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STATISTICAL APPROACH TO TRANSPORT OF MODULATION INSTABILITIES IN OPTIC FIBRES

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Abstract

In the present paper we prove equality of wave equation for Modulation Instability and
Vlasov kinetic equation. The quantum analogue of wave equation for Modulation Instability is
defined. On the basis of BBGKY hierarchy of quantum kinetic equations the kinetic equations
for any number Modulation Instabilities are defined and the method solution for these equations
are proposed.

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Introduction

It is known [1],[2] that dynamics of Modulation Instabilities (MI) in optic fibers is described by wave equation for MI. In our work the equality of wave equations for MI to the Vlasov kinetic equation [3] is shown. Using the quantum kinetic equation of von Neumann [4], which is a quantum analogue of the Vlasov equation, we can generalize the wave equation for MI for the quantum case. Then, using the relation between kinetic equation for one particle and kinetic equation for a many-particle system [5],[6] we can define the quantum kinetic equations for any number of MI, which describes the real system of MI. We can also define the exact solution of this kinetic equation for MI and the Hamiltonian for any number Modulation Instabilities.

The results can be useful for describing transport phenomena of any number of Modulation Instabilities in optic fibers.

Derivation of the hierarchy of kinetic equations for correlation matrices

Consider the Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY)’s hierarchy of quantum kinetic equations [7]-[12]:

\[ i \frac{\partial}{\partial t} f_n(t) = [H_n, f_n(t)] + \frac{1}{v} \int \sum_{1 \leq i \leq n} \{ \varphi(x_i - x), f_{n+1}(t) \} dx, \]

where \( f_n \) is the density matrix, \( x \) is the particle coordinate, \([,]\) is the Poisson bracket, \( 2m = 1, h = 1 \). \( 0 \leq t \) is time, \( n \in N \), \( N \) is the number of particles, \( V \) - the volume of the system, \( N \to \infty, V \to \infty, v = \frac{1}{N} = \text{const} \) is volume per particle, and \( H \) is the Hamiltonian:

\[ H_n = \sum_{1 \leq i \leq n} T_i + \sum_{1 \leq i < j \leq n} \varphi(x_i - x_j), \quad T_i = -\frac{\partial^2}{2\partial x_i^2}. \]

On the basis of the relation [13]:

\[ f(t) = \Gamma \varphi(t) = I + \varphi(t) + \frac{\varphi(t) \ast \varphi(t)}{2!} + \cdots + \frac{(\ast \varphi(t))^n}{n!} + \cdots, \]

we’ll receive quantum kinetic equations for correlation matrices [14],[5],[6]:

\[ i \frac{\partial}{\partial t} \varphi(t) = H \varphi(t) + \frac{1}{2} N(\varphi(t), \varphi(t)) + S p_x A_y D_x \varphi(t) + S p_y A_x D_y \varphi(t) \ast D_x \varphi(t) dx, \]

On the basis of an argument similar to those in [15],[16],[17],[5],[6]:

\[ \varphi(x_i - x_j) = v \theta(x_i - x_j) \]

and applying the substitution:

\[ \varphi_n(t) = v^{n-1} \psi_n(t) \]
to (3) we obtain
\[
\frac{\partial}{\partial t} \psi_n(t, X; X') = \left[ \sum_{1 \leq i \leq n} T_i \psi_n(t, X; X') \right] + v (U\psi)_n(t, X; X') + \frac{v}{2} (W\psi, \psi)_n(t, X; X'^2 S_{p_x}(A_x D_x \psi)_n(t, X; X') \right.
+ v S_{p_x}(A_x \psi \ast D_x \psi)_n(t, X; X')
\]

The solution we’ll search in the form [5],[6]:

\[
\psi_n(t, X; X') = \sum_\mu \psi_{n\mu}(t, X; X') \quad n = 1, 2, 3, \ldots \quad \mu = 0, 1, 2, \ldots
\]

Substituting the series (7) to (6) and equality coefficients on powers \(v\) we receive set of homogeneous and inhomogeneous equations

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + L_1 \right) \psi_1^0(t) &= 0, \\
\left( \frac{\partial}{\partial t} + L_1 + L_2 \right) \psi_2^0(t) &= S_2^0(t), \\
\vdots \\
\left( \frac{\partial}{\partial t} + \sum_{i=1} L_i \right) \psi_n^\mu(t) &= S_n^\mu(t),
\end{align*}
\]

where we have introduced the notation

\[
\begin{align*}
L_1 \psi_1^0(t, x_1, p_1) &= p_1 \frac{\partial}{\partial x_1} \psi_1^0(t, x_1; p_1) - \frac{1}{(2\pi)^3} \int \exp(-i\tau_1 p_1) \times \\
& \times S_{p_x}[(\Theta(x_1 - x - \frac{\tau_1}{2}) - \Theta(x_1 - x + \frac{\tau_1}{2})) \psi_1^0(t, x_1 - \frac{\tau_1}{2}; x_1 + \frac{\tau_1}{2}) \psi(t, x, x)] d\tau_1,
\end{align*}
\]

\[
\begin{align*}
L_4 \psi_2^0(t, x_1, x_2; x_1', x_2') &= p_1 \frac{\partial}{\partial x_1} \psi_2^0(t, x_1, x_2; p_1, p_2) - \frac{1}{(2\pi)^6} \int \exp(-i\tau_1 p_1 - i\tau_2 p_2) \times \\
& \times S_{p_x}[(\Theta(x_1 - x - \frac{\tau_1}{2}) - \Theta(x_1 - x + \frac{\tau_1}{2})) \psi_2^0(t, x_1, x_2; x_1 + \frac{\tau_1}{2}, x_2 + \frac{\tau_2}{2}) \psi(t, x, x)] d\tau_1 d\tau_2,
\end{align*}
\]

\[
S_2^0(t) = \frac{1}{(2\pi)^6} \int \exp(-i\tau_1 p_1 - i\tau_2 p_2) [(\Theta(x_1 - \frac{\tau_1}{2} - x_2 + \frac{\tau_2}{2}) - \Theta(x_1 + \frac{\tau_1}{2} - x_2 - \frac{\tau_2}{2}) \times \\
\times \psi_1^0(t, x_1 - \frac{\tau_1}{2}; x_1 + \frac{\tau_1}{2}, x_2 - \frac{\tau_2}{2}; x_2 + \frac{\tau_2}{2}) + S_{p_x}[((\Theta(x_1 - \frac{\tau_1}{2} - x) - \Theta(x_1 + \frac{\tau_1}{2} - x)) \times \\
\times \psi_1^0(t, x_1 - \frac{\tau_1}{2}; x_1 + \frac{\tau_1}{2}, x_2 - \frac{\tau_2}{2}, x; x_2 + \frac{\tau_2}{2}, x) + (\Theta(x_2 - \frac{\tau_2}{2} - x) - \Theta(x_2 + \frac{\tau_2}{2} - x)) \times \\
\times \psi_1^0(t, x_2 - \frac{\tau_2}{2}, x_2 + \frac{\tau_2}{2}) \psi_1^0(t, x_1 - \frac{\tau_1}{2}; x; x_1 + \frac{\tau_1}{2}, x))] d\tau_1 d\tau_2,
\]

\[
\begin{align*}
L_i \psi_n^\mu(t, X; P) &= p_i \frac{\partial}{\partial x_i} \psi_n^\mu(t, X; P) - \frac{1}{(2\pi)^{3n}} \int \exp(-i \sum_{1 \leq i \leq n} p_i \tau_i) \times \\
& \times \left[ \sum_{1 \leq i \leq n} T_i \psi_n(t, X; X') \right] + v (U\psi)_n(t, X; X') + \frac{v}{2} (W\psi, \psi)_n(t, X; X'^2 S_{p_x}(A_x D_x \psi)_n(t, X; X') \right.
+ v S_{p_x}(A_x \psi \ast D_x \psi)_n(t, X; X')
\]
\]

\[
\frac{\partial}{\partial t} \psi(t, X; X') = \left[ \sum_{1 \leq i \leq n} T_i \psi(t, X; X') \right] + v (U\psi)(t, X; X') + \frac{v}{2} (W\psi, \psi)(t, X; X'^2 S_{p_x}(A_x D_x \psi)(t, X; X') \right.
+ v S_{p_x}(A_x \psi \ast D_x \psi)(t, X; X')
\]

\[
\psi(t, X; X') = \sum_\mu \psi_{n\mu}(t, X; X') \quad n = 1, 2, 3, \ldots \quad \mu = 0, 1, 2, \ldots
\]
where

$$\psi^\mu_n(t) = \frac{1}{(2\pi)^{n+1}} \int \prod_{1 \leq i \leq n} \tau_i \exp(-i \sum_{1 \leq i \leq n} p_i \tau_i) \left( U \psi^\mu_n(t) \right)_{n}(t, X - \frac{\tau}{2}, X + \frac{\tau}{2}) +$$

$$+ \frac{1}{\delta_1 + \delta_2 = \mu - 1} \sum_{\delta_1 + \delta_2 = \mu - 1} (V(\psi^{\delta_1}, \psi^{\delta_2}))(t, X - \frac{\tau}{2}, X + \frac{\tau}{2}) + \nu \psi^\mu_n(A_x D_x \psi^\mu_n(t)_{n}(t, X - \frac{\tau}{2}, X + \frac{\tau}{2})) +$$

$$+ v S_{\psi}(A_x \psi^\mu_n(t)_{n}(t, Y - \frac{\tau'}{2}, Y + \frac{\tau'}{2})(D_x \psi^{\delta_2})(t, X - \frac{\tau}{2}, Y - \frac{\tau'}{2}, X + \frac{\tau}{2}, Y + \frac{\tau'}{2})))\}.$$ 

The first equation (8) is the Wigner equation for "quantum distribution function" $\psi^0_0$ (in our (8) it is "quantum correlation function" $\psi^0_1$). It is a quasi-quantum analogue of classical Vlasov equation [3] for the distribution function.

The equality of wave kinetic equation for modulation instability to Vlasov equation

The Vlasov kinetic equation has the form [3]:

$$\frac{\partial \psi_1(t, q_1, p_1)}{\partial t} = -p_1 \frac{\partial}{\partial q_1} \psi_1(t, q_1, p_1) - c \int \frac{\partial \theta(q_1 - q)}{\partial q_1} \psi_1(t, q_1, p_1) \psi_1(t, q, p) d\theta d\eta,$$

where q-coordinate and p-impulse of particle. The wave equation for modulation instability is [1]:

$$\frac{\partial \psi(t)}{\partial t} + \nu \frac{\partial}{\partial z} \psi(t) + w_0 \frac{\partial E^2}{\partial z} \frac{\partial \psi(t)}{\partial p} = 0,$$

where

$$\frac{\partial E^2}{\partial z} = \int \psi(t) dx.$$ 

Substituting in the Vlasov equation (12): $p_1 = \nu, q_1 = z, c = \frac{2n_0}{w_0 n_2}$, and $\theta(q_1 - q) = C \delta(q_1 - q)$ we obtain the equation (13). This is the proof of the equality of Vlasov (12) and the wave (13) kinetic equations.

Using this equivalence we can find the quasi-quantum form of wave equation for MI. It has the form of Wigner equation [6],[18]:

$$\frac{\partial \psi_1^0(t, x_1, p_1)}{\partial t} = -p_1 \frac{\partial}{\partial x_1} \psi_1^0(t, x_1, p_1) - \frac{1}{(2\pi)^3} \int \exp(-i \tau p_1_1) \times$$

$$\times S_{\psi_1}([-\Theta(x_1 - x - \frac{\tau}{2}) - \Theta(x_1 - x + \frac{\tau}{2})]) \psi_1^0(t, x_1 - \frac{\tau}{2}, x_1 + \frac{\tau}{2}) \psi(t, x, x) d\tau_1,$$

where $p_1 = \nu, q_1 = z, c = \frac{2n_0}{w_0 n_2}$ and $\theta(q_1 - q) = c \delta(q_1 - q)$. 


The solution of Wigner Equation

Now we can approach the wave equation (13) from quantum physics. This is gives not only the development of quantum approach to modulation instabilities, but gives the possibility to find the quantum kinetic equation which describes the many Modulation Instabilities, as well.

We can also find the Hamiltonian of these modulation instabilities.

For the solution of the Wigner equation we consider the quantum von Neumann kinetic equation:

\[
i \frac{\partial \psi_0^0(t, x_1, x'_1)}{\partial (t)} = (\Delta_x - \Delta_{x'})\psi_0^0(t, x_1, x'_1) + c S_{px}(\Theta(x_1 - x) - \Theta(x'_1 - x)) \times
\]

\[
\times \psi_0^0(t, x_1, x'_1) \psi_0^0(t, x, x),
\]

where \( \psi_0^0(t, x_1, x'_1) \) is the density matrix, \( S_{px} \) is the density matrix.

Substituting in this equation the density matrix \( \psi_0^0(t, x_1, x'_1) \) in the form:

\[
\psi_0^0(t, x_1, x'_1) = \Phi_1(t, x_1) \Phi_1^*(t, x'_1),
\]

and substituting (14) into (11), taking \( \theta(x_1 - x_j) \) in the form of delta \( \delta(x_1 - x_j) \) we get [19]:

\[
i \frac{\partial \Phi_1(t, x_1)}{\partial t} = -\frac{\partial^2 \Phi_1(t, x_1)}{\partial x_1^2} + 2c \Phi_1(t, x_1) | \Phi_1(t, x_1) |^2, \quad \Phi_1(t, x_1) \big|_{t=0} = \Phi_1(x_1),
\]

\[
i \frac{\partial \Phi_1^*(t, x'_1)}{\partial t} = -\frac{\partial^2 \Phi_1^*(t, x'_1)}{\partial x'_1^2} + 2c \Phi_1^*(t, x'_1) | \Phi_1^*(t, x'_1) |^2, \quad \Phi_1^*(t, x'_1) \big|_{t=0} = \Phi_1^*(x'_1).
\]

Equations (15),(16) are nonlinear Schrödinger’s equations. As known from [19],[20] for the one dimensional D=1 case, at \( c > 0 \) the solution of (15) has the following form

\[
\Phi_1(t, x_1) = \sqrt{\frac{2}{c}} \left( \lambda + i \nu \right)^2 + \exp[2\nu(x_1 - x_0 - 2\lambda t)]
\]

where \( \nu \) is the velocity and parameter \( \lambda \) characterizes the amplitude. The velocity \( \nu \) is expressed via parameter \( \lambda \) as \( \nu = \sqrt{1 - \lambda^2} \).

It should be noted that two relations are valid:

\[
\frac{c}{2} |\Phi_1(t, x_1)|^2 = 1 - \frac{\nu^2}{c \nu^2(x_1 - x_0 - 2\lambda t)}
\]

and

\[
\int |\Phi_1(t, x_1)|^2 dx_1 = N = 1,
\]

where \( N \) - number of particles in a system.

So, using the Wigner transform[21]:

\[
\psi_1^0(t, x_1, p_1) = \frac{1}{(2\pi)^3} \int \exp(-i\tau_1 p_1) \times
\]

\[
\times \psi_1^0(t, x_1 - \frac{\tau_1}{2}; x_1 + \frac{\tau_1}{2}) d\tau_1
\]
and (15) we find the solution of the Wigner equation (13) in the form
\[
\psi_1^0(t, x_1, p_1) = \frac{1}{(2\pi)^3} \int \exp(-i\tau_1 p_1) \Phi_1(t, x_1 - \frac{\tau_1}{2}) \Phi_1^*(t, x_1 + \frac{\tau_1}{2}) d\tau_1
\]

The von Neumann equation, which describes the Modulation Instabilities, can be derived on the basis of Hamiltonian [22]-[25]
\[
H_n = \sum_{1 \leq i \leq n} T_i + C \sum_{1 \leq i < j \leq n} \delta(x_i - x_j), \quad T_i = -\frac{\partial^2}{\partial x_i^2}, \quad C = \frac{w_0 n_2}{2n_0}.
\]

Generalization of the result for consideration of the any number modulation instabilities

Our method gives the ability to find the form of any number modulation instabilities.

If we define the solution of the von Neumann equation through nonlinear Schrödinger Equation (NLSE), we can find on the basis of equations (8)-(10) all \(\psi_n^\mu(t,X)\) using
\[
\psi_n^\mu(t,X) = \int dx_1' \cdots \int dx_n' \int_{-\infty}^{t} dt' S_n^\mu(t, t', x_1', \ldots, x_n') \cap G(t-t', x, x_i') ,
\]
where G satisfies the Cauchy problem [17],[6]:
\[
\frac{\partial G(t-t', x_i, y_i, p_i, p_i')}{\partial t} = -p_i \frac{\partial}{\partial x_i} G(t-t', x_i, y_i, p_i, p_i') + \frac{1}{(2\pi)^6} \int \exp(-i\tau p_i - i\tau' p_i') \times
\]
\[
\times \{Sp_x[\theta(x_i - \frac{\tau_1}{2} - x) - \theta(x_i + \frac{\tau_1}{2} - x)]\psi_1^0(t, x_i - \frac{\tau_1}{2}; x_i + \frac{\tau_1}{2}, y_i + \frac{\tau_1}{2}) + \}
\]
\[
\times \{Sp_x[\theta(x_i - \frac{\tau_1}{2} - x) - \theta(x_i + \frac{\tau_1}{2} - x)]\psi_1^0(t, x_i; x) G(t-t', x_i - \frac{\tau_1}{2}, y_i - \frac{\tau_1}{2}; x_i' + \frac{\tau_1}{2}, y_i' + \frac{\tau_1}{2})\} d\tau_1 d\tau'_1 dx
\]
with the initial condition
\[
G(0; x_1, y_1, p_1, p_1') = \delta(x_1 - y_1)\delta(p_1 - p_1').
\]

On the basis of formulas (18),(4),(5),(2)(0 we can also define the solution of BBGKY’s hierarchy through soliton solution (17) of NLSE (15).

Description of Modulation Instability of the quasi-particles

For a description of Modulation Instabilities of the any number quasi-particles we start from the BBGKY chain of classical kinetic equations(1) and using relation [5]:
\[
f(t) = \Gamma \varphi(t) \star \Gamma \rho(t),
\]
where
\[ \varphi(t) = \varphi_0 + \rho(t), \]
\( \varphi_0 \)-equilibrium state and \( \rho(t) \)- is non-equilibrium perturbation of equilibrium correlation matrices. We suppose that \( \rho(t) \) is of \( v \) order, so (5) will change to [5],[6]:
\[ \varphi(t) = v^{n-1}(\varphi_0 + v\rho(t)). \]
On the basis of (1),(19), we receive the chain of quantum kinetic equations for perturbations [5],[6]:
\[ \frac{\partial}{\partial t}\rho(t) = H\rho(t) + \frac{1}{2}W(\rho(t),\rho(t)) + W(\varphi,\rho(t)) + \int (A_x D_x \rho(t) + A_x \rho(t) \ast D_x \varphi + A_x \varphi \ast D_x \rho(t) + A_x \rho(t) \ast D_x \rho(t)) dx. \]
To investigate our system on the basis of arguments similar to those in [7], we can choose as expansion parameter \( v \), setting (4) and making substitution (24). The smallness of the perturbation from equilibrium \( \varphi \) can be taken into account by setting \( \varphi_n(t) = \varphi_n + v\rho_n(t) \), and thus regarding \( \rho(t) \) as the first approximation in the parameter \( v \). Then (24) can be expressed in terms of
\[ \varphi_n(t) = \varphi_n + v\rho_n(t) = v^{n-1}\psi_n(t) = v^{n-1}(\psi(n) + v g_n(t)), \]
where
\[ \varphi_n = v^{n-1}\psi_n, \quad \rho_n(t) = v^{n-1} g_n(t). \]
We assume that the equilibrium correlation functions (24), where
\[ \varphi_1(p) = \psi_1(p) \]
is the Maxwell distribution, are known and are a solution of equation
\[ H\varphi + \frac{1}{2}W(\varphi,\varphi) + \int (A_x D_x \varphi + A_x \varphi \ast D_x \varphi) dx. \]
for \( n \) quasi-particles.
Under this assumption, on the basis of (4) and (24), and also the relation [26]
\[ \int \frac{\partial \theta(q_i - q)}{\partial q_i} dq = 0 \]
Eq.(23) for \( n \) quasi-particles takes the form
\[ \frac{\partial}{\partial t}g_n(t,X) = \left[ \sum_{1 \leq i \leq n} T_i g_n(t,X) \right] + v (Ug)_n(t,X) + (W\psi,g)_n(X) + \]
\[ + \frac{v^2}{2} (Wg(t),g(t))_n(X) + v^2 \int (A_x D_x g(t))_n(X) dx + v \int (A_x g(t) \ast D_x \psi)_n(X) dx + \]
\[ + v \int (A_x \psi \ast D_x g(t))_n(X) dx + v \int (A_x g(t) \ast D_x g(t))_n(X) dx, \]
where \( Y = (x_{(n')}); n = 1, 2, 3, \ldots \) Here and also in what follows in the symbols \( U, W, A \) the interaction is replaced by \( \theta \).

To solve Eq.(25), we apply the perturbation theory. We shall seek a solution in the form of the series [5]

\[
g_n(t, X) = \sum_{\mu} v^\mu g_n^\mu(t, X; X'). \quad n = 1, 2, 3, \ldots, \quad \mu = 0, 1, 2, \ldots
\]

Substituting the series (23) in Eq.(25) and equating the coefficients of equal powers of \( v \), we obtain

\[
(\frac{\partial}{\partial t} + L_1)g_1^0(t) = 0
\]

\[
(\frac{\partial}{\partial t} + \Sigma_i L_1)\rho_i = S(\rho_{n-1}, \ldots, g)
\]

where

\[
L_1g_1^0(t) = p_1 \frac{\partial}{\partial q_1} g_1^0(t; x_1) - \int \frac{\partial \theta(|q_1 - q|)}{\partial q_1} \frac{\partial \psi(p_1)}{\partial p_1} g_1^0(t; x)dx
\]

\[
L_i g_0^\mu(t) = \sum_{\mu} p_{i} \frac{\partial}{\partial q_{i}} g_0^\mu(t; X) - \int (A_x \psi)(x_1)(D_x g^\mu)_{n-1}(t; X \setminus x_1)dx + \sum_{y \subset X} \int (A_x D_x g^\mu(t))_{n-1}(X \setminus Y)(D_x \psi)(Y)dx + \sum_{Z \subset X} \int (A_x \psi)(Z)(D_x g^\mu(t))(X \setminus Z)dx + \int (A_x) \sum_{v_1 + v_2 = \mu - 1} g^{v_1}(t) * (D_x g^{v_2}(t))_{n}(X)dx.
\]

Here and in what follows, \( Z = (x_{n''}); n'' = 2, 3, \ldots \) Thus, the solution of Eq.(25) reduces to the solution \( g_0^0 \) of homogeneous (27) linearized Vlasov Equation

\[
\frac{\partial g_1^0(t, q_1, p_1)}{\partial t} = -p_1 \frac{\partial}{\partial q_1} g_1^0(t, q_1, p_1) - C \int \frac{\partial \theta(|q_1 - q|)}{\partial q_1} \psi(p_1) g_0^0(t, q, p)dqdp,
\]

and inhomogeneous linear equations (28) for \( g_0^\mu \) on the basis of the formula

\[
g_0^\mu(t, X) = \int dx_1' \cdots \int dx_\mu' \int_{-\infty}^{t} dt' S_n^\mu(t, x_1', \cdots, x_\mu') \bigcup_{1 \leq i \leq n} G(t - t, x_i'),
\]

In (30) the Green function \( G \) is the solution

\[
F^{-1} \Lambda^{-1} G(w; k_1, p_1, k_1', p_1') = \frac{\delta(p_1 - p_1')}{w - ikp_1} - \frac{ik\theta(k)}{w - ikp_1} \frac{1}{(w - ikp_1)(w - ikp_1')}(k, w)
\]

of the Cauchy problem [17]

\[
(\frac{\partial}{\partial t} + p_1 \frac{\partial}{\partial q_1})G(t - t_0; x_1, x_1') = \frac{\partial \psi(p_1)}{\partial p_1} \int \frac{\partial \theta(|q_1 - q|)}{\partial q_1} G(t - t_0; x_1, x_1')dx = 0
\]

\[
G(0; x_1, x_1') = \delta(x_1 - x_1').
\]
With the initial condition $g^0_1(0, q_1, p_1) = 0$ equation (29) may be solved [1] through the technique of Fourier transform:

$$g^0_1(t, k_1, p_1) = \int g^0_1(t, q_1, p_1)e^{i q_1 k_1} dq_1$$

and the one-sided Fourier transform of $g^0_1(t, k_1, p_1)$, with respect to time

$$g^0_1(w, k_1, p_1) = \int g^0_1(t, k_1, p_1)e^{iw t} dt$$

As we see from [17], the one-sided Fourier transform is equivalent to the Laplace transform. From (29), we have

$$g^0_1(w, k_1, p_1) = \frac{Ck_1}{w - k_1 p_1} \frac{\partial \psi_1(p_1)}{\partial p_1} \int g^0_1(w, k_1, p) dp.$$ 

We noticed that the formula (29) is an analogue linearized formula (4.87) of [1] and the formula (30) is analogue of (4.88) in [1]:

$$(f(k_1) = \frac{\omega_0 n_0}{r n_0} \frac{k}{\Omega - k v_\varphi} \frac{\partial f_0}{\partial k} |E|^2)$$

for one modulation instability, if: $f_0 \equiv \psi^0_0(p)$, and $g^1_0(t, q, p) = \frac{1}{\omega f_0} e^{i(kz - \Omega t)}$ in (29) and $p_1 = \nu$, $q_1 = z$, $C = \frac{\omega_0 n_0}{2n_0}$, and $\theta(q_1 - q) = C\delta(q_1 - q)$, $\int f(w, k_1, p) dp = \frac{|E|^2}{\omega_0}$ in (30).

Therefore for one MI our result coincide with the results of [1].

For two quasi-particles equation (28) is

$$\frac{\partial g^0_1(t, q_1, p_1) g^0_2(t, q_2, p_2)}{\partial t} + \sum_{1 \leq i \leq 2} p_i \frac{\partial g^0_i(t, q_1, p_1) g^0_j(t, q_2, p_2)}{\partial q_i}$$

$$- \int \frac{\partial \theta(q_1 - q)}{\partial q_1} \frac{\partial \psi^0_1(p_1)}{\partial p_1} g^0_1(t, q_2, p_2) g^0_2(t, q, p) dqdp - \int \frac{\partial \theta(q_2 - q)}{\partial q_2} \frac{\partial \psi^0_2(p_2)}{\partial p_2} g^0_1(t, q_1, p_1) g^0_2(t, q, p) dqdp = 0.$$ 

The solution of this equation is a superposition of two solutions

$$g^0_2(t, q_1, q_2, p_1, p_2) = \prod_i g^0_i(t, q_i, p_i)$$

and dispersion relation for two quasi-particles is:

$$C_1 \frac{dp_1}{dp} \frac{k^2}{w - k p_1^2} + C_2 \frac{dp_2}{dp} \frac{k^2}{w - k p_2^2} = 1,$$

Formula (32) coincides with the dispersion relation for two quasi-particles, given by formula (4.94) in [1]:

$$\frac{k''_{1} v_{g1}^3 w_{g0} n_{0}}{2n_0} \frac{K^2}{\Omega - K v_{g1}^2} + \frac{k''_{2} v_{g2}^3 w_{g0} n_{0}}{2n_0} \frac{K^2}{\Omega - K v_{g2}^2} = 1,$$

if $C_1 = \frac{\omega_0 n_0}{2n_0}$, $C_2 = \frac{\omega_0 n_0}{2n_0}$, $\frac{dp_1}{dp} \equiv \frac{dv_{g1}}{dk} = -k''_{1} v_{g1}^3$, $\frac{dp_2}{dp} \equiv \frac{dv_{g2}}{dk} = -k''_{2} v_{g2}^3$

In (33) $v_{g1}$ and $v_{g2}$ are the group velocities at wave numbers $k_1$ and $k_2$ [3].
The Green function $G$ satisfies equation [17]

$$G(t-t_0,x_1,y_1) = \mathcal{F}^{-1}L^{-1}G(w;k,p_1,p_1') = \mathcal{F}^{-1}L^{-1}\left(\frac{i\delta(p_1-p_1')}{w-ikp_1} - ik\theta(k)\times \frac{\partial\psi(p_1)}{\partial p_1}\right)\frac{1}{(w-ikp_1)(w-ikp_1')\varepsilon(k,w)}$$

where

$$(\mathcal{F}G)(w,x_1,y_1') = \int d(x_1-y_1')G(w,x_1,y_1'\mu_{k_1}(x_1-y_1'))$$

So we can conclude that:

1. The classical analogue of kinetic equation for Modulation Instability process (4.84) of [1] coincides with the Vlasov equation with potential in the form of delta function.

2. The quantum analogue of this process is described by the von Neumann kinetic equation for density matrix or by the Winger equation for quantum distribution function, when potential is in the form of delta function.

3. The process of any particle modulation instability can be described by BBGKY’s chain of classical or quantum kinetic equations.

4. The generalization of results [1] for the quantum case will open to the ability to describe this quantum process using quantum density matrices or quantum distribution functions.

5. The quantum approach is introduced in this paper in order to explain the generation of transport of soliton-like pulses.

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