TWO-PARAMETER QUANTUM AFFINE ALGEBRA $U_{r,s}(\widehat{\mathfrak{sl}}_n)$, 
DRINFELD REALIZATION AND QUANTUM AFFINE LYNDON BASIS

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Abstract

We further find the defining structure of a two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$ ($n > 2$) in the sense of Benkart-Witherspoon [BW1] after the work of [BGH1], [HS] and [BH], which turns out to be a Drinfeld double. Of more importance for the “affine” cases is that we work out the compatible two-parameter version of the Drinfeld realization as a quantum affinization of $U_{r,s}(\mathfrak{sl}_n)$ and establish the Drinfeld isomorphism Theorem in the two-parameter setting via developing a new remarkable combinatorial approach — quantum “affine” Lyndon basis with an explicit valid algorithm, based on the Drinfeld realization.

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1. Introduction

1.1 In 2001, Benkart-Witherspoon investigated the structures of two-parameter quantum groups $U_{r,s}(g)$ for $g = gl_n$, or $sl_n$ in [BW1] originally obtained by Takeuchi [T], and the finite-dimensional weight representation theory in [BW2], and further obtained some new finite-dimensional pointed Hopf algebras in [BW3] when $rs^{-1}$ is a root of unity, which possesses new ribbon elements under some conditions (and will yield new invariants of knots and links). These show that the two-parameter quantum groups in the Benkart-Witherspoon’s sense have their own research value.

1.2 In 2004, Bergeron-Gao-Hu [BGH1] gave the structures of two-parameter quantum groups $U_{r,s}(g)$ for $g = so_{2n+1}, sp_{2n}, so_{2n}$, and developed in [BGH2] the highest weight representation theory when $rs^{-1}$ is not a root of unity. Especially, [BGH1] explored the environment condition upon which the Lusztig’s symmetries exist for the classical simple Lie algebras $g$, namely, found that a striking feature of these symmetries is that they exist as $Q$-isomorphisms between $U_{r,s}(g)$ and the associated object $U_{s^{-1},r^{-1}}(g)$ only when rank $(g) = 2$, and in the case when rank $(g) > 2$, the sufficient and necessary condition for the existence of Lusztig’s symmetries between $U_{r,s}(g)$ and its associated object forces $U_{r,s}(g)$ to take the “one-parameter” form $U_{q,q^{-1}}(g)$ where $r = s^{-1} = q$. In other words, when rank $(g) > 2$, the Lusztig’s symmetries exist only for the one-parameter quantum groups $U_{q,q^{-1}}(g)$ as $Q(q)$-automorphisms (rather merely as $Q$-isomorphisms). In this case, these symmetries give rise to, with respect to modulo some identification on group-like elements, the usual Lusztig symmetries on quantum groups $U_q(g)$ of Drinfel’d-Jimbo type. The Lusztig symmetry property indicates that there indeed exist the remarkable differences between the two-parameter quantum groups in question and the one-parameter quantum groups of Drinfeld-Jimbo type. Afterward, Hu-Shi [HS] and Bai-Hu [BH] studied the two-parameter quantum groups for type $G_2$ and type $E$ cases. Through these work, we found that the treatments in two-parameter cases are frequently more subtle to follow combinatorial approaches only, for instance, the description of the convex PBW-type basis (cf. [BH]) has to appeal to the use of Lyndon words (see [R2] and references therein) because of no braid group.

Thereby so far, it seems desirable to extend these kind of the two-parameter quantum groups in the Benkart-Witherspoon’s sense in finite cases to the affine cases. On one hand, the present paper is aimed to this purpose for the affine type $A_n^{(1)}$ case at first. We give the defining structure of $U_{r,s}(sl_n)$ ($n > 2$).

1.3 On the other hand, as is well-known, the importance of Drinfeld generators (in Drinfeld realization) for quantum affine algebras is just like that of loop generators (in loop realization) for affine Kac-Moody algebras (see [Ga], [K]). Early in 1987, Drinfeld [Dr2] put forward his famous new (conjectural) realization of quantum affine algebras $U_q(\hat{g})$ with $g$ semisimple because he recognized that the study of finite dimensional representations of $U_q(\hat{g})$ is made easier by the use of this realization on the set of Drinfeld generators, which is called Drinfeld realization of
\(U_q(\hat{\mathfrak{g}})\) or the Drinfeld quantum affinization of \(U_q(\mathfrak{g})\). Besides this, Drinfeld realization also finds its most important contribution to the construction of the vertex representations for quantum affine algebras \(U_q(\hat{\mathfrak{g}})\) (see [FJ], [J1], [DI2], etc.), which is as does the loop realization in the vertex representation theory of affine Kac-Moody algebras (see [K]). In 1993, Khoroshkin-Tolstoy [KT] constructed Drinfeld realization for untwisted types using a Cartan-Weyl generator system with no proof. The first perfect proof of the Drinfeld isomorphism only for untwisted types was given by Beck [B2] till 1994 making use of his extended braid group actions, based on the work of Damiani [Da], Levendorskii-Soibel’man-Stukopin [LSS] on the case \(U_q(\hat{\mathfrak{sl}_2})\). In 1998, Jing [J2] basically adopted the inverse map suggested by Beck in untwisted types (see the final remark in [B2, Section 4]) and gave another proof for the Drinfeld isomorphism in untwisted types (from the opposite direction compared to [B2]).

1.4 In order to explore further and enrich the structure and representation theory of the two-parameter quantum affine algebras later on, the another main result of this paper is to work out the Drinfeld realization for \(U_{r,s}(\hat{\mathfrak{sl}_n})\) \((n > 2)\). Its definition depends on the self-compatible defining system (Definition 3.1), which in the two-parameter setting varies dramatically relative to one-parameter cases (see [Dr2], or [B2, Thereom 4.7]) and is really nontrivial to match up here and there the whole relations together. Indeed, to invent the two-parameter version of Drinfeld realization needs some insights, e.g., from the antisymmetric point of view via the \(Q\)-algebra antiautomorphism \(\tau\), based on some experiences from the combinatorial description of the convex PBW-type basis via the Lyndon words (see [R2], BH), and also, the proof of the Drinfeld isomorphism in our case depends completely on our adopted combinatorial approach with specific techniques to design those defining relations to fit the compatibilities in the whole system. If the readers go with us into the details, they will find how our technical calculations (in somehow a bit tedious) work well and necessarily for exactly verifying the compatibilities of the defining system. The reasoning is that our method expanded here, to some extent, essentially follows the approach to a kind of description of the quantum “affine” Lyndon basis. Actually, we can construct explicitly all quantum real and imaginary root vectors using our method (see Lemmas 4.7 & 4.8, together with Definition 3.9). It is worth to mention that a similar approach with no consideration of Lyndon basis recently adapts to further confirm the Drinfeld (conjectural) realization of \(U_q(\hat{\mathfrak{g}})\) for twisted types in the one-parameter setting (this is the first proof in twisted cases, see a recent preprint of Zhang-Jing [ZJ]).

1.5 The paper is organized as follows. We first give the structure of two-parameter quantum affine algebra \(U_{r,s}(\hat{\mathfrak{sl}_n})\) \((n > 2)\) in the sense of Hopf algebra in Section 2, which is new. We prove that two-parameter quantum affine algebra \(U_{r,s}(\hat{\mathfrak{sl}_n})\) is characterized as Drinfeld double \(\mathcal{D}(\hat{\mathcal{B}}, \hat{\mathcal{B}}')\) of Hopf subalgebras \(\hat{\mathcal{B}}, \hat{\mathcal{B}}'\) with respect to a skew-dual pairing we give. In Section 3, we describe the two-parameter Drinfeld quantum affinization of \(U_{r,s}(\hat{\mathfrak{sl}_n})\) \((n > 2)\), which affords the Drinfeld realization in two-parameter case that is antisymmetric with respect to the \(Q\)-algebra antiautomorphism \(\tau\). In the case when \(rs = 1\), i.e., \(r = s^{-1} = q\), our result modulo identifications
yields the usual Drinfeld realization of quantum affine algebra $U_q(\widehat{sl}_n)$ of Drinfeld-Jimbo type (see [Dr2], [B2], [DI1], [J2], etc.). Owing to Beck’s extended braid group action approach being invalid for our cases, we adapt the Lyndon words approach, together with the quantum Lie bracket operation, to develop a combinatorial trick which can be utilized in the construction of all the quantum root vectors (including real and imaginary ones), so that we can formulate and prove the quantum “affine” Lyndon basis (in a more explicit form than that of [B1]) for $U_{r,s}(\widehat{n}^\pm)$ based on the Drinfeld realization in Section 3, and further prove the Drinfeld isomorphism using our combinatorial algorithm in Section 4. In fact, our proof also provides a concrete process of how to construct the Drinfeld generators using the Chevalley-Kac-Lusztig generators.

2. Quantum Affine Algebra $U_{r,s}(\widehat{sl}_n)$ and Drinfeld Double

2.1 Let $\mathbb{K} = \mathbb{Q}(r, s)$ denote a field of rational functions with two-parameters $r$, $s$ ($r \neq \pm s$). Assume $\Phi$ is a finite root system of type $A_{n-1}$ with $\Pi$ a base of simple roots. Regard $\Phi$ as a subset of a Euclidean space $E = \mathbb{R}^n$ with an inner product $(\cdot, \cdot)$. Set $I = \{1, \ldots, n-1\}$, $I_0 = \emptyset \cup I$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ denote an orthonormal basis of $E$, then we can take $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i \in I\}$ and $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in I\}$. Let $\delta$ denote the primitive imaginary root of $\widehat{sl}_n$. Take $\alpha_0 = \delta - (\varepsilon_1 - \varepsilon_n)$, then $\Pi’ = \{\alpha_i \mid i \in I_0\}$ is a base of simple roots of affine Lie algebra $\widehat{sl}_n$.

Let $A = (a_{ij})$ $i, j \in I_0$ be a generalized Cartan matrix associated to affine Lie algebra $\widehat{sl}_n$. Let $\mathfrak{h}$ be a vector space over $\mathbb{K}$ with a basis $\{ h_0, h_1, \ldots, h_{n-1}, d \}$ and define the linear action of $\alpha_i$ ($i \in I_0$) on $\mathfrak{h}$ by

$$a_i(h_j) = a_{ji}, \quad a_i(d) = \delta_i, \quad \text{for} \quad j \in I_0.$$

Let $Q = \mathbb{Z}\alpha_0 + \cdots + \mathbb{Z}\alpha_{n-1}$ denote the root lattice of $\widehat{sl}_n$. The standard nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ satisfies

$$(\alpha_i, \alpha_j) = a_{ij}, \quad (\delta, \alpha_i) = (\delta, \delta) = 0, \quad \forall \ i, j \in I_0.$$

**Definition 2.1.** Let $U = U_{r,s}(\widehat{sl}_n)$ $(n > 2)$ be the unital associative algebra over $\mathbb{K}$ generated by the elements $e_j, f_j, \omega_j^{\pm 1}$ $(j \in I_0)$, $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1}, D'^{\pm 1}$ (called the Chevalley-Kac-Lusztig generators), satisfying the following relations:

(A1) $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}$ are central with $\gamma = \omega_0$, $\gamma’ = \omega_0’$, such that $\omega_i \omega_i^{-1} = \omega_i’ \omega_i'^{-1} = 1 = DD^{-1} = D'D'^{-1}$, and

$$[\omega_i^{\pm 1}, \omega_j^{\pm 1}] = [\omega_i^{\pm 1}, D^{\pm 1}] = [\omega_j^{\pm 1}, D^{\pm 1}] = [\omega_i^{\pm 1}, D'^{\pm 1}] = 0,$$

$$[\omega_i^{\pm 1}, \omega_j^{\pm 1}] = [\omega_j^{\pm 1}, D'^{\pm 1}] = [D'^{\pm 1}, D^{\pm 1}] = [\omega_i'^{\pm 1}, \omega_j'^{\pm 1}].$$

(A2) For $i \in I_0$ and $j \in I$,

$$De_i D^{-1} = r^{\delta_{ii}} e_i, \quad Df_i D^{-1} = r^{\delta_{ii}} f_i,$$

$$\omega_j e_i \omega_j^{-1} = r^{(\varepsilon_j, \alpha_i)} s^{(\varepsilon_{j+1}, \alpha_i)} e_i, \quad \omega_j f_i \omega_j^{-1} = r^{-(\varepsilon_j, \alpha_i)} s^{-(\varepsilon_{j+1}, \alpha_i)} f_i,$$

$$\omega_0 e_i \omega_0^{-1} = r^{(\varepsilon_{i+1}, \alpha_0)} s^{(\varepsilon_1, \alpha_i)} e_i, \quad \omega_0 f_i \omega_0^{-1} = r^{(\varepsilon_{i+1}, \alpha_0)} s^{-(\varepsilon_1, \alpha_i)} f_i.$$
(A3) For \( i \in I_0 \) and \( j \in I \),
\[
\begin{align*}
D' e_i D'^{-1} &= s^{\delta_{0i}} e_i, \\
\omega'_j e_i \omega'^{-1}_j &= s^{(\epsilon_j, \alpha_i)(\epsilon_j+1, \alpha_i)} e_i, \\
\omega'_0 e_i \omega'^{-1}_0 &= s^{-(\epsilon_i+1, \alpha_0)(\epsilon_i, \alpha_i)} e_i,
\end{align*}
\]
\[
\begin{align*}
D' f_i D'^{-1} &= s^{-\delta_{0i}} f_i, \\
\omega'_j f_i \omega'^{-1}_j &= s^{-(\epsilon_j, \alpha_i)(\epsilon_j+1, \alpha_i)} f_i, \\
\omega'_0 f_i \omega'^{-1}_0 &= s^{(\epsilon_i+1, \alpha_0)(\epsilon_i, \alpha_i)} f_i.
\end{align*}
\]

(A4) For \( i, j \in I_0 \), we have
\[
[e_i, f_j] = \frac{\delta_{ij}}{r-s} (\omega_i - \omega'_j).
\]

(A5) For \( i, j \in I_0 \), but \((i, j) \notin \{(0, n-1), (n-1, 0)\}\) with \( a_{ij} = 0 \), we have
\[
[e_i, e_j] = 0 = [f_i, f_j].
\]

(A6) For \( i \in I_0 \), we have the \((r, s)\)-Serre relations:
\[
\begin{align*}
& \epsilon_i^2 e_{i+1} - (r+s) \epsilon_i e_{i+1} e_i + (rs) e_i e_{i+1} e_i^2 = 0, \\
& \epsilon_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + (rs) e_{i+1}^2 e_i = 0, \\
& \epsilon_{n-1}^2 e_0 - (r+s) e_{n-1} e_0 e_{n-1} + (rs) e_0 e_{n-1}^2 e_0 = 0, \\
& \epsilon_{n-1}^2 e_0^2 - (r+s) e_0 e_{n-1} e_0 + (rs) e_0^2 e_{n-1} = 0.
\end{align*}
\]

(A7) For \( i \in I_0 \), we have the \((r, s)\)-Serre relations:
\[
\begin{align*}
& f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + (r^{-1} s^{-1}) f_{i+1} f_i f_i = 0, \\
& f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + (r^{-1} s^{-1}) f_{i+1} f_i f_i = 0, \\
& f_{n-1}^2 f_0 - (r^{-1} + s^{-1}) f_{n-1} f_0 f_{n-1} + (r^{-1} s^{-1}) f_0 f_{n-1}^2 = 0, \\
& f_{n-1} f_0^2 - (r^{-1} + s^{-1}) f_{n-1} f_0 f_{n-1} + (r^{-1} s^{-1}) f_0^2 f_{n-1} = 0.
\end{align*}
\]

\( U_{r,s}(\mathfrak{sl}_n) \) is a Hopf algebra with the coproduct \( \Delta \), the counit \( \varepsilon \) and the antipode \( S \) defined below: for \( i \in I_0 \), we have
\[
\begin{align*}
\Delta(\gamma^{\pm \frac{1}{2}}) &= \gamma^{\pm \frac{1}{2}} \otimes \gamma^{\pm \frac{1}{2}}, \\
\Delta(\gamma'^{\pm \frac{1}{2}}) &= \gamma'^{\pm \frac{1}{2}} \otimes \gamma'^{\pm \frac{1}{2}}, \\
\Delta(D^{\pm 1}) &= D^{\pm 1} \otimes D^{\pm 1}, \\
\Delta(D'^{\pm 1}) &= D'^{\pm 1} \otimes D'^{\pm 1}, \\
\Delta(w_i) &= w_i \otimes w_i, \\
\Delta(w'_i) &= w'_i \otimes w'_i, \\
\varepsilon(e_i) &= \varepsilon(f_i) = 0, \\
\varepsilon(\gamma^{\pm \frac{1}{2}}) &= \varepsilon(\gamma'^{\pm \frac{1}{2}}), \\
\varepsilon(D^{\pm 1}) &= \varepsilon(D'^{\pm 1}) = \varepsilon(w_i) = \varepsilon(w'_i) = 1, \\
S(\gamma^{\pm \frac{1}{2}}) &= \gamma^{\mp \frac{1}{2}}, \\
S(\gamma'^{\pm \frac{1}{2}}) &= \gamma'^{\mp \frac{1}{2}}, \\
S(D^{\pm 1}) &= D^{-1}, \\
S(D'^{\pm 1}) &= D'^{-1}, \\
S(e_i) &= -w_i^{-1} e_i, \\
S(f_i) &= -f_i w'_i^{-1}, \\
S(w_i) &= w_i^{-1}, \\
S(w'_i) &= w'_i^{-1}.
\end{align*}
\]

2.2 In what follows, we give the skew-pairing and the Drinfeld double structure.
DEFINITION 2.2. A bilinear form \((\cdot, \cdot) : \mathfrak{B} \times \mathfrak{A} \to \mathbb{K}\) is called a skew-dual pairing of two Hopf algebras \(\mathfrak{A}\) and \(\mathfrak{B}\) (see [KS, 8.2.1]), if it satisfies

\[
\langle b, 1_\mathfrak{A} \rangle = \varepsilon_\mathfrak{B}(b), \quad \langle 1_\mathfrak{B}, a \rangle = \varepsilon_\mathfrak{A}(a),
\]

\[
\langle b, a_1a_2 \rangle = \langle \Delta_\mathfrak{B}(b), a_1 \otimes a_2 \rangle, \quad \langle b_1b_2, a \rangle = (b_1 \otimes b_2, \Delta_\mathfrak{B}(a)),
\]

for all \(a, a_1, a_2 \in \mathfrak{A}\) and \(b, b_1, b_2 \in \mathfrak{B}\), where \(\varepsilon_\mathfrak{A}, \varepsilon_\mathfrak{B}\) denote the counites of \(\mathfrak{A}\), \(\mathfrak{B}\), respectively, and \(\Delta_\mathfrak{A}, \Delta_\mathfrak{B}\) are the respective coproducts.

DEFINITION 2.3. For any two skew-paired Hopf algebras \(\mathfrak{A}\) and \(\mathfrak{B}\) by \((\cdot, \cdot)\), there exists a Drinfeld quantum double \(D(\mathfrak{A}, \mathfrak{B})\) which is a Hopf algebra whose underlying coalgebra is \(\mathfrak{A} \otimes \mathfrak{B}\) with the tensor product coalgebra structure, whose algebra structure is defined by

\[
\langle a \otimes b)(a' \otimes b' \rangle = \sum \langle S_\mathfrak{B}(b_{(1)}), a_1'(a_1) \rangle \langle b_3, a_3'(a_3) \rangle a_3' (2) \otimes b_2 b',
\]

for \(a, a' \in \mathfrak{A}\) and \(b, b' \in \mathfrak{B}\), and whose antipode \(S\) is given by

\[
S(a \otimes b) = (1 \otimes S_\mathfrak{B}(b))(S_\mathfrak{A}(a) \otimes 1).
\]

Let \(\hat{B}\) (resp. \(\hat{B}'\)) denote the Hopf (Borel-type) subalgebra of \(U_{r,s}(sI_n)\) generated by \(e_j, \omega_j^{\pm 1}, \gamma^{\pm 1}, D^{\pm 1}\) (resp. \(f_j, \omega_j^{\prime \pm 1}, \gamma^{\prime \pm 1}, D^{\prime \pm 1}\)) with \(j \in I_0\).

PROPOSITION 2.4. There exists a unique skew-dual pairing \((\cdot, \cdot) : \hat{B}' \times \hat{B} \to \mathbb{K}\) of the Hopf subalgebras \(\hat{B}\) and \(\hat{B}'\) such that:

(1) \[
\langle f_i, e_j \rangle = \delta_{ij} \frac{1}{s-r}, \quad (i, j \in I_0)
\]

(2) \[
\langle \omega_i', \omega_j \rangle = \begin{cases} r^{(\varepsilon_j, \alpha_i)}g(\varepsilon_j, \alpha_i), & (i \in I_0, j \in I) \\ r^{-\varepsilon_j}g(\varepsilon_j, \alpha_i), & (i \in I_0, j = 0) \end{cases}
\]

(3) \[
\langle \omega_i^{\pm 1}, \omega_j^{-1} \rangle = \langle \omega_i^{\pm 1}, \omega_j \rangle^{-1} = \langle \omega_i', \omega_j \rangle^\mp 1, \quad (i, j \in I_0)
\]

(4) \[
\langle \gamma^{\prime \pm 1}, \gamma \rangle = \langle \gamma', \gamma^{\prime \pm 1} \rangle = \langle \gamma', \gamma \rangle = 1,
\]

(5) \[
\langle D', D \rangle^{\pm 1} = \langle D', D \rangle = \langle D', D \rangle^{\pm 1} = 1,
\]

(6) \[
\langle \gamma^{\prime \pm 1}, \omega_i \rangle^{\pm 1} = 1 = \langle \omega_i^{\prime \pm 1}, \gamma^{\prime \pm 1} \rangle, \quad (i \in I_0)
\]

(7) \[
\langle D^{\pm 1}, \omega_i \rangle = \langle D', \omega_i^{\pm 1} \rangle = s^{\pm \delta_0}, \quad \langle \omega_i^{\pm 1}, D \rangle = \langle \omega_i', D \rangle^{\pm 1} = r^{\pm \delta_0}, (i \in I_0)
\]

(8) \[
\langle D', \gamma^{\prime \pm 1} \rangle = \langle D', \gamma^{\prime \pm 1} \rangle = s^{\mp \delta_0}, \quad \langle \gamma^{\prime \pm 1}, D \rangle = \langle \gamma', D \rangle^{\pm 1} = r^{\pm \delta_0},
\]

and all others pairs of generators are 0. Moreover, we have \(\langle S(b'), S(b) \rangle = \langle b', b \rangle\) for \(b' \in \hat{B}', b \in \hat{B}\).

PROOF. The uniqueness assertion is clear, as any skew-dual pairing of bialgebras is determined by the values on the generators. We proceed to prove the existence of the pairing.

The pairing defined on generators as (1)—(8) may be extended to a bilinear form on \(\hat{B}' \times \hat{B}\) in a way such that the defining properties in Definition 2.2 hold. We will verify that the relations in \(\hat{B}\) and \(\hat{B}'\) are preserved, ensuring that the form is well-defined and is a skew-dual pairing of \(\hat{B}\) and \(\hat{B}'\).
At first, it is straightforward to check that the bilinear form preserves all the relations among the $\omega_{i}^{\pm 1}$, $D^{\pm 1}$ in $\hat{B}$ and the $\omega_{i}^{\prime \pm 1}$, $D^{\prime \pm 1}$ in $\hat{B}'$. Next, we observe that the identities hold: for $i, j \in I$,

$$\langle \Delta(2), (2) \rangle \quad (\varepsilon, \alpha_i) = -(\varepsilon_{i+1}, \alpha_j), \quad (\varepsilon, \alpha_0) = -(\varepsilon_1, \alpha_j),$$

which ensure the compatibility of the form defined above with the relations of (A2) and (A3) in $\hat{B}$ or $\hat{B}'$ respectively. This fact is easily checked by definition (see (1)—(8)). So we are left to verify that the form preserves the $(r, s)$-Serre relations in $\hat{B}$ and $\hat{B}'$.

For $1 \leq i < n$, $(r, s)$-Serre relations in $\hat{B}$ and $\hat{B}'$ have been checked in [BW1]. Here we need only to verify the relations involving index $i = 0$ in $\hat{B}$ and $\hat{B}'$. It suffices to consider the following case (the remaining case is similar)

$$\langle X, e_0^2 e_{n-1} - (r^{-1} + s^{-1})e_0 e_{n-1} e_0 + (rs)^{-1}e_{n-1} e_0^2 \rangle,$$

where $X$ is any word in the generators of $\hat{B}'$. By definition, this equals

$$\langle \Delta^{(2)}(X), e_0 \otimes e_0 \otimes e_{n-1} - (r^{-1} + s^{-1})e_0 \otimes e_{n-1} \otimes e_0 + (rs)^{-1}e_{n-1} \otimes e_0 \otimes e_0 \rangle,$$

where the $\Delta$ corresponds to $\Delta^{op}_{\hat{B}'}$. In order for any one of these terms to be nonzero, $X$ must involve exactly two $f_0$ factors, one $f_{n-1}$ factor, and arbitrarily many $\omega_j^{\prime \pm 1}$ ($j \in I_0$), $\gamma^{\prime \pm 2}$, or $D^{\prime \pm 1}$ factors. For simplicity, we first consider three key cases:

(i) If $X = f_0^2 f_{n-1}$, then $\Delta^{(2)}(X)$ is equal to

$$(\omega_0' \otimes \omega_0' \otimes f_0 + \omega_0' \otimes f_0 \otimes 1 + f_0 \otimes 1 \otimes 1)^2 (\omega_{n-1}' \otimes \omega_{n-1}' \otimes f_{n-1} + \omega_{n-1}' \otimes f_{n-1} \otimes 1 + \omega_{n-1}' \otimes 1 \otimes 1).$$

The relevant terms of $\Delta^{(2)}(X)$ are

$$f_0 \omega_0' \omega_{n-1}' \otimes f_0 \omega_{n-1}' \otimes f_{n-1} + \omega_0' f_0 \omega_{n-1}' \otimes f_0 \omega_{n-1}' \otimes f_{n-1} + f_0 \omega_0' \omega_{n-1}' \otimes \omega_0' f_{n-1} \otimes f_0 + \omega_0' f_0 \omega_{n-1}' \otimes \omega_0' f_{n-1} \otimes f_0 + \omega_0^2 \omega_0' \otimes f_0 + \omega_0^2 f_{n-1} \otimes \omega_0' f_0 \otimes f_0.$$
Therefore (2.2) becomes
\[
\langle f_0 \omega'_0 | \omega'_{n-1}, e_0 \rangle \langle f_0 \omega'_n, e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \\
+ \langle \omega'_0 f_0 | \omega'_{n-1}, e_0 \rangle \langle f_0 \omega'_n, e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \\
- (r^{-1} + s^{-1}) \left( \langle f_0 \omega'_0 | e_0 \rangle \langle f_0 \omega'_{n-1}, e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \langle f_0, e_0 \rangle \\
+ \langle \omega'_0 f_0 | \omega'_{n-1}, e_0 \rangle \langle \omega'_0 f_0 | e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \langle f_0, e_0 \rangle \right) \\
+ (rs)^{-1} \left( \langle \omega'_0^2 f_{n-1}, e_{n-1} \rangle \langle f_0 | \omega'_0, e_0 \rangle \langle f_0, e_0 \rangle \\
+ \langle \omega'_0^2 f_{n-1}, e_{n-1} \rangle \langle f'_0 | e_0 \rangle \langle f_0, e_0 \rangle \right)
\]
\[
= \frac{1}{(s-r)^3} \left( 1 + \langle \omega'_0 | \omega_0 \rangle - (r^{-1} + s^{-1}) \left( \langle \omega'_0 | \omega_0 \rangle \langle \omega'_0 | \omega_{n-1} \rangle \langle \omega'_0 | e_0 \rangle \right) \langle \omega'_0 | e_0 \rangle \langle \omega'_0 | e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \langle f_0, e_0 \rangle \right) \\
+ (rs)^{-1} \left( \langle \omega'_0 | \omega_{n-1} \rangle^2 + \langle \omega'_0 | \omega_0 \rangle^2 \langle \omega'_0 | e_0 \rangle \langle \omega'_0 | e_0 \rangle \langle \omega'_0 | e_0 \rangle \langle \omega'_0 | e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \langle f_0, e_0 \rangle \right)
\]
\[
= \frac{1}{(s-r)^3} \left( 1 + rs^{-1} - (r^{-1} + s^{-1}) (s + rs^{-1} s) + (rs)^{-1} (s^2 + s^2 rs^{-1} s) \right) \\
= 0.
\]

(ii) When \( X = f_0 f_{n-1} f_0 \), it is easy to get the relevant terms of \( D^{(2)}(X) \):
\[
\omega'_0 \omega'_{n-1} f_0 \otimes f_0 \omega'_{n-1} \otimes f_{n-1} + f_0 \omega'_{n-1} \omega'_0 \otimes \omega'_{n-1} f_0 \otimes f_{n-1} \\
+ \omega'_0 \omega'_{n-1} f_0 \otimes \omega'_0 f_{n-1} \otimes f_0 + f_0 \omega'_{n-1} \omega'_0 \otimes f_{n-1} \omega'_0 \otimes f_0 \\
+ \omega'_0 f_{n-1} \omega'_0 \otimes \omega'_0 f_0 \otimes f_0 + \omega'_0 f_{n-1} \omega'_0 \otimes f_0 \omega'_0 \otimes f_0.
\]

Thus, (2.2) becomes
\[
\frac{1}{(s-r)^3} \left\{ \langle \omega'_0 | \omega_0 \rangle \langle \omega'_{n-1}, e_0 \rangle + \langle \omega'_{n-1} | \omega_0 \rangle \\
- (r^{-1} + s^{-1}) \langle \omega'_0 | \omega_0 \rangle \langle \omega'_0 | \omega_{n-1} \rangle \langle \omega'_0 | e_0 \rangle \langle \omega'_0 | e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \langle f_0, e_0 \rangle \right\} \\
+ (rs)^{-1} \left\{ \langle \omega'_0 | \omega_{n-1} \rangle + \langle \omega'_0 | \omega_0 \rangle \langle \omega'_0 | e_0 \rangle \langle \omega'_0 | e_0 \rangle \langle f_{n-1}, e_{n-1} \rangle \langle f_0, e_0 \rangle \right\}
\]
\[
= \frac{1}{(s-r)^3} \left\{ rs^{-1} \cdot r^{-1} + r^{-1} - (r^{-1} + s^{-1}) (s r^{-1} + s^{-1} + s) + (rs)^{-1} (s r^{-1} + s) \right\}
\]
\[
= 0.
\]

(iii) If \( X = f_{n-1} f_0^2 \), one can similarly get that (2.2) vanishes.

Finally, if \( X \) is any word involving exactly two \( f_0 \) factors, one \( f_{n-1} \) factor, and arbitrarily many factors \( \omega'_j \ (j \in I_0) \), \( \gamma' \pm \frac{1}{2} \) and \( D' \pm \frac{1}{2} \), then (2.2) will just be a scalar multiple of one of the quantities we have already calculated, and then will be 0.

Analogous calculations show that the relations in \( \hat{B}' \) are preserved. \[ \square \]

**Theorem 2.5.** \( D(\hat{B}, \hat{B}') \) is isomorphic to \( U_{r,s}(\mathfrak{sl}_n) \) as Hopf algebras.

**Proof.** We will denote the image \( e_i \otimes 1 \) of \( e_i \) in \( D(\hat{B}, \hat{B}') \) by \( \hat{e}_i \) and similarly for \( \omega'_i, \gamma'_i \pm \frac{1}{2}, D' \pm 1 \), denote the image \( 1 \otimes f_i \) of \( f_i \) in \( D(\hat{B}, \hat{B}') \) by \( \hat{f}_i \) and similarly for \( \omega'_{i, \pm 1}, \gamma'_{i, \pm \frac{1}{2}}, D' \pm 1 \). Define a map \( \varphi : D(\hat{B}, \hat{B}') \rightarrow U_{r,s}(\mathfrak{sl}_n) \) by
\[
\varphi(\hat{e}_i) = e_i, \quad \varphi(\hat{f}_i) = f_i, \quad \varphi(\omega_i^{\pm 1}) = \omega_i^{\pm 1}, \quad \varphi(\gamma_i^{\pm 1}) = \gamma_i^{\pm 1},
\]
\[
\varphi(\gamma_{i, \pm 1}^{\pm 1}) = \gamma_{i, \pm 1}^{\pm 1}, \quad \varphi(D^{\pm 1}) = D^{\pm 1}, \quad \varphi(D') = D'.
\]
The remaining argument is analogous to that of [BGH1, Theorem 2.5].

**Remark 2.6.** (1) Up to now, we have completely solved the compatibility problem on the defining relations of our two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}_n})$ $(n > 2)$. This is done in two steps: the proof of Theorem 2.5 indicates that the cross relations between $\widehat{B}$ and $\widehat{B}'$ are half of the relations (A1)–(A4), and the proof of Proposition 2.4 shows the remaining relations, including the remaining half of relations (A1)–(A4) and the $(r, s)$-Serre relations (A5)–(A7).

(2) When $r = s^{-1} = q$, the Hopf algebra $U_{q,q^{-1}}(\widehat{\mathfrak{sl}_n})$ modulo the Hopf ideal generated by the set $\{\omega_i' - \omega_i' (i \in I_0), \gamma^{-\frac{1}{2}} - \gamma^{-\frac{1}{2}}, D' - D^{-1}\}$ is the usual quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$ of Drinfeld-Jimbo type.

Let $U^0 = \mathbb{K}[\omega_0^\pm, \cdots, \omega_n^\pm, \omega_0'^\pm, \cdots, \omega_n'^\pm], U_0 = \mathbb{K}[\omega_0^\pm, \cdots, \omega_n^\pm], \text{ and } U'_0 = \mathbb{K}[\omega_0^\pm, \cdots, \omega_n^\pm]$ denote the Laurent polynomial subalgebras of $U_{r,s}(\widehat{\mathfrak{sl}_n}), \widehat{B}$, and $\widehat{B}'$ respectively. Clearly, $U^0 = U_0U'_0U_0$. Furthermore, let us denote by $U_{r,s}(\widehat{\mathfrak{n}})$ (resp. $U_{r,s}(\widehat{\mathfrak{n}}^-)$) the subalgebra of $\widehat{B}$ (resp. $\widehat{B}'$) generated by $e_i$ (resp. $f_i$) for all $i \in I_0$. Thus, by definition, we have $\widehat{B} = U_{r,s}(\widehat{\mathfrak{n}}) \times U_0$, and $\widehat{B}' = U'_0 \cong U_{r,s}(\widehat{\mathfrak{n}}^-)$, so that the double $D(\widehat{B}, \widehat{B}') \cong U_{r,s}(\widehat{\mathfrak{n}}) \otimes U^0 \otimes U_{r,s}(\widehat{\mathfrak{n}}^-)$, as vector spaces. On the other hand, if we consider $\langle, \rangle : \widehat{B}' \times \widehat{B} \rightarrow \mathbb{K}$ by $\langle b', b \rangle := (S(b'), b)$, the convolution inverse of the skew-dual paring $\langle , \rangle$ in Proposition 2.4, the composition with the flip mapping $\sigma$ then gives rise to a new skew-dual paring $\langle | \rangle := \langle , \rangle \circ \sigma : \widehat{B} \times \widehat{B}' \rightarrow \mathbb{K}$, given by $\langle b'|b \rangle = (S(b'), b)$.

As a byproduct of Theorem 2.5, similar to [BGH1, Coro. 2.6], we get the standard triangular decomposition of $U_{r,s}(\widehat{\mathfrak{sl}_n})$.

**Corollary 2.7.** $U_{r,s}(\widehat{\mathfrak{sl}_n}) \cong U_{r,s}(\widehat{\mathfrak{n}}^-) \otimes U^0 \otimes U_{r,s}(\widehat{\mathfrak{n}})$, as vector spaces.

**Corollary 2.8.** For any $\zeta = \sum_{i=0}^n \zeta_i \alpha_i \in Q$ (the root lattice of $\widehat{\mathfrak{sl}_n}$), the defining relations (A2) and (A3) in $U_{r,s}(\widehat{\mathfrak{sl}_n})$ take the form:

$$
\omega_\zeta e_i \omega_\zeta^{-1} = \langle \omega_i', \omega_\zeta \rangle e_i, \quad \omega_\zeta f_i \omega_\zeta^{-1} = \langle \omega_i', \omega_\zeta \rangle^{-1} f_i,
$$

$$
\omega'_\zeta e_i \omega'_\zeta^{-1} = \langle \omega'_i, \omega_i \rangle^{-1} e_i, \quad \omega'_\zeta f_i \omega'_\zeta^{-1} = \langle \omega'_i, \omega_i \rangle f_i.
$$

$U_{r,s}(\widehat{\mathfrak{n}}^\pm) = \bigoplus_{\eta \in Q^+} U^{\eta\eta}_{r,s}(\widehat{\mathfrak{n}}^\pm)$ is then $Q^\pm$-graded with

$$
U^{\eta\eta}_{r,s}(\widehat{\mathfrak{n}}^\pm) = \left\{ a \in U_{r,s}(\widehat{\mathfrak{n}}^\pm) \mid \omega_\zeta a \omega_\zeta^{-1} = \langle \omega_\eta, \omega_\zeta \rangle a, \omega'_\zeta a \omega'_\zeta^{-1} = \langle \omega'_\eta, \omega_\zeta \rangle^{-1} a \right\},
$$

for $\eta \in Q^+ \cup Q^-$. Furthermore, $U = \bigoplus_{\eta \in Q} U^{\eta\eta}_{r,s}(\widehat{\mathfrak{n}})$ is $Q$-graded with

$$
U^{\eta\eta}_{r,s}(\widehat{\mathfrak{n}}) = \left\{ \sum F_\alpha \omega_\mu \omega_\nu E_\beta \in U \mid \omega_\zeta (F_\alpha \omega_\mu \omega_\nu E_\beta) \omega_\zeta^{-1} = \langle \omega'_\beta - \alpha, \omega_\zeta \rangle F_\alpha \omega_\mu \omega_\nu E_\beta, \right. \right.
$$

$$
\omega'_\zeta (F_\alpha \omega_\mu \omega_\nu E_\beta) \omega'_\zeta^{-1} = \langle \omega'_\beta - \alpha, \omega_\zeta \rangle^{-1} F_\alpha \omega_\mu \omega_\nu E_\beta, \text{ with } \beta - \alpha = \eta \left\},
$$

where $F_\alpha$ (resp. $E_\beta$) runs over monomials $f_1 \cdots f_l$ (resp. $e_j \cdots e_m$) such that $\alpha_1 + \cdots + \alpha_i = \alpha$ (resp. $\alpha_j + \cdots + \alpha_j = \beta$).

□

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subject to the following defining relations: the elements $x_\pm(k), a_i(\ell), \omega_i^\pm, \omega_i'^\pm, \gamma^\pm\frac{k}{2}, \gamma'^\pm\frac{k}{2}, D^\pm_1, D'^\pm_1 (i \in I, k, k' \in \mathbb{Z}, \ell, \ell' \in \mathbb{Z}\{0\}),$ subject to the following defining relations:

\begin{align*}
\text{(D1)} & \quad \gamma^\pm\frac{k}{2}, \gamma'^\pm\frac{k}{2} \text{ are central, } \omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1 = DD^{-1} = D'D'^{-1} (i \in I), \text{ and for } i, j \in I, \text{ one has} \\
& \quad [\omega_i^\pm, \omega_j^\pm] = [\omega_i'^\pm, D^\pm_1] = [\omega_i'^\pm, D'^\pm_1] = [\omega_i^\pm, D'^\pm_1] = [\omega_i'^\pm, D^\pm_1] = 0 \\
& \quad [\omega_i^\pm, \omega_j'^\pm] = [\omega_i'^\pm, \omega_j^\pm] = [D^\pm_1, D^\pm_1] = [D'^\pm_1, D'^\pm_1] = 0.
\end{align*}

\begin{align*}
\text{(D2)} & \quad [a_i(\ell), a_j(\ell')] = \delta_{\ell + \ell', 0} (rs)^{\frac{e_{ij}^r}{2}} [a_i(\ell)] \cdot \frac{\gamma^{|\ell|} - \gamma'^{|\ell|}}{r - s}, \quad [n] = \frac{r^n - s^n}{r - s}; \\
\text{(D3)} & \quad [a_i(\ell), \omega_j'^\pm] = [a_i(\ell), \omega_j^\pm] = 0. \\
\text{(D4)} & \quad D x_i^\pm(k) D^{-1} = r^k x_i^\pm(k), \quad D' x_i^\pm(k) D'^{-1} = s^k x_i^\pm(k), \\
& \quad D a_i(\ell) D^{-1} = r^\ell a_i(\ell), \quad D' a_i(\ell) D'^{-1} = s^\ell a_i(\ell). \\
\text{(D5)} & \quad \omega_i x_j^\pm(k) \omega_i^{-1} = (j, i) ^{\pm1} x_j^\pm(k), \quad \omega_i' x_j^\pm(k) \omega_i'^{-1} = (i, j) ^{\pm1} x_j^\pm(k),
\end{align*}

where we briefly write $(i, j) := (\omega_i', \omega_j')$.

\begin{align*}
\text{(D6_1)} & \quad [a_i(\ell), x_j^\pm(k)] = \pm (rs)^{\frac{e_{ij}^r}{2}} [a_i(\ell)] \gamma^{|\ell|} x_j^\pm(\ell + k), \quad \text{for } \ell < 0, \\
\text{(D6_2)} & \quad [a_i(\ell), x_j^\pm(k)] = \pm (rs)^{\frac{e_{ij}^r}{2}} [a_i(\ell)] \gamma^{|\ell|} x_j^\pm(\ell + k), \quad \text{for } \ell > 0.
\end{align*}
\[
x_i^+(k+1) x_j^+(k') - \langle j, i \rangle^\pm x_j^+(k') x_i^+(k+1) \\
= -\left( \langle i, j \rangle (i, j)^{-1} \right)^{\pm \frac{1}{2}} \left( x_j^+(k'+1) x_i^+(k) - \langle i, j \rangle^\pm x_i^+(k) x_j^+(k'+1) \right).
\]

(D7)

\[
[x_i^+(k), x_j^-(k')] = \frac{\delta_{ij}}{r-s} \left( \gamma^{-k} \gamma^{-\frac{k+k'}{2}} \omega_i(k+k') - \gamma^{k'} \gamma^{\frac{k+k'}{2}} \omega'_i(k+k') \right),
\]

where \( \omega_i(m), \omega'_i(m) \) (\( m \in \mathbb{Z}_{\geq 0} \)) with \( \omega_i(0) = \omega_i \) and \( \omega'_i(0) = \omega'_i \) are defined by:

\[
\sum_{m=0}^{\infty} \omega_i(m) z^{-m} = \omega_i \exp \left( (r-s) \sum_{\ell=1}^{\infty} a_i(\ell) z^{-\ell} \right);
\]

\[
\sum_{m=0}^{\infty} \omega'_i(-m) z^m = \omega'_i \exp \left( -(r-s) \sum_{\ell=1}^{\infty} a_i(-\ell) z^\ell \right),
\]

with \( \omega_i(-m) = 0 \) and \( \omega'_i(m) = 0, \forall m > 0. \)

(D91)

\[
x_i^\pm(m) x_j^\pm(k) = x_j^\pm(k) x_i^\pm(m), \quad \text{for } a_{ij} = 0,
\]

(D92)

\[
\text{Sym}_{m_1, m_2} \left( x_j^\pm(m_1) x_i^\pm(m_2) x_j^\pm(k) - (r^\pm + s^\pm) x_j^\pm(m_1) x_i^\pm(k) x_j^\pm(m_2) \right) = 0, \quad \text{for } a_{ij} = -1, 1 \leq i < j < n,
\]

(D93)

\[
\text{Sym}_{m_1, m_2} \left( x_j^\pm(m_1) x_i^\pm(m_2) x_j^\pm(k) - (r^\pm + s^\pm) x_j^\pm(m_1) x_i^\pm(k) x_j^\pm(m_2) \right) = 0, \quad \text{for } a_{ij} = -1, 1 \leq j < i < n,
\]

Sym denotes symmetrization with respect to the indices \((m_1, m_2)\).

As one of crucial observations of considering the compatibilities of the defining system above, we have

**Proposition 3.2.** There exists the \(\mathbb{Q}\)-algebra anti-automorphism \(\tau\) of \(\mathcal{U}_{r,s}(\mathfrak{sl}_n)\) (\(n > 2\)) such that \(\tau(r) = s, \tau(s) = r, \tau((\omega'_i, \omega_j)^\pm) = (\omega'_i, \omega_j)^\mp\) and

\[
\tau(\omega_i) = \omega'_i, \quad \tau(\omega'_i) = \omega_i,
\]

\[
\tau(\gamma) = \gamma', \quad \tau(\gamma') = \gamma,
\]

\[
\tau(D) = D', \quad \tau(D') = D,
\]

\[
\tau(a_i(\ell)) = a_i(-\ell),
\]

\[
\tau(x_i^+(m)) = x_i^-(m),
\]

\[
\tau(\omega_i(m)) = \omega'_i(-m), \quad \tau(\omega'_i(-m)) = \omega_i(m),
\]

and \(\tau\) preserves each defining relation (Dn) in Definition 3.1 for \(n = 1, \ldots, 9.\) \(\square\)

**Remark 3.3.** (1) Note that the defining relations (D1)—(D5), (D7), (D8), and (D91)—(D93) are self-compatible each relative to the \(\mathbb{Q}\)-algebra anti-automorphism \(\tau\), while the couple of the defining relations ((D61), (D62)) is compatible with each other with respect to \(\tau\). Using
such a \( \tau \), it is sufficient to consider the compatibility for half of the relations, e.g., those relations involving in \(+\)-parts for \( x^\pm(m) \), or in positive \( \ell \)'s for \( a_i(\ell) \) (for instance, see (D62)).

(2) As a glimpse of the compatibility of (D2) with (D61), (D62) and (D8), we have the following: By (D8), we get 
\[
a_i(1) = \omega_{i}^{-1}\gamma^{1/2} \left[ x_i^+(0), x_i^-(1) \right]
\]
and 
\[
a_i(-1) = \omega_{i}^{-1}\gamma^{1/2} \left[ x_i^-(1), x_i^+(0) \right].
\]

Then using one of these expressions of \( a_i(\pm1) \) and using (D61) (or (D62)) and (D8) again, we may expand the Lie bracket \( [a_i(1), a_j(-1)] \) in two manners to get to the same formula as (D2). One manner in replace of \( a_i(1) \) is as follows (in terms of (D61) & (D8))
\[
[a_i(1), a_j(-1)] = \omega_{i}^{-1}\gamma^{1/2} \left[ [x_i^+(0), x_i^-(1)], a_j(-1) \right]
\]
\[
= \omega_{i}^{-1}\gamma^{1/2} \left[ [x_i^+(0), a_j(-1)], x_i^-(1) \right] + [x_i^+(0), [x_i^-(1), a_j(-1)]]
\]
\[
= \omega_{i}^{-1}\gamma^{1/2}(rs) \left[ -a_{ij} \right] \left[ \gamma^{-1/2} - \gamma^{1/2} \right] \gamma^{-1} \left[ x_i^+(0), x_i^-(0) \right]
\]
\[
= (rs) \left[ -a_{ij} \right] \gamma^{-1} \left[ \frac{\gamma' - \gamma}{r-s} \right] = (rs) \gamma^{-1} \left[ a_{ij} \right] \gamma^{-1} \left( \frac{\gamma' - \gamma}{r-s} \right)
\]
Expanding it in another way in replace of \( a_j(-1) \), we can get the same result. More compatibilities will be clearer in the proof of the Drinfeld isomorphism theorem.

(3) Another observation is the following: When \( r = s^{-1} = q \), the algebra \( \mathcal{U}_{q,q^{-1}}(\widehat{sl}_n) \) modulo the ideal generated by the set \{ \( \omega_i - \omega_i^{-1} \) \( i \in I \) \}, \( \gamma^{\frac{1}{2}} \gamma^{\frac{1}{2}} \), \( D' - D^{-1} \) is just the usual Drinfeld realization \( \mathcal{U}_{q}(\widehat{sl}_n) \) defined as follows (cf. [B2]).

The unital associative algebra \( \mathcal{U}_{q}(\widehat{sl}_n) \) over \( \mathbb{Q}(q) \) is generated by the elements \( x_i^\pm(k), a_i(\ell), \omega_i^\pm, \gamma^\pm, D^\pm, (i \in I, k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}) \) subject to the following defining relations:

(d1) \( \gamma^{\pm} \) are central, \( \omega_i\omega_i^{-1} = 1 = DD^{-1} \( i \in I \) \), and for \( i, j \in I \), one has
\[
[\omega_i^\pm, \omega_j^\pm] = [\omega_i^\pm, D^\pm] = 0.
\]

(d2) 
\[
[a_i(\ell), a_j(\ell')] = \delta_{\ell+\ell',0} \left[ \frac{\ell a_{ij}}{\ell} \right], \quad \gamma^\ell - \gamma^{-\ell} = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \left( [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \right).
\]

(d3) 
\[
[a_i(\ell), \omega_j^\pm] = 0.
\]

(d4) 
\[
D x_i^\pm(k) D^{-1} = q^k x_i^\pm(k), \quad D a_i(\ell) D^{-1} = q^\ell a_i(\ell).
\]

(d5) 
\[
\omega_i x_j^\pm(k) \omega_i^{-1} = q^{\pm a_{ij}} x_j^\pm(k).
\]

(d6) 
\[
[a_i(\ell), x_j^\pm(k)] = \pm \left[ \frac{\ell a_{ij}}{\ell} \right] \gamma^{\pm i\ell} x_j^\pm(\ell+k).
\]

(d7) 
\[
x_i^\pm(k+1)x_j^\pm(k') - q^{\pm a_{ij}} x_j^\pm(k')x_i^\pm(k+1)
\]
\[
= q^{\pm a_{ij}} x_i^\pm(k)x_j^\pm(k'+1) - x_j^\pm(k'+1)x_i^\pm(k).
\]

(d8) 
\[
[x_i^+(k), x_j^-(k')] = \frac{\delta_{ij}}{q - q^{-1}} \left( \frac{q^{k-k'} - q^{-k'} \omega_i(k + k') - q^{k-k'} \omega_i^{-1}(k + k')}{} \right),
\]
where $\omega_i(m)$ and $\omega_i^{-1}(-m)$ ($m \in \mathbb{Z}_{\geq 0}$) with $\omega_i(0) = \omega_i$ and $\omega_i^{-1}(0) = \omega_i^{-1}$ are defined by:

$$
\sum_{m=0}^{\infty} \omega_i(m)z^{-m} = \omega_i \exp \left( (q-q^{-1}) \sum_{\ell=1}^{\infty} a_i(\ell)z^{-\ell} \right), \quad (\omega_i(-m) = 0, \ \forall \ m > 0);
$$

$$
\sum_{m=0}^{\infty} \omega_i^{-1}(-m)z^{m} = \omega_i^{-1} \exp \left( -(q-q^{-1}) \sum_{\ell=1}^{\infty} a_i(-\ell)z^{\ell} \right), \quad (\omega_i^{-1}(m) = 0, \ \forall \ m > 0).
$$

\[(d9_1)\]

$$
x_i^\pm (m)x_j^\pm (k) = x_j^\pm (k)x_i^\pm (m), \quad \text{for} \quad a_{ij} = 0,
$$

\[(d9_2)\]

$$
\text{Sym}_{m_1, m_2} \left( x_i^\pm (m_1)x_j^\pm (m_2)x_j^\pm (k) - (q^\pm_1 + q^\pm_1) x_i^\pm (m_1)x_j^\pm (k)x_i^\pm (m_2) \right)
\quad + \ x_j^\pm (k)x_i^\pm (m_1)x_i^\pm (m_2) \right) = 0, \quad \text{for} \quad a_{ij} = -1, \ 1 \leq i < j < n,
$$

\[(d9_3)\]

$$
\text{Sym}_{m_1, m_2} \left( x_i^\pm (m_1)x_j^\pm (m_2)x_j^\pm (k) - (q^\pm_1 + q^\pm_1) x_i^\pm (m_1)x_j^\pm (k)x_i^\pm (m_2) \right)
\quad + \ x_j^\pm (k)x_i^\pm (m_1)x_i^\pm (m_2) \right) = 0, \quad \text{for} \quad a_{ij} = -1, \ 1 \leq j < i < n.
$$

3.2 Before showing that the $\mathbb{Q}(r, s)$-algebra $\mathcal{U}_{r, s}(\mathfrak{sl}_n)$ ($n > 2$) in Definition 3.1 is exactly the Drinfeld realization of the two-parameter quantum affine algebra $U_{r, s}(\mathfrak{sl}_n)$ ($n > 2$) defined in Definition 2.1, that is, putting forward the Drinfeld isomorphism theorem, we need to make some preliminaries on Lyndon words, as well as to adapt a definition of quantum Lie bracket that appears to be regardless to degrees of relative elements (see the properties (3.3) & (3.4) below). This a bit generalized quantum Lie bracket compared to the one used in the usual construction of the quantum Lyndon basis (for definition, see [R2]), which is consistent with the cases when adding the bracketing on those corresponding Lyndon words, is crucial to our proving later on.

DEFINITION 3.4. ([J2]) The quantum Lie bracket $[a_1, a_2, \cdots, a_s]_{(q_1, q_2, \cdots, q_{s-1})}$ is defined inductively by

$$
[a_1, a_2]_{q} = a_1a_2 - qa_2a_1, \quad \text{for} \quad q \in \mathbb{K}\backslash\{0\},
$$

$$
[a_1, a_2, \cdots, a_s]_{(q_1, q_2, \cdots, q_{s-1})} = [a_1, [a_2, \cdots, a_s]_{(q_1, \cdots, q_{s-2})}]_{q_{s-1}}, \quad \text{for} \quad q_i \in \mathbb{K}\backslash\{0\}.
$$

The following identities follow from the definition.

\[(3.1)\]

$$
[a, bc]_v = [a, b]_x c + x [a, c]_b, \quad x \neq 0,
$$

\[(3.2)\]

$$
[ab, c]_v = a [b, c]_x + x [a, c]_b, \quad x \neq 0,
$$

\[(3.3)\]

$$
[a, [b, c]_x]_v = [[a, b]_x, c]_\frac{v^x}{x} + x [b, [a, c]_x]_b, \quad x \neq 0,
$$

\[(3.4)\]

$$
[[a, b]_x, c]_v = [a, [b, c]_x]_\frac{v^x}{x} + x [[a, c]_x, b]_\frac{v^x}{x}, \quad x \neq 0.
$$

\[(3.5)\]

$$
[a, b_1, \cdots, b_s]_{(v_1, \cdots, v_{s-1})} = \sum_{i} [b_1, \cdots, [a, b_i], \cdots, b_s]_{(v_1, \cdots, v_{s-1})},
$$

\[(3.6)\]

$$
[a, a, b]_{(u, v)} = [a, a, b]_{(v, u)} = a^2b - (u + v) aba + (uv) ba^2.
$$

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Definition 3.5. For the generators system of the algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$, we define the $\hat{Q}$-gradation (where $\hat{Q}$ is the root lattice of $\mathfrak{sl}_n$) as follows:

$$\deg(\omega_i^{\pm1}) = \deg(\omega_i^{\pm1}) = \deg(\gamma_i^{\pm1}) = \deg(\gamma_i^{\pm1}) = \deg(D^{\pm1}) = \deg(D^{\pm1}) = 0,$$

$$\deg(a_i(\pm\ell)) = 0, \quad \deg(x_i^+(k)) = \pm\alpha_i.$$  

Hence, the defining relations (D1)–(D9) ensure that $U_{r,s}(\widehat{\mathfrak{sl}}_n)$ has a triangular decomposition:

$$U_{r,s}(\widehat{\mathfrak{sl}}_n) = U_{r,s}(\widehat{\mathfrak{n}}^-) \otimes U_{r,s}(\widehat{\mathfrak{sl}}_n) \otimes U_{r,s}(\widehat{\mathfrak{n}}),$$

where $U_{r,s}(\widehat{\mathfrak{n}}^\pm) = \bigoplus_{\alpha \in \mathbb{Q}\pm} U_{r,s}(\widehat{\mathfrak{n}}^\pm)_\alpha$ is generated respectively by $x_i^+(k)$ ($i \in I$), and $U_{r,s}(\widehat{\mathfrak{sl}}_n)$ is the subalgebra generated by $\omega_i^{\pm1}$, $\omega_i^{\pm1}$, $\gamma_i^{\pm1}$, $\gamma_i^{\pm1}$, $D^{\pm1}$, $D^{\pm1}$ and $a_i(\pm\ell)$ for $i \in I$, $\ell \in \mathbb{N}$. Namely, $U_{r,s}(\widehat{\mathfrak{sl}}_n)$ is generated by the toral subalgebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)^0$ and the quantum Heisenberg subalgebra $H_{r,s}(\widehat{\mathfrak{sl}}_n)$ generated by those quantum imaginary root vectors $a_i(\pm\ell)$ ($i \in I$, $\ell \in \mathbb{N}$).

Definition 3.6. For $\alpha, \beta \in \hat{Q}^+$ (positive root lattice of $\mathfrak{sl}_n$), and $x_i^+(k)$, $x_i^+(k') \in U_{r,s}(\widehat{\mathfrak{n}}^\pm)$, we define their “affine” quantum Lie bracket as follows:

$$[x_i^+(k), x_j^+(k')]_{(\omega_i, \omega_j)^{\pm1}} := x_i^+(k)x_j^+(k') - (\omega_i, \omega_j)^{\pm1} x_j^+(k')x_i^+(k).$$

By the definition above, the formula (D7) will take the convenient form as

$$[x_i^+(k), x_j^+(k+1)]_{(i,j)^{\pm1}} = -\left((j,i)(i,j)^{-1}\right)^{\mp1} [x_j^+(k'), x_i^+(k+1)]_{(j,i)^{\pm1}}.\tag{3.8}$$

By (3.6), the $(r,s)$-Serre relations (D9) & (D9b) for $m_1 = m_2$ in the case of $a_{ij} = -1$ can be reformulated as:

$$[x_i^+(m), x_j^+(m), x_j^+(k)]_{(r^\pm1,s^\pm1)} = 0, \quad \text{for} \quad 1 \leq i < j < n,\tag{3.9}$$

$$[x_i^+(m), x_j^+(m), x_j^+(k)]_{(r^\mp1,s^\mp1)} = 0, \quad \text{for} \quad 1 \leq j < i < n.$$

Remark 3.7. (1) For any nonsimple root $\alpha (\neq \alpha_i)$ ($i \in I$), the meaning of notation $x_i^+(k)$ (resp. $x_i^-(k)$) in Definition 3.6 has a bit ambiguity, as is well-known even for quantum “classical” root vectors $x_i^+(0)$ which have different linearly-independent choices. However, the combinatorial approach of Lyndon words, together with the “affine” quantum Lie bracket, will give us a valid and specific choice for $x_i^+(k)$ which leads to a construction of quantum “affine” Lyndon basis for $U_{r,s}(\widehat{\mathfrak{n}})$, on which acting $\tau$ will yield a corresponding construction of quantum “affine” Lyndon basis for $U_{r,s}(\widehat{\mathfrak{n}}^-)$ (see Proposition 3.10 & Theorem 3.11 below).

(2) In fact, (3.8) describes a kind of consistent constrains of quantum affine root vectors defined by some Lyndon words of different levels (if say, $x_i^+(k)$ have level $k$) which obeys the defining rule of Lyndon basis (see below) via Lyndon words as like in the classical types, since from (3.8), we get

$$[x_i^+(k), x_j^+(k'+1)]_{(i,j)^{\mp1}} = (i,j)^{\mp1} x_j^+(k'+1) x_i^+(k) + (i,j)^{\mp1} x_j^+(k') x_i^+(k+1).\tag{3.10}$$

Based on this formula, we will see that it makes reasonable to give the definition of quantum affine root vector $x_i^+(k)$ as in (3.14) & (3.15) below such that the level $k$ completely concentrates
on the component of the lowest index, from the ordered constituents of Lyndon basis. This will be clear from the proof of Proposition 3.10.

(3) Let \( \mathcal{U}_{r,s}(n) \) denote the subalgebra of \( \mathcal{U}_{r,s}(\mathfrak{n}) \), generated by \( x_i^+(0) \) \( (i \in I) \). By definition, it is clear that \( \mathcal{U}_{r,s}(n) \cong U_{r,s}(\mathfrak{n}) \), the subalgebra of \( U_{r,s}(\mathfrak{sl}_n) \) generated by \( e_i \) \( (i \in I) \) (see [BGG1, Remarks (2), p. 391]). Now let us recall the construction of a Lyndon basis. The natural ordering \( < \) in \( I \) gives a total ordering of the alphabet \( A = \{x_1^+(0), \cdots , x_{n-1}^+(0)\} \). Let \( A^* \) be the set of all words in the alphabet \( A \) (including the vacuum 1) and let \( u < v \) denote that word \( u \) is lexicographically smaller than word \( v \). Recall that a word \( \ell \in A^* \) is a Lyndon word if it is lexicographically smaller than all its proper right factors (cf. [LR], [R2], [BH]). Let \( \mathbb{K}[A^*] \) be the associative algebra of \( \mathbb{K} \)-linear combinations of words in \( A^* \) whose product is juxtaposition, namely, a free \( \mathbb{K} \)-algebra. Let \( J \) be the \( (r,s) \)-Serre ideal of \( \mathbb{K}[A^*] \) generated by elements \( \{(ad_i x_j^+(0))^{1-a_{ij}}(x_k^+(0)) \mid 1 \leq i \neq j \leq n-1\} \). Clearly, \( \mathcal{U}_{r,s}(n) = \mathbb{K}[A^*/J \). Now given another ordering \( \preceq \) in \( A^* \) with introducing a usual length function \( | \cdot | \) for each word \( u \in A^* \). We say \( u \preceq w \), if \( |u| < |w| \) or \( |u| = |w| \) and \( u \geq w \). Then we call a (Lyndon) word to be good w.r.t. the \( (r,s) \)-Serre ideal \( J \) if it cannot be written as a sum of strictly smaller words modulo \( J \) w.r.t. the ordering \( \preceq \). From [R2], the set of quantum Lie brackets (or say, \( q \)-bracketings) of all good Lyndon words consists of a system of quantum root vectors of \( \mathcal{U}_{r,s}(n) \). More precisely, we have a construction for any quantum root vector \( x_{\alpha}^+(0) \) with \( \alpha \in \Delta^+ \) (the positive root system of \( \mathfrak{sl}_n \)) in the following.

Take a corresponding ordering (compatible with the natural ordering \( < \) on \( I \)) of \( \Delta^+ = \{\alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} = \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\} \) with \( \alpha_{i,i+1} = \alpha_i \) as follows (see [H, p. 533]):

\[
\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{24}, \alpha_{34}, \cdots, \alpha_{1n}, \alpha_{2n}, \cdots, \alpha_{n-1,n}
\]

which is a convex ordering on \( \Delta^+ \) (see [R2, Section 6]). Hence, for each \( \alpha = \alpha_{ij} \in \Delta^+ \), by [R2], we can construct the quantum root vector \( x_{\alpha_{ij}}^+(0) \) as a \( (r,s) \)-bracketing of a good Lyndon word in an inductive fashion:

\[
x_{\alpha_{ij}}^+(0) := [x_{\alpha_{i,j-1}}^+(0), x_{j-1}^+(0)]_{\omega_{\alpha_{i,j-1},\omega_{j-1}}}^{-1} \\
= \cdots [x_{i}^+(0), x_{i+1}^+(0)]_{(i+1-1)}^{-1}, \cdots , x_{j-1}^+(0)]_{\omega_{\alpha_{i,j-1},\omega_{j-1}}}^{-1} \\
= \cdots [x_{i}^+(0), x_{i+1}^+(0)]_{r}, \cdots , x_{j-1}^+(0)]_{s}.
\]

Applying \( \tau \) to (3.12), we can obtain the definition of quantum root vector \( x_{\alpha_{ij}}^-(0) \) as below:

\[
x_{\alpha_{ij}}^-(0) = \tau(x_{\alpha_{ij}}^+(0)) = [x_{j-1}^-(0), \cdots , [x_{i+1}^-(0), x_{i}^-(0)]_s \cdots ]_s.
\]

**Theorem 3.8.** (i) The set

\[
\left\{ x_{\alpha_{n-1,n}}^+(0)^{f_{n-1,n}} \cdots x_{\alpha_{2n}}^+(0)^{f_{2n}} x_{\alpha_{1n}}^+(0)^{f_{1n}} \cdots x_{\alpha_{13}}^+(0)^{f_{13}} x_{\alpha_{12}}^+(0)^{f_{12}} \mid f_{ij} \geq 0 \right\}
\]

is a Lyndon basis of \( \mathcal{U}_{r,s}(n) \).
(ii) The set
\[
\left\{ x_{α_{1}}^{-}(0)x_{α_{2}}^{-}(0)\cdots x_{α_{n}}^{-}(0)x_{α_{1}}^{+}(0)x_{α_{2}}^{+}(0)\cdots x_{α_{n}}^{+}(0)x_{α_{1}}^{-}(0)\cdots x_{α_{n}}^{-}(0)\mid \ell_{ij} \geq 0 \right\}
\]
is a Lyndon basis of $U_{r,s}(\mathfrak{n}^{-})$.

**Definition 3.9.** For $α_{ij} ∈ \hat{Δ}^{+}$, we define the quantum affine root vectors $x_{α_{ij}}^{±}(k)$ of nontrivial level $k$ by
\[
\begin{align*}
& (3.14) & x_{α_{ij}}^{+}(k) := [\cdots [x_{i}^{+}(k), x_{i+1}^{+}(0)], \cdots, x_{j-1}^{+}(0)]_{\tau}, \\
& (3.15) & x_{α_{ij}}^{-}(k) := [x_{j-1}^{-}(0), \cdots, [x_{i}^{-}(0), x_{i}^{-}(k)]_{\tau}, \cdots],
\end{align*}
\]
where $τ(x_{α_{ij}}^{±}(±k)) = x_{α_{ij}}^{±}(±k)$.

For each fixed $α ∈ \hat{Q}^{+}$, let us denote $U_{r,s}^{(k)}(\mathfrak{n})_{α}$ by the subspace of $U_{r,s}(\mathfrak{n})_{α}$, consisting of elements of level $k$. Hence, $U_{r,s}(\mathfrak{n})_{α} = \bigoplus_{k∈\mathbb{Z}} U_{r,s}^{(k)}(\mathfrak{n})_{α}$. When $α = α_{i} ∈ \hat{Δ}^{+}$ is a simple root, by definition, $\dim U_{r,s}^{(k)}(\mathfrak{n})_{α} = 1$ for any level $k$. However, for any nonsimple root $α ≠ α_{i} (i ∈ I)$, $\dim U_{r,s}^{(k)}(\mathfrak{n})_{α} = ∞$ for any level $k$. In this case, given a positive root $α = α_{ij} ∈ \hat{Δ}^{+}$, we call a tuple $(β_{1}, \cdots, β_{ν}) (ν ≥ 1)$ to be a partition of the root $α_{ij}$ if $β_{1} < \cdots < β_{ν}$ as in the ordering given in (3.11) such that $β_{1} + \cdots + β_{ν} = α_{ij}$. If $ν > 1$, we say this partition to be proper. Denote by $\mathfrak{P}^{+}(α)$ the set of all proper partitions of root $α$. Obviously, we have $U_{r,s}^{(k_{1})}(\mathfrak{n})_{β_{1}} \cdots U_{r,s}^{(k_{ν})}(\mathfrak{n})_{β_{ν}} ⊆ U_{r,s}^{(k_{1})}(\mathfrak{n})_{α}$ if $k_{1} + \cdots + k_{ν} = k$. Now we write
\[
Ω^{(k)}_{α}(\mathfrak{n}) := \bigoplus_{(β_{1}, \cdots, β_{ν}) ∈ \mathfrak{P}^{+}(α)} U_{r,s}^{(k_{1})}(\mathfrak{n})_{β_{1}} \cdots U_{r,s}^{(k_{ν})}(\mathfrak{n})_{β_{ν}} ⊆ U_{r,s}(\mathfrak{n})_{α}
\]
for the subspace of $U_{r,s}^{(k)}(\mathfrak{n})_{α}$ spanned by basis elements’ products of level $k$ from those proper partitions pertaining to $α$. Using the $\mathbb{Q}$-antiautomorphism $τ$ on $Ω^{(-k)}_{α}(\mathfrak{n})$, we get
\[
Ω^{(k)}_{α}(\mathfrak{n}^{-}) := τ(Ω^{(-k)}_{α}(\mathfrak{n})).
\]
Then we have the following description on $U_{r,s}^{(k)}(\mathfrak{n}^{-})_{α}$ for $α ∈ \hat{Δ}^{+}$, whose proof shows that Definition 3.9 makes sense.

**Proposition 3.10.** For $1 ≤ i < j ≤ n$ and $α_{ij} ∈ \hat{Δ}^{+}$ (a positive root system of $\mathfrak{so}(n)$), we have
\[
\begin{align*}
& (i) & U_{r,s}^{(k)}(\mathfrak{n})_{α_{ij}} = \mathbb{K} x_{α_{ij}}^{+}(k) \bigoplus Ω^{(k)}_{α_{ij}}(\mathfrak{n}), \\
& & U_{r,s}^{(k)}(\mathfrak{n}^{-})_{α_{ij}} = \mathbb{K} x_{α_{ij}}^{-}(k) \bigoplus Ω^{(k)}_{α_{ij}}(\mathfrak{n}^{-}).
\end{align*}
\]

**Proof.** (i) We will use an induction on rank $n$, where $n ≥ 2$. Assume that $i < j$ and $k' > 0$, then by (3.10), we have
\[
\begin{align*}
& (3.16) & [x_{i}^{+}(k), x_{j}^{+}(k')]_{(i,j)}^{-1} = ⟨i, i⟩^{−\mathfrak{n}'} \left\{ [x_{i}^{+}(k+1), x_{j}^{+}(k'-1)]_{(i,j)}^{-1} + (⟨i, j⟩^{-1} - ⟨j, i⟩) x_{j}^{+}(k'-1) x_{i}^{+}(k+1) \right\}, \\
& & [x_{i}^{+}(k), x_{j}^{+}(-k')]_{(i,j)}^{-1} = ⟨i, i⟩^{+ \mathfrak{n}'} \left\{ [x_{i}^{+}(k-1), x_{j}^{+}(-k'+1)]_{(i,j)}^{-1} + (⟨j, i⟩ - ⟨i, j⟩^{-1}) x_{j}^{+}(-k'+1) x_{i}^{+}(k-1) \right\}.
\end{align*}
\]

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When \( n = 2 \), for any \( k' \in \mathbb{N} \), repeatedly using (3.16) & (3.17), we get

\[
\left[ x_1^+(k), x_2^+(k') \right]_r = \langle 1, 1 \rangle \sum_{t=1}^{k'} \langle 1, 1 \rangle^t \omega^{k+k'-t+1} (r-s) x_2^+(t-1) x_1^+(k+k'-t+1)
\]

\[
\equiv \langle 1, 1 \rangle^t \omega^{k+k'} \mod \Omega^{(k+k')}(\tilde{n}),
\]

\[
\left[ x_1^+(k), x_2^+(-k') \right]_r = \langle 1, 1 \rangle \sum_{t=1}^{k'} \langle 1, 1 \rangle^{-t} \omega^{k+k'} (s-r) x_2^+(-t) x_1^+(k-k'+t)
\]

\[
\equiv \langle 1, 1 \rangle^{-t} \omega^{k+k'} \mod \Omega^{(k-k')}(\tilde{n}),
\]

which means that in both cases, we have

\[
\left[ x_1^+(k), x_2^+(k') \right]_r = \langle 1, 1 \rangle \sum_{t=1}^{k'} \langle 1, 1 \rangle^t \omega^{k+k'} \mod \Omega^{(k+k')}(\tilde{n}), \quad \text{for any } k' \in \mathbb{Z}.
\]

Therefore, in rank 2 case, any elements (except for \( x_2^+(k')x_1^+(k) \)) of degree \( \alpha_{13} \) generated by \( x_1^+(k) \) and \( x_2^+(k') \) are of form: \( \left[ x_1^+(k), x_2^+(k') \right]_r \) for any \( a \in \mathbb{K} \), however,

\[
\left[ x_1^+(k), x_2^+(k') \right]_a = \left[ x_1^+(k), x_2^+(k') \right]_r + (r-a) x_2^+(k') x_1^+(k)
\]

\[
\equiv (rs^{-1})^t \omega^{k+k'} \mod \Omega^{(k+k')}(\tilde{n}).
\]

This fact shows that

\[
\mathcal{U}^{(k)}_{r,s}(\tilde{n})_{\alpha_{13}} = \mathbb{K} x_{\alpha_{13}}^{+}(k) \bigoplus \Omega^{(k)}(\tilde{n})
\]

as vector spaces. Dually, we also have \( \mathcal{U}^{(k)}_{r,s}(\tilde{n}^-)_{\alpha_{13}} = \mathbb{K} x_{\alpha_{13}}^{-}(k) \bigoplus \Omega^{(k)}(\tilde{n}^-) \) as vector spaces.

Now we assume that we have proved the results for rank \( n < n \), that is, for those \( \alpha_{ij} \) with \( 1 \leq i < j < n \). For rank \( n \) case, owing to the ordering given in (3.11), we are left to prove the remaining cases: \( \mathcal{U}^{(k)}_{r,s}(\tilde{n}^\pm)_{\alpha_{in}} \) with \( 1 \leq i < j = n \).

In view of the same observation as (3.18), we need only to consider the following elements of degree \( \alpha_{in} \) and level \( k+k' \) generated by \( x_{\alpha_{i,n-1}}^+(k) \) and \( x_{\alpha_{n-1}}^+(k') \) for \( 1 \leq i < n \):

\[
\left[ x_{\alpha_{i,n-1}}^+(k), x_{\alpha_{n-1}}^+(k') \right]_{\omega_{\alpha_{i,n-1}, \omega_{n-1}}} = \left[ x_{\alpha_{i,n-1}}^+(k), x_{\alpha_{n-1}}^+(k') \right]_r.
\]

By definition (see (3.14)) and
using (3.4), (3.5) & (3.1), we have

\[
\left[ x_{\alpha,i,n-1}^+(k), x_{n-1}^+(k') \right]_r = \left[ x_{\alpha,i,n-2}^+(k), x_{n-2}^+(0) \right]_r, x_{n-1}^+(k')_r \quad \text{(using (3.4))}
\]

\[
= \left[ x_{\alpha,i,n-2}^+(k), x_{n-2}^+(0), x_{n-1}^+(k')_r \right]_r + r\left[ x_{\alpha,i,n-2}^+(k), x_{n-1}^+(k'), x_{n-2}^+(0) \right]_r
\]

(2nd term = 0 by (3.5) & (D9))

\[
= \left[ x_{\alpha,i,n-2}^+(k), x_{n-2}^+(0), x_{n-1}^+(k')_r \right]_r \quad \text{(using (3.18): rank 2 case)}
\]

\[
= (rs^{-1})^{k(n-1)-1} \left[ x_{\alpha,i,n-2}^+(k), x_{n-2}^+(k')_r \right]_r, x_{n-1}^+(0)_r
\]

(by definition)

\[
+ \sum_t s_t (r-s) x_{n-1}^+(t) x_{\alpha,i,n-2}^+(k), x_{n-2}^+(k'-t)_r
\]

(using the inductive hypothesis)

\[
\equiv (rs^{-1})^{k(n-1)-1} \left[ x_{\alpha,i,n-2}^+(k+k'), x_{n-1}^+(0)_r \right]_r \quad \text{(by definition)}
\]

\[
= (rs^{-1})^{k(n-1)-1} x_{\alpha,i,n-2}^+(k+k') \mod \Omega_{\alpha,i,n-1}^{k+k'}(\bar{n})
\]

+ \sum_t s_t (r-s) x_{n-1}^+(t) x_{\alpha,i,n-1}^+(k+k'-t) \mod x_{n-1}^+(t) \Omega_{\alpha,i,n-1}^{k+k'-t}(\bar{n})

\[
\equiv (rs^{-1})^{k(n-1)-1} x_{\alpha,i,n-2}^+(k+k') \mod \Omega_{\alpha,i,n}^{k+k'}(\bar{n}),
\]

where in the 1st “\equiv”, we used the following fact:

\[
\left[ x_{\alpha,i,n-2}^+(k), x_{n-1}^+(t) x_{n-2}^+(k'-t) \right]_r = x_{n-1}^+(t) \left[ x_{\alpha,i,n-2}^+(k), x_{n-2}^+(k'-t) \right]_r
\]

+ \left[ x_{\alpha,i,n-2}^+(k), x_{n-1}^+(t) \right] x_{n-2}^+(k'-t)

(2nd term = 0 by (3.5) & (D9)))

\[
= x_{n-1}^+(t) \left[ x_{\alpha,i,n-2}^+(k), x_{n-2}^+(k'-t) \right]_r;
\]

while in the 2nd “\equiv”, we used the facts:

\[
\left[ \Omega_{\alpha,i,n-1}^{k+k'}(\bar{n}), x_{n-1}^+(0)_r \right]_r \subseteq \Omega_{\alpha,i,n}^{k+k'}(\bar{n}),
\]

\[
x_{n-1}^+(t) \Omega_{\alpha,i,n-1}^{k+k'-t}(\bar{n}) \subseteq \Omega_{\alpha,i,n}^{k+k'}(\bar{n}).
\]

The latter is clear, due to the definition of \(\Omega_{\alpha,i,n}^{k+k'}(\bar{n})\). As for the first inclusion, we have the following argument provided that we notice the basis elements’ constituents of \(\Omega_{\alpha,i,n-1}^{k+k'}(\bar{n})\).

Indeed, for any basis element

\[
x_{\alpha,i,n-1}^+(k_1) x_{\alpha,i,n-2}^+(k_2) \cdots x_{\alpha,i,\ell}^+(k_1) \in \Omega_{\alpha,i,n-1}^{k+k'}(\bar{n})
\]
of level $k + k'$ pertaining to a partition of $\alpha_{i,n-1}$, using (3.2), we have

$$
\left[ x_{\alpha_{\ell_\nu,-1}}^+ (k_\nu) x_{\alpha_{\ell_{\nu-1},\ell_\nu}}^+ (k_{\nu-1}) \cdots x_{\alpha_{i,\ell_1}}^+ (k_1), x_{n-1}^+ (0) \right]_r
= \left[ x_{\alpha_{\ell_\nu,-1}}^+ (k_\nu), x_{n-1}^+ (0) \right]_r \left[ x_{\alpha_{\ell_{\nu-1},\ell_\nu}}^+ (k_{\nu-1}) \cdots x_{\alpha_{i,\ell_1}}^+ (k_1) \right]_r \quad \text{(by definition)}
+ x_{\alpha_{\ell_\nu,-1}}^+ (k_\nu) \left[ x_{\alpha_{\ell_{\nu-1},\ell_\nu}}^+ (k_{\nu-1}) \cdots x_{\alpha_{i,\ell_1}}^+ (k_1), x_{n-1}^+ (0) \right]_r 
$$

(2nd term $= 0$ by (3.5) & (D9) since $\ell_1 < \cdots < \ell_\nu < n - 1$)

$$
= x_{\alpha_{\ell_\nu,-1}}^+ (k_\nu) x_{\alpha_{\ell_{\nu-1},\ell_\nu}}^+ (k_{\nu-1}) \cdots x_{\alpha_{i,\ell_1}}^+ (k_1)
\in \Omega_{\alpha_{in}}^{(k+k')} \left( \hat{n} \right),
$$

here $k_1 + \cdots + k_\nu = k + k'$.

Up to now, we have finished the proof of (i). Using $\tau$ to (i), we can get the second statement (ii).

The argument of Proposition 3.10 in fact implies the important conclusions below about the quantum affine Lyndon basis.

**Theorem 3.11.**

(i) The set

$$
\left\{ x_{\alpha_{n-1,n}}^+ (k_{n-1,n})^{\ell_{n-1,n}} \cdots x_{\alpha_{1,n}}^+ (k_1)^{\ell_{1,n}} \cdots x_{\alpha_{12}}^+ (k_{12})^{\ell_{12}} \left| \ell_{ij} \geq 0, k_{ij} \in \mathbb{Z} \right. \right\}
$$

is an “affine” Lyndon basis of $\mathcal{U}_{r,s}(\hat{n})$.

(ii) The set

$$
\left\{ x_{\alpha_{12}}^- (k_{12})^{\ell_{12}} \cdots x_{\alpha_{1,n}}^- (k_1)^{\ell_{1,n}} \cdots x_{\alpha_{n-1,n}}^- (k_{n-1,n})^{\ell_{n-1,n}} \left| \ell_{ij} \geq 0, k_{ij} \in \mathbb{Z} \right. \right\}
$$

is an “affine” Lyndon basis of $\mathcal{U}_{r,s}(\hat{n}^-)$.

**3.3** The following main theorem establishes the Drinfeld isomorphism between the two-parameter quantum affine algebra $U_{r,s}(\hat{\mathfrak{sl}}_n)$ (in Definition 2.1) and the $(r, s)$-analogue of Drinfeld quantum affinization of $U_{r,s}(\mathfrak{sl}_n)$ (in Definition 3.1), which affords the two-parameter Drinfeld realization of $U_{r,s}(\hat{\mathfrak{sl}}_n)$ as we required.

**Theorem 3.12. (Drinfeld Isomorphism)** For Lie algebra $\mathfrak{sl}_n$ with $n > 2$, let $\theta = \alpha_{1n}$ be the maximal positive root. Then there exists an algebra isomorphism $\Psi : U_{r,s}(\hat{\mathfrak{sl}}_n) \rightarrow \mathcal{U}_{r,s}(\hat{\mathfrak{sl}}_n)$.
defined by: for each \( i \in I \),

\[
\begin{align*}
\omega_i & \mapsto \omega_i \\
\omega'_i & \mapsto \omega'_i \\
\omega_0 & \mapsto \gamma'^{-1} \omega_0^{-1} \\
\omega'_0 & \mapsto \gamma^{-1} \omega'_0^{-1} \\
\gamma^{\pm \frac{1}{2}} & \mapsto \gamma^{\pm \frac{1}{2}} \\
\gamma'^{\pm \frac{1}{2}} & \mapsto \gamma'^{\pm \frac{1}{2}} \\
D^{\pm 1} & \mapsto D^{\pm 1} \\
D'_{\pm 1} & \mapsto D'_{\pm 1} \\
e_i & \mapsto x^+_i(0) \\
f_i & \mapsto x^-_i(0) \\
e_0 & \mapsto x^-_{\alpha_1n}(1) \cdot (\gamma'^{-1} \omega_0^{-1}) = x^-_0(1) \cdot (\gamma'^{-1} \omega_0^{-1}) \\
f_0 & \mapsto (\gamma^{-1} \omega_0^{-1}) \cdot x^+_{\alpha_1n}(-1) = \tau(x^+_{\alpha_1n}(1) \cdot (\gamma'^{-1} \omega_0^{-1}))
\end{align*}
\]

where \( \omega_0 = \omega_1 \cdots \omega_{n-1} \), \( \omega'_0 = \omega'_1 \cdots \omega'_{n-1} \).

Since the Lusztig’s symmetry of the braid group for the two-parameter cases is no more available when the rank of \( \mathfrak{g} \) is bigger than 2 (see [BGH1, Section 3]). This means the Beck’s approach using the extended braid group actions (see [B2]) to prove the Drinfeld Isomorphism Theorem is not yet valid for the two-parameter cases at least in a slightly big size of rank. Our treatment in the next section in fact develops a valid and interesting algorithm, which, as the reader will see, is a successful application of the combinatorial approach of the quantum “affine” Lyndon basis based on the Drinfeld generators we introduced above. In some sense, our method also affords another new combinatorial proof via the quantum “affine” Lyndon basis even in one-parameter setting.

### 4. Proof of the Drinfeld Isomorphism Theorem

#### 4.1 Let \( E_i, F_i, \omega_i, \omega'_i \) denote the images of \( e_i, f_i, \omega_i, \omega'_i \) \((i \in I_0)\) in the algebra \( U_{r,s}(\hat{sl}_n) \) under the mapping \( \Psi \), respectively.

Denote by \( U'_{r,s}(\hat{sl}_n) \) the subalgebra of \( U_{r,s}(\hat{sl}_n) \) generated by \( E_i, F_i, \omega_i^{\pm 1}, \omega'_i^{\pm 1} \) \((i \in I_0)\), \( \gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1} \) and \( D'_{\pm 1} \), that is,

\[
U'_{r,s}(\hat{sl}_n) := \left\langle E_i, F_i, \omega_i^{\pm 1}, \omega'_i^{\pm 1}, \gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1}, D'_{\pm 1} \mid i \in I_0 \right\rangle.
\]

Thereby, to prove the Drinfeld Isomorphism Theorem (Theorem 3.12) is equivalent to prove the following three Theorems:

**Theorem 4.1.** \( \Psi : U_{r,s}(\hat{sl}_n) \longrightarrow U'_{r,s}(\hat{sl}_n) \) is surjective.

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Theorem 4.2. $U'_{r,s}(\widehat{sl}_n) = U_{r,s}(\widehat{sl}_n)$.

Theorem 4.3. $\Psi : U_{r,s}(\widehat{sl}_n) \longrightarrow U_{r,s}(\widehat{sl}_n)$ is injective.

4.2 Proof of Theorem 4.1. We shall check that the elements $E_i$, $F_i$, $\omega_i$, $\omega'_i$ ($i \in I_0$), $\gamma^{\pm \frac{1}{2}}$, $\gamma'^{\pm \frac{1}{2}}$, $D^\pm, D'^\pm$ satisfy the defining relations of (A1)–(A7) of $U_{r,s}(\widehat{sl}_n)$. At first, the defining relations of $U_{r,s}(\widehat{sl}_n)$ imply that $E_i$, $F_i$, $\omega_i$, $\omega'_i$ ($i \in I$) generate a subalgebra $U_{r,s}(\widehat{sl}_n)$ of $U_{r,s}(\widehat{sl}_n)$, which is isomorphic to $U_{r,s}(\widehat{sl}_n)$. So we are left to check the relations involving $i = 0$.

Obviously, the relations of (A1) hold, according to the defining relations of $U_{r,s}(\widehat{sl}_n)$.

For (A2): we just check the following three relations involving $i = 0$, the remaining relations in (A2) are parallel to check. Using (D4), we get

$$DE_0D^{-1} = D x_{\theta}^- (1) D^{-1} : (\gamma'^{-1} \omega_{\theta}^{-1}) = r E_0.$$  

For $0 \leq j < n$, noting that $\langle \omega_{\theta}^{-1}, \omega_j \rangle = \langle \gamma'^{-1} \omega_{\theta}^{-1}, \omega_j \rangle = \langle \omega'_0, \omega_j \rangle$ (by Proposition 2.4), we have

$$\omega_j E_0 \omega_j^{-1} = \omega_j x_{\theta}^- (1) (\gamma'^{-1} \omega_{\theta}^{-1}) \omega_j^{-1} = \langle \omega'_{n-1}, \omega_j \rangle \cdots \langle \omega_1, \omega_j \rangle^{-1} E_0 = \langle \omega'_1, \omega_j \rangle E_0.$$  

For $0 \leq i < n$,

$$\omega_0 E_i \omega_0^{-1} = (\gamma'^{-1} \omega_{\theta}^{-1}) E_i (\gamma' \omega_{\theta}) = \omega_{\theta}^{-1} E_i \omega_{\theta},$$

when $i \neq 0$, since $\langle \omega'_i, \omega_0 \rangle^{-1} = \langle \omega'_i, \omega_0 \rangle$ (by Proposition 2.4), we obtain

$$\omega_{\theta}^{-1} E_i \omega_{\theta} = \omega_{\theta}^{-1} x_{\theta}^+ (0) \omega_{\theta} = \langle \omega'_i, \omega_0 \rangle^{-1} x_{\theta}^+ (0) = \langle \omega'_i, \omega_0 \rangle E_i;$$

and when $i = 0$, since $\langle \omega'_0, \omega_{\theta}^{-1} \rangle^{-1} = \langle \omega'_0, \omega_0 \rangle$ (by Proposition 2.4), we have

$$\omega_{\theta}^{-1} E_0 \omega_{\theta} = \omega_{\theta}^{-1} x_{\theta}^- (1) (\gamma'^{-1} \omega_{\theta}^{-1}) \omega_{\theta} = \langle \omega'_0, \omega_0 \rangle E_0.$$  

Similarly, one can verify the relations in (A3).

For (A4): at first, when $i \neq 0$, we see that

$$[E_0, F_i] = [x_{\theta}^- (1) \cdot (\gamma'^{-1} \omega_{\theta}^{-1}), x_{\theta}^- (0)] = -[x_{\theta}^- (0), x_{\theta}^- (1)]_{\omega'_i, \omega_{\theta}} (\gamma'^{-1} \omega_{\theta}^{-1}).$$

According to the corresponding cross relations held in $U_{r,s}(\widehat{sl}_n)$, we claim the following crucial Lemma whose proof is technical.

**Lemma 4.4.** $\left[ x_{\theta}^- (0), x_{\theta}^- (1) \right]_{\omega'_i, \omega_{\theta}} = 0$, for $i \in I$.

**Proof.** (I) When $i = 1$, $\langle \omega'_1, \omega_0 \rangle = \langle \omega'_2, \omega_1 \rangle = s$, and $\langle \omega'_1, \omega_{\theta} \rangle = s^{-1}$. By (3.8) & (3.9), we have

$$[x_{\theta}^- (0), x_{\alpha_{13}}^- (1)]_{s^{-1}} = [x_{\theta}^- (0), [x_{\theta}^- (0), x_{\theta}^- (1)]_{(2,1)}]_{s^{-1}}$$  

$$= -\left( (1,2)^{-1} (2,1) \right)^{\frac{1}{2}} [x_{\theta}^- (0), [x_{\theta}^- (0), x_{\theta}^- (1)]_{(1,2)}]_{s^{-1}}$$  

$$= - (r s)^{\frac{1}{2}} [x_{\theta}^- (0), x_{\theta}^- (0), x_{\theta}^- (1)]_{(r-1, s^{-1})} = 0.$$  

(by 3.8)
Hence, repeatedly using (3.3), we have

\[
\begin{align*}
[x_i^-(0), x_{α_{1,i}}(1)]_{s-1} &= \left[\left[x_i^-(0), x_{n-1}^-(0)\right], x_{α_{1,n-1}}^-\right]_s (0 \text{ by (D9}_1) \\
&\quad + \left[x_{n-1}^-(0), \left[x_i^-(0), x_{α_{1,n-1}}^-\right]_{s-1}\right]_s \quad \text{(by (3.3))} \\
&= \left[x_{n-1}^-(0), \left[x_i^-(0), x_{α_{1,n-1}}^-\right]_{s-1}\right]_s \\
&= \cdots \quad \text{(inductively using (3.3) & (D9}_1)) \\
&= \left[x_{n-1}^-(0), x_{n-2}^-(0), \cdots, \left[x_i^-(0), x_{α_{i}}^-\right]_{s-1}\right]_{s, \ldots, s} \\
&= 0.
\end{align*}
\]

(II) When \(i = n - 1\), \(\langle \omega'_n, \omega'_0 \rangle = r^{-1}\), that is, \(\langle \omega'_n, \omega'_0 \rangle = r\). By (3.3), (3.9) & (D9), we have

\[
\begin{align*}
[x_{n-1}^-(0), x_{α_{1,n}}^-]_r &\quad \text{(by definition)} \\
&= \left[x_{n-1}^-(0), \left[x_{n-1}^-(0), x_{n-2}^-(0), x_{α_{1,n-2}}^-\right]_s\right]_r \quad \text{(using (3.3))} \\
&= \left[x_{n-1}^-(0), \left[x_{n-1}^-(0), x_{n-2}^-(0)\right]_s, x_{α_{1,n-2}}^-\right]_r \quad \text{(this term using (3.3))} \\
&\quad + s \left[x_{n-1}^-(0), \left[x_{n-1}^-(0), x_{n-2}^-(0)\right]_s, x_{α_{1,n-2}}^-\right]_r \quad (0 \text{ by (3.5), (D9}_1) \\
&= \left[s \left[x_{n-1}^-(0), \left[x_{n-1}^-(0), x_{n-2}^-(0)\right]_s, x_{α_{1,n-2}}^-\right]_r \quad \text{(this term= 0 by (3.9))} \\
&\quad + r \left[x_{n-1}^-(0), x_{n-2}^-(0)\right]_s, x_{α_{1,n-2}}^-\right]_r \quad (0 \text{ by (3.5), (D9}_1) \\
&= 0.
\end{align*}
\]

(III) When \(1 < i < n - 1\), \(\langle \omega'_i, \omega'_0 \rangle = 1\), that is, \(\langle \omega'_i, \omega'_0 \rangle = 1\). In order to derive the required result, we first need to make two claims below:

**Claim (A):** \(\left[x_i^-(0), x_{α_{1,i+1}}^-\right]_{\omega'_i, ω'_1,i+1} = \left[x_i^-(0), x_{α_{1,i+1}}^-\right]_r = 0, \text{ for } i \geq 2\).

In fact, by (3.3), (3.9) & (D9), we have

\[
\begin{align*}
[x_i^-(0), x_{α_{1,i+1}}^-]_r &\quad \text{(by definition)} \\
&= \left[x_i^-(0), \left[x_i^-(0), x_{i-1}^-(0), x_{α_{1,i+1}}^-\right]_s\right]_r \quad \text{(using (3.3))} \\
&= \left[x_i^-(0), \left[x_i^-(0), x_{i-1}^-(0)\right]_s, x_{α_{1,i+1}}^-\right]_r \quad \text{(this term using (3.3))} \\
&\quad + s \left[x_i^-(0), \left[x_i^-(0), x_{i-1}^-(0)\right]_s, x_{α_{1,i+1}}^-\right]_r \quad (0 \text{ by (3.5), (D9}_1) \\
&= \left[s \left[x_i^-(0), \left[x_i^-(0), x_{i-1}^-(0)\right]_s, x_{α_{1,i+1}}^-\right]_r \quad \text{(0 by (3.9))} \\
&\quad + r \left[x_i^-(0), x_{i-1}^-(0)\right]_s, x_{α_{1,i+1}}^-\right]_r \quad (0 \text{ by (3.5), (D9}_1) \\
&= 0.
\end{align*}
\]

**Claim (B):** \(\left[x_i^-(0), x_{α_{1,i+2}}^-\right]_{\omega'_i, ω'_1,i+2} = \left[x_i^-(0), x_{α_{1,i+2}}^-\right] = 0 \text{ (i \geq 2), if } r \neq -s.\)
At first, by definition, we note that \( [b, a]_u = -u [a, b]_{u-1} \). Therefore, we get
\[
[x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{(s,r-1)} = -s [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{(s-1,r-1)} \quad \text{(by (3.6))}
\]
\[
= -s [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{(r-1,s-1)} \quad \text{(by (3.9))}
\]
\[
= 0.
\]

We then consider the following deduction
\[
[x_i^-(0), x_{i+1}^-]_{r-1} = [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{r-1} \quad \text{(by (3.3))}
\]
\[
= [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{r-1} \quad \text{(using (3.3))}
\]
\[
+ s [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{r-1} \quad \text{(this term = 0 by (3.5), (D9))}
\]
\[
= [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{r-1} \quad \text{(this term = 0 by the above)}
\]
\[
+ r^{-1} [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{r-1} \quad \text{(using (3.4))}
\]
\[
= r^{-1} [x_i^-(0), x_{i+1}^-(0), x_i^-(0), x_{i+1}^-(0)]_{r-1} + [x_{i+1}^-(1), x_i^-(0)]_{r-1}
\]
\[
= [x_{i+1}^-(1), x_i^-(0)]_{r-1} \quad \text{(1st term = 0 by Claim (A))}
\]

Expanding both sides of the above equation according to definition, we easily get
\[
(1 + r^{-1}s) [x_i^-(0), x_{i+1}^-] = 0.
\]

Thus the required result is obtained under the assumption.

Next applying (3.5), we can get
\[
[x_i^-(0), x_{i}^-] = [x_{i-1}^-(0), \ldots, x_{i+2}^-(0), x_i^-(0), x_{i+1}^-(0)]_{(s,\ldots,s)}
\]
\[
= 0. \quad \text{(by Claim (B))}
\]

We complete the proof of Lemma 4.4. \(\square\)

Next, we turn to check the relation below, whose argument is crucial to our verification on compatibilities of the defining relations system of \( U_{s,t}(s) \) mentioned in Remark 3.3.

**Proposition 4.5.** \( [E_0, F_0] = \frac{\omega_0 - \omega_1'}{r-s} \).

**Proof.** Using (D1) & (D5), we have
\[
[E_0, F_0] = [x_{i+1}^-(1) \gamma'^{-1} \omega_0^{-1}, \gamma^{-1} \omega_0'^{-1} x_{i+1}^- (-1)]
\]
\[
= [x_{i+1}^-(1), x_{i+1}^- (-1)] \cdot (\gamma'^{-1} \omega_0^{-1} \omega_0'^{-1}).
\]

Note that for \( j \geq 1 \), we have
\[
[x_{j+1}^-(0), \omega_j']_s = (r-s) \omega_j x_{j+1}^-(0), \quad [x_{j+1}^-(0), \omega_j]_s = 0,
\]
\[
[\omega_j' X, x_{j+1}^+(k)]_r = \omega_j' [X, x_{j+1}^+(k)],
\]
\[
[x_j^+(k), Y\omega_j+1]_r = [x_j^+(k), Y] \omega_j+1.
\]
So (4.2) implies that there hold

\begin{align}\label{eq:4.5}  
\left[ x_{j+1}^-(0), [ x_j^-(0), x_j^+(0) ] \right]_{s} &= \omega_j' x_{j+1}^-(0), \\
\left[ x_2^+(0), [ x_1^-(1), x_1^+(1) ] \right]_{s} &= \gamma \omega_1' x_2^+(0), \\
\left[ [ x_{j+1}^+(0), x_{j+1}^+(0) ], x_j^-(0) \right]_{s} &= -x_j^-(0) \omega_{j+1}.  
\end{align}

Now let us write briefly

\[
\left[ x_1^+(-1), x_2^+(0), \cdots, x_{i-1}^+(0) \right]_{(r,\cdots,r)} := \left[ [ \cdots [ x_1^+(-1), x_2^+(0) ]_r, \cdots, x_{i-1}^+(0) ]_r. 
\]

Thus, by (3.5), we have

\[
\left[ x_{\alpha_{i1}}^-(1), x_{\alpha_{i1}}^+(1) \right] = \left[ x_{\alpha_{i1}}^-(1), [ x_{\alpha_{i1}}^+(1), x_2^+(0), \cdots, x_{i-1}^+(0) ]_{(r,\cdots,r)} \right] \\
= \left[ [ x_{\alpha_{i1}}^-(1), x_{\alpha_{i1}}^+(1) ], x_2^+(0), \cdots, x_{i-1}^+(0) ]_{(r,\cdots,r)} \\
+ \sum_{j=2}^{i-1} \left[ x_1^+(-1), x_2^+(0), \cdots, [ x_{\alpha_{i1}}^-(1), x_{\alpha_{i1}}^+(1) ], \cdots \right]_{(r,\cdots,r)}.
\]

(i) For \( j = 1 \), by (3.5), (D8) & (4.6), we have

\[
\left[ x_{\alpha_{i1}}^-(1), x_1^+(1) \right] = \left[ x_{\alpha_{i1}}^-(1), \cdots, [ x_2^+(0), [ x_1^-(1), x_1^+(1) ] ]_{(s,\cdots,s)} \right]_{(r,\cdots,r)} (\text{by } (4.6)) \\
= \gamma \omega_1' x_{\alpha_{i1}}^-(0), \quad (i > 2)
\]

so that

\[
M(i) := \left[ [ x_{\alpha_{i1}}^- (1), x_1^+ (-1) ], x_2^+(0), \cdots, x_{i-1}^+(0) \right]_{(r,\cdots,r)} \\
= \gamma \left[ [ \omega_1' x_{\alpha_{i1}}^- (0), x_2^+(0) ]_r, \cdots, x_{i-1}^+(0) \right]_{(r,\cdots,r)} \\
= \gamma \omega_1' \left[ [ x_{\alpha_{i1}}^- (0), x_2^+(0) ], \cdots, x_{i-1}^+(0) \right]_{(r,\cdots,r)} \\
= \gamma \omega_1' \left[ [ \omega_2' x_{\alpha_{i1}}^- (0), x_3^+(0) ], \cdots, x_{i-1}^+(0) \right]_{(r,\cdots,r)} \\
= \gamma \omega_1' \omega_2' \left[ [ x_{\alpha_{i1}}^- (0), x_3^+(0) ], \cdots, x_{i-1}^+(0) \right]_{(r,\cdots,r)} \\
= \cdots \\
= \gamma \omega_1' \cdots \omega_{i-2}' \left[ x_{i-1}^-(0), x_{i-1}^+ (0) \right] \\
= \gamma \omega_1' \cdots \omega_{i-2}' \frac{\omega_{i-1}' - \omega_{i-1}}{r - s}, \quad (i > 2)
\]

where we used the following identities, respectively

\[
\left[ \omega_{j-1}' x_{\alpha_{j1}}^- (0), x_j^+(0) \right]_r = \omega_{j-1}' \left[ x_{\alpha_{j1}}^- (0), x_j^+(0) \right], \quad (\text{by } (4.3)) \\
\left[ x_{\alpha_{j1}}^- (0), x_j^+(0) \right] = \omega_j' x_{\alpha_{j1+1}, s}^- (0), \quad (\text{by } (3.13) & (4.5)).
\]
(ii) For \( j = i - 1 \), again by (3.5), (3.3) & (4.7), we get

\[
[x_{\alpha_1}(1), x_{i-1}^+(0)] = \left[ [x_{i-1}^-(0), x_{i-1}^+(0)], [x_{i-2}^-(0), x_{\alpha_1,i-2}(1)] \right]_s \quad \text{(by (3.3))}
\]

\[
= \left[ \left[ [x_{i-1}^-(0), x_{i-1}^+(0)], x_{i-2}^-(0) \right] \right]_s, x_{\alpha_1,i-2}(1) \quad \text{(by (4.7))}
\]

\[
+ s \left[ x_{i-2}^-(0), \left[ [x_{i-1}^-(0), x_{i-1}^+(0)], x_{\alpha_1,i-2}(1) \right] \right] = 0
\]

where we notice that (*): \( \left[ [x_{i-1}^-(0), x_{i-1}^+(0)], x_{\alpha_1,i-2}(1) \right] = 0 \).

Thereby, we further obtain

\[
N(i) := [x_1^-(1), x_2^+(0), \ldots, [x_{\alpha_1,i}^-(1), x_{i-1}^+(0)], \ldots]_{(r, \ldots, r)}
\]

\[
= -[x_{\alpha_1,i-1}^+(1), x_{\alpha_1,i-1}(1) \omega_{i-1}] \quad \text{(by (4.4))}
\]

\[
= -[x_{\alpha_1,i-1}^-(1), x_{\alpha_1,i-1}(1)] \omega_{i-1}
\]

\[
= [x_{\alpha_1,i-1}^+(1), x_{\alpha_1,i-1}^-(1)] \omega_{i-1}
\]

(iii) For \( 1 < j < i - 1 \), by (3.5), (3.3), (4.7) & (D91), we obtain

\[
[x_{\alpha_1}(1), x_j^+(0)] = [x_{j-1}^-(0), \ldots, [x_j^-(0), x_j^+(0)], [x_{j-1}^-(0), x_{\alpha_1,j}(1)] \right]_s (s, \ldots, s)
\]

\[
(\text{by (3.3))}
\]

\[
= [x_{j-1}^-(0), \ldots, [x_j^-(0), x_j^+(0)], x_{j-1}^-(0)]_s, x_{\alpha_1,j-1}(1) \right]_{(s, \ldots, s)}
\]

\[
(= 0 \text{ by (*)})
\]

\[
= -[x_j^-(0), \ldots, x_{j+1}^-(0), [x_{j-1}^-(0) \omega_j, x_{\alpha_1,j-1}(1)]_s]_{(s, \ldots, s)}
\]

\[
= -[x_j^-(0), \ldots, x_{j+1}^-(0), x_{\alpha_1,j}(1) \omega_j]_s]_{(s, \ldots, s)}
\]

\[
= -[x_j^-(0), \ldots, x_{j+1}^-(0), x_{\alpha_1,j}(1)] \omega_j (s, \ldots, s) \quad \text{(by (3.5), (D91))}
\]

\[
= 0
\]

where in forth and fifth equality “=” we used the following identities, respectively

\[
[x_{j-1}^-(0) \omega_j, x_{\alpha_1,j-1}(1)]_s = [x_{j-1}^-(0), x_{\alpha_1,j-1}(1)]_s \omega_j = x_{\alpha_1,j}(1) \omega_j,
\]

\[
[x_{j+1}^-(0), x_{\alpha_1,j}(1)] \omega_j]_s = [x_{j+1}^-(0), x_{\alpha_1,j}(1)] \omega_j.
\]
As a result of (i), (ii) & (iii), (4.8) becomes

\[
\left[ x_{α_{11}}^-(1), x_{α_{11}}^+(1) \right] = M(i) + N(i)
= M(i) + \left[ x_{α_{11},i}^-(1), x_{α_{11},i}^+(1) \right] \omega_{i-1}
= M(i) + M(i-1) \omega_{i-1} + \left[ x_{α_{11},i-2}^-(1), x_{α_{11},i-2}^+(1) \right] \omega_{i-2} \omega_{i-1}
= \cdots
= M(i) + M(i-1) \omega_{i-1} + M(i-2) \omega_{i-2} \omega_{i-1} + \cdots
+ M(3) \omega_3 \cdots \omega_{i-1} + \left[ x_{α_{12}}^-(1), x_{α_{12}}^+(1) \right] \omega_2 \cdots \omega_{i-1}
\]

\[
\gamma \omega_{α_{11}}' - γ' \omega_{α_{11}}, \quad (i > 1),
\]

where we used (D8) to get \( \left[ x_{α_{12}}^-(1), x_{α_{12}}^+(1) \right] = \gamma \omega_{1} - γ' \omega_{1} \).

Therefore, by (4.9), (4.1) takes the required formula:

\[
\left[ E_0, F_0 \right] = \frac{γ' - 1 \omega_{θ} - γ^{-1} \omega_{θ}'}{r - s}.
\]

The proof of Proposition 4.5 is completed. □

For (A5): we need only to verify that \( [E_0, E_j] = 0 \) and \( [F_0, F_j] = 0 \) for \( 1 < j < n - 1 \). Actually, in the proof of Proposition 4.5, the fact that \( \left[ x_{α_{11}}^-(1), x_{α_{11}}^+(0) \right] = 0 \) for \( 1 < j < i - 1 \) implies the first identity (taking \( i = n \)) since \( \left[ E_0, E_j \right] = \left[ x_{θ}^-(1), x_{θ}^+(0) \right] \cdot γ' - 1 \omega_{θ}^{-1} \). The second can be obtained utilizing \( τ \) on the first one. □

For (A6): when \( i \cdot j \neq 0 \), (D9a) implies that the corresponding generators satisfy exactly those \((r,s)\)-Serre relations in \( U_{r,s}(sl_n) \), so it is enough to check the \((r,s)\)-Serre relations involving \( i \cdot j = 0 \).

**Lemma 4.6.** (1) \( E_0 E_1^2 - (r + s) E_1 E_0 E_1 + (r s) E_1^2 E_0 = 0 \),
(2) \( E_0^2 E_1 - (r + s) E_0 E_1 E_0 + (r s) E_1 E_0^2 = 0 \),
(3) \( E_{n-1}^2 E_0 - (r + s) E_{n-1} E_0 E_{n-1} + (r s) E_0 E_{n-1}^2 = 0 \),
(4) \( E_{n-1} E_0^2 - (r + s) E_0 E_{n-1} E_0 + (r s) E_0^2 E_{n-1} = 0 \),
(5) \( F_1^2 F_0 - (r + s) F_1 F_0 F_1 + (r s) F_0 F_1^2 = 0 \),
(6) \( F_0^2 F_1 - (r + s) F_0 F_1 F_0 + (r s) F_0^2 F_1 = 0 \),
(7) \( F_0 F_{n-1}^2 - (r + s) F_{n-1} F_0 F_{n-1} + (r s) F_{n-1}^2 F_0 = 0 \),
(8) \( F_0^2 F_{n-1} - (r + s) F_{n-1} F_0 F_{n-1} + (r s) F_{n-1} F_0^2 = 0 \).

**Proof.** The proofs for the relations of (5)—(8) follow from taking \( τ \) on the first four relations (1)—(4). We shall demonstrate the first two \((r,s)\)-Serre relations, the third and forth ones are similar to the first two relations (1) & (2), which are left to the reader.
we have

\[ E_0 E_1^2 - (r + s) E_1 E_0 E_1 + (rs) E_1^2 E_0 \]

\[ = (rs) \left( E_1^2 x_\theta^-(1) - (1 + r^{-1}s) E_1 x_\theta^-(1) E_1 + (r^{-1}s) x_\theta^-(1) E_1^2 \right) (\gamma'^{-1} \omega^{-1}) \]

\[ = (rs) \left[ E_1, \left[ E_1, x_\theta^-(1) \right] \right]_{r^{-1}s} (\gamma'^{-1} \omega^{-1}) \quad \text{(by (3.6))} \]

\[ = -\gamma^{-1}(r^{-1}s)^{-1} \frac{1}{2}(rs) \left[ x_\theta^+(0), \omega_1 x_{\alpha_{2n}}^-(1) \right]_{r^{-1}s} (\gamma'^{-1} \omega^{-1}) \]

\[ = -\gamma^{-1}(r^{-1}s)^{-1} \frac{1}{2} \omega_1 \left[ x_\theta^+(0), x_{\alpha_{2n}}^-(1) \right] (\gamma'^{-1} \omega^{-1}) \]

\[ = 0. \quad \text{(by (3.5), (D8))} \]

(2) Using the formula of \( [E_1, x_\theta^-(1)] \) derived in (1) above, we have

\[ E_0 E_1^2 - (r + s) E_0 E_1 E_0 + (rs) E_1 E_0^2 \]

\[ = (rs) \left[ E_0, \left[ E_0, x_\theta^-(1), E_1 \right] \right]_{(1, r, s^{-1})(\gamma'^{-2} \omega^{-2})} \]

\[ = (rs^3)^{\frac{1}{2}} \gamma^{-1} \left[ x_\theta^-(1), \omega_1 x_{\alpha_{2n}}^-(1) \right]_{rs^{-1}} (\gamma'^{-2} \omega^{-2}) \]

\[ = (rs)^{\frac{1}{2}} \gamma^{-1} \omega_1 \left[ E_0, x_{\alpha_{2n}}^-(1) \right]_{s^{-1}} (\gamma'^{-2} \omega^{-2}) \]

\[ = -\left( rs^3 \right)^{\frac{1}{2}} \gamma^{-1} \omega_1 \left[ x_{\alpha_{2n}}^-(1), x_\theta^-(1) \right]_{s^{-1}} (\gamma'^{-2} \omega^{-2}) \]

\[ = 0, \quad \text{(by Claim (C) below)} \]

where we used the following claim:

**Claim (C):** \( [x_{\alpha_{2n}}^-(1), x_{\alpha_{2n}}^-(1)] \) = 0, for \( n > 2 \) and \( r \neq -s \).

The argument for Claim (C) is technical. Indeed, by induction on \( n \), we have:

When \( n = 3 \), by (3.8), one gets

\[ [x_\theta^-(1), x_{\alpha_{13}}^-(1)]_s = [x_\theta^-(1), [x_\theta^-(0), x_\theta^-(1)]_s]_s \quad \text{(by (3.8))} \]

\[ = -(rs)^{\frac{1}{2}} \left[ x_\theta^-(1), [x_\theta^-(0), x_\theta^-(1)]_{r^{-1}} \right]_s \]

\[ = (rs^{-1})^{-\frac{1}{2}} \left[ x_\theta^-(1), x_\theta^-(1), x_\theta^-(0) \right]_{(r,s)} \]

\[ = (rs^{-1})^{-\frac{1}{2}} \left[ x_\theta^-(1), x_\theta^-(1), x_\theta^-(0) \right]_{(s,r)} \quad \text{(by (3.6))} \]

\[ = 0, \quad \text{(by (3.9))} \]

which is exactly the \( (r, s) \)-Serre relation (see (3.9)).

While for \( n > 3 \), we first notice the fact:

\[ [x_{n-1}^-(0), x_{\alpha_{2n}}^-(1)]_{(\omega_{n-1}, \omega_{2n})} = [x_{n-1}^-(0), x_{\alpha_{2n}}^-(1)]_r = 0, \quad \text{for } n > 3, \]
which can be proved in terms of the same method of the proof of (II) in Lemma 4.4. We thus have
\[
[x_{\alpha_{2n}}^{-1}(1), x_{\alpha_{2n+1}}^{-1}(1)] = \left[\left[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} \right] \right]_r (by (3.4))
\]
\[
+ \left[ \left[ \left[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} \right] \right] \right]_s (0 by Claim (A))
\]
\[
= \left[ \left[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} \right] \right]_r = \left[ \left[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} \right] \right]_s (by (3.3))
\]
\[
+ s^{-1}[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} ] (2nd summand = 0 using induction hypothesis)
\]
\[
= \left[ \left[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} \right] \right]_r (3.3)
\]
\[
= \left[ \left[ x_{\alpha_{2n}-1}(1), x_{\alpha_{2n+1}^{-1}(1)} \right] \right]_s (0 by (4.10))
\]
\[
- r s^{-1}[ x_{\alpha_{2n}}^{-1}(1), x_{\alpha_{2n+1}^{-1}(1)} ] (by definition)
\]
\[
= - r s^{-1}[ x_{\alpha_{2n}}^{-1}(1), x_{\alpha_{2n+1}^{-1}(1)} ] (r - 1 s^2)
\]
By definition, expanding both sides of the above identity gives us
\[
(1 + r s^{-1}) x_{\alpha_{2n}}^{-1}(1) x_{\alpha_{2n+1}^{-1}(1)} = (r + s) x_{\alpha_{2n}}^{-1}(1) x_{\alpha_{2n+1}^{-1}(1)}
\]
which means
\[
[x_{\alpha_{2n}}^{-1}(1), x_{\alpha_{2n+1}^{-1}(1)}] = 0,
\]
under the assumption \( r \neq -s \).

For (A7): the verification is analogous to that of (A6).

4.3. Proof of Theorem 4.2. We shall show that the algebra \( \mathcal{U}_{r,s}(\hat{sl}_n) \) is generated by
\( E_i, F_i, \omega_i^{\pm 1}, \omega_i, \gamma^{\pm \frac{1}{2}}, \gamma', \gamma'' \), and \( D_i \) (\( i \in I_0 \)).

To this end, we need to prove the following results.

**Lemma 4.7.**

1.\( x^{-1}_1(1) = [E_2, E_3, \cdots, E_{n-1}, E_0]_{(r, \cdots, r)} \gamma^{-1} \omega_1^1 \in \mathcal{U}_{r,s}(\hat{sl}_n), \) then for any
\( i \in I, \ x^{-1}_i(1) \in \mathcal{U}_{r,s}(\hat{sl}_n). \)

2.\( x^{-1}_1(-1) = r \left( [E_2, E_3, \cdots, E_{n-1}, E_0]_{(r, \cdots, r)} \gamma^{-1} \omega_1 \right) = \gamma^{-1} \omega_1 \left( E_0, F_{n-1}, \cdots, F_3, F_2 \right) \in \mathcal{U}_{r,s}(\hat{sl}_n), \) then for any \( i \in I, \ x^{-1}_i(-1) \in \mathcal{U}_{r,s}(\hat{sl}_n). \)

**Proof.**

1. Set \( \tilde{E}(i) = x_{\alpha_{i+1}}^{-1}(1) \omega_{i+1} \cdots \omega_n \gamma^{-1} \omega_1 \) for \( i \geq 1, \) where \( \tilde{E}(n-1) = E_0. \)

Observing that \( [x^+_i(0), x_{\alpha_{i+1}}^{-1}(1)] = x_{\alpha_{i+1}}(1) \omega_i \) in the proof (see, case (ii)) of Proposition 4.5, we get an important recursive relation:
\[
[x^+_i(0), x_{\alpha_{i+1}}^{-1}(1)] = x_{\alpha_{i+1}}^{-1}(1) \omega_{i+1} \cdots \omega_n \gamma^{-1} \omega_1
\]
(4.11)
Recursively using the above relations, we obtain

\[
x^+_1(1) = \tilde{E}(1)\gamma^1\omega_1 = \begin{bmatrix} E_2, \tilde{E}(2) \end{bmatrix} \gamma^1\omega_1 = \cdots
\]
\[
= \begin{bmatrix} E_2, \cdots, E_{n-1}, \tilde{E}(n-1) \end{bmatrix}_{(r,\cdots,r)} \gamma^1\omega_1
\]
\[
= \begin{bmatrix} E_2, \cdots, E_{n-1}, E_0 \end{bmatrix}_{(r,\cdots,r)} \gamma^1\omega_1
\]
\[
\in U'_{r,s}(\mathfrak{sl}_n).
\]

(4.12)

Now suppose that we already have \(x^+_i(1) \in U'_{r,s}(\mathfrak{sl}_n)\) for \(i \geq 1\). Notice that
\[
x^-_{i+1}(1) = (rs) \left[ \begin{bmatrix} x^+_i(0), x^-_i(0) \end{bmatrix}, x^-_{i+1}(1) \right]_{(r-1)} \omega_i^{-1}
\]
\[
= (rs) \left[ x^+_i(0), x^-_i(0), x^-_{i+1}(1) \right]_{(r-1,1)} \omega_i^{-1}
\]
(4.13)
\[
= -(rs)^{1/2} \left[ x^+_i(0), x^-_{i+1}(0), x^-_i(1) \right]_{(s,1)} \omega_i^{-1}
\]
\[
= (rs)^{1/2} \left[ \left[ F_{i+1}, x^-_i(1) \right], E_i \right] \omega_i^{-1}
\]
\[
\in U'_{r,s}(\mathfrak{sl}_n),
\]

which gives rise to the recursive construction of some basic quantum real root vectors of level 1. Hence, we obtain the required result.

(2) Set \(\tilde{F}(i) = \tau(\tilde{E}(i)) = \gamma^{-1} \omega_i^{-1} \omega_{i-1} \cdots \omega_{i+1} x^-_{i+1}(-1)\) for \(i \geq 1\), where \(\tilde{F}(n-1) = F_0\). Applying \(\tau\) to (4.11), we see that \(\left[ \tilde{F}(i), F_i \right]_s = \tilde{F}(i-1)\) and \(\tilde{F}(1) = \gamma^{-1} \omega_1^{-1} x^+_1(-1)\), which implies the first claim.

The remaining claim follows from the fact that
\[
x^+_{i+1}(-1) = \tau(x^-_{i+1}(1)) = (rs)^{1/2} \omega_i^{-1} \left[ F_i, x^+_i(-1), E_{i+1} \right]_{r}
\]
(4.14)
\[
\in U'_{r,s}(\mathfrak{sl}_n).
\]

We observe that Lemma 4.7, together with (4.12), (4.13) & (4.14), gives the construction of the Drinfeld generators of level 1. Furthermore, the first conclusion of the following Lemma gives the the construction of the quantum imaginary root vectors of any level (\(\neq 0\)), while the second gives the construction of some basic quantum real root vectors of any level. Actually, as a result of Definition 3.9 and Lemma 4.8 below, this approach also gives the construction of all quantum real root vectors of any level.

**Lemma 4.8.**

1. \(a_i(\ell) \in U'_{r,s}(\mathfrak{sl}_n), \) for \(\ell \in \mathbb{Z}\setminus\{0\}\).
2. \(x^+_i(k) \in U'_{r,s}(\mathfrak{sl}_n), \) for \(k \in \mathbb{Z}\).

**Proof.**

(1) At first, it follows from (D8) that
\[
a_i(1) = \omega_i^{-1} \gamma^{1/2} \left[ x^+_1(0), x^-_1(1) \right] \in U'_{r,s}(\mathfrak{sl}_n),
\]
(4.15)
\[
a_i(-1) = \omega_i^{-1} \gamma^{1/2} \left[ x^+_1(-1), x^-_1(0) \right] = \tau(a_i(1)) \in U'_{r,s}(\mathfrak{sl}_n).
\]

Suppose that we have already \(a_i(\pm \ell') \in U'_{r,s}(\mathfrak{sl}_n)\) for all \(\ell' \leq \ell\) and some \(\ell \geq 1\). Now using (D6\(_n\)) & (D8), we have the following expansion (in fact, the expansions of both sides are
the same which also show the compatibility between (D6) and (D8) for \( n = 1, 2 \):

\[
\mathcal{U}'_{r,s}(\hat{s}_n) \ni \left[ x_i^+ (0), \left[ a_i (\ell), x_i^- (1) \right] \right] \\
= \left[ \left[ x_i^+ (0), a_i (\ell) \right], x_i^- (1) \right] + \left[ a_i (\ell), \left[ x_i^+ (0), x_i^- (1) \right] \right] \\
= * \gamma \frac{\ell}{2} \left[ x_i^+ (\ell), x_i^- (1) \right] \\
+ \left[ a_i (\ell), \gamma^{-\frac{\ell}{2}} \omega_i a_i (1) \right] \quad \text{(this term= 0 by (D2))} \\
= *(\gamma^r)^{-\frac{\ell}{2}} \gamma^{-\frac{\ell}{2}} \omega_i \left[ a_i (\ell+1) + \sum_{1 \leq p < \ell+1} s' (r-s)^{p-1} a_i (\ell_j) \cdots a_i (\ell_p) \right],
\]

where scalars \(*, s' \in \mathbb{K}\{0\}\). So \( a_i (\ell+1) \in \mathcal{U}'_{r,s}(\hat{s}_n) \).

Applying \( \tau \) on the above formula, we can get \( a_i (- (\ell+1)) \in \mathcal{U}'_{r,s}(\hat{s}_n) \). Therefore, \( a_i (\ell) \in \mathcal{U}'_{r,s}(\hat{s}_n) \), for any \( \ell \in \mathbb{Z}\{0\} \).

(2) follows from (D6) (setting \( i = j \) and \( k = 0 \)), together with (1).

4.4 Proof of Theorem 4.3. From 4.2 & 4.3, we actually get an algebra epimorphism \( \Psi : U_{r,s}(\hat{s}_n) \longrightarrow U_{r,s}(\hat{s}_n) \), since they have the essentially same generators system enjoying with the defining relations from the former.

Notice that both algebras \( U_{r,s}(\hat{s}_n) \) and \( U_{r,s}(\hat{s}_n) \) have commonly a natural Q-gradation structure (see Corollary 2.8), which is by definition preserved evidently under \( \Psi \). On the other hand, both toral subalgebras \( U_{r,s}(\hat{s}_n)^0 \) and \( U_{r,s}(\hat{s}_n)^0 \) generated by the same generator system of groups-like elements

\[ \{ \omega_i^{\pm 1}, \omega_i^{\pm 1} (i \in I_0), \gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1}, D'^{\pm 1} \} \]

are obviously isomorphic with respect to \( \Psi^0 := \Psi|_{U_{r,s}(\hat{s}_n)} \).

Assigned to the positive or negative nilpotent Lie subalgebra \( \hat{n}^{\pm} \) of \( \hat{s}_n \), there are two subalgebras \( U_{r,s}(\hat{n}^{\pm}) \) and \( U_{r,s}(\hat{n}^{\pm}) \), which are both generated by \( \hat{n}^{\pm} \) in \( U_{r,s}(\hat{s}_n) \) and \( U_{r,s}(\hat{s}_n) \) respectively. If denote \( \Psi^{\pm} := \Psi|_{U_{r,s}(\hat{n}^{\pm})} \), the fact that the double structure of \( U_{r,s}(\hat{s}_n) \) in Theorem 2.5 implies its triangular decomposition structure \( U_{r,s}(\hat{n}^-) \otimes U_{r,s}(\hat{s}_n)^0 \otimes U_{r,s}(\hat{n}^+) \) (by Corollary 2.7) then indicates \( \Psi \) has a corresponding decomposition \( \Psi^- \otimes \Psi^0 \otimes \Psi^+ \). That means we are left to show \( \Psi^{\pm} \) are isomorphic. It suffices to consider the epimorphism \( \Psi^{\pm} : U_{r,s}(\hat{n}^+) \longrightarrow U_{r,s}(\hat{n}^+) \).

Observe that \( U_{r,s}(\hat{n}^+) \) (resp. \( U_{r,s}(\hat{n}^+) \)) is generated by elements \( e_i \) (resp. \( E_i \)) for \( i \in I_0 \) and subject to \((r, s)\)-Serre relations (A5) & (A6). To check that \( \Psi^{\pm} \) is an isomorphism, now we fix \( r = q \) and specialize \( s \) at \( q^{-1} \) as follows.

Let \( A \subset \mathbb{Q}(r, s) \) be the localization of ring \( \mathbb{Q}[r^{\pm 1}, s^{\pm 1}] \) at the maximal ideal \((r s - 1)\). Let \( U_{r,s}^+ \) be the \( A \)-subalgebra of \( U_{r,s}(\hat{n}^+) \) generated by \( e_i \) (\( i \in I_0 \)). Let \((r s - 1)U_{r,s}^+ \) be the left ideal generated by \((r s - 1)\) in \( U_{r,s}^+ \). Define the algebra \( U_q^+ \), the specialization of \( U_{r,s}(\hat{n}^+) \) at \( q^{-1} \), by \( U_q^+ = U_{r,s}^+/ (r s - 1)U_{r,s}^+ \). Obviously, \( U_q^+ \cong U_q(\hat{n}^+) \), the usual one-parameter quantum subalgebra
of $U_q(\widehat{sl}_n)$. However, in this case, $\Psi^+$ induces the isomorphism $\Psi^+: U_q(\widehat{n}^+) \rightarrow U_q(\widehat{n}^+)$ given by the Drinfeld isomorphism in one-parameter case (see [B2] or [J2]).

Since specialization doesn’t change the root multiplicities, $\Psi^+: U_{r,s}(\widehat{n}^+) \rightarrow U_{r,s}(\widehat{n}^+)$ is an isomorphism.

From 4.2—4.4, we finally have established the Drinfeld isomorphism in our two-parameter case.

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