DEFECT INDUCED INTERMITTENCY IN THE TRANSIT TIME DYNAMICS GENERATES $\frac{1}{f}$ NOISE IN A TRIMER DESCRIBED BY THE DISCRETE NONLINEAR SCHROEDINGER EQUATION

C.L. Pando L.

*IFUAP, Universidad Autónoma de Puebla,
Apdo. Postal J-48, Puebla, Pue. 72570, México*

*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy*

*and*

E.J. Doedel

*Concordia University, 1455 Boulevard de Maisonneuve W.,
Montreal, Quebec, H3G 1M8 Canada.*

MIRAMARE – TRIESTE

August 2006
Abstract

We investigate the nonlinear dynamics in a trimer, described by the one-dimensional discrete nonlinear Schrödinger equation (DNLSE), with periodic boundary conditions in the presence of a single on-site defect. We make use of numerical continuation to study different families of stationary and periodic solutions, which allows us to consider suitable perturbations. Taking into account a Poincaré section, we are able to study the dynamics in both a thin stochastic layer solution and a global stochasticity solution. We find that the time series of the transit times, the time intervals to traverse some suitable sets in phase space, generate $\frac{1}{f}$ noise for both stochastic solutions. In the case of the thin stochastic layer solution, we find that transport between two almost invariant sets along with intermittency in small and large time scales are relevant features of the dynamics. These results are reflected in the behaviour of the standard map with suitable parameters. In both chaotic solutions, the distribution of transit times has a maximum and a tail with exponential decay in spite of the presence of long-range correlations in the time series. We motivate our study by considering a ring of weakly-coupled Bose-Einstein condensates (BEC) with attractive interactions, where inversion of populations between two spatially symmetric sites and phase locking take place in both chaotic solutions.
1. INTRODUCTION.

Typical Hamiltonian systems with two degrees of freedom have a divided phase space consisting of regular and chaotic regions, where the former regions are organized in a hierarchical way [1]. One important property of chaotic trajectories is the trapping that these undergo in the neighbourhood of hierarchical structures of KAM islands and Cantori [2]. Cantori are invariant Cantor sets that work as partial barriers to the transport close to KAM islands [3]. The stickiness to these invariant structures brings about intermittency in the behaviour of a chaotic trajectory [4]. This phenomenon is characterized by long-range correlations [2, 4]. In the present article, we will explore the onset of intermittencies and long-range correlations in the discrete nonlinear Schroedinger equation (DNLSE).

An important Hamiltonian model that describes a lattice of coupled anharmonic oscillators is the DNLSE [5]. Indeed, the DNLSE has been derived from the discrete Klein-Gordon equation in the limits of small-amplitude oscillations and weak inter-site couplings [6]. The DNLSE describes a large class of discrete nonlinear systems such as optical fibers [7, 8], polarons [9], small molecules such as benzene [9], and, more recently, dilute Bose-Einstein condensates (BEC) trapped in a multiwell periodic potential [10]. The evolution of BEC is governed by the Gross-Pitaevskii equation and can be mapped, in the tight binding approximation, to a DNLSE [10]. We will consider the nonlinear dynamics of the DNLSE describing a ring consisting of three sites, the trimer, in the presence of a single defect. In a BEC, defects can be created with additional lasers or magnetic fields.

The DNLSE is a vast subject with many different and relevant issues such as the dynamics of discrete breathers in one or two-dimensional infinite lattices [5]. Motivated by the dynamics of BEC, we concentrate on the special issue concerning the onset of chaos in the DNLSE with a small number of oscillators. The DNLSE is a nonlinear Hamiltonian system with $M$ degrees of freedom, where $M$ refers to the number of oscillators. The DNLSE, as is well known, has two constants of motion [5]. Therefore, when $M = 2$, the DNLSE is integrable. However, when $M \geq 3$, the DNLSE can exhibit an amazing degree of complexity. The chaotic dynamics for $M = 3$ and $M = 4$ was studied, to our knowledge, for the first time almost two decades ago [11]. These authors underlined the fact that the DNLSE is generic in the spirit of modern nonlinear science, putting forward several molecular models described by the DNLSE. The nonlinear dynamics for a small number of oscillators has been considered since then [12]. Most of these studies, however, did not take into account the presence of a defect, which is central to our study. For $M = 3$, symmetry arguments show that the dynamics is that of an area-preserving two-dimensional map. As a result, it provides a rich behaviour, which is of interest, in particular, for both theory and experiment of BEC. When $M = 3$, the passage of a trajectory from one stochastic region in phase space to another is blocked by KAM surfaces [1]. When $M > 3$, however, Arnold diffusion can take place,
that is, stochastic layers of different resonances intersect. Therefore, generically the motion will spread out over an entire system of intersecting layers [1]. Indeed, in the context of DNLSE, Arnold diffusion was considered for $M = 4$ [13].

In recent work [14, 15], we have studied the nonlinear dynamics of the DNLSE with $M = 7$ and a single one-site defect. In [14, 15], we found several families of stationary solutions, whose unstable orbits trigger interesting complex dynamics. Moreover, relevant information on the chaotic behaviour was investigated considering suitable return maps of Poincaré cycles, which, to the best of our knowledge, was considered for the first time in multi-dimensional conservative systems in [15]. These Poincaré cycles give a better understanding of robust properties of this system. On the one hand, these cycles enable us to study the self-trapping chaotic regime, where some peculiar behaviour, typical of Arnold diffusion, arises. On the other hand, in continuous-time, the difference of actions in certain pairs of oscillators switch sign almost simultaneously [14–16]. In this regime, the statistics of the Poincaré cycles suggest, surprisingly, an almost Markovian behaviour in spite of the fact that the oscillators are to a good extent phase locked.

In this work, we study two different types of chaotic solutions of the DNLSE with periodic boundary conditions and a single on-site defect when $M = 3$. In the study of these solutions we made use of numerical continuation, which allowed us to find families of stationary and periodic solutions. This has been central to assess the linear stability properties of different solutions. One type of solution displays thin stochastic layer behaviour over a wide range of defect parameters, while the other solution exhibits global stochasticity, where the size of the stochastic sea and that of the islands are of the same order of magnitude. We use a suitable Poincaré section, which, being related to the geometry of the solutions, suggests the set of variables to consider. The thin stochastic layer (TSL) solution displays intermittency along with transport between two almost invariant regions, while both solutions show $\frac{1}{f}$ noise in the time series of the transit times to cross suitable sets in phase space. The $\frac{1}{f}$ noise phenomenon has attracted the attention of physicists for several decades. One of the reasons is that the variable of the time series can have an infinite mean square deviation. Our results suggest that the distribution of the transit times gives a finite mean and variance, however, these data have long range correlations, which in turn are responsible for the $\frac{1}{f}$ behaviour in their power spectra. In Hamiltonian systems, a generic mechanism that gives rise to $\frac{1}{f}$ noise was associated with the presence of a self-similar hierarchy of Cantori [4].

This article has seven sections. We consider the DNLSE within the framework of BEC in Sec. II. In Sec. III, we study the linear stability of different families of stationary and periodic solutions using numerical continuation. In Sec. IV, we consider the Poincaré section of the two chaotic solutions under study. In Sec. V, we discuss intermittency and transport in the TSL solution. We also investigate the characteristic $\frac{1}{f}$ noise in the time series of the transit times to cross a suitable neighborhood. We compare both solutions, the TSL solution and the global stochasticity (GS) solution, in Sec. VI. This is done mainly by considering the time series of the transit times
to traverse an area related to a symmetry of the return map. Finally, in Sec. VII we give our conclusions.

II. THE MODEL

We study a ring of three coupled nonlinear oscillators with periodic boundary conditions in the presence of a single on-site defect. To motivate the physical background of our model in the context of BEC, we consider an array of weakly coupled condensates, whose equation of motion [10] is given by

\[ i \frac{\partial \Psi_m}{\partial t} + \Delta_m \Psi_m + K(\Psi_{m-1} + \Psi_{m+1}) + \rho |\Psi_m|^2 \Psi_m = 0, \tag{1} \]

where \( \Psi_m \) stands for the condensate complex amplitude in the mth well, \( \rho \) is the nonlinear coefficient arising from the interatomic interaction, \( K \) is proportional to the microscopic tunneling rate between adjacent sites, \( \Delta_m \) stands for the on-site defect and is proportional to an external field superimposed on the lattice and, finally, \( t \) is the time. By introducing the dimensionless amplitude \( \psi_m = \sqrt{\rho/2K} \Psi_m \exp(-i(\Delta + 2K)t) \), Eq.(1) transforms into the discrete nonlinear Schrödinger equation (DNLSE) given by

\[ i \frac{\partial \psi_m}{\partial \tau} + \delta_m \psi_m + (\psi_{m-1} + \psi_{m+1} - 2\psi_m) + 2 |\psi_m|^2 \psi_m = 0, \tag{2} \]

where \( \delta_m = (\Delta_m - \Delta)/K \) stands for the defects, \( \tau = Kt \) and \( \Delta \) is any arbitrary number. As a result, the only explicit parameters of the DNLSE in Eq.(2) correspond to the defects \( \delta_m \). The positive sign before the nonlinear term indicates that we are considering an attractive interatomic interaction between the condensates, such as in the case of Lithium atoms [17].

There are two integrals of motion in Eq.(2). The first is the Hamiltonian, from which Eq.(2) is derived [5]. It is given by

\[ H = \sum_{m=1}^{M} (|\psi_m - \psi_{m+1}|^2 - |\psi_m|^4 - \delta_m |\psi_m|^2). \tag{3} \]

The second constant is the norm, which is given by

\[ N = \sum_{m=1}^{M} |\psi_m|^2. \tag{4} \]

Here \( M \) stands for the number of condensates.

Eq. (1) has been derived from the Gross-Pitaevskii equation when the height of the optical potential is strong enough. In this case, the tight-binding approximation holds and the lowest
band dynamics maps onto the DNLSE [10]. Under these conditions, the single-band Bose-Hubbard model describes this quantum system, which becomes the DNLSE when the number of trapped atoms is not small enough in the multiwell lattice [18].

We can rewrite the DNLSE by transforming into action-angle variables \((N_m, \theta_m)\), where \(\psi_m = \sqrt{N_m} \exp(-i\theta_m)\), to stress the physical meaning of the equations of motion. The equations for \(N_m \geq 0\) and \(\theta_m\) are the following:

\[
\begin{align*}
\frac{dN_m}{d\tau} &= 2\sqrt{N_mN_{m-1}} \sin(\theta_{m-1} - \theta_m) \\
&\quad + 2\sqrt{N_mN_{m+1}} \sin(\theta_{m+1} - \theta_m), \\
\frac{d\theta_m}{d\tau} &= 2 - \delta_m - \sqrt{N_{m-1}} \cos(\theta_{m-1} - \theta_m) \\
&\quad - \sqrt{N_{m+1}} \cos(\theta_{m+1} - \theta_m) - 2N_m.
\end{align*}
\]

\(N_m\) stands for the atomic population of site \(m\).

To find the DNLSE stationary solutions, we use the nonlinear map approach [19, 20]. This map is obtained by setting \(\frac{dN_n}{d\tau} = 0\), and \(\theta_n = \theta_m\), for any \(n \neq m\) in Eq.(5). Moreover, we can define the frequency of the resulting periodic orbit by setting \(\frac{d\theta_m}{d\tau} = \lambda\), where \(\lambda\) is a constant. Therefore, the stationary solutions have the form \(\psi_m(\tau) = \sqrt{N_m} \exp(-i\lambda\tau)\). As a result, the following cubic map (CM) is obtained:

\[
\begin{align*}
X_{n+1} &= Y_n, \\
Y_{n+1} &= (\Gamma - 2Y_n^2)Y_n - X_n,
\end{align*}
\]

where \(\Gamma_n = 2 - \lambda - \delta_n\) and \(Y_n = \sqrt{N_n}\). In the CM we will set \(\Gamma_n = \Gamma = 2 - \lambda\) for which \(\delta_n = 0\). The Jacobian \(J\) of this map is area preserving, i.e., \(J = 1\). The fixed points \((\pm \sqrt{\Gamma/2 - 1}, \pm \sqrt{\Gamma/2 - 1})\) of the CM are elliptic for \(2 < \Gamma < 4\). In this article, the use of this fixed point, to construct an initial condition for the DNLSE, give us the freedom to select the number of oscillators within the ring. The search for the stability of these solutions is given in the next section.

In our previous publications [14–16, 21], the number of oscillators in the ring was \(N = 6, 7\). Using the abovementioned map, we found new families of stationary solutions by considering stable periodic orbits that surround a suitable elliptic point of the map. In [14–16, 21], the number of oscillators in the ring is exactly determined by the periodicity of the orbit. Numerical continuation was further applied to find a very complex dependence of the solutions on the defect parameter [14, 15].
III. A FAMILY OF STATIONARY AND PERIODIC SOLUTIONS AND THEIR LINEAR STABILITY

The bifurcation diagram in Fig. 1a shows the real part of $\psi_1$, $\text{Re}(\psi_1)$, versus $\delta_3$ for stationary solution families of the DNLSE with three oscillators. The trivial solution, $\psi_i = 0$ where $i = 1, 2, 3$, is linearly stable, here meaning that the eigenvalues of the Jacobian evaluated at the stationary solutions all have zero real part, i.e., these solutions are linearly, neutrally stable. At the branch point, which is the bifurcation point with label 1, along the trivial family there are two eigenvalues equal to zero, giving rise to a secondary family of nontrivial solutions.

Along the secondary family we have $\psi_1 = \psi_2 \neq \psi_3$, except at the solution with label 1, where $\psi_1 = \psi_2 = \psi_3 = 0$, and at the solution with label 2, where $\delta_3 = 0$, and $\psi_1 = \psi_2 = \psi_3 \neq 0$. Also, along the secondary family there are always two eigenvalues equal to zero, in view of the rotational invariance of the solutions of this family. At the secondary branch point, which is the solution with label 4, there are four zero eigenvalues, and the family becomes unstable to the left of this point, where its solutions have two real eigenvalues of the form, $r$ and $-r$, where $r$ is real and where $r$ depends on $\delta_3$. Stationary solutions along the tertiary family that emanates from the secondary branch point, that is from the solution with label 4, have the property that $\psi_1$, $\psi_2$, and $\psi_3$ have different values, except at the solution with label 4, where $\psi_1 = \psi_2$. Again, along the tertiary family there are always two eigenvalues equal to zero.

The bifurcation diagram in Fig. 1b shows $\text{Re}(\psi_1)$ versus $T_0 = -2\pi/\lambda$. The stationary solution with label 2 in Fig. 1b corresponds to the same solution with label 2 in Fig. 1a, i.e., the solution with $\delta_3 = 0$ and $\psi_1 = \psi_2 = \psi_3 \neq 0$. Stationary solutions, along the family that passes through the solution with label 2 in Fig. 1b, have the property that $\psi_1 = \psi_2 = \psi_3 \neq 0$, and their Jacobians have a double zero eigenvalue. To the right of the solution with label 5 all eigenvalues lie on the imaginary axis. At the solution with label 5 all eigenvalues are zero, while to the left of the solution with label 5 there are four real eigenvalues of the form $r$ and $-r$, each being a double eigenvalue. Thus solutions of this family are unstable to the left of the solution with label 5. The latter is a branch point from which several families of stationary solutions arise. One of these families is represented in Fig. 1b, namely, the family that carries the solution with label 5. Solutions along this family have the property that $\psi_1 = \psi_2 \neq \psi_3$, except at the solution with label 5. Note that the family contains another branch point, that also carries the solution with label 6, and that part of the family consists of stable stationary solutions. As mentioned, the solution with label 5 is multiple bifurcation point. In addition to the bifurcating solution family described above, for which $\psi_1 = \psi_2 \neq \psi_3$, there are two additional families that emanate from the solution with label 5, namely families whose solutions satisfy $\psi_1 = \psi_3 \neq \psi_2$, and $\psi_2 = \psi_3 \neq \psi_1$, respectively. Thus the three families that emanate from the solution with label 5 are related through a permutation symmetry. In view of the Lyapunov Center Theorem [22], there are several families of periodic
solutions that emanate from each stationary point. These results are shown in Table I, where the counts may not be valid at a discrete set of points.

As an example, consider the two families that emanate from the stationary solution at $\delta_3 = -0.863333$, the solution with label 3, along the secondary family of stationary solutions. At this point the stationary solution has two zero eigenvalues and two pairs of purely imaginary eigenvalues with imaginary parts $\pm 3.20745$ and $\pm 0.778746$, giving rise to two families of periodic solutions, which we label as Family 1 and Family 2, respectively. Fig. 2a shows the imaginary part of $\psi_1$, Im($\psi_1$), versus Re($\psi_1$) for a selection of periodic solutions of Family 1. Fig. 2b shows Re($\psi_1$) versus (scaled) time for a selection of periodic solutions of Family 2. Small amplitude oscillations are stable along both families, but both families contain also unstable solutions.

### IV. THE POINCARÉ SECTION

We consider the DNLSE with a single defect : $\delta_3 < 0$, and $\delta_n = 0$ for $n \neq 3$, in a ring with three oscillators. We make use of the exact stationary DNLSE solutions of the previous sections. We added small random perturbations of the order of $10^{-7}$ to an exact stationary solution to obtain a set of initial conditions. The chosen stable stationary solution for $\delta_3 = 0$ is $\psi_i = \sqrt{\frac{1}{3}}$, for $i = 1, 2, 3$, where $\lambda = -\frac{2}{3}$ and $T_0 = 3\pi$. The stability of these stationary solutions are shown in Fig. 1a and Fig. 1b. Typically, as the parameter $\delta_3 < 0$ increases in absolute value, we find a transition from quasiperiodic to chaotic solutions for the set of initial conditions. The qualitative picture that emerges is, therefore, that of an initial condition moving away from the centers, which are given in Fig. 1a. as $\delta_3$ changes. As in typical Hamiltonian systems, this leads to a closer vicinity to thin stochastic layers where chaotic behaviour occurs. Thin stochastic layers (TSL) are separated from each other by KAM curves and motion from one layer to another is forbidden [1]. Notice that an increase of the absolute value of $\delta_3 < 0$ leads to an increase of the energy, while the norm of our solutions remains constant. We will also consider below another solution which displays global stochasticity (GS). In the latter many stochastic layers merge as a result of the overlap of neighboring resonances [1]. We make a comparison of these two chaotic solutions in Sec. VI. A larger region of phase space is involved in the dynamics of the GS solution whose positive Lyapunov exponent is substantially larger than that of the TSL solution.

<table>
<thead>
<tr>
<th>Table I: Families of Periodic Solutions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stationary Family</td>
</tr>
<tr>
<td>trivial</td>
</tr>
<tr>
<td>secondary (To the right of label 4)</td>
</tr>
<tr>
<td>secondary (To the left of label 4)</td>
</tr>
<tr>
<td>tertiary</td>
</tr>
</tbody>
</table>
In order to consider a suitable Poincaré section, we define the new variable \( \nu \equiv N_1 - N_2 \). The chaotic solutions that we study intersect the Poincaré section transversally when \( \nu = 0 \) and \( \frac{d\nu}{d\tau} > 0 \). In Fig. 3a and Fig. 3b we see the evolution in continuous time of \( \nu \) for the two aforementioned solutions as they define the Poincaré section at \( \nu = 0 \). In Fig. 3a, where \( \delta_3 = -\frac{16}{15} \), we observe that in continuous time the bursting behaviour of \( \nu \) is followed by laminar intervals. During these laminar intervals, we find that \( \nu \approx 0 \) and, moreover, \( \nu \) becomes zero several times as suggested by Fig. 3a. It is worth to notice that the phase difference \( \theta_1 - \theta_2 \) is roughly proportional to \( \nu \). As a result, during the laminar intervals \( \theta_1 - \theta_2 \approx 0 \) and, simultaneously, \( N_1 - N_2 \approx 0 \). The dynamics shown in Fig. 3a is typical for the TSL solutions in continuous time. Fig. 3b stands for the GS solution for which \( \delta_3 = -0.863333 \), where laminar intervals are not a feature of the dynamics.

The Poincaré return map of a Hamiltonian flow is also area-preserving [1]. The DNLSE is an autonomous Hamiltonian flow with two constants of motion and two dynamically relevant phases, namely \( \theta_1 - \theta_3 \). As a result, this system has four vanishing Lyapunov exponents and the associated map is two-dimensional. The two dimensional surface, where this Poincaré return map is located, is given by the following manifold:

\[
H - N(2 - \sin^2(\mu) \cos(2\theta_-)) - \frac{\left| \sin(2\mu) \right|}{\sqrt{2}} (\cos(\theta_+ + \theta_-) + \cos(\theta_+ - \theta_-)) - \frac{N}{2} \sin^4(\mu) - N \cos^4(\mu) - \delta_3 \cos^2(\mu) = 0, \tag{7}
\]

where \( \mu = \arccos(\sqrt{\frac{N_3}{N}}) \), \( \theta_+ = \frac{1}{2}(\theta_1 + \theta_2 - 2\theta_3) \), \( \theta_- = \frac{1}{2}(\theta_1 - \theta_2) \). We will use these variables in what follows. This two-dimensional surface has been obtained by considering the equations for \( H \), \( N \) and \( \nu = 0 \). We have numerically tested its validity.

In the context of a ring of weakly-coupled BEC with attractive interactions, both chaotic solutions are characterized by inversion of populations between two spatially symmetric sites as a result of the defect \( \delta_3 \). Moreover, the phases in these two different solutions are locked, namely, \( |\theta_{1,2} - \theta_3| < 0.8 \) in the TSL case and \( |\theta_{1,2} - \theta_3| < 1.3 \) in GS case. These properties of BEC take place in continuous time.

V. DEFECT INDUCED INTERMITTENCY AND TRANSPORT

We now focus on the dynamics at the Poincaré section. For values of \( \delta_3 \) roughly between \( \delta_3 = -\frac{14}{15} \) and \( \delta_3 = -\frac{19}{15} \) with the set of initial conditions of the last section, the dynamics of the Poincaré return map displays typically intermittency as well as transport between two distinct regions of phase space. In Fig. 3c and Fig. 3d we consider the projection of the return map onto the plane \((\mu, \theta_+)\). Fig. 3c and Fig. 3d, which we relate to region A and region B respectively, show
two consecutive long time intervals of this map when $\delta_3 = -\frac{16}{15}$. This is the TSL solution. Each interval is composed typically of roughly $10^5$ iterations. The return map suggests that there is a mechanism for transport between the two regions A and B. In an almost invariant region [23], such as regions A and B, typical points are mapped into the region itself with high probability. Therefore, one has to deal with transition probabilities for these two different sets. In spite of the fact that we have about $10^6$ iterations in the full times series of Fig. 3a and Fig. 3b, the transitions between these two regions occur just a few number of times. The concept of separatrix is useful to understand the transport mechanism between regions A and B. First, we point out that the separatrix partitions phase space into disjoint regions [24]. Technically, the separatrix is formed by taking the union of segments of the stable and unstable manifolds of a fixed point of the Poincaré return map [24]. A trajectory crosses the separatrix through the turnstile, which is the region of phase space mediating transport across the separatrix [24]. Specifically, our simulations suggest that there is a separatrix which defines the transition from region A to region B and vice versa. As for the corresponding fixed point, there is presumably a suitable hyperbolic fixed point in our Poincaré return map, whose manifolds bring about the aforementioned separatrix. The presence of this hyperbolic fixed point is suggested by a comparison of the behaviour of our model with that of the standard map. This is carried out at the end of this section.

Another important feature of the return map is the intermittency that the trajectory displays when crossing the two symmetric trapping regions (STR) of phase space. This is appreciated in Fig. 4d, where we have chosen a few hundreds consecutive iterations of $\mu$ at the Poincaré section. We will elucidate below the nature of these two STR, where $\mu$ shows almost regular motion. The time series of Fig. 4d alternates between almost regular intervals, also known as laminar, and disordered behaviour in an irregular fashion. During these disordered intervals, $\mu$ looks like a short and large burst when the orbit leaves the two symmetric trapping regions (STR). The continuous time series of Fig. 3a is the same orbit which we show in Fig. 4d, where the latter deals with the return map. In Fig. 4a, Fig. 4b and Fig. 4c, we consider the projection of the return map onto the planes $(\theta_-, \theta_+)$, $(\mu, \theta_+)$ and $(\mu, \theta_-)$ for $\delta_3 = -\frac{16}{15}$. The two STR, where the dynamics with laminar behaviour occurs, are localized around the intersections of the approximately smooth curves of Fig. 4a, Fig. 4b and Fig. 4c. These figures look like those of the complete time series of the return map, which consists of roughly $10^6$ iterations.

The existence of dynamical traps is a robust effect observed in several Hamiltonian systems. These traps influence to a great extent the transport properties of these systems [25]. These trapping domains can be attributed to the sticky domains that belong to boundaries of islands and Cantori [25]. These are so relevant that even untrapped orbits feel their influence [3]. We found numerical evidence that the residence time within these traps typically increases as the parameter $\delta_3 < 0$ increases in absolute value. This behaviour is characterized by sticky parts of the trajectory with almost regular oscillations of $\mu$. This is shown in Fig. 5. At the time scale of about $10^5$
iterations, we can better appreciate the presence of these trapping domains, while at a smaller
time scale the orbit has short laminar intervals as in Fig. 4d.

At a qualitative level, a better understanding of the intermittent behaviour is gained by the
observation that the trajectory is repeatedly reinjected back to the STR, which, in turn, leads
to laminar behaviour. This suggests to us that within the STR there is a hyperbolic fixed point
of the return map. It is well known that, any homoclinic or heteroclinic point and much of its
neighbourhood are reinjected back to the neighbourhood of the relevant hyperbolic fixed point of a
generic Hamiltonian system. These neighbourhoods are, in turn, pushed away from the hyperbolic
fixed point by the unstable manifold. This reinjection mechanism apparently induces laminar
intervals in short time scales. Moreover, in any neighbourhood of the relevant hyperbolic point
there are an infinite number of periodic points of the return map [1]. This suggests the observation
of the above mentioned trapping domains with almost regular behaviour, where the orbits traverse
repeatedly the STR. To support the above mentioned scenario, we compare below the results that
we get in our model with those of the standard map. Our model, therefore, displays intermittency
in the short time scale, as shown in Fig 4d. In contrast, the time scales of transport between
regions A and B, as shown in Fig 3c and Fig. 3d, along with those of the trapping domains, as
shown in Fig. 5, typically are much longer. This occurs when $\delta_3$ lies roughly between $\delta_3 = -\frac{13}{15}$
and $\delta_3 = -\frac{14}{15}$ with the aforesaid set of initial conditions. This suggests that our mechanism for
intermittency is robust in contrast to other mechanisms for which intermittency exist only near
isolated points in parameter space [1].

Consider the maps $(\mu(i), \mu(i + 1))$ and $(\tau(i + 1) - \tau(i), \tau(i + 2) - \tau(i + 1))$, shown in Fig. 6a and
Fig. 6b respectively, where we see that both STR are located, roughly speaking, in a neighbourhood
of the intersection point of the curves. Upon magnification, this curve has a complicated structure,
as we will see next. The width of the TSL solution is one of its main characteristics [26]. We trace
small circles located within the STR whose radii are $\rho = 10^{-4}$, $\rho = 10^{-3}$ and, finally, $\rho = 10^{-2}$.
Then we look for the probability density function (PDF) of the transit times for an orbit to traverse
these circles, i.e., we look for the distribution of the lengths of the laminar intervals $(\Delta i)_m$, where $m$
stands for the sequence of this time series. These histograms are depicted in Fig. 6c. The bigger the
circle, the more shifted to the right these distributions are, indicating that the trajectory can spend
longer time within the circle of larger radius. We point out that these PDF decay exponentially for
the largest laminar intervals $(\Delta i)_m$, while for smaller intervals the distributions have a maximum.
In any case, these PDF differ from the exponential distributions. It is known that the distribution
of the transit times for hyperbolic systems has an exponential decay [27]. Moreover, the average
transit time to traverse a set is proportional to some measure of this set [27]. That is precisely
what we see in Fig.6c as the radii of the circles become larger.

The power spectra $S(F)$ of the time series $(\Delta i)_m$ display a low-frequency dependence which
scales with $F$ over more than two orders of magnitude. Indeed, $S(F) \sim F^{-1.35}$ as shown in Fig. 6d.
In this figure the time series \((\Delta i)_m\) were generated using as radii \(\rho = 10^{-3}\) and \(\rho = 10^{-4}\). Our power spectra \(S(F)\) differ from those of other studies [4, 28], where some system variable was considered. Based on Fig. 6d, we may argue that to some extent there is self-similarity in the low-frequency components of the time series \((\Delta i)_m\). Indeed, that is what we see qualitatively if Fig. 7a for the time series \((\Delta i)_m\), where the value of \((\Delta i)_m\) with the largest probability appears with some degree of persistence, reflecting a long-range dependence. In Fig. 7a the time series was generated using as radius \(\rho = 10^{-4}\). Therefore, our model generates infrequent hopping between numerically different laminar intervals \((\Delta i)_m\). This kind of mechanism contains the essential features of systems whose power spectra display power-law dependence. A class of one-dimensional maps with a hopping mechanism, showing \(1/f\) noise, was considered some time ago [28].

To quantify self-similarity, we have calculated the Hurst exponent \(h\), which is defined as the slope of the curve \(\log_{10} \sigma(L)\) versus \(\log_{10} L\), where \(\sigma(L) = \frac{1}{R-L} \sum^{R-L}_{m=1} \left( |(\Delta i)_{m+L} - (\Delta i)_m| \right)^{\frac{1}{2}}\). \(R\) is the amount of data and \(L\) is the corresponding lag. We find that \(h \sim 0.1\) for about two orders of magnitude of the lag \(L\) as shown in Fig. 7b. In this figure the time series were generated using as radii \(\rho = 10^{-3}\) and \(\rho = 10^{-4}\). For very large lag \(L\), the time series \((\Delta i)_m\) does not grow any further. That is why we see saturation of the variance \(\sigma(L)\). A value of the Hurst exponent \(h\) smaller than \(\frac{1}{2}\), as in our case, indicates that we are dealing with a subdiffusive process on scales where \(L < 10^3\).

The aforementioned self-similarity and the degree of persistence of the time series \((\Delta i)_m\) can be further studied by calculating the autocorrelation function (ACF) \(C(L)\), where \(C(L) = \frac{\sum_{m=1}^{R-S}((\Delta i)_m - <\Delta i>)(\Delta i)_{m+L} - <\Delta i>)}{\sum_{m=1}^{R-S}((\Delta i)_m - <\Delta i>)^2}\). In this equation, \(<\Delta i>\) is the sample average of \((\Delta i)_m\), \(L\) is the time lag, \(R\) is the number of data points, \(L < S\) and \(R \gg S\) [1]. As shown in Fig. 7c, the ACF \(C(L)\) has a long tail displaying a complicated nonexponential decay for about two orders of magnitude of \(L\). This is a clearcut indication of the long-range correlations. In Fig. 7c the time series were generated using as radii \(\rho = 10^{-3}\) and \(\rho = 10^{-4}\). We would like to point out that our PDF for the transit times \((\Delta i)_m\), having an asymptotic exponential decay, and the ACF \(C(L)\), having a power-law like dependence, do not have a straightforward relation as in [2]. Indeed, as pointed out in [4], there is not a unique general relation between the PDF of Poincaré recurrence times and the ACF. The power spectra \(S(F)\) and the ACF \(C(L)\) depend on topological details of the model and on the variable for which it is determined [4]. In Fig. 7d, we show a magnification of one of the STR projected onto the \((\mu, \theta_-)\) plane. We can appreciate the complexity of this region, where a divided phase space with regular and chaotic regions coexist. The presence of islands in this figure suggests the influence of self-similar structures in the dynamics of our system, where a trajectory can be trapped temporarily.

Consider a thin stochastic layer of the standard map to compare it with that of the DNLSE.
The standard map is defined on the two dimensional torus and is given by the following \[1\]:

\[
\begin{align*}
X_{n+1} &= X_n + P_{n+1}, \\
P_{n+1} &= P_n + K \sin(X_n).
\end{align*}
\] (8)

Both \(X_n\) and \(P_n\) are modulo \(2\pi\) and are given in the interval \((-\pi, \pi)\). If \(K > 0\) the map has a hyperbolic fixed point \((0, 0)\). The largest eigenvalue of the matrix of the linear part of this map is \(1 + \frac{K}{2} + \sqrt{K + \frac{K^2}{4}} > 0\). Therefore, the stable and unstable manifolds of \((0, 0)\) for \(K > 0\) already intersect creating the so-called thin stochastic layer (TSL). The width is one of the main characteristics of the TSL. It is conjectured that this width is of the order of \(O(\exp(-const/\sqrt{K}))\) for \(K \ll 1\) [26]. We choose the parameter \(K = 0.17\), therefore, most of the phase space is filled with KAM tori.

We trace a small circle of radius \(\rho = 10^{-2}\) centered at the hyperbolic point \((0, 0)\). The time series \((\Delta i)_{\text{in}}\) of the transit times for an orbit to traverse this circle is shown in Fig. 8a. The PDF of this time series is displayed in Fig. 8b. We see that both are reminiscent of the TSL of the DNLSE. Indeed, Fig. 8a exhibits persistence of \((\Delta i)_{\text{in}}\) for the standard map, exactly as Fig. 7a did for the DNLSE. The structure of the PDF in Fig. 8b is consistent with those PDF of Fig. 6c. The power spectrum \(S(F)\) in Fig. 8c for the standard map and those of Fig. 6d for the DNLSE display a power-law dependence on the frequencies \(F\) for about two orders of magnitude. Finally, a magnification in the neighbourhood of the hyperbolic point \((0, 0)\) for the standard map shows an island structure which arises also at the STR in the TSL solution of the DNLSE. By the same token, in the standard map we observe that the typical time scale of the laminar intervals \((\Delta i)_{\text{in}}\) is much smaller than that of the trapping domains. This reminds us of the time scales of Fig. 4d and Fig. 5 calculated for the DNLSE. This comparison is made in such a way that the number of data points of the time series in the standard map has the same order of magnitude as that in the DNLSE.

VI. A COMPARISON BETWEEN THE THIN STOCHASTIC LAYER AND GLOBAL STOCHASTICITY SOLUTIONS OF THE DNLSE

Now we will consider a global stochasticity (GS) solution to compare its dynamics with that of the TSL solution. To this end, we follow the evolution of a typical trajectory with the initial condition in a set of radius \(\rho = 10^{-13}\) centered at the unstable periodic orbit of the first family of periodic solutions, discussed in Sec. III, for which \(\delta_3 = -0.863333\). In Fig. 9a, Fig. 9b and Fig. 9c, we project the Poincaré map of this orbit onto the planes \((\theta_-, \theta_+)\), \((\mu, \theta_+)\) and \((\mu, \theta_-)\) respectively. This regime is characterized by an interconnected stochastic region where invariant islands are embedded in a stochastic sea. This connected stochasticity typically results from the merging of primary resonance layers according to the criteria of the overlapping of neighbouring
resonances as proposed by Chirikov [1]. We notice that in this regime, there is basically no evidence of robust laminar intervals. This is shown in Fig. 9d. Moreover, we can yet find that there are trapping domains where the trajectory sticks presumably near island borders, even though the typical duration of these time intervals is smaller than those of the TSL solution, which were shown in Fig. 5.

In the DNLSE with seven degrees of freedom and periodic boundary conditions [15], the time series of the actions \( N_i \) at the Poincaré section, on the one hand, and that of the difference of the Poincaré return times \( \tau(i+1) - \tau(i) \), on the other, did not have similar features in different chaotic regimes, as shown, for instance, by their power spectra. Motivated by this fact, we would like to explore this aspect for \( N = 3 \) by considering both the TSL and the GS solutions of the DNLSE. In Fig. 10a we can appreciate the power spectra \( S(F) \) of the time series \( \mu(i) \) and \( \tau(i+1) - \tau(i) \) for the TSL solution. Similarly, in Fig. 10b we plotted the power spectra for the GS solution. Both power spectra \( S(F) \) look the same except for a constant factor, which makes the difference in Fig. 10a. Instead, in Fig. 10b both power spectra \( S(F) \) differ in a relevant range of frequencies \( F \). To give a qualitative explanation, we first plot \( \mu(i) \) versus \( \tau(i+1) - \tau(i) \) in both cases. This is done in Fig. 10c and Fig. 10d. In Fig. 10c we find that the plot describes a seemingly smooth closed curve, the TSL, whose width is several orders of magnitude smaller than the typical scale of the full trajectory. Instead, in Fig. 10d, features of a divided phase space of island and stochastic regions appear. Our qualitative argument, to explain the likeness of the power spectra \( S(F) \) in Fig. 10a, is based on a condition imposed on the equations of smooth plane closed curves, whose variables are related to some expansion in terms of periodic functions, such as the epitrochoid [29].

The parametric equations for an epitrochoid are
\[
\begin{align*}
  x(w) &= (a + b) \cos(w) - c \cos\left(\frac{(a + b)w}{b}\right), \\
  y(w) &= (a + b) \sin(w) - c \sin\left(\frac{(a + b)w}{b}\right),
\end{align*}
\]
where \( a, b \) and \( c \) are positive constant parameters [29]. We notice that the amplitudes and frequencies of the trigonometric components of \( x(w) \) and \( y(w) \) are precisely the same. Now, we choose a given time series \( w_i \) such that the parametric equations, \( x(w_i) \) and \( y(w_i) \), cover most of the closed curve, roughly as it happens in Fig. 4a, Fig. 4b and Fig. 4c. Next, we look for a condition that brings about similar power spectra for \( x(w_i) \) and \( y(w_i) \). To this end, we calculate, first, the ACF \( C(L) \) for both \( x(w_i) \) and \( y(w_i) \). As a result, we find that if the following
relation holds:

\[
| < (a+b)^2 \cos(w_{i+L} - w_i) + c^2 \cos\left( \frac{a+b}{b} w_{i+L} - \frac{a+b}{b} w_i \right) - c(a+b) \cos(w_{i+L} - \frac{a+b}{b} w_i) - c(a+b) \cos\left( \frac{a+b}{b} w_{i+L} - w_i \right) > | \\
\gg | < (a+b)^2 \cos(w_{i+L} + w_i) + c^2 \cos\left( \frac{a+b}{b} w_{i+L} + \frac{a+b}{b} w_i \right) - c(a+b) \cos(w_{i+L} + \frac{a+b}{b} w_i) - c(a+b) \cos\left( \frac{a+b}{b} w_{i+L} + w_i \right) > |
\]

then the ACF \( C(L) \) for both \( x(w_i) \) and \( y(w_i) \) will be basically proportional, where the proportionality factor depends on the ratio of \( < x(w_i)^2 > \) to \( < y(w_i)^2 > \). \( < .. > \) stands for sample average. Since the right hand side of the inequality makes the difference between both ACF, our condition consists of making it as small as possible. As a result of the Wiener-Khintchine theorem [30], proportionality between the ACF \( C(L) \) for \( x(w_i) \) and \( y(w_i) \) implies proportionality between the power spectra \( S(F) \) for \( x(w_i) \) and \( y(w_i) \). That is precisely what we see in Fig. 10a. This observation suggests to us that for \( \mu(i) \) and \( \tau(i+1) - \tau(i) \) there is a dependence qualitatively similar to that of parametric curves, which approximates better at scales much larger than the width of the TSL. This approximate description, therefore, could not reproduce the TSL fine structure.

The PDF of \( \tau(i+1) - \tau(i) \) is shown in Fig. 11a for both chaotic solutions. The histogram for the TSL case, the dashed line, shows a clearcut global maximum at \( \tau(i+1) - \tau(i) \approx 0.4 \). This maximum corresponds to the return times when the laminar intervals take place, as in Fig. 4d. As for the GS solution, the continuous line in Fig. 11a, the PDF of \( \tau(i+1) - \tau(i) \) is by far less singular. The positive Lyapunov exponents \( \lambda_1 \) for both chaotic solutions are displayed in Fig. 11b. We notice that the Lyapunov exponent of the GS solution, which is given by the continuous line, is larger than that of the TSL solution, the dashed line.

Consider the transit times to traverse the region of phase space where \( \theta_+ < 0 \). According to Fig. 4a and Fig. 4b, related to the TSL solution, and Fig. 9a and Fig. 9b, related to the GS solution, the Poincaré map of both solutions has the symmetry \((\mu, \theta_+, \theta_-) \rightarrow (\mu, -\theta_+, \theta_-)\). This symmetry of the map presumably arises as a result of the mirror symmetry of our Hamiltonian, i.e., \( N_{1,2} \rightarrow N_{2,1}, \theta_{1,2} \rightarrow \theta_{2,1}, N_3 \rightarrow N_3, \theta_3 \rightarrow \theta_3 \). Notice that both variables, \( \theta_+ \) and \( \theta_- \), are modulo \( 2\pi \) and are defined in the interval \((-\pi, \pi)\). That is, to find the successive transit times, we count the number of consecutive iterations when \( \theta_+ < 0 \), \((\Delta i)_m\), which allows us to compare both chaotic solutions. Notice that now \((\Delta i)_m\) contains the intervals in the laminar phase as well as part of the bursting intervals. The time series of \((\Delta i)_m\) are shown in Fig. 11c and Fig. 11d for the
TSL and GS solutions respectively. We can see that the structure of these time series qualitatively resembles that of Fig. 7a. The histograms of \((\Delta i)_m\) for both chaotic solutions look like those of Fig. 6c, i.e., there is a maximum and a tail with exponential decay for large intervals \((\Delta i)_m\). This is shown in Fig. 12a. This infrequent hopping between numerically different intervals \((\Delta i)_m\) gives rise to long-range correlations. The average transit time is shorter in the case of the GS solutions, which accounts for a faster diffusion process in phase space. To assess the long-range correlations, we consider the ACF \(C(L)\) of the time series \((\Delta i)_m\), where \(L\) is the lag interval. The decay rates of \(C(L)\) for both chaotic solutions are not exponential, but rather appear to be almost algebraic. The ACF \(C(L)\) of the GS solution, the solid line in Fig. 12b, shows a clearcut power-law behaviour for more than two orders of magnitude of the lag \(L\), i.e., \(C(L) \sim L^\alpha\), where \(\alpha \sim -\frac{1}{2}\). As for the TSL solution, the dashed line in Fig. 12b, the ACF \(C(L)\) decays more slowly than that of GS solution and, apparently, has a more complicated structure. In the TSL case, the corresponding ACF \(C(L)\) in Fig. 12b almost matches that for the time series of the laminar intervals \((\Delta i)_m\), as shown in Fig. 7c. The power spectra \(S(F)\) for the time series \((\Delta i)_m\) of both chaotic solutions show that, at low frequencies \(F\), \(S(F) \sim F^\alpha\) for more than two orders of magnitude of \(F\). \(\alpha \sim -1.35\) in the TSL case, while \(\alpha \sim -1.0\) in the GS regime. This is shown in Fig. 12c. Again, in the TSL case, this exponent \(\alpha\) is very close to that for the time series of the laminar intervals \((\Delta i)_m\) shown in Fig. 6d. As mentioned in the previous section, the Hurst exponent checks for the range of self-similarity of the time series \((\Delta i)_m\). The Hurst exponent \(h\) in the case of the TSL solution is about \(h \sim 0.08\) and persists over more than two orders of magnitude of the lag \(L\). This is shown in Fig. 12d. In this case, the Hurst exponent \(h\) is close to that for the time series of the laminar intervals \((\Delta i)_m\) shown in Fig. 7b. Instead, the dependence of \(\sigma(L)\) on \(L\) looks somewhat more complicated for the GS solution and the associated Hurst exponent is, in any case, much smaller than that for the TSL case. As for the long-range memory in the TSL case, the closeness of the power spectra \(S(F)\), the ACF \(C(L)\) and the Hurst exponent \(h\) for the two different time series \((\Delta i)_m\) suggests both the relevant contribution of the laminar intervals and the robustness of our results.

VII. CONCLUSIONS

In our model we make use of numerical continuation to investigate the dependence of different families of stationary and periodic solutions on the defect parameter, which induces a mirror symmetry, and on the period of the solution. This allows us to consider suitable sets of initial conditions. We constructed a Poincaré section, where a two-dimensional manifold contains the Poincaré return map. The persistent time series of transit times, to cross some suitable sets in phase space in both a thin stochastic layer solution and a global stochasticity solution, generate \(\frac{1}{\sqrt{T}}\) noise over a wide range of parameters of the DNLSE. This is a robust property of the system in spite of the fact that the shape of the density distribution for the transit times has a maximum and a tail with
exponential decay. In the case of the thin stochastic layer solution, we find that transport between two almost invariant sets is a basic feature of this chaotic solution and occurs, typically, at large time scales. As a result of the trapping domains in the latter the phenomenon of intermittency takes place typically at large time scales. In contrast, the ubiquitous laminar intervals, the intermittency at small time scales, are presumably generated by the frequent reinjection mechanism into the neighbourhood of a hyperbolic fixed point of the return map of the system, where the return times are almost constant. Further, we suggest a peculiar coarse-grained description of the thin layer dynamics, where a suitable time series, the \( w_i \) of Sec.VI, accounts for the evolution. As for the long-range correlations in the thin stochastic layer case, we believe that the convergent results for the two different time series of transit times, studied in Sec. V and VI, suggest the relevant contribution of the laminar intervals. These long-memory effects are nicely reflected in the behaviour of suitable transit times in the standard map, which, as is well known, portrays many of the properties of real systems.

Acknowledgements

The work of EJD is also supported by a Discovery Grant from NSERC Canada. This work was also supported by CONACYT-México and the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, (ICTP).

References


FIG. 1: (a) Plot of $\text{Re}(\psi_1)$ versus $\delta_3$ for families of stationary solutions. Here $\lambda = -\frac{2\pi}{1.00529}$. (b) Plot of $\text{Re}(\psi_1)$ versus $T_0 = -\frac{2\pi}{\lambda}$ for families of stationary solutions, where $\delta_3 = 0$. The solid and dashed lines stand for stable and unstable solutions respectively.
FIG. 2: (a) Plot of Re($\psi_1$) versus Im($\psi_1$) for a selection of periodic solutions of Family 1. (b) Plot of Re($\psi_1$) versus the time in units of the normalized period for a selection of periodic solutions of Family 2. Both Family 1 and Family 2 contain stable and unstable orbits.
FIG. 3: (a) Continuous time plot of $\nu = N_1 - N_2$ versus $\tau$ for the TSL solution, where $\delta_3 = -\frac{16}{15}$. (b) Same as (a) but for the GS solution, see text. (c) Projection of the Poincaré return map onto the plane ($\mu, \theta_{+}$) for a long time interval. This is the projection of the region with label A. (d) Same as (c), but for a consecutive long time interval. This is the projection of the region with label B. Both projections belong to the TSL solution.
FIG. 4: Different projections for the Poincaré return map of the TSL solution. (a) Projection onto the plane \((\theta_-, \theta_+)\) for a time interval containing both regions A and B. (b) Projection onto the plane \((\mu, \theta_+)\). (c) Projection onto the plane \((\mu, \theta_-)\). (d) Plot of the time series of \(\mu\) versus the iterate \(i\) of the return map for the TSL solution. Here, we appreciate the laminar intervals.
FIG. 5: Plot of the time series of $\mu$ versus the iterate $i$ of the return map for the TSL solution for a long time interval. Here, we appreciate sticky parts of the trajectory having almost regular oscillations in the trapping domains.
FIG. 6: (a) Plot of $(\mu(i), \mu(i+1))$ for the return map of the TSL solution. (b) Plot of $(\tau(i+1) - \tau(i), \tau(i+2) - \tau(i+1))$ for the return map of the TSL solution. (c) Histograms of the transit time, $\log_{10} P[(\Delta_i)_m]$, for the TSL solution when the radius $\rho$ becomes $\rho = 0.0001$ (line 1), $\rho = 0.001$ (line 2) and $\rho = 0.01$ (line 3). (d) Plot of the power spectra $\log_{10} S(F)$ versus $\log_{10} F$ for the time series of the transit times $(\Delta_i)_m$ for the TSL solution when the radius $\rho$ becomes $\rho = 0.0001$ (dotted line) and $\rho = 0.001$ (solid line).
FIG. 7: (a) Plot of the time series of the laminar intervals $(\Delta i)_m$ in the Poincaré return map for the TSL solution when $\rho = 0.0001$. (b) Plot of $\log_{10} \sigma(L)$ versus $\log_{10} L$ for the time series of the laminar intervals $(\Delta i)_m$ when the radius $\rho$ becomes $\rho = 0.0001$ (dotted line) and $\rho = 0.001$ (solid line). The associated slope defines the Hurst exponent $h$. See the text. (c) Plot of $\log_{10} C(L)$ versus $\log_{10} L$ for the time series considered in (b). $C(L)$ is the ACF and $L$ is the time lag. (d) Magnification of a symmetric trapping region (STR) projected onto the $(\mu, \theta_-)$ plane for the return map of the TSL solution.
FIG. 8: (a) Plot of the time series of the laminar intervals $(\Delta i)_m$ in the standard map. Here, in the neighbourhood of the hyperbolic point $(0,0)$, we considered $\rho = 0.01$. (b) Histogram of the laminar intervals $(\Delta i)_m$ for the time series considered in (a). (c) Plot of the power spectra $\log_{10} S(F)$ versus $\log_{10} F$ for the time series considered in (a). (d) Magnification of the phase space neighbourhood of the hyperbolic fixed point considered in (a).
FIG. 9: Different projections for the Poincaré return map of the GS solution. (a) Projection onto the plane $(\theta_-, \theta_+)$. (b) Projection onto the plane $(\mu, \theta_+)$. (c) Projection onto the plane $(\mu, \theta_-)$. (d) Plot of the time series of $\mu$ versus the iterate $i$ for the return map of the GS solution.
FIG. 10: (a) Plot of the power spectra $\log_{10} S(F)$ versus $\log_{10} F$ for the TSL solution, where the time series are $\tau_{i+1} - \tau_i$ (dashed line) and $\mu(i)$ (solid line). (b) Plot of the power spectra $\log_{10} S(F)$ versus $\log_{10} F$ for the GS solution, where the time series are $\tau_{i+1} - \tau_i$ (dashed line) and $\mu(i)$ (solid line). (c) Plot of $\tau_{i+1} - \tau_i$ versus $\mu(i)$ for the return map of the TSL solution. (d) Same as (c) but for the GS solution.
FIG. 11: (a) Histogram of the Poincaré cycles for $\tau_{i+1} - \tau_i$, $\log_{10} P(\tau_{i+1} - \tau_i)$, for the TSL solution (dashed line) and the GS solution (solid line). (b) Plot of the largest Lyapunov exponent $\lambda_1$ versus time at the Poincaré section $\tau(i)$ for the TSL solution (dashed line) and the GS solution (solid line). (c) Plot of the time series of the transit time intervals $(\Delta i)_m$ to traverse the region $\theta_+ < 0$ for the TSL solution. (d) Same as (c), but for the GS solution.
FIG. 12: (a) Histogram of the transit time intervals $\langle \Delta t \rangle_m$ to traverse the region $\theta_+ < 0$ for the TSL solution (label 2) and the GS solution (label 1). (b) Plot of $\log_{10} C(L)$ versus $\log_{10} L$ of the time series in (a) for the TSL solution (label 1) and the GS solution (label 2). $C(L)$ is the ACF and $L$ is the time lag. (c) Plot of the power spectra $\log_{10} S(F)$ versus $\log_{10} F$ of the time series in (a) for the TSL solution (dashed line) and the GS solution (solid line). (d) Plot of $\log_{10} \sigma(L)$ versus $\log_{10} L$ of the time series in (a) for the TSL solution (dashed line) and the GS solution (solid line). The associated slope defines the Hurst exponent $h$. See the text.