and \(\eta\) is the “new” structure conformal to the original one. In fact, in this section \(\tilde{\eta}\) will stand for the standard quaternionic contact structure on the quaternionic sphere. The Yamabe problem in this case is to find all structures \(\eta\), which are conformal to \(\tilde{\eta}\) and have constant scalar curvature equal to \(16n(n + 2)\), see Corollary 4.13. The Yamabe equation is given by (5.14) and the problem is to find all solutions of this equation.

**Proposition 8.2.** Let \((M, \tilde{\eta}, \mathbb{Q})\) be a compact qc-Einstein manifold of dimension \((4n + 3), n > 1\). Let \(\tilde{\eta} = \frac{1}{2\eta}\eta\) be a conformal deformation of the qc-structure \(\tilde{\eta}\) on \(M\). Suppose \(\eta\) has constant scalar curvature. Then any one of the following two conditions

i) the vertical space of \(\eta\) is integrable;

ii) the QC structure \(\eta\) is qc-pseudo Einstein;

implies that \(\eta\) is a qc-Einstein structure.

**Proof.** The proof follows the steps of the solution of the Riemannian Yamabe problem on the standard unit sphere, see \[\text{LP}\]. Theorem 1.3 tells us that \(\tilde{\eta}\) is a qc-Einstein structure. Theorem 3.12 and equations (5.13), (5.11), and (5.12) imply

\[
\begin{align*}
[Ric_0]_{[-1]}(X,Y) &= (2n + 2)T(X,Y) = \frac{2n + 2}{h}[
abla dh]_{[\text{sym}]}[-1](X,Y) \\
[Ric_0][3](X,Y) &= 2(2n + 5)U(X,Y) = -\frac{2n + 5}{h}[
abla dh - \frac{2}{h}dh \otimes dh][3][0](X,Y).
\end{align*}
\]
Furthermore, when the scalar curvature of $\eta$ is a constant then Theorem 4.8 gives

\[(8.4) \quad \nabla^* T^0 = (n + 2)A, \quad \nabla^* U = \frac{(1 - n)}{2}A.\]

If either the vertical space of $\eta$ is an integrable distribution or $\eta$ is qc-pseudo Einstein then (8.4) together with Proposition 6.2 show that the divergences of $T^0$ and $U$ vanish

\[\nabla^* T^0 = 0 \quad \text{and} \quad \nabla^* U = 0.\]

We shall see that in fact $T^0$ and $U$ vanish, i.e., $\eta$ is also qc-Einstein. Consider first the $[-1]$ component. Taking norms, multiplying by $h$ and integrating, the divergence formula gives

\[
\int_M h \left| [Ric_0][-1] \right|^2 \eta \wedge \omega^{2n} = (2n + 2) \int_M \langle [Ric_0][-1], \nabla dh \rangle \eta \wedge \omega^{2n}
\]

\[= (2n + 2) \int_M (\nabla^* [Ric_0][-1], \nabla h) \eta \wedge \omega^{2n} = 0.\]

Thus, the $[-1]$ component of the qc-Einstein tensor vanishes $| [Ric_0][-1] | = 0$. Define $h = \frac{1}{2\pi}$, inserting (6.2) into (8.3) one gets

\[| [Ric_0][3] | = 2(2n + 5)U = -(2n + 5)|\nabla du|_3\]

from where, arguing as before we get $| [Ric_0][3] | = 0$. Theorem 1.3 completes the proof. 

Corollary 8.3. Let $\bar{\eta} = \frac{1}{2\pi} \eta$ be a conformal deformation of a compact qc-Einstein manifold of dimension $(4n+3), n > 1$. If $\bar{\eta}$ is with constant qc-scalar curvature and either the gradient $\nabla h$ or the gradient $\nabla \left( \frac{1}{h} \right)$ is a QC vector fields then $h$ is a constant.

Proof. Suppose $\nabla h$ is a QC-vector field. Corollary 7.15 b) yields $[\nabla dh]_{[sym]}[-1] = 0$ since the torsion of Biquard connection vanishes due to Proposition 4.2. Then Proposition 8.2 and a) in Corollary 7.15 imply that on $H$ we have

\[dh \otimes dh + d_1h \otimes d_1h + d_2h \otimes d_2h + d_3h \otimes d_3h = \frac{|dh|^2}{n}g.\]

Hence, $dh|_H = 0$, since $n > 1$, which implies $dh = 0$ using the bracket generating condition.

Suppose $\nabla \left( \frac{1}{h} \right)$ is a QC vector field. Then Proposition 8.2 (6.2) combined with b) in Corollary 7.15 show that on $H$ we have

\[3dh \otimes dh - d_1h \otimes d_1h - d_2h \otimes d_2h - d_3h \otimes d_3h = 0.\]

Define $X = I_1X, Y = I_1Y$ etc. to get $dh \otimes dh = d_1h \otimes d_1h = d_2h \otimes d_2h = d_3h \otimes d_3h$. Hence, $dh|_H = 0$ since $\dim Ker dh = 4n - 1$ and $dh = 0$ as above. \[\square\]
8.3. Proof of Theorem \[1.2\]

Proof. We start the proof with the observation that from Proposition 8.2 the new structure \( \eta \) is also qc-Einstein. Next we bring into consideration the quaternionic Heisenberg group. Let us identify \( G(\mathbb{H}) \) with the boundary \( \Sigma \) of a Siegel domain in \( \mathbb{H}^n \times \mathbb{H} \),

\[ \Sigma = \{(q', p') \in \mathbb{H}^n \times \mathbb{H} : \Re p' = |q'|^2\}, \]

by using the map \((q', \omega') \mapsto (q', |q'|^2 - \omega')\). The standard contact form, written as a purely imaginary quaternion, is given by (cf. (5.22))

\[(8.5) \tilde{\Theta} = \frac{1}{2} (d\omega - q' \cdot d\bar{q}' + dq' \cdot \bar{q}'), \]

where \( \cdot \) denotes the quaternion multiplication. Since

\[(8.6) \quad dp' = q' \cdot d\bar{q}' + dq' \cdot \bar{q}' - d\omega', \]

under the identification of \( G(\mathbb{H}) \) with \( \Sigma \) we have also

\[(8.7) \quad \tilde{\Theta} = -\frac{1}{2} dp' + dq' \cdot q'. \]

Taking into account that \( \tilde{\Theta} \) is purely imaginary, the last equation can be written also in the following form

\[2 \tilde{\Theta} = \frac{1}{2} (dp' - dp') + dq' \cdot q' - q' \cdot dq'.\]

Now, consider the Cayley transform as the map

\[ \mathcal{C} : S \mapsto \Sigma \]

from the sphere \( S = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H} \) minus a point to the Heisenberg group \( \Sigma \), with \( \mathcal{C} \) defined by

\[ (q', p') = \mathcal{C}((q, p)), \]

where

\[ q' = (1 + p)^{-1} q \quad \quad p' = (1 + p)^{-1} (1 - p) \]

and with an inverse map \((q, p) = \mathcal{C}^{-1}((q', p')) \) given by

\[ q = \frac{1}{2} (1 + p') q' \quad \quad p = (1 + p')^{-1} (1 - p'). \]

The Cayley transform maps \( S \) minus a point to \( \Sigma \) since

\[ \Re p' = \Re \left( \frac{(1 + p)(1 - p)}{|1 + p|^2} \right) = \Re \left( \frac{1 - |p|}{|1 + p|^2} \right) = \frac{|q|^2}{|1 + p|^2} = |q'|^2. \]
Writing the Cayley transform in the form

\[(1 + p)q' = q, \quad (1 + p)p' = 1 - p,\]

gives

\[dp \cdot q' + (1 + p) \cdot dq' = dq, \quad dp \cdot p' + (1 + p) \cdot dp' = -dp,\]

from where we find

\[(8.8) \quad dp' = -2(1 + p)^{-1} \cdot dp \cdot (1 + p)^{-1}
\]

The Cayley transform is a quaternionic contact conformal diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic contact structure \(\tilde{\Theta}\) and the sphere minus a point with its standard structure \(\tilde{\eta}\), a fact which can be seen as follows.

Equations (8.8) imply the following identities

\[(8.9) \quad 2 C^* \tilde{\Theta} = -(1 + \bar{p})^{-1} \cdot d\bar{p} \cdot (1 + p)^{-1} + (1 + p)^{-1} \cdot dp \cdot (1 + p)^{-1} + (1 + p)^{-1} \cdot dq - dp \cdot (1 + p)^{-1} \cdot q \cdot (1 + \bar{p})^{-1}
\]

\[= (1 + p)^{-1} \left[ dp \cdot (1 + p)^{-1} \cdot (1 + \bar{p}) - |q|^2 dp \cdot (1 + p)^{-1} \right] (1 + p)^{-1}
\]

\[+ (1 + p)^{-1} \left[ - (1 + p) \cdot (1 + \bar{p})^{-1} \cdot d\bar{p} + |q|^2 (1 + p)^{-1} dp \right] (1 + p)^{-1}
\]

\[= \frac{1}{1 + p} \lambda \tilde{\eta} \tilde{\lambda},\]

where \(\lambda = |1 + p| (1 + p)^{-1}\) is a unit quaternion and \(\tilde{\eta}\) is the standard contact form on the sphere,

\[(8.10) \quad \tilde{\eta} = dq \cdot \bar{q} + dp \cdot \bar{p} - q \cdot d\bar{p} - p \cdot dp.
\]

Since \(|1 + p| = \frac{2}{|1 + p'|}\) we have \(\lambda = \frac{1 + p'}{|1 + p'|}\) equation (8.9) can be put in the form

\[\lambda \cdot (C^{-1})^* \tilde{\eta} \cdot \tilde{\lambda} = \frac{8}{1 + p' |2} \tilde{\Theta}.\]

We see that up to a constant multiplicative factor and a quaternionic contact automorphism the forms \((C^{-1})^* \tilde{\eta}\) and \(\tilde{\Theta}\) are conformal to each other. It follows that the same is true for \((C^{-1})^* \eta\) and \(\tilde{\Theta}\).
In addition, $\tilde{\Theta}$ is $qc$-Einstein by definition, while $\eta$ and hence also $(\mathcal{C}^{-1})^*\eta$ are $qc$-Einstein as we observed at the beginning of the proof. Now we can apply Theorem 1.1 according to which up to a multiplicative constant factor the forms $(\mathcal{C}^{-1})^*\tilde{\eta}$ and $(\mathcal{C}^{-1})^*\eta$ are related by a translation or dilation on the Heisenberg group. Hence, taking also into account Lemma 2.4, we conclude that up to a multiplicative constant, $\eta$ is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism. □

Let us note that the Cayley transform defined in the setting of groups of Heisenberg type is also a conformal transformation on $H$, see cf. [ACD, Lemma 2.5]. One can write the above transformation formula in this more general setting.

Acknowledgments. This project has been funded in part by the National Academy of Sciences under the [Collaboration in Basic Science and Engineering Program 1 Twinning Program] supported by Contract No. INT-0002341 from the National Science Foundation. The contents of this publication do not necessarily reflect the views or policies of the National Academy of Sciences or the National Science Foundation, nor does mention of trade names, commercial products or organizations imply endorsement by the National Academy of Sciences or the National Science Foundation. S. Ivanov is a Senior Associate of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. He thanks ICTP for providing the support and an excellent research environment during the final stages of the paper. I. Minchev is a member of the Junior Research Group ”Special Geometries in Mathematical Physics” founded by the Volkswagen Foundation” The authors would like to thank The National Academies for the support. It is a pleasure to acknowledge the role of Centre de Recherches Mathématiques, Montréal where the project initiated. The authors also would like to thank University of California, Riverside and University of Sofia for hosting the respective visits of the authors.


