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ASYMPTOTIC RESULTS FOR THE SEMI-MARKOVIAN
RANDOM WALK WITH DELAY

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Abstract

In this study, the semi-Markovian random walk with a discrete interference of chance \( X(t) \) is considered and under some weak assumptions the ergodicity of this process is discussed. Characteristic function of the ergodic distribution of \( X(t) \) is expressed by means of the probability characteristics of the boundary functionals \( (N,S_N) \). Some exact formulas for first and second moments of ergodic distribution of the process \( X(t) \) are obtained when the random variable \( \xi - s \), which is describing a discrete interference of chance, has Gamma distribution on the interval \([0, \infty)\) with parameter \((\alpha, \lambda)\). Based on these results, the asymptotic expansions with three terms for the first two moments of the ergodic distribution of the process \( X(t) \) are obtained, as \( \lambda \rightarrow 0 \).
1. Introduction

It is known that numerous interesting problems of reliability, queuing, inventory theories, mathematical insurance, financial mathematics, mathematical biology and physics are given in terms of random walks with various types of barriers. Some important studies exist on this topic in the literature (see, for example, Afanaseva L. G., Bulinskaya E. V. [1]; Alsmeyer G. [2]; Aras G., Woodroofe M. [3]; Borovkov A. A. [4]; Brown M., Solomon H. A. [5]; Kemperman J. [10]; Khaniyev T.A. and et al. [11-16]; Lotov V. I. [17]; Nasirova T. I. [18]; Rogozin B. A. [19]; Skorohod A. V., Slobodenyuk, N.P. [20]; Spitzer F. [21]; Wolfgang P., Jorg B. [24], etc.). In these studies, absorbing, delaying, reflecting and other known barriers are usually used. However, some theoretical and practical problems are expressed by the process with a discrete interference of chance, as in the banking systems or insurance companies’ working (see, for example [9], [21], [22]).

In the present study, semi-Markovian random walk with a discrete interference of chance is considered. This model can be described as follows:

Suppose that, a system is in state \( z \in (s, \infty) \) at the initial epoch \( t = 0 \). Signals of demands and supplies are included to the system at the random epochs \( T_n = \sum_{i=1}^{n} \xi_i, \ n \geq 1 \). The system passes from one state to another by jumping in the moments \( T_n \), according to quantities of demands and supplies \( \{\eta_i\}, \ n \geq 1 \). This “natural variation” of system continues until certain random moment \( \tau_1 \), where \( \tau_1 \) is the first passage time to the control level \( s > 0 \) which is defined before. The system is stopped at level \( s > 0 \). Then, as a consequence of external interference, the system is brought from the control level \( s > 0 \) to the state process having delay random \( \theta_1 \) and \( \zeta_1 \), which is a positive random variable having a certain distribution in the interval \( [s, \infty) \) and expresses a discrete interference of chance. Thus, the first period has been completed. Afterwards, the system will continue its function similar to the preceding period. Under some weak assumptions, after a long time this process will have stationary probability characteristics. Investigation of these characteristics is important, from the point of view of the theory of stochastic processes and their applications. It is obvious that these characteristics depend on the one’s of the random variables \( \zeta_1 \) and \( \eta_1 \).
2. Mathematical Construction of the Process X(t)

Let \( (\xi_n, \eta_n, \theta_n, \zeta_n) \), \( n \geq 1 \), be a sequence of independent and identically distributed triples of random variables defined on any probability space \( (\Omega, \mathcal{F}, P) \), such that \( \xi_i \) and \( \theta_i \) are non-negative, \( \eta_i \)'s take on negative values as well as positive one's, \( \zeta_i \)'s are concentrated in the interval \([s, \infty)\), and \( s \) is any fixed positive number. Suppose \( \xi_i, \eta_i, \theta_i, \zeta_i \) are independent random variables and their distribution functions are known.

Define renewal sequence \( \{T_n\} \) and random walk \( \{S_n\} \) as follows:

\[
T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = S_0 = 0, \quad n = 1, 2, \ldots.
\]

and a sequence of integer valued random variables \( \{N_n\} \) as:

\[
N_0 = 0, \quad N_1 = N(x) = \inf \{n \geq 1 : S_n \geq x\}, \quad x = z - s; z \geq s;
\]

\[
N_{n+1} = \inf \{k \geq N_n + 1 : \zeta_n - S_k + S_{N_n} < s\}, \quad n = 1, 2, \ldots, \quad \inf \{\emptyset\} = +\infty \text{ is stipulated.}
\]

Put \( \tau_n = T_{N_n}, \gamma_n = T_n + \theta_n, \quad n \geq 0 \) and define the \( v(t) \) as:

\[
v(t) = \max \{n \geq 0 : T_n \leq t\}.
\]

We can now construct the desired stochastic process \( X(t) \) as follows:

\[
X(t) = \max \{k, \zeta_n - S_{v(t)} + S_{N_n}\}, \quad \text{if } \gamma_n \leq t < \gamma_{n+1}, \quad n = 0, 1, 2, \ldots, \zeta_0 = z \in [s, \infty).
\]

The process \( X(t) \) is called “The semi-Markovian random walk with a discrete interference of chance with delay”’. The main purpose of this study is to investigate the asymptotic expansions as \( \lambda \to 0 \), for the first two moments of the ergodic distribution of the process \( X(t) \), when the random variable \( \zeta_1 - s \) has Gamma distribution with parameters \( (\alpha, \lambda) \), \( \alpha > 0, \lambda > 0 \).

Analogically the problem was studied in [14], for the Gaussian random walk with two barriers, when \( \theta_n = 0 \) and \( P\{\zeta_n = a\} = 1 \), and in [15], when the random variable \( \zeta_1 - s \) has a third-order Erlang distribution.

3. Preliminary Discussions

We will consider the random walk \( \{S_n\} \), \( n \geq 1 \), with initial state \( S_0 = 0 \). Let \( N_1 \) be a first moment of the exit of the random walk \( \{S_n\} \), \( n \geq 1 \) from interval \([s, \infty)\), i.e. \( N_1 = N(x) = \min \{n \geq 1 : S_n \geq x\}, \quad x = z - s \geq 0 \).

Let \( v_1^+ = \min \{n \geq 1 : S_n > 0\}, \quad \chi_1^+ = S_{v_1^+} \).

Note that, the random variables \( v_1^+ \) and \( \chi_1^+ \) are called the first strict ascending ladder moment and height of the random walk \( \{S_n\} \), \( n \geq 0 \), respectively (see, Feller W., [7], p.391).
Let $H(x) = \min \left\{ n \geq 1 : \sum_{i=1}^{n} \chi_{i}^{+} \geq x \right\}$, $x \geq 0$.

Note that $H(x)$ is a renewal process, which is generated by means of the positive valued random variables $\chi_{n}^{+}$, $n \geq 1$. It can be shown that in this case

$$N(x) = \sum_{i=1}^{H(x)} \nu_{i}^{+} \quad \text{and} \quad S_{N(x)} = \sum_{i=1}^{H(x)} \chi_{i}^{+}.$$

First state the following proposition on the ergodicity of $X(t)$, as $t \to \infty$.

**Lemma 3.1.** Let the initial sequence of random variables $\{\xi_{n}, \eta_{n}, \theta_{n}, \varsigma_{n}\}$, $n \geq 1$, satisfy the following supplementary conditions:

1) $\sum_{i=1}^{\infty} \nu_{i}^{+} < \infty$; 2) $\sum_{i=1}^{\infty} \theta_{i} < \infty$; 3) $\sum_{i=1}^{\infty} \eta_{i} > 0$; 4) $\eta_{i}$ is a non-arithmetic random variable.

Then the process $X(t)$ is ergodic.

**Proof.** We remind that the process $X(t)$ belongs to a wide class of processes which are known in literature as the class of Semi-Markov processes with a discrete interference of chance. Moreover, the ergodic theorem of the type of Smith’s “key renewal theorem” exists in the literature for the processes with a discrete chance interference (see, [8], p.243). Consequently, the proof of Lemma 3.1 can be extracted from this general ergodic theorem. ◊

**Remark 3.1.** Let’s put $\phi_{X}(u) \equiv \lim_{t \to \infty} E \{\exp(iuX(t))\} , \quad u \in \mathbb{R}$. Using the basic identity for random walks (see, [7], p.600), we can now state the following Lemma 3.2.

**Lemma 3.2.** Under the assumptions of Lemma 3.1, the characteristic function $\phi_{X}(u)$ of the ergodic distribution of the process $X(t)$ can be expressed by means of the probability characteristics of the pair $(N(x), S_{N(x)})$ and random variable $\eta_{1}$ as follows:

$$\phi_{X}(u) = \frac{1}{EN(\varsigma_{1} - s) + K} \left\{ \int_{\varsigma_{1}}^{\infty} e^{iu \varphi_{S_{N(x)}}}(z) dz - 1 \right\} + \int_{\varsigma_{1}}^{\infty} e^{iu \varphi_{S_{N(x)}}}(u) - 1\pi(z) + K \int_{\varsigma_{1}}^{\infty} e^{iu \varphi_{S_{N(x)}}}(z) - u\pi(z) \right\}$$

where $\varphi_{S_{N(x)}}(u) = E \exp(iuS_{N(x)})$, $\varphi_{\eta}(u) = E \exp(iu \eta_{1})$, $u \in \mathbb{R}$; $\pi(z) = P\{\varsigma_{1} \leq z\}$, $K = \frac{\Theta_{i}}{E\varsigma_{1}}$.

### 4. Main Results

In this section we will investigate the asymptotic expansion of expectation $E(\mathcal{X})$ of the ergodic distribution of the process $X(t)$, as $\lambda \to 0$.

Now we introduce the following notations:
\[ m_k = \mathbb{E}(\eta_1^k), \quad M_k(x) = \mathbb{E}(S_{N(x)}^k), \quad k = 1, 2, 3, \quad x > 0. \]

and for the shortness of expressions we put
\[ m_{k1} = \frac{m_k}{m_1}, \quad M_{k1}(x) = \frac{M_k(x)}{M_1(x)}, \quad k = 2, 3; \quad \mathbb{E}(X^k) = \lim_{t \to \infty} \mathbb{E}(X^k(t)), \quad k = 1, 2. \]

Moreover, assume that \( \bar{X}(t) = X(t) - s \).

We can now get exact formulas for the first two moments of the ergodic distribution of \( X(t) \).

**Theorem 4.1.** Let the initial random variables \( \xi_1, \eta_1, \theta_1, \zeta_1 \) satisfy the following supplementary conditions:

1) \( E\xi_1 < \infty \); 2) \( E\theta_1 < \infty \); 3) \( E\eta_1 > 0 \) and \( E[\eta_1^2] < \infty \);

4) \( \eta_1 \) is a non-arithmetic random variable;

5) random variable \( \zeta_1 - s \) has a Gamma distribution with parameter \( (\alpha, \lambda) \).

Then the first and second moments of the ergodic distribution of the process \( X(t) \) exist and can be expressed by means of the characteristics of the boundary functional \( S_{N(x)} \) and random variable \( \eta_1 \) as follows:

\[
\mathbb{E}(\bar{X}) = \frac{1}{\mathbb{E}(M_1(\zeta_1)) + Km_1 \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha}} \left\{ \mathbb{E}(\zeta_1 M_1(\zeta_1)) - \frac{1}{2} \mathbb{E}(M_2(\zeta_1)) + \frac{m_{21}}{2} \mathbb{E}(M_1(\zeta_1)) + \left( \frac{m_{21}}{2} - Km_1 \right) E(M_1(\zeta_1)) + Km_1 \cdot \frac{\Gamma(\alpha + 1)}{\lambda^{\alpha+1}} \right\}, \tag{4.1}
\]

\[
\mathbb{E}(\bar{X}^2) = \frac{1}{\mathbb{E}(M_1(\zeta_1)) + Km_1 \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha}} \left\{ m_{21} \mathbb{E}(\zeta_1 M_1(\zeta_1)) - \frac{1}{2} m_{21} \mathbb{E}(M_2(\zeta_1)) + \mathbb{E}(\zeta_1^2 M_1(\zeta_1)) - \mathbb{E}(\zeta_1 M_2(\zeta_1)) + \frac{3m_{21}^2 - 2m_{21}}{6} \mathbb{E}(M_1(\zeta_1)) + \frac{1}{3} \mathbb{E}(M_3(\zeta_1)) + Km_1 [-2\mathbb{E}(\zeta_1 M_1(\zeta_1)) + \mathbb{E}(M_2(\zeta_1)) + \frac{\Gamma(\alpha + 2)}{\lambda^{\alpha+2}}] \right\}. \tag{4.2}
\]

**Proof.** From (3.1), we have:
\[
\varphi_X(u) = \frac{e^{iuX}}{\ln(\lambda) + K} \int_0^{\infty} x^{n-1} e^{-\lambda x} e^{iuX} \frac{\varphi_{\rho_n}(-u)}{\varphi_{\rho_n}(-u)} - 1 \, dx + K \int_0^{\infty} x^{n-1} e^{-\lambda x} e^{iuX} \varphi_{\rho_n}(-u) \, dx, \quad (4.3)
\]

where \( \ln(\alpha, \lambda) = \int_0^{\infty} x^{n-1} e^{-\lambda x} \ln(\alpha) \, dx. \)

Note that the conditions of Theorem 4.1 provide the existence and finiteness of the first three moments of \( S_{N(x)} \) (see, [7], p. 514). Therefore, Taylor expansions of the characteristic functions of variables \( \eta_1 \) and \( S_{N(x)} \) can be written as follows, as \( u \to 0 \):

\[
\varphi_{\rho_n}(-u) - 1 = -iu \ln_1 \left[ 1 - \frac{i u}{2} m_{21} + \frac{(iu)^2}{6} m_{31} + o(u^2) \right], \quad (4.4)
\]

\[
\varphi_{S_{N(x)}}(-u) - 1 = -iu M_1(x) \left[ 1 - \frac{i u}{2} M_{21}(x) + \frac{(iu)^2}{6} M_{31}(x) + o(u^2) \right]. \quad (4.5)
\]

From this, it is easily seen that under the conditions of Theorem 4.1 the first second moments of the ergodic distribution of the process \( X(t) \) exist. Substituting asymptotic expansions (4.4) and (4.5) in (4.3) after some calculations we can get the exact formulas for the \( E(X) \) and \( E(X^2) \).

This completes the proof of Theorem 4.1. \( \square \)

Now state some auxiliary Lemmas for the investigation of the asymptotic behavior of the moments \( E(X) \) and \( E(X^2) \):

**Lemma 4.1.** Let \( E(\chi_1^+) \times < \infty \). Then we can write the following asymptotic expansion for the first three moments of boundary functional \( S_{N(x)} \) as \( x \to \infty \):

\[
M_1(x) = x + \frac{1}{2} \mu_{21} + o \left( \frac{1}{x} \right); \quad (4.6)
\]

\[
M_2(x) = x^2 + x \mu_{21} + \frac{1}{3} \mu_{31} + o(1); \quad (4.7)
\]

\[
M_3(x) = x^3 + \frac{3}{2} \mu_{31} x^2 + \mu_{31} x + o(x), \quad (4.8)
\]

where \( \mu_{k1} = \frac{\mu_k}{\mu_1}, \quad \mu_k = E(\chi_1^+) \), \( k = 1, 2, 3 \);

the random variable \( \chi_1^+ \) is the first ascending ladder height of random walk \( \{S_n\}, n \geq 0 \).
Lemma 4.2. Let \( g(x) \) be a continuous function and \( \lim_{x \to \infty} g(x) = 0 \). Then for each \( \alpha > 1 \) the following relation is true: 
\[ \lim_{\lambda \to 0} \int_0^\infty t^{\alpha-1} e^{-t} g \left( \frac{t}{\lambda} \right) dt = 0. \]

Proof. From the condition of Lemma, for any \( \varepsilon > 0 \) exist \( m(\varepsilon) > 0 \), for any \( x > m(\varepsilon) \) holds \( |g(x)| < \varepsilon \). Choose \( b > 0 \), so that \( \int_0^b t^{\alpha-1} e^{-t} dt < \varepsilon \). Since \( g(x) \) continuous function, then for every finite interval \([0, b]\) the function \( g(x) \) is bounded, and for the \( \lambda < \frac{b}{m(\varepsilon)} \), we have:

\[
\left| \int_0^\infty t^{\alpha-1} e^{-t} g \left( \frac{t}{\lambda} \right) dt \right| \leq \int_0^b t^{\alpha-1} e^{-t} \left| g \left( \frac{t}{\lambda} \right) \right| dt + \int_b^\infty t^{\alpha-1} e^{-t} \left| g \left( \frac{t}{\lambda} \right) \right| dt \\
\leq \max_{[0, b]} |g(x)| \int_0^b t^{\alpha-1} e^{-t} dt + \varepsilon \int_b^\infty t^{\alpha-1} e^{-t} dt \leq \varepsilon M + \varepsilon \int_0^\infty t^{\alpha-1} e^{-t} dt = \varepsilon (M + \Gamma(\alpha)),
\]

where \( M = \max_{[0, b]} |g(x)| \).

Since \( M \) and \( \Gamma(\alpha) \) finite and \( \varepsilon > 0 \) are any number, then we obtain the statement of Lemma 4.2. \( \Diamond \)

Lemma 4.3. Under the conditions of Theorem 4.1 we can write the following asymptotic expansions, as \( \lambda \to 0 \):

1. \( E(M_1(\zeta_1)) = E(\zeta_1) + \frac{1}{2} \mu_{21} + o(\lambda) \),
2. \( E(\zeta_1 M_1(\zeta_1)) = E(\zeta_1^2) + \frac{1}{2} \mu_{21} E(\zeta_1) + o(1) \),
3. \( E(\zeta_1^2 M_1(\zeta_1)) = E(\zeta_1^3) + \frac{1}{2} \mu_{21} E(\zeta_1^2) + o \left( \frac{1}{\lambda} \right) \),
4. \( E(M_2(\zeta_1)) = E(\zeta_1^2) + \mu_{21} E(\zeta_1) + \frac{1}{3} \mu_{31} + o(1) \),
5. \( E(\zeta_1 M_2(\zeta_1)) = E(\zeta_1^3) + \mu_{21} E(\zeta_1^2) + \frac{1}{3} \mu_{31} E(\zeta_1) + o \left( \frac{1}{\lambda} \right) \),
6. \( E(M_3(\zeta_1)) = E(\zeta_1^3) + \frac{3}{2} \mu_{21} E(\zeta_1^2) + \mu_{31} E(\zeta_1) + o \left( \frac{1}{\lambda} \right) \).
Proof. By using Theorem 1 from paper [19] and Lemma 4.2 we may integrate the asymptotic expansions (4.6)–(4.8) for \( M_k(x) \), \( k = 1,2,3 \) and after some corresponding calculations, we get the asymptotic expansions 1–6 as \( \lambda \to 0 \).

We can now state the main result of this study as follows:

**Theorem 4.2.** Let the conditions of Theorem 4.1 be satisfied. Then the following asymptotic expansion can be written for the first and second moments of the ergodic distribution of the process \( X(t) \), for each \( \alpha > 1 \), as \( \lambda \to 0 \):

\[
E(\overline{X}) = \frac{\alpha + 1}{2} \frac{1}{\lambda} + \frac{m_{12}}{2} - \frac{\mu_{21} + 2Km_{1}}{4} \frac{\alpha + 1}{\alpha} + \\
+ \frac{(\mu_{21} + 2Km_{1})^2}{8} \frac{\alpha + 1}{\alpha^2} - \frac{1}{2} \left( \frac{\mu_{11}}{3} - Km_{1}(m_{21} - \mu_{21}) \right) \frac{1}{\alpha} \lambda + o(\lambda),
\]

\[
E(\overline{X}^2) = \frac{(\alpha + 1)(\alpha + 2)}{3} \frac{1}{\lambda^2} + \frac{m_{12}}{2} (\alpha + 1) - \frac{\mu_{21} + 2Km_{1}}{6} \frac{(\alpha + 1)(\alpha + 2)}{\alpha} \frac{1}{\lambda} + d_3 + o(1),
\]

where

\[
d_3 = \frac{\mu_{31}}{3} (\alpha + 1) - \frac{m_{12} (\mu_{21} + 2Km_{1})}{4} \frac{\alpha + 1}{\alpha} + \frac{(\mu_{21} + 2Km_{1})^2}{12} \frac{(\alpha + 1)(\alpha + 2)}{\alpha^2} + \frac{3m_{12}^2 - 2m_{31}}{6}.
\]

**Proof.** Substituting the asymptotic expansions 1–6, in exact formulas (4.1) and (4.2) for \( E(X) \) and \( E(X^2) \), consequently, after some calculations we get asymptotic expansions for the first and second moments of ergodic distribution of the process \( X(t) \), as \( \lambda \to 0 \).

5. Conclusions
In this study, we considered the semi-Markovian random walk with a discrete interference of chance and obtained the exact formulas and asymptotic expansions for the first two moments of the process \( X(t) \), whenever the random variable \( \zeta_1 \), which describes discrete interference of chance, has a Gamma distribution with parameter \( (\alpha,\lambda) \), as \( \lambda \to 0 \). Taking the second and third terms in the asymptotic expansions, in addition to the first term, allows us to approximate the exact expressions for the moments of \( X(t) \) by some approximation formulas that have sufficiently high accuracy. The evident and clear forms of the asymptotic expansions with three terms allowed us to observe how the initial random variables \( \xi_1, \eta_1, \zeta_1 \) and \( \theta_1 \) influence the stationary characteristics of the process.
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References