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LOJASIEWICZ EXPONENTS AND NEWTON POLYHEDRA

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Abstract
In this paper we obtain the exact value of the Lojasiewicz exponent at the origin of analytic
map germs on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) under the Newton non-degeneracy condition, using information
from their Newton polyhedra. We also give some conclusions on Newton non-degenerate analytic
map germs. As a consequence, we obtain a link between Newton non-degenerate ideals and
their integral closures, thus leading to a simple proof of a result of Saia. Similar results are also
considered to polynomial maps which are Newton non-degenerate at infinity.

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1. Introduction

1. Let \( f := (f_1, f_2, \ldots, f_k) : (K^n, 0) \to (K^k, 0) \) be an analytic map germ, where \( K = \mathbb{R} \) or \( K = \mathbb{C} \). We define the Lojasiewicz exponent \( L_0(f) \) of the germ \( f \) as the greatest lower bound of the set of all real numbers \( l > 0 \) which satisfy the condition: there exists a positive constant \( c \) such that

\[
\max_{i=1,2,\ldots,k} |f_i(x)| \geq c \|x\|^l \quad \text{for} \quad \|x\| \ll 1.
\]

If the set of all the exponents is empty we put \( L_0(f) = +\infty \). It is well known that \( L_0(f) < +\infty \) if and only if \( f \) has an isolated zero at the origin in \( K^n \).

Calculating explicitly the Lojasiewicz exponent \( L_0(f) \) is important in the theory of singularities. There are some previous works which give an upper estimate for this number. For instance, when \( g \) is a complex analytic function of two variables, some formulae for \( L_0(\text{grad } g) \) are obtained as in [20]. In the papers [23], [11], [3], [4], [1], estimates for the Lojasiewicz exponent \( L_0(\text{grad } g) \) in terms of the Newton diagram of Newton non-degenerate complex analytic function germs \( g \) (in the sense of [18]) are also given. It seems more difficult to obtain effective estimates in the real case (see [19], [10], [15], [16], [17], [2]).

The first aim of this paper is to calculate the Lojasiewicz exponent \( L_0(f) \) in terms of the Newton polyhedron of an analytic map germ \( f \), under the Newton non-degeneracy condition. Moreover, motivated by the works of Yoshinaga [28], Saia [27] and Biviá-Ausina [4], we also extract some conclusions on Newton non-degenerate analytic map germs. As a consequence, we obtain a connection between Newton non-degenerate ideals and their integral closures. In particular, we retrieve a result of Saia in [27].

Our method is actually different from the argument of the previous authors: the proof, based on the ideas of Kuo and Lojasiewicz, uses only the Curve Selection Lemma as a tool.

2. We next suppose that \( f := (f_1, f_2, \ldots, f_k) : K^n \to K^k \) is a polynomial mapping. We define the Lojasiewicz exponent at infinity \( L_\infty(f) \) of the map \( f \) as the smallest upper bound of the set of all real numbers \( l > 0 \) which satisfy the condition: there exists a positive constant \( c \) such that

\[
\max_{i=1,2,\ldots,k} |f_i(x)| \geq c \|x\|^l \quad \text{for} \quad \|x\| \gg 1.
\]

If the set of all the exponents is empty we put \( L_\infty(f) = -\infty \).

In the case \( n = 2 \), Hà [12] (see also [22]) gave an exact formula for the Lojasiewicz exponent at infinity \( L_\infty(\text{grad } g) \) of the gradient of a complex polynomial \( g \), and he showed a link between \( L_\infty(\text{grad } g) \) and the singularities at infinity of \( g \). In the papers [6], [7] Chadzynski and Krasinski described the Lojasiewicz exponent at infinity of a polynomial mapping \( f : \mathbb{C}^2 \to \mathbb{C}^2 \). In particular, they obtained a characterization of a component of a polynomial automorphism of \( \mathbb{C}^2 \) from a characterization of \( L_\infty(f) \). Kollár [17] (see also [9]) gave an effective estimate for the Lojasiewicz exponent at infinity of real polynomial maps with a compact zero set.
It is worth noting that if \( f \) is generic in the sense of Kouchnirenko (see [18]).

Newton diagram of \( \Gamma(f) \) uniquely determines \( \nu \), which is essential ly standard, and was established in [18]. Let \( \mathbb{N} \subset \mathbb{R}_+ \subset \mathbb{R} \) be the sets of all nonnegative integers, all nonnegative real numbers, and all real numbers respectively. Let \( J \subseteq \{1,2,\ldots,n\} \). We write \( \mathbb{K}^J := \{ \alpha \in \mathbb{K}^n \mid \alpha_j = 0 \text{ if } j \notin J \} \). For any map germ \( f: (\mathbb{K}^n,0) \to (\mathbb{K}^k,0) \) we denote by \( f|_{\mathbb{K}^J} \) the map germ \( f \) where the indeterminate \( x_j \) is zero whenever \( j \notin J \).

Let \( f := (f_1,f_2,\ldots,f_k): (\mathbb{K}^n,0) \to (\mathbb{K}^k,0) \) be an analytic map germ. If the Taylor expansions of \( f_i \) are \( \sum_{\alpha \in \mathbb{N}^n} a_{\alpha}(i)x^\alpha, i = 1,2,\ldots,k \) (where \( x^\alpha \) denotes the monomial \( x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \)), the support \( \text{supp}(f) \) is defined to be \( \bigcup_{i=1}^k \{ \alpha \in \mathbb{N}^n \mid a_{\alpha}(i) \neq 0 \} \). We define \( \Gamma_+(f) \) to be the convex hull of the set \( \bigcup_{\alpha \in \text{supp}(f)} (\alpha + \mathbb{R}_+^n) \). For any \( m \in \mathbb{R}_+^n, m \neq 0 \), we consider a supporting hyperplane \( \{ \alpha \in \mathbb{R}^n \mid \langle m,\alpha \rangle = \nu \} \) of \( \Gamma_+(f) \) such that

\[
\langle m,\alpha \rangle \geq \nu \quad \text{for all } \alpha \in \Gamma_+(f)
\]

These conditions determine \( \nu \) uniquely, while \( \Gamma_+(f) \) is given by the system of inequalities\(^2\)

\[
\langle m,\alpha \rangle \geq \nu, m \in \mathbb{R}_+^n.
\]

A face of the boundary of the Newton polyhedron \( \Gamma_+(f) \) is an intersection of \( \Gamma_+(f) \) with some supporting hyperplane. The union of the compact faces of \( \Gamma_+(f) \) is called the Newton diagram \( \Gamma(f) \) of \( f \).

For a face \( \gamma \in \Gamma(f) \) we put \( f_{i,\gamma}(x) := \sum_{\alpha \in \gamma} a_{\alpha}(i)x^\alpha, i = 1,2,\ldots,k \). We say that \( f \) is Newton non-degenerate if for any face \( \gamma \in \Gamma(f) \), the functions \( f_{i,\gamma} \) have no common zero in \( (\mathbb{K} - \{0\})^n \). It is easily seen from Sard’s lemma that the Newton non-degenerate condition is generic in the sense of Kouchnirenko (see [18]).

The germ \( f \) is said to be convenient if the Newton diagram \( \Gamma(f) \) meets all coordinate axes. It is worth noting that if \( f \) has isolated zero at the origin then \( f \) is convenient. In this case let \( l_j, j = 1,2,\ldots,n \), be the length from the origin to the intersection point of \( \Gamma(f) \) and the \( \alpha_j \)-axis.

\(^2\)The system of inequalities is infinite; however, there exists a finite number of inequalities of which the remaining inequalities are a consequence.
Define a positive integer number \( l_0(f) \) by the following formula
\[
l_0(f) := \max_{j=1,2,\ldots,n} l_j.
\]
A version of the following result for the case \( n = 2 \) can be found in [13].

**Theorem 2.1.** Let \( f := (f_1,f_2,\ldots,f_k) : (\mathbb{K}^n,0) \to (\mathbb{K}^k,0) \) be a convenient analytic map germ. If \( f \) is Newton non-degenerate, then \( l_0(f) \) is equal to the Lojasiewicz exponent \( L_0(f) \) of \( f \).

Here we assume that \( f := (f_1,f_2,\ldots,f_k) : \mathbb{K}^n \to \mathbb{K}^k \) is a polynomial map. We express \( f \) as follows: \( f_i(x) := \sum_{|\alpha| \leq d_i} a_\alpha(i)x^\alpha, i = 1,2,\ldots,k, \) (where \( d_i := \deg f_i \) is the degree of \( f_i \)). The support \( \text{supp}(f) \) is defined to be \( \bigcup_{i=1}^k \{ |\alpha| \leq d_i \mid a_\alpha(i) \neq 0 \} \). We define \( \Gamma_- (f) \) to be the convex hull of the set \( \{0\} \cup \text{supp}(f) \). The Newton diagram at infinity of \( f \), denoted by \( \Gamma_\infty(f) \), is the polyhedron formed by the closed faces of \( \Gamma_-(f) \) which do not contain the origin. As before, for each closed face \( \gamma \) of the polyhedron \( \Gamma_\infty(f) \) we denote by \( f_{i,\gamma} \) the polynomial \( \sum_{\alpha \in \gamma} a_\alpha(i)x^\alpha \). \( f \) is called Newton non-degenerate at infinity if for each face \( \gamma \in \Gamma_\infty(f) \), the polynomial functions \( f_{i,\gamma} \) have no common zero in \( (\mathbb{K} - \{0\})^n \).

The polynomial map \( f \) is said to be convenient if the Newton diagram \( \Gamma_\infty(f) \) meets all coordinate axes. In this case let \( l_{j,\infty}, j = 1,2,\ldots,n, \) be the length from the origin to the intersection point of \( \Gamma_\infty(f) \) and the \( \alpha_j \)-axis. We define \( l_\infty(f) \) by
\[
l_\infty(f) := \min_{j=1,2,\ldots,n} l_{j,\infty}.
\]
The next theorem was proved in [13] for \( n = 2 \), that is, for polynomial maps in two variables.

**Theorem 2.2.** Let \( f := (f_1,f_2,\ldots,f_k) : \mathbb{K}^n \to \mathbb{K}^k \) be a convenient polynomial map. Suppose that \( f \) is Newton non-degenerate at infinity. Then \( l_\infty(f) \) is equal to the Lojasiewicz exponent at infinity \( L_\infty(f) \) of \( f \).

2.2. **Proofs.** We prove only Theorem 2.1. The proof of Theorem 2.2 follows by entirely analogous arguments but instead of working in a small sphere we work in the complement of a large sphere.

The proof of the following lemma is clear from the definitions.

**Lemma 2.3.** Let \( f := (f_1,f_2,\ldots,f_k) : (\mathbb{K}^n,0) \to (\mathbb{K}^k,0) \) be an analytic map germ. Suppose that \( f \) is convenient. For every \( \emptyset \neq J \subset \{1,2,\ldots,n\} \) then

(i) \( f|_{\mathbb{K}^J} \) is convenient. Moreover, if the germ \( f \) is Newton non-degenerate, then so is \( f|_{\mathbb{K}^J} \).

(ii) \( \Gamma_+(f|_{\mathbb{K}^J}) = \Gamma_+(f) \cap \mathbb{R}^J \).

Let \( \{\alpha \in \mathbb{R}^n \mid \langle m,\alpha \rangle = \nu \} \) be the supporting hyperplane of a given face \( \gamma \in \Gamma(f) \). The next lemma indicates a convenient way to determine \( f_{i,\gamma} \) from \( f_i \).

**Lemma 2.4.** Let \( x \in \mathbb{K}^n, x \neq 0 \). We have
\[
f_i(t^m \bullet x) = t^\nu f_{i,\gamma}(x) + o(t^\nu) \quad \text{as} \quad t \to 0,
\]
where \( t^m \bullet x := (t^{m_1}x_1,t^{m_2}x_2,\ldots,t^{m_n}x_n) \).
Proof. By definition, \( (m, \alpha) \geq \nu \) for all \( \alpha \in \Gamma_+(f) \) with equality if and only if \( \alpha \in \gamma \). Moreover, it is obvious that

\[
f_{i,\gamma}(t^m \cdot x) = t^\nu f_{i,\gamma}(x).
\]

This implies the lemma. \( \square \)

Lemma 2.5. Let \( f := (f_1, f_2, \ldots, f_k): (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0) \) be a convenient analytic map germ. If \( f \) is Newton non-degenerate, then the origin in \( \mathbb{K}^n \) is an isolated zero of \( f \).

Proof. This proof is due to Wall [30] (see also [29], [14]).

Suppose that the claim does not hold. Then, by the Curve Selection Lemma [25], there exists an analytic curve

\[
\varphi: [0, \epsilon) \to \mathbb{K}^n, \quad t \mapsto (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)),
\]

such that \( \varphi(t) = 0 \) if and only if \( t = 0 \), and

\[
f_1[\varphi(t)] = f_2[\varphi(t)] = \cdots = f_k[\varphi(t)] = 0 \quad \text{for} \ t \in [0, \epsilon).
\]

Let \( J \) be the set of all the indices \( j \in \{1, 2, \ldots, n\} \) such that \( \varphi_j \) does not vanish identically. For \( j \in J \), expand the coordinate \( \varphi_j \) in terms of the parameter: say

\[
\varphi_j(t) = x^j_0 t^{m_j} + \text{higher order terms in } t,
\]

where \( x^j_0 \) is non-zero number and \( m_j > 0 \).

We consider the set \( \Gamma' \), obtained by intersecting the Newton diagram \( \Gamma(f) \) and the subspace \( \mathbb{R}^J \). By Lemma 2.3, \( \Gamma' \neq \emptyset \) and \( \Gamma' \) is the Newton diagram of \( f|_{\mathbb{K}^J} \). Let \( \nu \) be the least value attained by the linear function \( \alpha \mapsto \sum_{j \in J} m_j \alpha_j \) on \( \Gamma_+(f|_{\mathbb{K}^J}) \). Let \( \gamma \) denote the face of \( \Gamma_+(f|_{\mathbb{K}^J}) \) along which this value is attained. Then, by Lemma 2.4, the functions \( f_i, i = 1, 2, \ldots, k \), restricted on \( \varphi \) have the form

\[
f_i[\varphi(t)] = t^\nu f_{i,\gamma}(x^0_1, x^0_2, \ldots, x^0_n) + \text{higher order terms in } t,
\]

where \( x^0_j := 1 \) whenever \( j \notin J \). (Note that the functions \( f_{i,\gamma} \) are independent on the variables \( x_j \) for all \( j \notin J \).)

But by hypothesis, all functions \( f_i \) vanish along \( \varphi \). So in particular,

\[
f_{i,\gamma}(x^0_1, x^0_2, \ldots, x^0_n) = 0 \quad \text{for } i = 1, 2, \ldots, k,
\]

which contradicts Newton non-degeneracy of \( f \) because \( (x^0_1, x^0_2, \ldots, x^0_n) \in (\mathbb{K} - \{0\})^n \). This completes the proof. \( \square \)

Proof of Theorem 2.1. By Lemma 2.5, the Lojasiewicz exponent \( L_0(f) \) is finite.

Without loss of generality we may assume \( l_0(f) = l_1 \)-the length from the origin to the intersection point of \( \Gamma(f) \) and the \( \alpha_1 \)-axis. Let \( H \) denote the \( \alpha_1 \)-axis. Then it easy to check that

\[
\min_{i=1,2,\ldots,k} O(f_i|_H) = l_1,
\]
where $O(f_i|_H)$ is the multiplicity of the restriction of $f_i$ to $H$. Hence
\[ L_0(f) \geq l_1 = l_0(f). \]

By the above inequality, one has only to prove that $L_0(f) \leq l_0(f)$. By the definition of $L_0(f)$, it suffices to show that
\[ \max_{i=1,2,\ldots,k} |f_i(x)| \geq c \|x\|^{l_0(f)} \]
for $\|x\|$ sufficiently small and for $c > 0$. Suppose that this is not the case. By standard argument, based again on the Curve Selection Lemma [25], there exists an analytic curve
\[ \varphi : [0, \epsilon) \to \mathbb{K}^n, \quad t \mapsto (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)), \]
passing through the origin, such that
\[ \max_{i=1,2,\ldots,k} |f_i(\varphi(t))| \ll \|\varphi(t)\|^{l_0(f)} \quad \text{as} \quad t \to 0. \]

Let $J$ be the set of all the indices $j \in \{1, 2, \ldots, n\}$ such that $\varphi_j$ does not vanish identically. For $j \in J$, expand the coordinate $\varphi_j$ in terms of the parameter: say
\[ \varphi_j(t) = x_j^0 t^{m_j} + \text{higher order terms in } t, \]
where $x_j^0$ is the non-zero number and $m_j > 0$. Let $m_* := \min_{j \in J} m_j > 0$. Then, we have, asymptotically as $t \to 0$,
\[ \|\varphi(t)\| \simeq |t|^{m_*}. \]

We consider the set $\Gamma'$, obtained by intersecting the Newton diagram $\Gamma(f)$ and the subspace $\mathbb{R}^J$. By Lemma 2.3, $\Gamma' \neq \emptyset$ and $\Gamma'$ is the Newton diagram of $f|_{\mathbb{K}^J}$. Let $\nu$ be the least value attained by the linear function $\alpha \mapsto \sum_{j \in J} m_j \alpha_j$ on $\Gamma_+(f|_{\mathbb{K}^J})$. Let $\gamma \in \Gamma'$ denote the face of $\Gamma_+(f|_{\mathbb{K}^J})$ along which this value is attained. Then it is obvious that $\nu/m_*$ is equal to the maximum of the lengths from the origin to the intersection points of $\Gamma'$ and the $\alpha_j$-axis, $j \in J$. From this observation, together with the definition of $l_0(f)$, we obtain the following inequality
\[ \nu \leq m_* l_0(f). \]

On the other hand, from Lemma 2.4 we get
\[ f_i[\varphi(t)] = t^\nu f_i,\gamma(x_1^0, x_2^0, \ldots, x_n^0) + o(t^\nu), \]
where $x_j^0 := 1$ if $j \notin J$. Since $f$ is Newton non-degenerate, not all the values $f_i,\gamma(x_1^0, x_2^0, \ldots, x_n^0)$ are zero. Consequently,
\[ \max_{i=1,2,\ldots,k} |f_i(\varphi(t))| \simeq |t|^\nu \quad \text{as} \quad t \to 0. \]
It follows from (1), asymptotically as $t \to 0$, that
\[ |t|^\nu \ll \|\varphi(t)\|^{l_0(f)} \simeq |t|^{m_* l_0(f)}. \]
Therefore
\[ \nu > m_* l_0(f), \]
which contradicts (2). This ends the proof. \[\square\]
Example 2.6. (See [27]). Let \( f := (f_1, f_2) : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0) \), where \( f_1 = x^8 + xy^5 \) and \( f_2 = y^8 + yx^5 \). Then \( f \) is convenient and \( l_0(f) = 8 \). The 1-dimensional compact faces of \( \Gamma_+(f) \) are \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) with vertices \( \{(0,8),(1,5)\}, \{(1,5),(5,1)\} \) and \( \{(5,1),(8,0)\} \) respectively. The ideal \( I \) is Newton non-degenerate since there is no common solution in \( (\mathbb{K} - \{0\})^2 \) for the equations \( f_{1,\gamma_i} = f_{2,\gamma_i} = 0, i = 1, 2, 3. \) By Theorem 2.1, \( L_0(f) = l_0(f) = 8 \).

Remark 2.7. (i) After the preparation of this paper we have learnt that Theorem 2.1 was also proved in the case \( \mathbb{K} = \mathbb{C} \) by Bivià-Ausina [4] using a different argument.

(ii) In general, as we see in the next example, the conditions \( f^{-1}(0) = \{0\} \) and \( L_0(f) = l_0(f) \) do not imply that \( f \) is Newton non-degenerate.

Example 2.8. Let \( f := (f_1, f_2) : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0) \), where \( f_1 = xy - y^2 \) and \( f_2 = x^3 \). The 1-dimensional compact faces of \( \Gamma_+(f) \) are \( \gamma_1 \) and \( \gamma_2 \) with vertices \( \{(0,2),(1,1)\} \) and \( \{(1,1),(3,0)\} \) respectively. The germ \( f \) is non Newton non-degenerate since any point \( (t,t) \) with \( t \neq 0 \) is a solution of the equations \( f_{1,\gamma_1} = f_{2,\gamma_1} = 0 \). On the other hand, it is not difficult to verify that \( f^{-1}(0,0) = \{(0,0)\} \) and \( L_0(f) = l_0(f) = 3 \). This works both over \( \mathbb{R} \) and \( \mathbb{C} \).

3. The Newton Non-degenerate condition

We are motivated by the works of Yoshinaga [28], Saia [27] and Bivià-Ausina [4] on the characterization of Newton non-degenerate complex analytic map germs. In this section, following this procedure, we will give some conclusions of the class of (complex or real) analytic map germs which are Newton non-degenerate. However, our arguments are based on other ideas, more precisely, we use only the Curve Selection Lemma. Firstly, we have

Theorem 3.1. Let \( f := (f_1, f_2, \ldots, f_k) : (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0) \) be a convenient analytic map germ. Then the following conditions are equivalent:

(i) \( f \) is Newton non-degenerate.

(ii) Take any monomial \( x^\alpha \) with \( \alpha \in \Gamma_+(f) \). There exists a positive constant \( c \) such that

\[
\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x^\alpha\| \quad \text{for} \quad \|x\| \ll 1.
\]

Proof. Suppose, by contradiction, that there exists a monomial \( x^{\alpha_0} \) with \( \alpha_0 \in \Gamma_+(f) \) such that the claim (ii) does not hold. Then, by the Curve Selection Lemma [25], there exists an analytic curve

\[
\varphi : [0, \varepsilon) \to \mathbb{K}^n, \quad t \mapsto (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)),
\]

such that \( \varphi(t) = 0 \) if and only if \( t = 0 \), and

\[
\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll \|\varphi(t)^{\alpha_0}\| \quad \text{as} \quad t \to 0.
\]

Let \( J \) be the set of all the indices \( j \in \{1,2,\ldots,n\} \) such that \( \varphi_j \) does not vanish identically. It is clear that \( J \neq \emptyset \) and for \( j \not\in J \) the \( j \)-th component, \( \alpha_j^0 \), of \( \alpha^0 \) is zero.
For $j \in J$, expand the coordinate $\varphi_j$ in terms of the parameter: say

$$\varphi_j(t) = x_j^0 t^{m_j} + \text{higher order terms in } t,$$

where $x_j^0$ is non-zero number and $m_j > 0$.

Let $\nu$ be the least value attained by the linear function $\alpha \mapsto \sum_{j \in J} m_j \alpha_j$ on $\Gamma_+(f|_{\mathbb{K}^n})$. Let $\gamma$ denote the face of $\Gamma_+(f|_{\mathbb{K}^n})$ along which this value is attained. Then, by Lemma 2.4, the functions $f_i, i = 1, 2, \ldots, k$, restricted on $\varphi$ have the form

$$f_i[\varphi(t)] = t^\nu f_i,\gamma(x_1^0, x_2^0, \ldots, x_n^0) + \text{higher order terms in } t.$$

Since $f$ is Newton non-degenerate, not all the values $f_i,\gamma(x_1^0, x_2^0, \ldots, x_n^0)$ are zero. Thus,

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \simeq |t|^\nu \quad \text{as } t \to 0.$$

Then, it follows from (3), asymptotically as $t \to 0$, that

$$|t|^\nu \ll \|(\varphi(t))^\alpha\| \simeq |t|^{(m,\alpha^0)}.$$

This gives $\nu > (m,\alpha^0)$, which contradicts the fact that $\alpha^0 \in \Gamma_+(f)$. 

Suppose now that (i) fails. Then there exists $\gamma \in \Gamma(f)$ and $x^0 \in (\mathbb{K} - \{0\})^n$ such that

$$(4) \quad f_1,\gamma(x^0) = f_2,\gamma(x^0) = \cdots = f_k,\gamma(x^0) = 0.$$

Let $J \subset \{1, 2, \ldots, n\}$ be the smallest set of indices such that the subspace $\mathbb{R}^J$ contains $\gamma$. Let $\{\alpha \in \mathbb{R}^J \mid \langle m, \alpha \rangle = \nu\}$ be a supporting hyperplane of $\gamma \subset \Gamma_+(f|_{\mathbb{K}^n})$. That means $\langle m, \alpha \rangle \geq \nu$ for all $\alpha \in \Gamma_+(f|_{\mathbb{K}^n})$ with equality if and only if $\alpha \in \gamma$. Define the monomial curve $\varphi: [0, \epsilon) \to \mathbb{K}^n, t \mapsto \varphi(t)$, by

$$\varphi_j(t) = \begin{cases} x_j^0 t^{m_j} & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Take any $\alpha \in \gamma \cap \text{supp}(f) \subset \Gamma_+(f) \cap \mathbb{N}^n$. By definition, it is clear that

$$\|\varphi(t)^\alpha\| \simeq |t|^{(m,\alpha)} = |t|^\nu.$$

On the other hand, it follows from Lemma 2.4 and the relation (4) that

$$f_i[\varphi(t)] = t^\nu f_i,\gamma(x^0) + o(t^\nu) = o(t^\nu) \quad \text{for } i = 1, 2, \ldots, k.$$

These imply that

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll \|\varphi(t)^\alpha\| \quad \text{as } t \to 0,$$

which contradicts (ii). This completes the proof.

We denote by $\Lambda(f)$ the convex hull in $\mathbb{R}^n_+$ of the set

$$\{\alpha \in \mathbb{N}^n \mid \exists c > 0 \text{ such that } \max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x^\alpha\| \text{ for } \|x\| \ll 1\}.$$

A version of the following lemma for the case $\mathbb{K} = \mathbb{C}$ can be found in [27].

**Lemma 3.2.** Let $f := (f_1, f_2, \ldots, f_k): (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. Then $\Lambda(f) \subset \Gamma(f)$. 

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Proof. Suppose that $\alpha^0 := (\alpha^0_1, \alpha^0_2, \ldots, \alpha^0_n) \notin \Gamma_+(f) \cap \mathbb{N}^n$. Let $J := \{j \mid \alpha^0_j > 0\}$. It is obvious that $\alpha^0 \notin \Lambda(f)$ when $J = \emptyset$. Hence one has only to consider the case $J \neq \emptyset$. Then there exists a supporting hyperplane $\{\alpha \in \mathbb{R}^J \mid \langle m, \alpha \rangle = \nu\}$ of $\Gamma_+(f|_K)$ such that $\langle m, \alpha^0 \rangle < \nu \leq \langle m, \alpha \rangle$ for all $\alpha \in \Gamma_+(f|_R)$. Define the monomial curve $\varphi : [0, \epsilon) \to \mathbb{K}^n, t \mapsto \varphi(t)$, by $\varphi_j(t) = \begin{cases} t^{m_j} & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$ By a direct calculation, then $||[\varphi(t)]^{\alpha^0}|| = |t|^{\langle m, \alpha^0 \rangle} \gg |t|^{\nu}$, $f_i[\varphi(t)] = t^\nu f_i t^{-\gamma}(1, 1, \ldots, 1) + o(t^\nu)$ for $i = 1, 2, \ldots, k$. These give $\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll ||[\varphi(t)]^{\alpha^0}||$. As a consequence, $\alpha^0 \notin \Lambda(f)$. The lemma is proved. \hfill \Box

From Theorem 3.1 and Lemma 3.2 we obtain immediately:

**Corollary 3.3.** Let $f := (f_1, f_2, \ldots, f_k) : (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. Then the following conditions are equivalent:

(i) $f$ is Newton non-degenerate.

(ii) $\Gamma_+(f) = \Lambda(f)$.

**Remark 3.4.** It is worth noting that the proofs of the above results are based on the Curve Selection Lemma. Therefore, by entirely analogous arguments but instead of working in a small sphere we work in the complement of a large sphere and then using the Curve Selection Lemma at infinity [26], we may obtain similar results for polynomial maps $f : \mathbb{K}^n \to \mathbb{K}^k$. We will leave to the reader to verify these facts.

In the rest of this note, we establish a relation between Newton non-degenerate ideals and their integral closures. In order to do this, we need some definitions.

Let $A(\mathbb{K}^n)$ be the ring of analytic function germs from $(\mathbb{K}^n, 0)$ onto $(\mathbb{K}, 0)$. If $S \subset A(\mathbb{K}^n)$, we denote by $V(S)$ the zero set germ at the origin of $S$ in $\mathbb{K}^n$.

Let $I$ be an ideal of $A(\mathbb{K}^n)$ and $g \in A(\mathbb{K}^n)$ such that $V(I) \subseteq V(g)$. Let $f_1, f_2, \ldots, f_k \in A(\mathbb{K}^n)$ be a system of generators of $I$. By [24] (see also [5]), we can consider the greatest lower bound of those $l > 0$ such that $\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|g(x)\|^l$ for $\|x\| \ll 1$, with some positive constant $c$. We denote this number by $L_0(g, I)$. We observe that this definition does not depend on the chosen system of generators of $I$.

By the works of Lejeune-Teissier [21] and Bochnak-Risler [5], $L_0(g, I)$ is a positive rational number.
We will say that \(g\) is integral over \(I\) when \(L_0(g, I) \leq 1\) (see [8], [3]). The set of integral elements over \(I\) forms an ideal of \(A(K^n)\) called the integral closure of \(I\). This ideal is denoted by \(\bar{I}\). Clearly, we have the inclusion \(I \subseteq \bar{I}\).

The ideal \(I := \langle f_1, f_2, \ldots, f_k \rangle\) is said to be Newton non-degenerate if the germ
\[(f_1, f_2, \ldots, f_k): (K^n, 0) \to (K^k, 0)\]
is Newton non-degenerate. It is easy to check that the above definition does not depend on the chosen system of generators of \(I\).

We say that \(I\) has finite codimension in \(A(K^n)\) if \(\dim_K A(K^n)/I < \infty\). This is equivalent to saying that \(V(I) = \{0\}\).

The following is an characterization of the integral closure of an ideal of \(A(K^n)\).

**Corollary 3.5.** Let \(I\) be an ideal of finite codimension in \(A(K^n)\). Then the following conditions are equivalent:

(i) \(I\) is Newton non-degenerate.

(ii) The integral closure \(\bar{I}\) is equal to the ideal generated by the monomials \(x^\alpha\) such that \(\alpha \in \Gamma_+(f)\).

**Proof.** Let \(f_1, f_2, \ldots, f_k \in A(K^n)\) be a system of generators of \(I\). Let \(f: (K^n, 0) \to (K^k, 0)\) be the germ such that its components are \(f_1, f_2, \ldots, f_k\). Then \(f\) is convenient because \(V(I) = \{0\}\). Therefore the claim comes from Corollary 3.3. \(\square\)

**Remark 3.6.** To end this paper, we consider the case \(K = \mathbb{C}\). Let \(I\) be an ideal of \(A(\mathbb{C}^n)\) and \(g \in A(\mathbb{C}^n)\). It is proved by Lejeune and Teissier [21] that the following conditions are equivalent:

(i) \(g \in \bar{I}\).

(ii) \(g\) satisfies an equation of the form
\[g^d + a_1 g^{d-1} + \cdots + a_{d-1} g + a_d = 0,\]
where \(a_i \in I^i, i = 1, 2, \ldots, d\), for some \(d \geq 1\).

Therefore, in the case \(K = \mathbb{C}\), Corollary 3.5 is just the result of Saia [27, Theorem 3.4].

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