UNSTABLE PATTERNS AND ROBUST SYNCHRONIZATION
IN A MODEL OF MOTOR PATHWAY IN BIRDSONG

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Abstract

This paper investigates the fundamental dynamical mechanism responsible for transition to chaos in periodically modulated Duffing-Van der Pol oscillator. It is shown that a modulationally unstable pattern appears into an initially stable motionless state. A further spatiotemporal transition occurs with a sharp interface from the selected stable pattern to a stabilized pattern or chaotic state. Also, the synchronization of the chaotic state of the model is investigated. The results are discussed in the context of generalized synchronization. The main idea is to construct an augmented dynamical system from the synchronization error system, which is itself uncertain. The advantage of this method over existing results is that the synchronization time is explicitly computed. Numerical simulations are provided to verify the operation of the proposed algorithm.
1 Introduction

Most natural phenomena have nonlinear behavior and, the equations that model them contain nonlinear terms [1-12]. They are responsible for making a system very sensitive to initial conditions of operation that produce erratic and unpredictable behavior. By unpredictable, we mean that one path cannot be inferred from another regardless of how close they were initially. This fundamental property called chaos, exists only in nonlinear dynamical systems. Chaotic systems have many fascinating properties, and there is a good deal of evidence that much of nature is chaotic; the solar system, for instance (this is actually, by a long and devious story, where dynamical systems theory comes from). It raises a lot of neat and nasty problems about how to understand dynamics from observations, and about what it means to make a good mathematical model of something. Biological systems are capable to generate an extremely rich variety of motor commands. Most impressively, in many cases, these articulated commands are learned through experience. The dynamical processes involved in learning are the focus of extensive research, both in order to gain knowledge on how living systems operate and as an inspiration for the design of artificial systems capable of adaptation and learning.

Pattern formation is an area of active research in a wide variety of extended physical systems. It occurs in a wide variety of natural contexts, from animal coat markings to convection cells in the Sun. Experimentally, patterns have been studied in many different systems, including Rayleigh-Bénard convection, solidification, chemical reactions and Faraday waves. One of the most studied systems that model practically many physical, chemical, or biological systems is the Van der Pol equation. The Van der Pol oscillator is predictable for instance, but perturbs its hinge while swinging and it becomes chaotic. The extended version of the system contains a cubic term and is sometimes called the Duffing-Van der Pol equation. As in many extended systems, the role played by the extended term is not completely understood.

On the other hand, when coupled, even weakly, oscillators interact via adjustment, i.e., their timing, often leading to synchronization. Synchronization phenomena of coupled nonlinear oscillators are of fundamental interest, as they are encountered in various areas of science [1-3]. Recently, chaos synchronization phenomena of coupled chaotic systems have attracted much attention, since their well-controllable parameters and their well-studied nonlinear behavior [12-14] makes them ideally suited for studies on the fundamental synchronization phenomena of coupled nonlinear systems. The understanding of the synchronization scenario related to variations of the system parameters is of high relevance for both aspects.

Our prime concern in the present work is to understand how a typical nonlinear system, namely, the Duffing-Van der Pol oscillator, responds to a single frequency periodically driven forcing and exhibits different dynamics. The mechanism of the transition to chaos and the development of synchronized dynamics by making a series of measurements focusing on both analytical and numerical computations is particularly interesting. First, we analyzed the patterns formation in the
parameter spaces and the bifurcation diagrams. Second, the synchronization in the coupled system is studied. The main idea is to construct an augmented dynamical system from the synchronization error system, which is itself uncertain. The advantage of this method over existing results is that the feedback controller has a predictable synchronization delay.

2 Patterns of the transition to chaos in periodically modulated Duffing-Van der Pol oscillator

The animal model inspiring our dynamical model is the motor pathway in oscine birds. Part of this pathway is the nucleus RA (robustus nucleus of the archistriatum) containing excitatory neurons, some of which enervate respiratory nuclei and others enervate the nucleus nXIIIs, which projects to the muscles in the syrinx [2-5]. These two populations are segregated into different regions of the RA structure.

Recently, the study of the avian vocal organ allowed to associate acoustical features of the song with properties of the muscle instructions necessary to generate the song. The production of repetitive syllables requires a cyclic expiratory gesture and a cyclic gesture of the syringeal muscles [3, 5]. Sound is produced by labia located at the junction between bronchii and tract, obstructing periodically the airflow. The model mentioned above describes the departure of the midpoint of a labium from the prephonatory position $x$ [2-5]:

$$\ddot{x} + \xi_i(t)x - \xi_{ii}(t)\dot{x} + \beta x^3 + \gamma x^2 \dot{x} = \xi(t)$$

where $\xi_i(t) = \mu(1 + \epsilon \cos \omega t)$ is a function of the activity of ventral muscles, whereas $\xi_{ii}(t) = \nu(1 + \epsilon \cos \omega t)$ is a function of the bronchial pressure. This model has been tested by using experimentally recorded $\xi_i(t)$ and $\xi_{ii}(t)$. The resulting $x(t)$ was remarkably similar to the one produced while the physiological data had been recorded [2-5]. The phase difference between the $\xi_i(t)$ and $\xi_{ii}(t)$, responsible for important acoustic features of song (such as the dynamics of the syllabic fundamental frequency), originates in RA. Recent works have unveiled that direct connections between respiratory nuclei and nXIIIs can affect the final value of the phase difference. The term $\beta x^3$ introduce to take in account the nonlinear activity of the ventral muscles and the incoming noise $\xi(t)$, due to external interactions. The oscillator has proven to be a very useful tool in the description of self-oscillating state of many physical and biological systems. The objective of the present section is to understand how typical nonlinear system, namely the Duffing-Van der Pol oscillator responds to periodically driven forcing and exhibits different dynamical transitions.

2.1 Unstable Patterns in the two-parameter $(\epsilon, \beta)$ phase diagrams

As in many extended systems, the role played by the incoming nonlinear and linear terms are of great importance. In order, to investigate the size of the chaotic regions in the Duffing-Van der Pol oscillator, we plotted the largest Lyapunov exponent as a function of the two main parameters, namely the amplitude $\epsilon$ of the periodic driven and, the coefficient $\beta$ of the Duffing term.
The set of Lyapunov exponents \( \lambda_k \) provides an intuitively appealing and yet a very powerful measure of sensibility to initial conditions which are required for a chaotic system. All \( \lambda_k \) originates from linear stability analysis. In this approximation, all solutions are of the form \( e^{\lambda_k t} \), \( k = 1, 2, \ldots, M \). The maximum Lyapunov exponent \( \lambda_{\text{max}} \) is defined as:

\[
\lambda_{\text{max}} = \lim_{\tau \to \infty} \frac{1}{\tau} \ln(||\Phi(\tau)||),
\]

where \( ||\Phi(\tau)|| = \sqrt{\delta x^2 + \delta \dot{x}^2} \) is obtained in the Poincaré cross-section by solving numerically the variational equation

\[
\delta \ddot{x} + \xi_i(t) \delta x - \xi_{ii}(t) \delta \dot{x} + 3\beta x^2 \delta x + 2\gamma x \delta \dot{x} + \gamma x^2 \delta \dot{x} = 0
\]

simultaneously with the system (1). A positive Lyapunov exponent is a signature of chaos while zero and negative values of the exponent is an indication of a marginally stable or quasi-periodic orbit and periodic orbit, respectively. In the parameter ranges pertaining to darker blue regions in Fig. 1, quasi-periodic behavior is found and middle blue regions correspond to chaotic states. Region colored in light blue shades correspond to periodic dynamical states. In general, chaotic behavior is found for a large range values of the Duffing term when \( \varepsilon \) is also sufficiently large. Moreover, the size of the quasi-periodic motion increase with the increase of the size of the chaotic motion.

In comparison, in Fig. 1(i), the map is plotted when the systems exhibit the subharmonic oscillations (i.e., \( \omega = \frac{1}{3} \sqrt{\mu} \)). We observe that, the pattern of the chaotic motion presents a self-assembled structures. In the map sharp, the basin of attraction for chaotic motions develops a series of fingers, a scenario well known in Duffing oscillators. The consequence of the presence of these fingers is that, the number of intervals in which periodic and chaotic motions intersect each other increases as the Duffing parameter increases. This fingers disappears and lives place to a broad band chaotic attractor when the system inter in resonance with the external excitation (Fig. 1(iii)). When the system shifts around the resonance, the size of the quasi-periodic motion increases with the increase of the size of the chaotic motion (see Fig. 1(ii)). The two parameters diagram of Fig. 1(iv) is the contrary of that described before. The figure is plotted in the case of the super-harmonic motions; that is for \( \omega = 3 \sqrt{\mu} \). The map sharp presents a condensed structure with a sharp band of chaotic motions. The formation of fingers observed in Fig. 1(i) are condensed here on the values of \( \varepsilon < 2 \). In particular, in this case, we observe a broad band of periodic motion.

### 2.2 Bifurcations and route to chaos

For our system, we observe three characteristic regions in the parameter space, namely quasiperiodic, periodic, and chaotic regions. In order to locate these regions, we investigate the parameter space using the bifurcation diagrams. Such diagrams show the evolution of the attractor for dissipative system with a change of the systems parameters. We choose to plot a projection of the Poincaré section on the \( x \) axis against the system’s parameter \( \varepsilon \) which represent the amplitude
of the periodically modulated signal, while keeping the other parameters fixed as: $\mu = 1.0434$, $\nu = \gamma = 0.1$, $\omega = 1$ and $\xi(t) = 0$. We repeat this procedure for four values of the Duffing term $\beta$.

With the increase of $\varepsilon$ from 0 on, the motion of the system is quasi-periodic, which is observed as a smeared region in the bifurcation graphs (Figs. 2 and 3). It is noted from these patterns of bifurcation that the smeared region increases as the value of $\beta$ change increasingly. For certain value of $\varepsilon$ (depending on the value of $\beta$), the motion suddenly becomes periodic, and with the further increase of the amplitude of the periodically modulated signal $\varepsilon$, the system reaches chaos through sequence of period doubling. However, for the values of the Duffing parameters $\beta = 0$, and $\beta = 1$ the system reached the chaotic state after sequences of periodic windows remain there. In the case of $\beta = 0.1$, and $\beta = 0.5$ the system returns to periodic state as a final destination. For $\beta = 0.1$, periodic motion are dominant over the range of $\varepsilon$ and possible chaotic behavior are found only in narrow isolated regions just before the value of $\varepsilon = 4$ is reached.

Accuracy of the bifurcation diagrams can be made using the maximum Lyapunov exponent. This give us, besides qualitative information about a system behavior, a quantitative measure of the system stability. Evolution of the maximum Lyapunov exponent with a change of parameters gives us information about the system behavior in addition to that from bifurcation graphs. Combined these two methods are reliable tools for examining a system behavior and routes to chaos. Fig. 2 and Fig. 3 gives a comparison between the bifurcation diagram and maximum Lyapunov exponent evolution for a range of $\varepsilon$. As presented in the figures, there is a good accuracy between the bifurcation graph and the maximum Lyapunov exponent $\lambda_{\text{max}}$.

In order to go deep with our analysis, we look for the attractors of the Poincaré maps of different regions for $\beta = 1$. From the bifurcation diagram and the Lyapunov exponent of Fig. 4(ii), different windows are identified, one for quasi-periodic motion, three for periodic and chaotic motions. In the quasi-periodic region, we set $\varepsilon = 2$ and the poincaré map showed in Fig. 4(i) is a limit cycle, confirming the prediction that we obtained with the bifurcation graph. For the chaotic regions, the attractor (Fig. 4(ii)), is separated by periodic and chaotic motions. Interestingly, it exhibits the self-similar structure at different scale. When we move to another windows, the Poincaré map presents a beautiful chaotic map with a fractal structure (see Fig. 4(iii)). Moving to another windows by increasing the value of $\varepsilon$, the fractal structure tends to disappear up to the system remain chaotic (Fig. 4(iv)).

3 Synchronization in coupled Duffing-Van der Pol oscillators

In this section, we consider the robust synchronization of the model of motor pathway in birdsong modelled by the periodically modulated Duffing-Van der Pol oscillator. The approach developed in this work considers incomplete state measurements and no detailed model of the Duffing-Van der Pol oscillator to guarantee robust stability (in fact, robust synchronization). Our approach includes a state/uncertainty observer and leads to a robust feedback control scheme. The
expression of the synchronization time is explicitly computed.

By using \( x_1 = x \) and \( x_2 = \dot{x} \), one may rewritten (1) in a matrix equation form as:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\xi(t)x_1 + \xi_{ii}(t)x_2 - \beta x_1^3 - \gamma x_1^2 x_2 + \xi(t).
\end{align*}
\]  

(4)

In order to observe the synchronization behavior in the Duffing-Van der Pol oscillator, we assume that the drive system is given by system (4) and the response is

\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -\hat{\xi}(t)y_1 + \hat{\xi}_{ii}(t)y_2 - \hat{\beta} y_1^3 - \hat{\gamma} y_1^2 y_2 + \hat{\xi}(t) + u,
\end{align*}
\]

(5)

where \( \hat{\xi}(t) = \tilde{\mu}(1 + \tilde{\epsilon} \cos \omega t), \hat{\xi}_{ii}(t) = \tilde{\nu}(1 + \tilde{\epsilon} \cos \omega t) \) and the added term \( u \) represents a feedback synchronization force. The parameters of the slave system are assumed to be different from the parameters of the drive system so that they cannot be synchronous. Thus, the control input \( u \) is to be determined for the purpose of synchronizing two coupled Duffing-Van der Pol oscillators with the same but unknown parameters in spite of the difference in parameters and initial conditions.

Now, assume that the outputs of the master and slave systems are respectively, \( y_m = x_1 \) and \( y_s = y_1 \). For further analysis of stability and synchronization, we define the state error vector between the master and slave Duffing-Van der Pol oscillators as \( e_1 = y_1 - x_1 \) and \( e_2 = y_2 - x_2 \). From Eqs. (4) and (5), the error dynamics is described by

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= \Theta(e, x_1, x_2, t) + u, \\
y_e &= e_1,
\end{align*}
\]

(6)

where \( e = (e_1, e_2)^T \), \( y_e \) is the error system output and

\[
\Theta(e, x_1, x_2, t) = -\hat{\xi}(t)e_1 + \hat{\xi}_{ii}(t)e_2 - \hat{\beta} e_1(e_1^3 + 3e_1 x_1 + 3x_1^2) \\
- \hat{\gamma}(e_1 e_2 + e_1 x_2 + e_2 x_1) + (\hat{\xi}(t) - \hat{\xi}(t))x_1 + (\hat{\xi}_{ii}(t) - \xi_{ii}(t))x_2 \\
+ (\beta - \hat{\beta})x_1^3 + (\gamma - \hat{\gamma})x_1 x_2 + \hat{\xi}(t) - \xi(t).
\]

The synchronization goal can be stated as follows: given the transmitted signal \( y_m \) and least prior information about the structure of the nonlinear filter, system (4), the problem is to design a receiver signal \( u \) such that

\[
\lim_{t \to T_s} e(t) = 0,
\]

(7)

where \( T_s \) is the synchronization time.

To describe the new design and analysis, the following hypothesis are needed.

**H1:** Only the system output \( y_e = e_1 \) is available for feedback.

**H2:** The function \( \Theta(e, x_1, x_2, t) \) is uncertain.
Some comments regarding the above hypothesis are in order. Assumption 1 is realistic. For instance, in the secure communication case, only the transmitted signal and receiver signal are available for feedback from measurements. Another example can be found in neuron synchronization where master neuron transmits a scalar signal. The slave neuron tracks the signal of the master neuron. Assumption 2 refers to a general and practical situation because the term $\Theta(e, x_1, x_2, t)$ involves the uncertainties in the system. Hence, the nonlinear function $\Theta(e, x_1, x_2, t)$ is unknown and it is clear that it cannot be directly used in a linearizing-type of feedback.

The idea for dealing with the uncertain term $\Theta(e, x_1, x_2, t)$ in Eq. (6) is to lump it into a new state $\eta$. Thus, let $\eta = \Theta(e, x_1, x_2, t)$. In this way, system (6) can be rewritten as the following extended dynamically equivalent system [15, 16]:

$$
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= \eta + u, \\
\dot{\eta} &= \Xi(e, x_1, x_2, \eta, t, u), \\
y_e &= e_1, \\
\end{align*}
$$

where

$$
\Xi(e, x_1, x_2, \eta, t, u) = e_2 \partial \Theta(e, x_1, x_2, t)/\partial e_1 + (\eta + u) \partial \Theta(e, x_1, x_2, t)/\partial e_2 \\
+ x_2 \partial \Theta(e, x_1, x_2, t)/\partial x_1 + (\dot{\xi}(t) - \xi(t)) \partial \Theta(e, x_1, x_2, t)/\partial t \\
+ (\xi_i(t)x_1 - \xi_{ii}(t)x_2 - \beta x_1^4 - \gamma x_1^3 x_2 + \xi(t)) \partial \Theta(e, x_1, x_2, t)/\partial x_2.
$$

It should be pointed out that system (8) is dynamically equivalent to system (6), that is, system (8) has the same solution as the system (6).

For the sake of compactness, we introduce the following alternative description for system (8):

$$
\begin{align*}
\dot{e} &= Ae + B(\eta + u), \\
\dot{\eta} &= \Xi(e, x_1, x_2, \eta, t, u),
\end{align*}
$$

where

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

Let $M(\alpha, \beta_1)$ be a matrix which elements are given by

$$
M_{ij}(\alpha, \beta_1) = \frac{(-1)^{i+j} \left( \frac{\alpha}{\beta_1} \right)^{5-i-j}}{(2-i)!(2-j)!(\alpha+1)\cdots(\alpha+5-i-j)}, \quad i, j = 1, 2
$$

where $\alpha$ and $\beta_1$ are positive constants. Also, let $N(\theta)$ be the matrix:

$$
N(\theta) = \frac{\alpha}{\beta_1} \theta^\frac{1}{2} \int_0^1 (1-t)^\alpha e^{-\frac{\theta}{\beta_1}} A^T \tilde{\theta}^t \tilde{\theta}^t B B^T e^{-\frac{\theta}{\beta_1}} A^T \tilde{\theta}^t \tilde{\theta}^t dt
$$
Note that $N(\theta)$ is symmetric, positive definite and is the solution of the differential matrix equation:

$$
\frac{dX}{d\theta} = \frac{1}{\beta_1^3} \theta^{-\frac{3}{2}} \left[ -A^T X - X A - \beta_1 \theta^{-\frac{3}{2}} X + BB^T \right].
$$

Equation (12) will be useful in proving that the dynamics of the closed-loop system is asymptotically stable. This is motivated by the fact that the proposed control scheme is based on the use of bounded positive functions that are nonincreasing along the solutions of the closed-loop system. Moreover, we have

$$
N_{ij}(\theta) = \frac{(-1)^{i+j} \left( \frac{\alpha}{\beta_1} \theta^{\frac{3}{2}} \right)^{5-i-j} (4-i-j)!}{(2-i)!(2-j)!(\alpha+1) \cdots (\alpha+5-i-j)!}, \quad i, j = 1, 2.
$$

(13)

$\theta = \theta(z)$ is of class $C^1$ and is the unique positive solution of

$$
\theta^{1+\frac{3}{\alpha}} = \sum_{i,j=1}^{2} \tilde{M}_{ij}(\alpha, \beta_1) \theta^{\frac{1}{2}(i+j-2)} z_i z_j,
$$

(14)

where $\tilde{M}_{ij}(\alpha, \beta_1)$ are elements of $M^{-1}$, the inverse matrix of $M$.

Now, let us consider the following linearizing-like control law:

$$
u = -\eta - \frac{1}{2} B^T N^{-1}(\theta) e,
$$

(15)

where $N^{-1}(\theta)$ is the inverse matrix of $N(\theta)$.

Substitution of the linearizing-like controller (15) into (9) leads to

$$
\begin{aligned}
\dot{e} &= \left( A - \frac{1}{2} B B^T N^{-1}(\theta) \right) e, \\
\dot{\eta} &= \Xi(e, x_1, x_2, \eta, t, u).
\end{aligned}
$$

(16)

Now, we can establish the following result.

**Theorem 1**: Let $e_0 = e(0)$ be the initial condition of $e(t)$. If $e_0 \neq 0$, $\alpha \geq 1$ and $\beta_1 > 0$, then the synchronization error $e(t)$ converges asymptotically to zero at a finite time

$$
T_s = \frac{\alpha}{\beta_1} \theta_{\beta_1}(e_0).
$$

(17)

Without loss of generality, we first assume that $e(t)$ is defined in the interval $[0, T_s]$ so that $\theta \neq 0$ and the matrices $N(\theta)$ and $N^{-1}(\theta)$ exist. Define as the Lyapunov candidate function, the function $\theta$ defined in Eq. (14) which can be rewritten as:

$$
\theta(e) = e^T N^{-1}(\theta) e.
$$

(18)

Now, consider the following function $F(e, \theta) = \theta(e) - e^T N^{-1}(\theta) e$. From Eq. (18), one has that

$$
\frac{dF}{dt} = F' \frac{\partial \theta}{\partial e} + F e = 0 \quad \text{where} \quad F' = \frac{\partial F}{\partial e} = -2N^{-1}(\theta) e \quad \text{and} \quad F' = \frac{\partial F}{\partial \theta} = e^T \left[ \frac{1}{\theta} N^{-1}(\theta) - \frac{d}{d\theta} N^{-1}(\theta) \right] e.
$$
Then, one can deduce that \( \frac{\partial \theta}{\partial e} = -\frac{E'_{e}}{E_{\theta}} \). With this in mind, the time derivative of (18) along the trajectories of (16) satisfies

\[
\dot{\theta}(e) = \left< \frac{\partial \theta}{\partial e}, \left( A - \frac{1}{2} N^{-1}(\theta)BB^T \right)e \right>,
\]

\[
= -\left< \frac{E'_{e}}{E_{\theta}}, \left( A - \frac{1}{2} N^{-1}(\theta)BB^T \right)e \right>,
\]

\[
= \frac{1}{E_{\theta}} e^T \left( A^TN^{-1}(\theta) + N^{-1}(\theta)A^T - N^{-1}(\theta)BB^TN^{-1}(\theta) \right)e,
\]

where \(<.,.>\) is the inner product. Now using Eq. (12), one may easily prove that

\[
A^TN^{-1}(\theta) + N^{-1}(\theta)A^T - N^{-1}(\theta)BB^TN^{-1}(\theta) = \beta_1 \theta^{1-a-1} \left[ \frac{d}{d\theta} N^{-1}(\theta) - \frac{1}{\theta} N^{-1}(\theta) \right].
\]

Then, we get

\[
\dot{\theta}(e) = -\beta_1 \theta^{1-a-1}(e),
\]

which is negative definite if \( \alpha \geq 1 \) and \( \beta_1 > 0 \). This means that if \( \alpha \geq 1 \) and \( \beta_1 > 0 \), the synchronization error \( e(t) \) converges asymptotically to zero. Since \( \theta \) is a continuous function, applying LaSalle invariance principle [18], one can easily prove that the origin is the largest invariant set contained in \( E = \{ e \in \mathbb{R}^2, \dot{\theta}(e) = 0 \} \). Thus, the synchronization error remains at zero for all \( t \geq t_0 \geq 0 \).

The convergence of \( \eta(t) \) to zero follows from the fact that the closed-loop system is in cascade form. This guarantees the convergence to the origin of the closed-loop system (16).

To compute the synchronization time, we have to follow the time trajectory of the closed-loop system (16). In this case, synchronization is achieved when \( e(t) \) is zero for all \( t \geq T_s \geq 0 \). So, let us integrate Eq. (21) to get \( \theta^\frac{1}{a} = \frac{1}{a}(-\beta_1 t + c) \), where \( c \) is an integration constant. Since \( e_0 \neq 0 \) on has that \( \dot{\theta}(e_0) \neq 0 \). In this case, one can deduce that \( c = \alpha \theta^{\frac{1}{a}}(e_0) \). With this in mind and using the fact that \( \dot{\theta}(e) = 0 \) at \( t = T_s \), one may easily prove that the synchronization time is defined as in Eq. (17). This completes the proof.

\[\triangle\]

**Remark 1** Given the feedback parameters \( \alpha \) and \( \beta_1 \), it is not immediately apparent how one chooses the function \( \theta \) so that the synchronization objective (7) is satisfied. Furthermore, it is not easy to find the analytic solutions of Eq. (14). Fortunately, this equation can be solved numerically.

Nevertheless, the linearizing-like feedback (15) is not physically realizable because it requires a perfect knowledge of the non-linear term \( \Theta(e, x_1, x_2, t) \). Because of Assumptions 1 and 2, the linearizing-like feedback (15) must be modified in such a way as to encompass consideration of modeling errors and parameter perturbations. We therefore use the estimation of \( \Theta(e, x_1, x_2, t) \) in
such a way that the main characteristics of the linearizing-like feedback (15) are retained. As it has been established in [17], the problem of estimating \((\epsilon, \eta)\) can be addressed by using a high-gain observer. Thus, we are interested in a dynamical output feedback of the form:

\[
\begin{align*}
\dot{\hat{e}}_1 &= \dot{\hat{e}}_2 + \rho \beta_1 (e_1 - \hat{e}_1), \\
\dot{\hat{e}}_2 &= \dot{\eta} + \rho^2 \beta_2 (e_1 - \hat{e}_1) + u, \\
\dot{\eta} &= \rho^3 \beta_3 (e_1 - \hat{e}_1),
\end{align*}
\]

\(u(\hat{e}) = \text{Sat}\left\{ \frac{-(\alpha + 2)(\alpha + 3)\hat{e}_1}{(\beta_1^{\frac{1}{\alpha}})^2} \frac{\alpha \theta_1}{\beta_1^1} + \frac{\alpha + 2}{(\beta_1^{\frac{1}{\alpha}})^2} \right\},\)  

where \(\hat{e}_1, \hat{e}_2\) and \(\hat{\eta}\) are respectively, the estimated values of \(e_1, e_2\) and \(\eta,\) and \(\rho > 0\) is the so-called high-gain parameter, which can be interpreted as the uncertainties estimation rate and often be chosen as a constant [15], and

\[
\text{Sat}\{\} = \left\{ \begin{array}{ll}
= u_{\text{max}}, & \text{if } u > u_{\text{max}}, \\
= -\frac{1}{2} B^T N^{-1}(\theta) \hat{e}, & \text{if } -u_{\text{max}} \leq u \leq u_{\text{max}}, \\
= -u_{\text{max}}, & \text{if } u < -u_{\text{max}},
\end{array} \right.
\]

\(\theta\) is the unique positive solution of the equation:

\[
\beta_1^{\frac{1}{\alpha}} \beta_1^{\frac{1}{\alpha}} = \frac{2\beta_1^{\alpha + 3}}{\alpha} (\alpha + 2) \beta_2 e_2 \right\}
\]

\[
\theta_1^{\frac{1}{\alpha}} = \frac{2\beta_1^{\alpha + 3}}{\alpha} (\alpha + 2) \beta_2 e_2 \right\}
\]

Now, the following result can be claimed:

**Theorem 2**: Let \(\hat{e}_0 = \hat{e}(0)\) be the initial condition of \(\hat{e}(t)\). If \(\hat{e}(0) \neq 0, \alpha \geq 1\) and \(\beta_1 > 0\), then the synchronization error \(e(t)\) converges asymptotically to zero at a finite time

\[
T_{\epsilon} = \frac{\alpha}{\beta_1} \theta_1^1 (\hat{e}_0).
\]

**Proof**: Let \(\bar{e} \in \mathbb{R}^{2}\) be an estimation error vector whose components are defined as follows: \(\bar{e}_i = \rho^{2-i}(e_i - \hat{e}_i), i = 1, 2\) and \(\bar{e}_3 = \eta - \hat{\eta}\). Substitution of the robust feedback controller (23) and the dynamics of the above defined estimation error into (9) yields

\[
\begin{align*}
\dot{\bar{e}} &= \Lambda(e, \bar{e}, \eta, u), \\
\dot{\eta} &= \Xi(e, \bar{e}, x_1, x_2, \eta, t, u), \\
\dot{\bar{e}} &= \rho D\bar{e} + B' \Xi(e, \bar{e}, x_1, x_2, \eta, t, u),
\end{align*}
\]

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where \( \Lambda(e, \tilde{e}, \eta, u) = Ae + B(\eta + u) \) with \( u = u(e_i - \rho^{i-2} \tilde{e}_i, \eta - \tilde{e}_3) \),

\[
D = \begin{bmatrix}
-k_1 & 1 & 0 \\
-k_2 & 0 & 1 \\
-k_3 & 0 & 0
\end{bmatrix}
\text{ and } B' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Since the saturation function is a bounded function, there exists a continuous function \( \gamma(||\tilde{e}||) \) such that \( ||\Lambda(e, \tilde{e}, \eta, u)|| \leq \gamma(||\tilde{e}||) \). In addition, since \( \eta = \Theta(e, x_1, x_2, t) \) and \( u = \text{Sat} \left\{ -\tilde{\eta} - \frac{1}{2} B^T N^{-1}(\theta) \tilde{\theta} \right\} \), one can obtain the contraction \( \eta = Z(e, \tilde{e}, x_1, x_2, \eta, t, u) \) (which can be computed from the first integral of the second equation of system (26), i.e., \( \eta = \int \Xi(e, \tilde{e}, x_1, x_2, \eta, t, u)dt \)). Then, according to the Contraction Mapping Theorem, the state \( \eta \) can be expressed globally and uniquely as a function of the coordinates \( (e, \tilde{e}) \).

On the other hand, since the matrix \( D \) is Hurwitz by construction, the nominal system \( \dot{\tilde{e}} = \rho D \tilde{e} \) is quadratically stable. This implies that the Lyapunov equation \( D^T P + P D = -I_3 \) (where \( I_3 \) is the identity matrix of dimension 3) has a positive definite solution \( P \). Since the nonlinear function \( \Xi(e, \tilde{e}, x_1, x_2, \eta, t, u) \) is bounded, the last equation of (26) is quadratically stable.

From this, and the boundedness of \( \Lambda(e, \tilde{e}, \eta, u) \), one can conclude that, given a compact set of initial conditions \( \Omega_0 \subset \mathbb{R}^3 \) containing the origin, there exists an upper bound \( u_{\text{max}} \), with \( |\text{Sat}\{\cdot\}| \leq u_{\text{max}} \) and a high-gain estimation \( \rho \) such that \( \Omega_0 \) is contained in the attraction basin \( \Omega_\varepsilon \times \Omega_{\tilde{\varepsilon}0} \) of system (26). Hence, system (26) is semi globally practically stable, i.e., \( (e, \eta) \rightarrow (0, 0) \). Then, since the solution of (10) is the projection of system (6), one can conclude that \( e(t) \rightarrow 0 \), via module \( \Pi(e, \eta) \) (where \( \Pi(e, \eta) \) is the projection of system (9) into system (6) for all \( t \geq 0 \)). Therefore, \( e(t) \rightarrow 0 \) as \( t \rightarrow T_s \) and this achieves the proof.

\( \triangle \)

4 Numerical simulations

In order to validate the performance of the proposed control law, we will show a series of numerical experiments to demonstrate the effectiveness of the proposed synchronization scheme.

Without lost of generality, one can consider that initial conditions and parameters of the drive system are those of Fig. 4(ii) for \( t \leq 0 \). Also, parameters of the slave system are chosen with a difference of 1% from the parameters of the master system so that \( \hat{\mu} = 0.0534 \), \( \hat{\epsilon} = 4.93 \), \( \hat{\beta} = 1.01 \), \( \hat{\nu} = \hat{\gamma} = 0.11 \), \( \omega = 1 \) and \( \hat{\xi}(t) = 0.01 \).

The initial conditions for \( (\hat{e}_1, \hat{e}_2, \hat{\eta}) \) were chosen as follows: \( (1, 0, -2.5) \) so that

\[
\theta(\hat{e}_0) = \left[ \frac{\beta^2}{\alpha^3} (\alpha + 2)^2 (\alpha + 3) \right]^{\frac{1}{(\alpha+3)}}.
\]

Hence, the synchronization time (25) can be expressed as:

\[
T_s = \frac{1}{\beta^1_{(\alpha+3)}} \left[ \alpha^\alpha (\alpha + 2)^2 (\alpha + 3) \right]^{\frac{1}{(\alpha+3)}}.
\]
Remark 2 The $\alpha, \beta_1$-parameterization of the feedback coupling (23) provides a simple tuning procedure. From the above equation, one can observe that for $\beta_1$ fixed, if $\alpha$ increases, then $\theta_0^{1/2}(e_0)$ decreases so that $\alpha \theta_0^{1/2}(e_0)$ increases. This means that the synchronization time $T_s$ increases with $\alpha$. Also, according to Eq. (27), for $\alpha$ fixed, if $\beta_1$ increases, then the synchronization time $T_s$ decreases. Hence, the analysis shows how the synchronization time can be minimized. This is of great practical interest, since the synchronization process can be affected as fast as desired, just depending on the feedback parameters $\alpha$ and $\beta_1$.

It should be pointed out that the value of $\theta$ is obtained via numerical simulations. The high gain parameter is chosen to be $\rho = 10$ and $u_{\text{max}} = 20$. The estimation constants $k_i$, $i = 1, 2, 3$ are $[k_1, k_2, k_3] = [3, 3, 1]$ so that the eigenvalues of matrix $D$ are located at $-1$.

Figures 5(a) and 5(b) present respectively, the synchronization time as a function of $\beta_1$ for three different values of $\alpha$, and as a function of $\alpha$ for three different values of $\beta_1$. From Fig. 5(a), it clearly appears that the synchronization time decreases when $\beta_1$ increases while in Fig. 4(b), the synchronization time increases with $\alpha$.

Figure 6 shows the state trajectories of the drive Duffing-Van der Pol oscillator (solid line) and the slave Duffing-Van der Pol oscillator (dashed line) performed with $\alpha = \beta_1 = 1$. It clearly appears that after a short transient oscillatory period, the states $x_1(t)$ and $y_1(t)$, $x_2(t)$ and $y_2(t)$ evolve together in an almost synchronous way. This implies that the response system (5), (22), (3) globally synchronizes the drive Duffing-Van de Pol oscillator (4), even though there exist parametric perturbation. Note that in this case, the analytical value of the synchronization is about 2.445 sec.

Figure 7 presents the time evolution of the synchronization error. From this figure, it clearly appears that the synchronization error is stabilized at the origin by the output-feedback controller (23) in spite of the fact that both master and slave systems have different parameters and initial conditions. From Figs. 7(a) and 7(b), one can see that a fairly good convergence of $e \in \mathbb{R}^2$ is obtained in about 2.4 sec which corresponds to the analytical value of the synchronization time.

In order to add evidence of the effectiveness and efficiency of the proposed adaptive synchronization scheme, we have plotted $x_1(t)$ versus $y_1(t)$ and $x_2(t)$ versus $y_2(t)$ without and with the output feedback controller (23). The projections of the attractors from the fourth-dimensional phase space onto the planes $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$, respectively, for $u = 0$ are shown on the left hand side of Fig. 8 (uncontrolled evolution). These projections clearly indicate that these oscillations are not synchronized, neither in phase nor in frequency. The relationship between the states of the drive and response systems under the output feedback controller (23) is depicted on the right hand side of Fig. 8. Note that the phases of the master and slave systems are locked, which is a common measure of the degree of synchronization. Thus, it is clearly evident that the manifolds $x_1(t) = y_1(t)$ and $x_2(t) = y_2(t)$ are stable, and one can conclude that the chaotic oscillations of the drive and response systems are synchronized in the complete sense and our synchronization objective has been attained.
5 Conclusion

In this work, on one hand, the minimum possible parameter combinations in which chaos is possible is obtained. On the other hand, it is possible to use the Lyapunov exponents, Poincaré map, bifurcation diagram to heuristically argue out a sufficient condition for the onset of steady-state chaos. The synchronization problem was addressed as one of chaos suppression. A robust adaptive feedback is developed such that two Duffing-Van der Pol chaotic oscillators can be synchronized. A state observer is used to estimate the system’s uncertainties and the unmeasurable states based on the measurable synchronization error. An explicit expression of the synchronization time was given in terms of two parameters for which an arbitrary convergence rate of the synchronization error can be prescribed. Simulation results demonstrate that the proposed strategy is able to achieve the synchronization of two chaotic models of motor pathway in birdsong.

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References


Figure 1: The domains of periodic, quasi-periodic and chaotic solution in the two-parameters diagram ($\varepsilon, \beta$). This diagram is determined numerically by finding the largest Lyapunov exponent of the model. The parameters fixed as: $\mu = 1.0434, \nu = \gamma = 0.1$, and $\xi(t) = 0$; (i) $\omega = \frac{1}{3}\sqrt{\mu}$; (ii) $\omega = 1$; (iii) $\omega = \sqrt{\mu}$; (iv) $\omega = 3\sqrt{\mu}$.
Figure 2: Graphical representation of the bifurcation diagram and largest Lyapunov exponent of the system in function of $\varepsilon$ with the parameters: $\mu = 1.0434, \ \nu = \gamma = 0.1, \ \omega = 1$ and $\xi(t) = 0$ : (i) $\beta = 0$; (ii) $\beta = 0.1$

Figure 3: Idem with Fig. 2, with (i) $\beta = 0.5$; (ii) $\beta = 1$
Figure 4: Typical attractors for the parameters taken from Fig. 3(ii) with fixed values of the driven amplitude $\varepsilon$. Shown are quasi-periodic attractor for (i) $\varepsilon = 2$ and chaotic attractors for (ii) $\varepsilon = 2.92$, (iii) $\varepsilon = 4.92$, (iv) $\varepsilon = 9.2$

Figure 5: Synchronization time $T_s$: (a) As a function of $\alpha$ when $\beta_1 = 1$. (b) As a function of $\beta_1$ when $\alpha = 1$. 
Figure 6: State trajectories of $x_1(t)$, $y_1(t)$ (solid line) and $x_2(t)$, $y_2(t)$ (dashed line), respectively.

Figure 7: Time evolution of the synchronization error. (a) $e_1(t) = y_1(t) - x_1(t)$ and (b) $e_2(t) = y_2(t) - x_2(t)$ when $\alpha = \beta_1 = 1$. 

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Figure 8: Relation between the drive and response systems: without coupling terms (Figures on the left hand side) and with coupling terms (Figures on the right-hand side).