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COMMUTING GRAPHS OF MATRIX ALGEBRAS

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Abstract

The commuting graph of a ring $R$, denoted by $\Gamma(R)$, is a graph whose vertices are all non-central elements of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = yx$. The commuting graph of a group $G$, denoted by $\Gamma(G)$, is similarly defined. In this paper we investigate some graph theoretic properties of $\Gamma(M_n(F))$, where $F$ is a field and $n \geq 2$. Also we study the commuting graphs of some classical groups such as $GL_n(F)$ and $SL_n(F)$. We show that $\Gamma(M_n(F))$ is a connected graph if and only if every field extension of $F$ of degree $n$ contains a proper intermediate field. We prove that apart from finitely many fields, a similar result is true for $\Gamma(GL_n(F))$ and $\Gamma(SL_n(F))$. Also we show that for two fields $E$ and $F$ and integers $m, n \geq 2$, if $\Gamma(M_m(E)) \simeq \Gamma(M_n(F))$, then $m = n$ and $|E| = |F|$. 
1. Introduction

For a ring $R$, we denote the center of $R$ by $Z(R)$. If $X$ is either an element or a subset of $R$, then $C_R(X)$ denotes the centralizer of $X$ in $R$. For any non-commutative ring $R$, we associate a graph with the vertex set $R \setminus Z(R)$ and join two vertices $x$ and $y$ if and only if $x \neq y$ and $xy = yx$. This graph was introduced in [2] and is called the commuting graph of $R$ and denoted by $\Gamma(R)$. If $F$ is a field and $n$ is a natural number, then $M_n(F)$ denotes the ring of $n \times n$ matrices over $F$, $\text{GL}_n(F)$ and $\text{SL}_n(F)$ denote the group of all invertible matrices in $M_n(F)$ and the group of all matrices with determinant 1 in $M_n(F)$, respectively. In this article, we denote the finite field of order $q$ by $\mathbb{F}_q$. For any field $F$, we set $F^* = F \setminus \{0\}$, that is the multiplicative group of $F$. For any $i, j$, $1 \leq i, j \leq n$, we denote by $E_{ij}$, that element in $M_n(F)$ whose $(i, j)$-entry is 1 and whose other entries are 0. Also, $I$ and $I_r$ denote the identity matrix and the identity matrix of size $r$, respectively. For a matrix $A \in M_n(F)$, $F[A]$ denotes the $F$-subalgebra generated by $A$ and $I$. Also for any matrix $A \in M_n(F)$ and $\alpha \in F^n$, the $A$-annihilator of $\alpha$ is a polynomial with minimum degree, say $f(x)$, such that $f(A)\alpha = 0$. The matrix $A \in M_n(F)$ is said to be cyclic if there exists a vector $\alpha \in F^n$ such that $\{\alpha, A\alpha, \ldots, A^{n-1}\alpha\}$ is a basis for the vector space $F^n$ over $F$. It is easily checked that a matrix $A$ is cyclic if and only if the minimal and the characteristic polynomials of $A$ coincide. For any two matrices $A \in M_n(F)$ and $B \in M_m(F)$, we define

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in M_{n+m}(F).$$

In a graph $G$, a path $\mathcal{P}$ is a sequence of distinct vertices $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$ in which every two consecutive vertices are adjacent. The graph $G$ is called connected if for every pair of vertices $u$ and $v$ in $G$, there exists a path with endpoints $u$ and $v$. In this paper, we study the connectivity problem of the graphs $\Gamma(M_n(F))$, $\Gamma(\text{GL}_n(F))$, and $\Gamma(\text{SL}_n(F))$. Recently, the connectivity problem of the other subgraphs of matrix rings over division rings have been studied in [5, 6]. For further information about the connectivity of the commuting graphs of finite simple groups, see [14].

For a vertex $v$ of a graph $G$, the degree of $v$ is the cardinality of the set of edges incident with $v$. A subset $X$ of the vertices of a graph $G$ is called a dominating set if each vertex of $G \setminus X$ is adjacent to at least one vertex of $X$. The minimum size of a dominating set in a graph $G$ is called the domination number of $G$ and denoted by $\gamma(G)$. Also a subset $Y$ of the vertices of a graph is called an independent set if the induced subgraph on $Y$ has no edges. The maximum size of an independent set in a graph $G$ is called the independence number of $G$ and denoted by $\alpha(G)$. In [2] it has been proven that for any finite field $F$ and $n \geq 2$, the following hold.

(i) If $n < |F|$, then $\alpha(\Gamma(M_n(F))) \geq (|F|^2 + |F| + 1)|F|^{\frac{2}{|F| - 2}}$.

(ii) If $n \geq |F|$, then $\alpha(\Gamma(M_n(F))) \geq (|F|^2 + |F| + 1)|F|(|F| - 1)(n - |F|)^{-1}$. 

2
Here we improve the lower bound showing that if $F$ is a finite field and $n \geq 2$, then $\alpha(\Gamma(M_n(F))) \geq |F|^{(n-1)^2+1}(|F| - 1)^{n-2}$.

2. Connectivity of $\Gamma(M_n(F))$

In [2] it has been shown that for any field $F$, $\Gamma(M_2(F))$ is not connected and each of its connected components is a complete graph. If $F$ is finite, then the number of connected components of $\Gamma(M_2(F))$ is $|F|^2 + |F| + 1$ and each of them has $|F|^2 - |F|$ vertices. In this section we prove that for any field $F$ and integer $n \geq 3$, the graph $\Gamma(M_n(F))$ is connected if and only if every field extension of $F$ of degree $n$ contains a proper intermediate field. We first recall the following beautiful lemma due to Frobenius.

**Lemma A.** [9, p. 111] Let $F$ be a field and $n \geq 2$. Suppose that $A_1 \oplus \cdots \oplus A_k$ is the rational form of a matrix $A \in M_n(F)$. If for each $i$, $n_i$ is the size of the matrix $A_i$ and $n_1 \geq \cdots \geq n_k$, then $\dim_F C_{M_n(F)}(A) = n_1 + 3n_2 + \cdots + (2k - 1)n_k$.

**Corollary 1.** Let $F$ be a field and $n \geq 2$. The matrix $A \in M_n(F)$ is cyclic if and only if $C_{M_n(F)}(A) = F[A]$.

**Lemma 2.** Let $F$ be a field and $n \geq 3$. If there is a field extension of $F$ of degree $n$ with no proper intermediate fields, then $\Gamma(M_n(F))$ is not a connected graph.

**Proof.** Let $K$ be a field extension of $F$ with no proper intermediate fields and $\dim_F K = n$. Let $f(x) \in F[x]$ be the minimal polynomial of some element $a \in K \setminus F$ and let $A \in M_n(F)$ be the companion matrix of $f(x)$. Clearly, $K \simeq F[A]$. Since $A$ is a cyclic matrix, by Corollary 1 we have $C_{M_n(F)}(A) = F[A]$. Suppose that $B$ is a non-scalar matrix in $F[A]$. Since there is no proper intermediate fields between $F$ and $K$, we conclude that $F[B] = F[A]$. Thus $\dim_F F[B] = n$ and $B$ is a cyclic matrix and moreover $C_{M_n(F)}(B) = F[B]$. This implies that $F[A] \setminus FI$ is a connected component of $\Gamma(M_n(F))$. So the proof is complete.

The proof of the previous lemma concludes the following corollary.

**Corollary 3.** Let $F$ be a field and $n \geq 3$. Suppose $A$ is a non-scalar matrix in $M_n(F)$ such that $F[A]$ is a field extension of $F$ of degree $n$ with no proper intermediate fields. Then $F[A] \setminus FI$ is a connected component of $\Gamma(M_n(F))$.

In the next lemma we investigate the existence of a non-cyclic matrix in the $F$-algebra $F[A]$, where $A \in M_n(F)$.

**Lemma 4.** Let $F$ be a field and $n \geq 3$ and $A \in M_n(F) \setminus FI$. Suppose that $F[A]$ contains at least one proper intermediate field, if $F[A]$ is a field extension of $F$ of degree $n$. Then $F[A] \setminus FI$ contains a non-cyclic matrix.
Proof. First suppose that the minimal polynomial of $A$ is reducible. Let $f(x)$ is the minimal polynomial of $A$ and assume that $f(x) = g_1(x)g_2(x)$, for two coprime polynomials $g_1(x)$ and $g_2(x)$ of degrees at least 1. Using the primary decomposition theorem [8, p.220], we find an integer $k$ and two matrices $B_1 \in M_k(F)$ and $B_2 \in M_{n-k}(F)$ such that the minimal polynomials of $B_1, B_2$ are $g_1(x), g_2(x)$, respectively, and $A$ is conjugate to $B_1 \oplus B_2$. Since $n \geq 3$, without loss of generality we may assume that $k \geq 2$. Clearly, $A' = g_1(A)$ has the form $0 \oplus C$, for some $C \in M_{n-k}(F)$. Now, if $\varphi$ is the minimal polynomial of $C$, then $A'\varphi(A') = 0$. This implies that the degree of the minimal polynomial of $A'$ is at most $n-k+1$. Thus $A'$ is a non-cyclic element of $F[A] \setminus FI$. Moreover, if $f(x) = p(x)^r$ for some irreducible polynomial $p(x)$ and $r \geq 2$, then $p(A)^{r/2}$ is a non-cyclic element contained in $F[A] \setminus FI$.

Next assume that the minimal polynomial of $A$ is irreducible. So $F[A]$ is a field extension of $F$. If $\dim_F F[A] < n$, then $A$ is not cyclic and so we are done. So suppose that $\dim_F F[A] = n$. By the hypothesis, there exists a matrix $B \in F[A]$ such that $FI \subsetneq F[B] \subsetneq F[A]$. Since $\dim_F F[B] < n$, $B$ is a non-cyclic element of $F[A] \setminus FI$. The proof is complete. □

Lemma 5. Let $F$ be a field and $n \geq 3$. Then for every two non-cyclic matrices $X$ and $Y$ in $M_n(F)$, there exists a path with non-invertible intermediate vertices between $X$ and $Y$ in $\Gamma(M_n(F))$.

Proof. Assume that $A$ is a non-cyclic matrix in $M_n(F)$. Clearly, it is enough to show that there exists a path with non-invertible intermediate vertices between $A$ and $E_{11}$. By Lemma A, we have $\dim_F C_{M_n(F)}(A) \geq n + 1$. Let $\{C_1, \ldots, C_{n+1}\}$ be a linearly independent subset of $C_{M_n(F)}(A)$, where $C_1 = I$. There exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that the matrix $A_1 = \sum_{i=1}^{n+1} \lambda_i C_i$ is a non-zero matrix whose first row is 0. Suppose that

$$A_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ X & B \end{bmatrix}. $$

Let $g(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$ be the $B$-annihilator of $X$. First assume that $g(x)$ is not equal to the minimal polynomial of $B$. It is easy to check that both $A_1$ and $E_{11}$ commute with $A_2 = 0_1 \oplus g(B)$. Hence $A - A_1 - A_2 - E_{11}$ is a path in $\Gamma(M_n(F))$ with non-invertible intermediate vertices, as desired. Next suppose that $g(x)$ is the minimal polynomial of $B$. We have

$$g(A_1) = \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ Y & 0_{n-1} \end{bmatrix},$$

where $Y = (B^{m-1} + b_{m-1}B^{m-2} + \cdots + b_1 I)X$. Since $Y \neq 0$, we have $g(A_1) \neq 0$. Moreover there exists a non-zero matrix $B' \in M_{n-1}(F)$ such that $B'Y = 0$. Now obviously the matrix $A'_2 = 0_1 \oplus B'$ commutes with $g(A_1)$. Therefore $A - A_1 - g(A_1) - A'_2 - E_{11}$ is a path in $\Gamma(M_n(F))$ with non-invertible intermediate vertices. The proof of the lemma is complete. □
Now we are in a position to prove the main theorem of this section.

**Theorem 6.** Let $F$ be a field and $n \geq 3$. The graph $\Gamma(M_n(F))$ is connected if and only if every field extension of $F$ of degree $n$ contains at least a proper intermediate field.

**Proof.** By Lemma 2, one direction is clear. For other direction, consider two vertices $A$ and $B$ in $\Gamma(M_n(F))$. By Lemma 4, we find two non-cyclic matrices $A' \in C_{M_n(F)}(A)$ and $B' \in C_{M_n(F)}(B)$. Now, using Lemma 5, there exists a path between $A'$ and $B'$ in $\Gamma(M_n(F))$. This completes the proof. □

**Corollary 7.** Suppose that $F$ is a finite field and $n \geq 2$. The graph $\Gamma(M_n(F))$ is connected if and only if $n$ is not a prime number.

**Remark 8.** In [11] it has been shown that for any $n$, there is a field extension of $\mathbb{Q}$ of degree $n$ with no proper intermediate fields, where $\mathbb{Q}$ is the field of rational numbers. Thus for each $n \geq 2$, the graph $\Gamma(M_n(\mathbb{Q}))$ is not connected. Moreover, if $F$ is either the field of real numbers or an algebraically closed field, then for $n \geq 3$, $\Gamma(M_n(F))$ is a connected graph.

### 3. Connectivity in $\Gamma(GL_n(F))$

In this section we would like to determine under which conditions does there exist a path with invertible intermediate vertices between two vertices of $\Gamma(M_n(F))$. The following lemma is used in our proofs frequently.

**Lemma B.** [13, Theorem 27.5.1] Suppose that $A \in M_r(F)$ and $B \in M_s(F)$ are two matrices such that the minimal polynomials of $A$ and $B$ are coprime. Then

$$C_{M_n(F)}(A \oplus B) = \{X \oplus Y \mid X \in C_{M_r(F)}(A) \text{ and } Y \in C_{M_s(F)}(B), \text{ where } n = r + s\}.$$

**Theorem 9.** Let $F$ be a field and $n \geq 3$. Suppose that two vertices $X$ and $Y$ are contained in the same connected component of $\Gamma(M_n(F))$. Then there is a path with invertible intermediate vertices between $X$ and $Y$, unless $n - 1$ is a prime number and $F = \mathbb{F}_2$.

**Proof.** Clearly, it suffices to show that for any two non-adjacent vertices $X$ and $Y$, if $X \rightarrow A \rightarrow Y$ is a path in $\Gamma(M_n(F))$, then there are invertible matrices $P_1, \ldots, P_\ell$ such that $X \rightarrow P_1 \rightarrow \cdots \rightarrow P_\ell \rightarrow Y$ is a path in $\Gamma(M_n(F))$. If there is an invertible matrix $P \in F[A] \setminus FI$, then there is nothing to prove.

So assume that all invertible matrices in $F[A]$ are scalar matrices. This implies that $F[A]$ has no non-zero nilpotent elements. Therefore the minimal polynomial of $A$ is the product of some distinct irreducible polynomials. By the primary decomposition theorem [8, p. 220] and
with no loss of generality, we may write $A = A_1 \oplus \cdots \oplus A_r$, where the minimal polynomials of matrices $A_1, \ldots, A_r$ are irreducible and distinct. Hence $F[A_1], \ldots, F[A_r]$ are fields and $F[A] \simeq F[A_1] \times \cdots \times F[A_r]$. Since each unit of $F[A]$ is a scalar matrix, $F[A_1]^*, \ldots, F[A_r]^*$ are trivial groups. This yields that $F = \mathbb{F}_2$ and $A$ is an idempotent matrix. With no loss of generality suppose that $A = I_k \oplus 0_{n-k}$, for some $k$. Since $X$ and $Y$ commute with $A$, we have $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$, where $X_1, Y_1 \in M_k(F)$ and $X_2, Y_2 \in M_{n-k}(F)$. Assume that $n-1$ is not a prime number. To complete the proof, we consider two cases as follows.

**Case 1.** $2 \leq k \leq n-2$. We claim that for any matrix $S \in M_m(F)$ with $m \geq 3$, there is a matrix $S' \in M_m(F)$ which is not idempotent and $SS' = S'S$. If $S$ is not idempotent, then put $S' = S$. Suppose that $S$ is an idempotent matrix. There is an invertible matrix $Q$ such that $QSQ^{-1} = I_s \oplus 0_{m-s}$, where $s = \text{rank } S$. If $s \geq 2$, then put $S' = Q^{-1}E_{12}Q$; otherwise $m - s \geq 2$ and in this case set $S' = Q^{-1}E_{m,m-1}Q$. So the claim is proved.

Since $n-1$ is not a prime number, we have $n \geq 5$. If $k \geq 3$, then there are two matrices $X'_1 \in M_k(F)$ and $Y'_1 \in M_k(F)$ which are not idempotent and commute with $X_1$ and $Y_1$, respectively. Now since $n - k \geq 2$,

$$X - X'_1 \oplus I_{n-k} - (I_n + E_{n,n-1}) - Y'_1 \oplus I_{n-k} - Y$$

is a path in $\Gamma(M_n(F))$. Furthermore, since the matrices $X'_1 \oplus I_{n-k}$ and $Y'_1 \oplus I_{n-k}$ are not idempotent, by the first step of the proof there are two invertible matrices $P_1 \in F[X'_1 \oplus I_{n-k}] \setminus FI$ and $P_2 \in F[Y'_1 \oplus I_{n-k}] \setminus FI$. Thus $X - P_1 - (I_n + E_{n,n-1}) - P_2 - Y$ is a path with invertible intermediate vertices. If $k = 2$, then $n - k \geq 3$ and a similar argument works.

**Case 2.** $k = 1$ or $k = n-1$. First assume that $k = 1$. Since $XY \neq YX$, $X_2$ and $Y_2$ are not scalar matrices and so they are two vertices of $\Gamma(M_{n-1}(F))$. Note that $n-1$ is not a prime number and $F = \mathbb{F}_2$, so by Corollary 7 there is a path $X_2 - B_1 - \cdots - B_t - Y_2$ in $\Gamma(M_{n-1}(F))$.

Now,

$$X - \lambda_1 I_1 \oplus B_1 - \cdots - \lambda_i I_1 \oplus B_i - Y$$

is a path in $\Gamma(M_n(F))$, with $\lambda_i = 0$ if $B_i$ is an idempotent matrix of rank at least 2, and otherwise $\lambda_i = 1$. For an index $j$, if $B_j$ is idempotent, then $\text{rank } B_j \neq n-1$, since $B_j \neq I_{n-1}$. Thus we have $2 \leq \text{rank } (\lambda_j I_1 \oplus B_j) \leq n-2$. Therefore, by Case 1, we can replace the vertex $\lambda_j I_1 \oplus B_j$ in (*) with a path whose vertices are invertible matrix. Also if $B_j$ is not an idempotent matrix, then $F[\lambda_j I_1 \oplus B_j] \setminus FI$ contains an invertible matrix. Hence in this case we also find a path between $X$ and $Y$ whose intermediate vertices are invertible. The case $k = n-1$ is proven similarly. So the proof is complete.

In the next remark we will show that if $n-1$ is a prime number, then $\Gamma(GL_n(\mathbb{F}_2))$ is not connected.

**Corollary 10.** Suppose that $F \neq \mathbb{F}_2$ is a field and $n \geq 3$. The graph $\Gamma(GL_n(F))$ is connected if and only if every field extension of $F$ of degree $n$ contains a proper intermediate field.
Proof. By Theorems 6 and 9, one direction is clear. Assume that there is a field extension of \( F \) of degree \( n \) with no proper intermediate fields. Using Corollary 3, there is an invertible matrix \( A \) such that \( F[A] \setminus FI \) is a connected component of \( \Gamma(M_n(F)) \) and each matrix contained in \( F[A] \setminus FI \) is cyclic. Now, \( B = I + E_{12} \) is an invertible matrix which is not cyclic and there is no path between \( A \) and \( B \). So \( \Gamma(GL_n(F)) \) is not a connected graph. This completes the proof. □

Combining Theorem 6 and Corollary 10 and using [2, Theorem 2], we obtain the following result.

**Corollary 11.** Suppose that \( F \neq \mathbb{F}_2 \) is a field and \( n \geq 2 \). The graph \( \Gamma(GL_n(F)) \) is connected if and only if the graph \( \Gamma(M_n(F)) \) is connected.

4. Connectivity in \( \Gamma(SL_n(F)) \)

In this section we determine whether two vertices of \( \Gamma(M_n(F)) \) are connected by a path whose intermediate vertices have determinant 1.

**Lemma C.** [12, Corollary 16.4.b] Let \( F \) be a field and \( n \) be a natural number. For any matrix \( A \in GL_n(F) \), there exist matrix \( C \in SL_n(F) \) such that \( A^n = (\det A)C \).

**Theorem 12.** Let \( F \) be a finite field and \( n \geq 3 \). Suppose that either \( n - 1 \) is not a prime number or \( n \) is not divisible by \( |F| - 1 \). Then for any two vertices \( X \) and \( Y \) which are contained in the same connected component of \( \Gamma(M_n(F)) \), there is a path between \( X \) and \( Y \) whose intermediate vertices have determinant 1.

**Proof.** Clearly, it suffices to prove that for any two non-adjacent vertices \( X \) and \( Y \), if \( X \rightarrow A \rightarrow Y \) is a path in \( \Gamma(M_n(F)) \), then there are matrices \( U_1, \ldots, U_s \in SL_n(F) \) such that \( X \rightarrow U_1 \rightarrow \cdots \rightarrow U_s \rightarrow Y \) is a path in \( \Gamma(M_n(F)) \). If there is a non-zero nilpotent matrix \( N \in F[A] \), then \( I + N \in C_{M_n(F)}(\{X,Y\}) \) has determinant 1 and so there is nothing to prove. So we may assume that \( F[A] \) has no non-zero nilpotent element. Therefore the minimal polynomial of \( A \) is the product of some distinct irreducible polynomials. By the primary decomposition theorem [8, p. 220] and with no loss of generality, we may write \( A = A_1 \oplus \cdots \oplus A_k \), where the minimal polynomials of matrices \( A_1, \ldots, A_k \) are irreducible and distinct. Since \( X \) and \( Y \) commute with \( A \), by Lemma B we have \( X = X_1 \oplus \cdots \oplus X_k \) and \( Y = Y_1 \oplus \cdots \oplus Y_k \), where for \( i = 1, \ldots, k \), the matrices \( X_i \) and \( Y_i \) commute with \( A_i \). We consider the following two cases.

**Case 1.** With no loss of generality, assume that \( A_1 \) is not a scalar matrix. Since the minimal polynomial of \( A_1 \) is irreducible, \( F[A_1] \) is a finite field. Let \( r \) be the size of \( A_1 \) and \( m = \dim_F F[A_1] \). Therefore the order of the multiplicative cyclic group \( F(A_1)^*/F^* \) is \( |F|^{m-1} + \cdots + |F| + 1 \) which is more than \( m \). Thus there is a matrix \( B \in F[A_1] \) such that \( B^m \) is not a scalar.
matrix. Assume that $A_1 = A_0 \oplus \cdots \oplus A_0$ is the rational form of $A_1$, where $A_0$ is a cyclic matrix of size $m$. Since $B \in F[A_1]$, we can write $B = B_0 \oplus \cdots \oplus B_0$, for some $B_0 \in M_n(F)$. Now $B_0^m$ is not a scalar matrix, so using Lemma C we find a matrix $C_0 \in F[B_0] \setminus FI$ with determinant 1. Set $C = C_0 \oplus \cdots \oplus C_0$, where the number of $C_0$ and the number of $A_0$ appearing in the rational form of $A_1$ are the same. Now, since $C \in F[A_1]$, $C \oplus I_{n-r}$ is a matrix with determinant 1 in $C_{M_n(F)}(\{X,Y\})$. So in this case, we are done.

Case 2. Suppose that $A_1, \ldots, A_k$ are scalar matrices. If $k = n$, then since $A_i \neq A_j$, for $i \neq j$, $X$ and $Y$ are diagonal matrices and so $XY = YX$, a contradiction. Thus $k < n$ and without loss of generality we may assume that $r$, the size of $A_1$, is more than 1. If $F = \mathbb{F}_2$, then the result follows from Theorem 9. So assume that $F \neq \mathbb{F}_2$.

Assume first that $r \leq n - 2$. If there exists a matrix $P \in GL_r(F)$ such that $PX_1P^{-1}$ is diagonal, then define $X' = P^{-1} \text{diag}(\lambda^2, \lambda^{r-n-2}, 1, \ldots, 1)P \oplus \lambda I_{n-r}$, for some $\lambda \neq 0, 1$. Otherwise, $X_1$ is not diagonalizable and so by Case 1 there is a non-scalar matrix $Q \in SL_r(F)$ which commutes with $X_1$. In this case set $X' = Q \oplus I_{n-r}$. Similarly define a non-scalar matrix $Y' \in SL_n(F)$ correspondence to $Y_2 \oplus \cdots \oplus Y_k$ and note that $X \rightarrow X' \rightarrow Y' \rightarrow Y$ is a path whose intermediate vertices have determinant 1. So the assertion is proved.

Now, assume that $r = n - 1$. If $n$ is not divisible by $|F| - 1$, then there is a scalar $\mu \in F^*$ with $\mu^n \neq 1$. Therefore $X - \mu I_r \oplus \mu^{1-n} I_1 \rightarrow Y$ is a path in $\Gamma(M_n(F))$, as desired. So assume that $n - 1$ is not a prime number. Since $XY \neq YX$, the matrices $X_1$ and $Y_1$ are not scalars and so they are two vertices in $\Gamma(M_{n-1}(F))$. Then by Theorem 9 we find a path $X_1 \rightarrow P_1 \rightarrow \cdots \rightarrow P_t \rightarrow Y_1$ in $\Gamma(M_{n-1}(F))$, where $P_i \in GL_{n-1}(F)$ for $i = 1, \ldots, t$. Now, $X \rightarrow P_1 \oplus (\det P_1)^{-1} I_1 \rightarrow \cdots \rightarrow P_t \oplus (\det P_t)^{-1} I_1 \rightarrow Y$ is a path in $\Gamma(M_n(F))$ whose intermediate vertices have determinant 1. This completes the proof.

\[ \square \]

Remark 13. Note that the converse of the previous theorem is also true. Assume that $n - 1$ is a prime number and $|F| - 1$ divides $n$. Let $S_0 \in M_{n-1}(F)$ be a cyclic matrix such that its minimal polynomial is irreducible and put $S = S_0 \oplus (\det S_0)^{-1} I_1$. By Corollary 1 and Lemma B, every vertex adjacent to $S$ has the form $Z \oplus \nu I_1$, \quad (*)$

where $Z \in F[S_0]$ and $\nu \in F$. Suppose that $T$ is a vertex not adjacent to $S$ and $S \rightarrow Z_0 \oplus \nu_0 I_1 \rightarrow T$ is a path in $\Gamma(M_n(F))$. We show that $Z_0$ is a scalar matrix. Working towards a contradiction, assume that $Z_0 \not\in FI$. Since $F[S_0]$ is a field with prime degree over $F$ and $Z_0 \in F[S_0]$, we have $F[Z_0] = F[S_0]$. This implies that $Z_0$ is a cyclic matrix and therefore every vertex adjacent to $Z_0 \oplus \nu_0 I_1$ has the form ($*$). But this is a contradiction, since $T$ is not adjacent to $S$. Hence every path in $\Gamma(M_n(F))$ between $S$ and a vertex which is not adjacent to $S$, contains a vertex of the form $\alpha I_{n-1} \oplus \beta I_1$, for some $\alpha, \beta \in F$. Now, since the $n$th powers of all elements of $F^*$ are
equal to 1, every matrix of the form $\alpha I_{n-1} \oplus \beta I_1$ in $SL_n(F)$ is scalar. Hence there is no path between $S$ and $T$ whose intermediate vertices have determinant 1.

**Corollary 14.** Suppose that $F$ is a finite field and $n \geq 2$. The graph $\Gamma(SL_n(F))$ is not a connected graph if and only if at least one of the following cases occurs.

(i) $n$ is a prime number.

(ii) $n-1$ is a prime number and $|F|-1$ divides $n$.

**Proof.** Suppose first that for two vertices $X$ and $Y$ in $\Gamma(SL_n(F))$, there is no path between $X$ and $Y$ in $\Gamma(SL_n(F))$. If there is no path between $X$ and $Y$ in $\Gamma(M_n(F))$, then by Corollary 7, Case (i) occurs. Otherwise, by Theorem 12, Case (ii) occurs.

For the other direction, assume first that $n$ is a prime number. Let $K$ be a field extension of $F$ of degree $n$. By [3, Corollary 1], there is an element $a \in K \setminus F$ such that $N_{K/F}(a) = 1$. Consider the map $\mathcal{L}_a : K \to K$ defined by $\mathcal{L}_a(x) = ax$. Thus the matrix representation of $\mathcal{L}_a$, say $A$, is contained in $SL_n(F) \setminus FI$. Hence $A$ is a cyclic matrix with determinant 1 and $\dim_F F[A]$ is a prime number. Now, using Corollary 3, we conclude that $F[A] \cap SL_n(F) \setminus FI$ is a connected component of $\Gamma(SL_n(F))$ and so this graph is not connected. Moreover, if $n-1$ is a prime number and $|F|-1$ divides $n$, then by Remark 13 we conclude that $\Gamma(SL_n(F))$ is not connected. 

**Theorem 15.** Let $F \neq \mathbb{F}_2$ be a field and $n \geq 2$. For every matrix $A \in M_n(F)$, the set $F[A] \setminus FI$ contains a matrix with determinant 1, unless $F[A]$ is a field, $F[A]/F$ is purely inseparable, and $\dim_F F[A]$ divides $n$. Furthermore, if $F$ has a purely inseparable field extension and $\text{char } F$ divides $n$, then there exists a matrix $B \in M_n(F)$ such that none of the elements of $F[B] \setminus FI$ has determinant 1.

**Proof.** If there is a non-zero nilpotent matrix $N \in F[A]$, then $I + N$ has determinant 1 and so there is nothing to prove. Thus we may assume that $F[A]$ is a reduced ring. Therefore there are fields $K_1, \ldots, K_r$ such that $F[A] \simeq K_1 \times \cdots \times K_r$. If $r \geq 2$, then the element of $F[A]$ corresponding to the element $(\lambda, \lambda^{-1}, 1, \ldots, 1)$, for some $\lambda \in F \setminus \{0,1\}$, is a non-scalar matrix with determinant 1. So assume that $r = 1$. If $F$ is finite, then using [3, Corollary 1], we can find an element of $F[A] \setminus FI$ with determinant 1. Hence suppose that $F$ is infinite. If there exists a matrix $X \in F[A]$ such that $X^n$ is not a scalar matrix, then by Lemma C, there exists a matrix $C \in F[A] \setminus FI$ with determinant 1. So assume that $F[A]^*/F^*$ is an Abelian group whose exponent divides $n$. Now [1, Lemma 13] implies that $F[A]$ is a purely inseparable extension of $F$. Let $\text{char } F = p$. Since $F[A]/F$ is a purely inseparable field extension, there is an integer $s$ such that $\dim_F F[A] = p^s$. Therefore the order of $AF^*$ in the group $F[A]^*/F^*$ is $p^s$. This implies that $\dim_F F[A]$ divides $n$, as desired.
To complete the proof, suppose that $F$ has a purely inseparable field extension $K$ and $p = \text{char } F$ divides $n$. Without loss of generality, we may suppose that $\dim_F K = p$. Let $E$ be the subfield of $F$ generated by all algebraic elements over $F_p$. Suppose that $\omega \in K \setminus F$ and $x^p - t$ is the minimal polynomial of $\omega$ over $F$. Since $K/F$ is a purely inseparable field extension, $t$ is not an algebraic element over $E$. Let $S \subseteq F$ be a transcendental basis of $F$ over $E$ which contains $t$. Let $F_0 = E(S \setminus \{t\})$, so we have $F = F_0(t)$ and $\omega$ is transcendental over $F_0$. Now, let $T$ be the companion matrix of the polynomial $x^p - t$. Put $B = T \oplus \cdots \oplus T$, where the number of $T$ is $n/p$. We show that none of the elements of $F[B] \setminus FI$ has determinant 1.

Let $Y = \lambda_1 B^{p-1} + \cdots + \lambda_{p-1} B + \lambda_p I$ be an arbitrary element of $F[B] \setminus FI$. Also put $Y_0 = \lambda_1 T^{p-1} + \cdots + \lambda_{p-1} T + \lambda_p I$. Since $p = \text{char } F$ and $T^p = t$,

$$Y_0^p - (\lambda_1^p t^{p-1} + \cdots + \lambda_{p-1}^p t + \lambda_p^p) I = 0. \quad (*)$$

This implies that the minimal polynomial of $Y_0$ as a matrix over $F$ has the form $(x - \alpha)^k$, where $F$ is the algebraic closure of $F$ and $k \geq 2$. Thus the characteristic polynomial of $Y_0$ is $(x - \alpha)^p$.

Now, using $(*)$, we find that $x^p - (\lambda_1^p t^{p-1} + \cdots + \lambda_{p-1}^p t + \lambda_p^p)$ is the characteristic polynomial of $Y_0$. So if $\det Y = 1$, then $(\lambda_1^p t^{p-1} + \cdots + \lambda_{p-1}^p t + \lambda_p^p)^n = 1$ and $(\lambda_1 \omega^{p-1} + \cdots + \lambda_{p-1} \omega + \lambda_p)^n = 1$.

On the other hand $\lambda_1, \ldots, \lambda_p \in F = F_0(\omega^p)$. So the latter equation shows that $\omega$ is an algebraic element over $F_0$, a contradiction. Hence the proof is complete. \(\square\)

**Theorem 16.** Let $F$ be an infinite field and $n \geq 2$. Suppose that two non-adjacent vertices $A$ and $B$ are contained in the same connected component of $\Gamma(M_n(F))$. Also suppose that $F$ does not have a purely inseparable extension of degree $n$. Then there exists a path between $A$ and $B$ whose intermediate vertices have determinant 1.

**Proof.** If $F[A]$ is a field extension of $F$ of degree $n$ with no proper intermediate fields, then using Corollary 3, $F[A] \setminus FI$ is a connected component of $\Gamma(M_n(F))$. Therefore the two vertices $A$ and $B$ are adjacent, a contradiction. So we may assume that if $F[A]$ is a field of degree $n$, then there is at least one field between $F$ and $F[A]$. Clearly, it is sufficient to find a path between $A$ and $M = \lambda^{1-n} I_1 \oplus \lambda I_{n-1}$ whose intermediate vertices have determinant 1, where $\lambda$ is a scalar such that $\lambda^n \neq 0, 1$.

**Case 1.** If either $F[A]$ is not a field or $A$ is a non-cyclic matrix, then by Lemmas 4 and 5, there is a path between $A$ and $M$ whose intermediate vertices are non-invertible matrices. Now, using Theorem 15, we can replace intermediate vertices of this path with vertices which have determinant 1.

**Case 2.** Assume that $A$ is a cyclic matrix and $F[A]$ is a field of degree $n$. So there is a matrix $A_0 \in F[A]$ such that $F[A_0] \neq F[A]$ and $F[A_0]$ is a separable extension of $F$. Hence by Theorem 15, there is a matrix $A_1 \in F[A_0] \setminus FI$ with determinant 1. Now since $A_1$ is not a cyclic matrix, using Case 1 we find a path between $A_1$ and $M$ whose intermediate vertices have determinant 1. \(\square\)
Corollary 17. Suppose that $F$ is an infinite field and $n \geq 3$. If every field extension of $F$ of degree $n$ contains at least a proper intermediate field, then $\Gamma(SL_n(F))$ is a connected graph.

Proof. Using Case 1 in the proof of Theorem 16, it sufficient to show that any cyclic matrix in $SL_n(F)$ is adjacent to at least a non-cyclic matrix in $SL_n(F)$. Assume that $A \in SL_n(F)$ is a cyclic matrix. If $F[A]$ is not a field, then by Lemma 4, there is a non-cyclic matrix $B \in F[A] \setminus FI$ which is not invertible. Therefore by Theorem 15, there is a matrix $B_1 \in F[B] \setminus FI$ with determinant 1. Since $B_1 \in SL_n(F)$ is a non-cyclic matrix which commutes with $A$, we are done. Now, suppose that $F[A]$ is a field. Working towards a contradiction, assume that $F[A]$ is a purely inseparable extension of $F$. Hence the minimal polynomial of $A$ is $x^n + (-1)^n$. Since $n$ is a power of char $F$, $(A - I)^n = 0$. On the other hand, $F[A]$ is a field, so we have $A = I$, a contradiction. Thus there exists a matrix $C \in F[A]$ such that $F[C] \nsubseteq F[A]$ and $F[C]$ is a separable extension of $F$. Hence by Theorem 15, there exists a matrix $C_1 \in F[C] \setminus FI$ with determinant 1. Now, $C_1 \in SL_n(F)$ is a non-cyclic matrix commuting with $A$ and the proof is complete.

5. Some graph theoretic parameters of commuting graphs

In this section we first improve the lower bound for $\alpha(\Gamma(M_n(F)))$ given in [2] and show that if $F$ is a finite field and $n \geq 2$, then $\alpha(\Gamma(M_n(F))) \geq |F|^{(n-1)^2+1}|F|^{-n-2}$. We begin with the following lemma.

Lemma D. [7, p. 91] Let $G$ be a graph with $n$ vertices. If $d_1, \ldots, d_n$ are the degrees of all vertices of $G$, then $\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}$.

Theorem 18. If $F$ is a finite field and $n \geq 2$, then $\alpha(\Gamma(M_n(F))) \geq |F|^{(n-1)^2+1}|F|^{-n-2}$.

Proof. If $A$ is a cyclic matrix in $M_n(F)$, then by Corollary 1 the degree of $A$ in $\Gamma(M_n(F))$ is $|F|^n - |F| - 1$. Let $A_1, \ldots, A_m$ be all cyclic matrices in $M_n(F)$ with distinct minimal polynomials. Clearly, we have $m = |F|^n$. Moreover, if $A \in M_n(F)$ is a cyclic matrix, then using Corollary 1, there are at most $|F|^n - 1$ invertible matrices commuting with $A$. Thus the number of conjugates of $A$ is at least $|GL_n(F)|/(|F|^n - 1)$. Therefore the total number of cyclic matrices is at least $t = |F|^n |GL_n(F)|/(|F|^n - 1)$. By Lemma D, we conclude that $\alpha(\Gamma(M_n(F))) \geq \sum_{i=1}^{t} \frac{1}{|F|^n - |F|}$. Thus we find that

$$\alpha(\Gamma(M_n(F))) \geq \frac{|F|^n}{(|F|^n - |F|)(|F|^n - 1)} \prod_{i=0}^{n-1} (|F|^n - |F|^i) = |F|^n \prod_{i=2}^{n-1} (|F|^n - |F|^i).$$

This implies that

$$\alpha(\Gamma(M_n(F))) \geq |F|^n (|F|^n - |F|^{n-1})^{n-2} = |F|^{(n-1)^2+1}|F|^{-n-2}.$$ 

So the proof is complete.
Now we state a result about the domination numbers of commuting graphs.

**Theorem 19.** Let $F$ be an infinite field and $n \geq 2$. Then $\gamma(\Gamma(M_n(F)))$, $\gamma(\Gamma(GL_n(F)))$, and $\gamma(\Gamma(SL_n(F)))$ are infinite.

**Proof.** It is well known that every vector space over an infinite field cannot be a union of finitely many of its proper subspaces. Thus for any vertices $A_1, \ldots, A_k$ of $\Gamma(M_n(F))$, we have $M_n(F) \neq C_{M_n(F)}(A_1) \cup \cdots \cup C_{M_n(F)}(A_k)$. Clearly, this implies that $\gamma(\Gamma(M_n(F)))$ is infinite.

Now, assume that $\gamma$ is a dominating set for $\Gamma(GL_n(F))$. Since $F$ is infinite, for any matrix $A \in M_n(F)$, there exists a scalar $\alpha \in F$ such that $\alpha I - A$ is invertible. Therefore there exists a vertex $X \in \gamma$ such that $X$ commutes with $\alpha I - A$. Thus if $A \neq X$, then $A - X$ is an edge of $\Gamma(M_n(F))$. This yields that $\gamma$ is also a dominating set for $\Gamma(M_n(F))$. So $\gamma(\Gamma(GL_n(F)))$ is infinite.

Finally, we prove that $\gamma(\Gamma(SL_n(F)))$ is infinite. To get a contradiction, suppose $\{S_1, \ldots, S_m\}$ is a finite dominating set for $\Gamma(SL_n(F))$. We have $SL_n(F) = C_{SL_n(F)}(S_1) \cup \cdots \cup C_{SL_n(F)}(S_m)$. It is well known that $Z(SL_n(F)) \subseteq FI$, so $C_{SL_n(F)}(S_i)$ is a proper subgroup of $SL_n(F)$, for $i = 1, \ldots, m$. By Neumann’s Lemma [10, p. 92], at least one of these subgroups is of finite index in $SL_n(F)$. Thus it is easy to see that $SL_n(F)$ contains a normal subgroup $N$ of finite index $r$. On the other hand, by [4, Theorem 11], we have $N \subseteq FI$. Hence for any $\lambda \in F^*$, $\text{diag}(\lambda, \lambda^{-1}, 1, \ldots, 1)^r$ is a scalar matrix. This implies that $\lambda^{2r} = 1$, for each $\lambda \in F^*$, a contradiction. Now, the proof is complete. \qed

6. A uniqueness theorem for commuting graphs

In [2] it has been proved that if $E$ and $F$ are two finite fields such that $\Gamma(M_n(E)) \cong \Gamma(M_n(F))$, $m, n \geq 2$, then $m = n$ and $|E| = |F|$. In the sequel, we will generalize this result for arbitrary fields.

**Lemma E.** [2, Lemma 2] Let $F$ be a field, $n \geq 1$, and $A, B \in M_n(F)$. If $C_{M_n(F)}(A) \subseteq C_{M_n(F)}(B)$, then there exists a polynomial $f(x) \in F[x]$ such that $B = f(A)$.

**Lemma 20.** Let $F$ be a field and $n \geq 1$. If $C_{M_n(F)}(A_1) \subsetneq \cdots \subsetneq C_{M_n(F)}(A_k)$ is a chain of centralizers, then $k \leq n$. Moreover, if $|F| \geq n$, then there exists a chain of distinct centralizers in $M_n(F)$ of length $n$.

**Proof.** Suppose that $C_{M_n(F)}(A) \subsetneq C_{M_n(F)}(B)$, for $A, B \in M_n(F)$. Then by Lemma E, we have $B \in F[A]$. Since $C_{M_n(F)}(A) \neq C_{M_n(F)}(B)$, we conclude that $F[B] \subsetneq F[A]$. Hence the degree of the minimal polynomial of $B$ is less than the degree of the minimal polynomial of $A$. Now, if $C_{M_n(F)}(A_1) \subsetneq \cdots \subsetneq C_{M_n(F)}(A_k)$ is a chain and $m_i$ is the degree of the minimal
polynomial of $A_i$, then $n \geq m_1 > \cdots > m_k \geq 1$. This implies that $k \leq n$. Furthermore, if $a_1, \ldots, a_n$ are distinct elements of $F$, then the following chain

$$C_{M_n(F)}(\text{diag}(a_1, a_2, \ldots, a_n)) \supsetneq C_{M_n(F)}(\text{diag}(a_2, a_2, a_3, \ldots, a_n)) \supsetneq \cdots \supsetneq C_{M_n(F)}(\text{diag}(a_{n-1}, \ldots, a_{n-1}, a_n)) \supsetneq C_{M_n(F)}(\text{diag}(a_n, \ldots, a_n))$$

has length $n$. This completes the proof. 

The uniqueness of the Wedderburn-Artin theorem states that if $E$ and $F$ are two fields and $M_m(E) \cong M_n(F)$, then $m = n$ and $E \cong F$. In the next theorem we prove a similar uniqueness theorem for the commuting graphs.

**Theorem 21.** If $E$ and $F$ are two fields, $m, n \geq 2$ and $\Gamma(M_m(E)) \cong \Gamma(M_n(F))$, then $m = n$ and $|E| = |F|$.

**Proof.** By [2, Corollary 2], we may assume that $E$ and $F$ are infinite. Since $M_m(E) \setminus EI$ is the vertex set of $\Gamma(M_m(E))$ and $E$ is infinite, the cardinal of the vertex set of $\Gamma(M_m(E))$ is $|E|$. Now, since $\Gamma(M_m(E)) \cong \Gamma(M_n(F))$, we have $|E| = |F|$. To complete the proof, for any vertex $A$ of $\Gamma(M_m(E))$, let $N(A)$ be the set of all vertices $X$ in $\Gamma(M_m(E))$ such that either $X = A$ or $X - A$ is an edge of $\Gamma(M_m(E))$. Assume that

$$N(A_1) \supsetneq \cdots \supsetneq N(A_k)$$

is a chain in $\Gamma(M_m(E))$ with maximum possible length. Then clearly, $C_{M_m(E)}(A_1) \supsetneq \cdots \supsetneq C_{M_m(E)}(A_k) \supsetneq C_{M_m(E)}(I)$ is a chain of centralizers in $M_m(E)$ with maximum length. By Lemma 20, we conclude that $k = m - 1$. Similarly, the size of every chain of type $(\ast)$ with maximum length in $\Gamma(M_n(F))$ is $n - 1$. Since $\Gamma(M_m(E)) \cong \Gamma(M_n(F))$, we have $m = n$, as desired. 

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