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ON BENNEY TYPE HYDRODYNAMICAL SYSTEMS
AND THEIR BOLTZMANN-VLASOV EQUATIONS KINETIC MODELS

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Abstract

Some years ago Zakharov and Gibbon observed a very nice relation between the Benney type equation in hydrodynamics and the Vlasov equation of kinetic theory. These equations are generalized and put into the framework of infinite-dimensional Lie algebras associated to Lie algebra structures on rings of functions on finite-dimensional manifolds. This gives rise to a complete description of the Hamiltonian structure of both types of equations under consideration. In particular, their Lax type representations together with an infinite involutive hierarchy of conservation laws are obtained in an exact form. Some applications to chaotic many-particle dynamical systems, turbulent fluid flows and swept volume analysis are considered.

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0. INTRODUCTION

It is well-known [1, 2] that the classical Boltzmann equation under the no correlation condition describes long waves in a dense gas with a short-range interaction potential. The same equation, which is called the Vlasov equation [3] in the one-dimensional case, is clearly equivalent to the hydrodynamic equations for long waves in an ideal incompressible liquid with a free surface under gravity. It also quite easy to see that this equation in the classical “random phase” approximation reduces to the completely integrable nonlinear Schrödinger equation [1, 2] on an axis $\mathbb{R}$. These equivalences for the hydrodynamic Benney type equations can be used for studying chaos in many-particle systems and turbulence arising in fluid flow. Yet the dynamical many-particle systems discussed in [1, 2] do not possess an important intrinsic property of particle motion in a liquid - convective mass transfer of particles in a fixed volume - which is known to always accompany a transition from laminar to turbulent flow and cause convective vortex motion. Moreover, these models do not possess an intrinsic “dry” viscosity for the particle flows.

To overcome in part the inadequacies noted for the Benney type hydrodynamic model, in this investigation we introduce a new generalized dynamical system for the flow of particles on an axis; namely its Boltzmann equation in the Vlasov approximation with no many-particle correlation, which describes the long waves in a dense gas of particles with a short-range interaction potential. Then the associated Benney type system of equations contains the convective terms in a form that is especially convenient for describing turbulence [5]. Moreover, the mathematical model of interacting particles on an axis $\mathbb{R}$ we choose is such that the associated Benney type system of equations is bi-Hamiltonian with an infinite hierarchy of polynomial conservation laws in involution. We also find an interesting connection between the Benney type system and a quasiclassical approximation of a nonlinear Schrödinger type dynamical system [6] on an axis.

1. BOLTZMANN EQUATION AND ASSOCIATED MOMENT PROBLEM

1.1 Let us consider a quantum dynamical system on $\mathbb{R}$ consisting of $N$ identical spinless particles with the singular Hamiltonian

$$
\hat{H} = -\frac{\hbar^2}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k) \left[ \beta + \alpha \frac{\hbar}{i} \left( \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right) \right],
$$

(1)

where $\alpha$ and $\beta$ are real parameters, $\hbar$ is Planck’s constant (divided by $2\pi$) and $\delta(x-y)$, $x, y \in \mathbb{R}$, is the Dirac delta-function. Then Wigner’s transformation [3, 7] to a quasiclassical limit as $\hbar \to 0$ yields $\hat{H} \xrightarrow{\hbar \to 0} H$, where the classical Hamiltonian function $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ has the form

$$
H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k) [\beta + \alpha (p_j + p_k)].
$$

(2)
Here the \( x_j \in \mathbb{R}, 1 \leq j \leq N \), are the coordinates of the particles on the axis \( \mathbb{R} \). The Heisenberg commutator for dynamical observables \([3]\) becomes the standard canonical Hamiltonian bracket \( \{\cdot, \cdot\} \), viz.

\[
\{\cdot, \cdot\} \xrightarrow{\hbar \to 0} \{\cdot, \cdot\},
\]

in accordance with the Bohr principle. Therefore, on the phase space \( M = T^*(\mathbb{R}^N) \) the Hamiltonian equations take the following form:

\[
dx_j/dt = \{H, x_j\} = \partial H/\partial p_j, \quad dp_j/dt = \{H, p_j\} = -\partial H/\partial x_j,
\]

where \( t \in \mathbb{R} \) is an evolution parameter and \((x_j, p_j) \in T^*(\mathbb{R})\), \( 1 \leq j \leq N \).

In view of the singularities in (2), the equations (4) are in general not solvable for arbitrary Cauchy data and large \( N \in \mathbb{Z}^+ \). Therefore because of our “hydrodynamical” interest in the motion of (4), we pass further to its statistical description \([3]\) using the Boltzmann-Bogoliubov distribution function \( F \): \((\mathbb{R} \times \mathbb{R}) \times \mathbb{R} \to D'((\mathbb{R} \times \mathbb{R})) \) defined by

\[
F(x, p; t) := \sum_{j=1}^{N} \delta(x - x_j(t)) \delta(p - p_j(t)).
\]

Here \((x, p) \in \mathbb{R} \times \mathbb{R} \) and \((x_j(t), p_j(t)) \in \mathbb{R} \times \mathbb{R}, 1 \leq j \leq N \), is a solution of the Hamiltonian equations (4). The distribution function (5) satisfies the standard Liouville-Hamilton equation

\[
dF/dt = \{F, H\}. \tag{6}
\]

1.2 Now we apply the averaging operator \(< \cdot >\) to the distribution function (5) assuming no many-particle correlation over all initial states of (6). The averaging operation on (6) results in the kinetic Boltzmann-Vlasov equation \([3, 7]\) of the form

\[
df/dt = <\{F, H\}> := \{\{F, H\}\}, \tag{7}
\]

where \( f = f(x, p; t) := <F(x, p; t)> \) is the statistically averaged distribution function (5) and \( \{\{\cdot, \cdot\}\} \) is the new “averaged” Poisson bracket on the infinite-dimensional functional space \( M(f) \subset C^\infty(\mathbb{R}^2, \mathbb{R}_+) \), which for a pair of functionals \( \gamma, \mu \in D(M(f)) \) has the form \([2, 7]\):

\[
\{\gamma, \mu\} = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dp f(x, p; t) \{\text{grad}\gamma, \text{grad}\mu\}(x, p; t) \tag{8}
\]

and is called the Lie-Poisson bracket \([2, 8]\). The Hamiltonian \( H \in D(M(f)) \) in (7) is given by

\[
H = \int_{\mathbb{R}} dx \left[ \int_{\mathbb{R}} dp \frac{p^2}{2m} f(x, p; t) + \frac{\beta}{2} \left( \int_{\mathbb{R}} dp f(x, p; t) \right)^2 + \alpha \int_{\mathbb{R}} df(x, p; t) \int_{\mathbb{R}} dq f(x, q; t) \right] \tag{9}
\]

and \text{grad} is the standard Euler variational derivative on \( D(M(f)) \).
To derive (8) let us consider on the phase space $M \subset T^*(\mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$ the canonical Poisson bracket $\{f, g\} = \partial_p f \partial_x g - \partial_p g \partial_x f$, where $f, g \in D(M)$. The space $D(M)$ of smooth functions on $M$ has the structure of a Lie algebra $G \cong (D(M); \{\cdot, \cdot\})$ with respect to this bracket.

Let $G^*$ be the adjoint or dual of $G$, i.e., the space of continuous linear functionals on $G$. The space $G$ is a Hilbert space with respect to the scalar product defined by

$$(f, g) := \int_\mathbb{R} dx \int_\mathbb{R} dp f(x, p) g(x, p).$$

Then $G^* \cong G$ follows from Riesz’s theorem and we note that the above scalar product is invariant with respect to the Poisson bracket $\{\cdot, \cdot\}$ in the sense that

$$(f, \{g, h\}) = (\{f, g\}, h)$$

for all $f, g, h \in G$. This structure enables us to determine the map $\text{grad}: D(G^*) \to G$ by means of the formula (grad$\gamma(f), g) = \frac{d}{d\varepsilon} \gamma(f + \varepsilon g) \big|_{\varepsilon=0}$ for arbitrary $f, g \in G^* \cong G$. Consequently, grad$\gamma(f) \in G$ is completely equivalent to the variational Euler derivative of the functional $\gamma \in D(G^*)$ at the point $f \in G^* \cong G$. For convenience we will also denote grad$\gamma(f)$ by $\nabla \gamma(f)$.

The canonical Hamiltonian structure $\{\cdot, \cdot\}$ on the manifold $G^*$ can now be expressed via the well-known Lie-Poisson formula [2, 8, 9]

$$\{\gamma, \mu\} = (f, \{\nabla \gamma(f), \nabla \mu(f)\})$$

which coincides with (8). To reveal the essence of the formula (7) we consider a coadjoint action of the Lie algebra on $G^*$ as follows: $df/dt = ad_{\nabla \gamma(f)} f$, where $t$ is a real evolution parameter and $\nabla \gamma(f) \in G$ at $f \in G^*$. Then owing to the invariance of the scalar product on $G$, the above vector field is equivalent to the following Lax type representation on $G$: $df/dt = \{f, \nabla \gamma(f)\}$, which in turn is equivalent to (8) after an identification $\gamma \equiv H \in D(M_f) \subset D(G^*)$.

It follows from (7) that the Hamiltonian function $H$ given by (9) is a conservation law for the Boltzmann-Vlasov system (7), i.e., $dH/dt = 0$ for all $t \in \mathbb{R}$. Apart from this conservation law, the dynamical system (7) possesses the following additional invariant functionals on $G^* \cong G$:

$$N = \int_\mathbb{R} dx \int_\mathbb{R} df(x, p; t), \quad P = \int_\mathbb{R} dx \int_\mathbb{R} dpf(x, p; t),$$

(10)

where $N$ is the number of particles and $P \in D(M_f)$ is the total particle impulse in the system.

Below we shall show that the Boltzmann-Vlasov system (7) with Hamiltonian (9) can be represented in the equivalent commutator form

$$df/dt = \{f, \text{grad}H(f)\},$$

(11)

where $f \in D(M) \cong G^* \cong G$, and has an infinite involutive (with respect to the Lie-Poisson bracket) hierarchy of conservation laws yielding complete integrability [8].

1.3 The exact form of the Boltzmann-Vlasov equation (11) is

$$f_t = -pf_x - \alpha(a_0 f)_x + (\beta a_0 + \alpha a_1)_x f_p + \alpha a_{0,x}(pf)_p,$$

(12)
where the momentum functionals $a_n^{(x)} \in M(\mathbb{Z}_+), n \in \mathbb{Z}_+, x \in \mathbb{R}$ are defined as follows:

$$a_n(x, t) := \int_\mathbb{R} dx p^n f(x, p; t).$$  \hspace{1cm} (13)

In momentum terms (13), the Hamiltonian (9) as a functional $H \in D(M(\mathbb{Z}_+))$ takes the form

$$H = \frac{1}{2} \int_\mathbb{R} dx (a_2 + \beta a_0^2 + 2\alpha a_0 a_1),$$  \hspace{1cm} (14)

with the Lie-Poisson bracket (8) on the manifold $M(\mathbb{Z}_+)$ being given by the expression [2, 9, 10]:

$$\{\gamma, \mu\}_{\mathcal{L}} := \int_\mathbb{R} dx \langle \text{grad}\gamma, \theta(a)\text{grad}\mu \rangle,$$  \hspace{1cm} (15)

where $\theta(a) := [ma_{m+n-1} \partial + n \partial a_{m+n-1}], m, n \in \mathbb{Z}_+$ is a skew-symmetric matrix operator on $M(\mathbb{Z}_+)$, grad(...) = $(\delta/\delta a_n(...))^T, n \in \mathbb{Z}_+$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on the space $l_2(\mathbb{R})$ of real square-summable sequences.

The dynamical system for the momentum functions corresponding to (12) is

$$\frac{da_n}{dt} = \{H, a_n\}_\theta = -a_{n+1,x} - \alpha a_n a_{0,x} - na_{n-1}(\beta a_0 + \alpha a_1)_x \alpha(a_0 a_n)_x,$$ \hspace{1cm} (16)

where $n \in \mathbb{Z}_+$, and $\alpha, \beta$ are arbitrary real parameters. The equations (16) are called the generalized Benney type momentum system. In particular, for $n = 0$ and $n = 1$ we obtain the following from (16):

$$\frac{da_0}{dt} = -a_{1,x} - \alpha(a_0^2)_x$$

$$\frac{da_1}{dt} = -a_{2,x} - 2\alpha(a_0 a_1)_x - \beta a_0 a_0 a_x.$$  \hspace{1cm} (17)

Substituting $f(x, p; t) = \rho(x, t)\delta(p - u(x, t)), (x, p) \in \mathbb{R}^2$ in (17), for the case of a hydrodynamic representation of the averaged distribution function (6) we obtain the following new nonlinear Benney type system of equations:

$$K[u, \rho] = \{ u_t = -uu_x - (\beta \rho + 2\alpha u \rho)_x, \rho_t = -(u \rho)_x - 2\alpha \rho \rho_x \},$$ \hspace{1cm} (18)

where $(u, \rho)^T \in M(u, \rho) \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R} \times \mathbb{R})$ owing to natural physical considerations for the system of particles.

2. COMPLETELY INTEGRABLE NONLINEAR SCHRODINGER TYPE DYNAMICAL SYSTEM ON AN AXIS AND ITS QUASICLASSICAL APPROXIMATION

2.1 We now consider the quantum manyparticle Hamiltonian operator (1) on $\mathbb{R}$ and obtain its second quantized [3, 6] representation in the Hilbertian Fock space $\Phi$ via the scheme

$$\left(L_2(\mathbb{R}^N; \mathbb{C}) \leftarrow L_2(\mathbb{R}^N; \mathbb{C}) : \hat{H} \right) \xrightarrow{\mathcal{H}} (H : \Phi \to \Phi).$$ \hspace{1cm} (19)
Here the operator $H : \Phi \to \Phi$ is representable by the expression

$$H = \frac{1}{2} \int_{\mathbb{R}} dx \left[ \psi_x^+ \psi + \beta \psi^+ \psi \psi + i \alpha (\psi^+ \psi_x \psi - \psi^+_x \psi^+ \psi \psi) \right], \quad (20)$$

where $\psi^+(x), \psi(x) : \Phi \to \Phi, x \in \mathbb{R}$ are, respectively, creation and annihilation operators of one-particle spinless states in the Fock space which satisfy the canonical commutation relations

$$[\psi(x), \psi^+(y)] = \delta(x - y), \quad x, y \in \mathbb{R}$$

$$[\psi(x), \psi(y)] = [\psi^+(x), \psi^+(y)] = 0 \quad (21)$$

In computing (20) we have for convenience set $\hbar = 1$, which can always be achieved by rescaling the axis $\mathbb{R}$. The Heisenberg dynamical equations [3, 5] for the operators $\psi(x), \psi^+(x)$ are given as

$$i \psi_t = [\psi, H], \quad i \psi^+_t = [\psi^+, H],$$

which in explicit form are

$$i \psi_t = -\frac{1}{2} \psi_{xx} + \beta \psi^+ \psi \psi + 2i \alpha \psi^+ \psi_x, \quad i \psi^+_t = \frac{1}{2} \psi^+_{xx} - \beta \psi^+ \psi^+ \psi^+ + 2i \alpha \psi^+ \psi^+_x \psi^+. \quad (22)$$

In the classical field-theoretic approximation [3, 11, 12] the quantum dynamical system (22) under the map

$$(\Phi \leftarrow \Phi : (\psi \psi^*)^T) \to ((\psi \psi^*)^T \in M(\psi, \psi^*) \subset C^\infty(\mathbb{R}/2\pi \mathbb{Z}; \mathbb{C}^2))$$

turns into the nonlinear Schrödinger type dynamical system

$$K[\psi, \psi^*] = \left\{ \begin{array}{ll}
\psi_t = -\frac{1}{2} \psi_{xx} - i \beta \psi^* \psi^* + 2 \alpha \psi^* \psi_x \\
\psi^*_t = \frac{1}{2} \psi^*_{xx} + i \beta \psi^* \psi^* + 2 \alpha \psi^*_x \psi^*
\end{array} \right. \quad (23)$$

on the functional manifold $M(\psi, \psi^*)$.

As shown in [6], the systems (22) and (23) are completely integrable both in the quantum and classical cases. The last property is very important because, as will be shown in the sequel, a quasiclassical approximation for the dynamical system (23) via the scheme in [1] yields the nonlinear hydrodynamic Benney type system (18) obtained earlier. Using the complete integrability of (23) we can show that the Benney type dynamical system (18) possesses on the manifold $M(u, \rho)$ an infinite hierarchy of commuting polynomial conservation laws in relation to two compatible implectic structures.

2.2 We use the change of variables

$$\psi(x) = \sqrt{\rho(x)} \exp \left( -i \int_{x_0}^x u(y) dy \right),$$

$$\psi^*(x) = \sqrt{\rho(x)} \exp \left( i \int_{x_0}^x u(y) dy \right), \quad (24)$$

where $\rho \in C^\infty(\mathbb{R}/2\pi \mathbb{Z}; \mathbb{R}_+)$ is a quasiclassical “density” of the fields (24), $u \in C^\infty(\mathbb{R}/2\pi \mathbb{Z}; \mathbb{R}_+)$ is a phase “velocity” and $x_0$ is an arbitrary point on $\mathbb{R}$. 

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Substituting (24) into (23) and carrying out the quasiclassical approximation \( \frac{d}{dt} \rightarrow \varepsilon \frac{d}{dt}, \frac{d}{dx} \rightarrow \varepsilon \frac{d}{dx}, \) after transition to the limit \( \varepsilon \rightarrow 0 \) we immediately obtain the hydrodynamic system (18) defined on the functional manifold \( \mathcal{M}_{(u,\rho)} \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}_+ \times \mathbb{R}_+) \).

It follows [6] from the complete integrability of the dynamical system (23) on \( \mathcal{M}_{(\psi,\psi^*)} \) that there exists an infinite hierarchy of commuting conservation laws, namely

\[
H_0 = \int_{0}^{2\pi} dx \psi^* \psi, \quad H_1 = i \int_{0}^{2\pi} dx (\psi_x^* \psi - \psi^* \psi_x), \\
H_2 = \int_{0}^{2\pi} dx (\psi_x^* \psi_x + \beta \psi^* \psi^* \psi + 2i \alpha \psi^* \psi \psi_x), ...
\]  

The corresponding Poisson bracket on \( \mathcal{M}_{(\psi,\psi^*)} \) is given by

\[
\{ \psi(x), \psi^*(y) \} = -\delta(x - y), \\
\{ \psi(x), \psi(y) \} = 0 = \{ \psi^*(x), \psi^*(y) \}
\]

for all \( x, y \in \mathbb{R} \).

After substituting (24) into (25) we get an infinite hierarchy of hydrodynamic conservation laws for the Benney type system (18):

\[
H_0 = \int_{0}^{2\pi} dx \rho, \quad H_1 = \int_{0}^{2\pi} dx u \rho, \\
H_2 = \frac{1}{2} \int_{0}^{2\pi} dx (u^2 + \beta \rho^2 + 2\alpha \rho^2 u), ...
\]  

The first three conservation laws in (26) have the following physical interpretations: \( H_0 \) is the mass conservation law, \( H_1 \) is the impulse conservation law and \( H_2 \) is the conservation of energy law for the liquid.

Analogously, the symplectic structure defined on \( \mathcal{M}_{(\psi,\psi^*)} \) carries over to a symplectic structure on the manifold \( \mathcal{M}_{(u,\rho)} \) and for all \( x, y \in \mathbb{R} \) the following hold:

\[
\{ u(x), u(y) \}_\theta = 0 = \{ \rho(x), \rho(y) \}_\theta \\
\{ u(x), \rho(y) \}_\theta = \delta'(x - y) = \{ \rho(x), u(y) \}_\theta.
\]  

Hence the system (18) on \( \mathcal{M}_{(u,\rho)} \) is the Hamiltonian flow representable in the form

\[
\frac{d}{dt} (u, \rho)^T = -\theta \text{grad}H_\theta = K[u, \rho],
\]

where, according to (27), the implectic operator \( \theta : T^*(\mathcal{M}_{(u,\rho)}) \rightarrow T(\mathcal{M}_{(u,\rho)}) \) is defined by the expression

\[
\theta := \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix},
\]

with the functional \( H_\theta = H_2 \in D(\mathcal{M}_{(u,\rho)}) \) being the Hamiltonian. This result is a natural consequence of the fact that in the quasiclassical limit of the Bäcklund transformation (24) [6], the Schrödinger type dynamical system (23) with an infinite hierarchy of symplectic structures for all \( \alpha, \beta \in \mathbb{R} \) transforms into the Hamiltonian system (18) with an infinite hierarchy of symplectic structures of hydrodynamic type [13]. Whence, in analogy to the statement in [14],
the Benney type system (18) is essentially bi-Hamiltonian with the compatible pair of implectic operators \((\theta, \eta)\) [6], where \(\eta : T^*(M(u,\rho)) \to T(M(u,\rho))\) is defined by

\[
\eta := \left[ \begin{array}{cc} \beta \partial \alpha + \alpha(u \partial + \partial u) & 2^{-1} \partial u + \alpha \rho \partial \\ \alpha \partial \rho + 2^{-1} u \partial & 2^{-1}(\rho \partial + \partial \rho) \end{array} \right]. \tag{30}
\]

Using the operator (30) with the analogy to (28), we may write

\[
\frac{d}{dt}(u,\rho)^T = -\theta \text{grad} H = K[u,\rho], \tag{31}
\]

where \(H_2 = H_1 \in D(M(u,\rho))\).

The hereditary compatibility of the implectic pair (29)-(30) of Noetherian operators \((u,\rho)\) implies that (18) possesses an infinite hierarchy of commuting conservation laws \(\gamma_j, j \in \mathbb{Z}_+\) which may be computed using the following formula:

\[
\text{grad} \gamma_j = (\theta^{-1} \eta)^j \text{grad} \gamma_0, \tag{32}
\]

where \(\gamma_0 := H_0\). In particular,

\[
\begin{align*}
\gamma_0 &= \int_0^{2\pi} dx \rho, & \bar{\gamma}_0 &= \frac{1}{2} \int_0^{2\pi} dx u, & \gamma_1 &= \int_0^{2\pi} dx u \rho, \\
\gamma_2 &= \frac{1}{2} \int_0^{2\pi} dx (u^2 \rho + \beta \rho^2 + 2\alpha \rho^2 u), \\
\gamma_3 &= \frac{1}{4} \int_0^{2\pi} dx (u^3 \rho + 6 \alpha u^2 \rho^2 + 4 \alpha^2 \rho^3 u + 3 \beta \rho^2 u + 2 \alpha \beta \rho^3), & \ldots,
\end{align*} \tag{33}
\]

where the functionals \(\bar{\gamma}_0, \gamma_0 \in D(M(u,\rho))\) are conservation laws obtained directly from the exact form of the hydrodynamic system (18).

2.3 The Benney type system (18) obviously admits the following manycomponent generalization which is different from that considered in [9]:

\[
K_N^N [u,\rho] = \left\{ \begin{array}{l}
\frac{du_n}{dt} = -u_n u_{n,x} - \frac{d}{dx} \left[ \sum_{k=1}^{N} \rho_k (\beta + 2\alpha u_n) \right] \\
\frac{d\rho_n}{dt} = -\rho_n u_{n,x} - 2\alpha \sum_{k=1}^{N} \rho_k \rho_n \\
\end{array} \right., \tag{34}
\]

where \(1 \leq n \leq N\) and \(N\) is an arbitrary positive integer.

It is easily seen that the dynamical system (34) is Hamiltonian on the manifold \(M(u,\rho)\) and

\[
\frac{d}{dt}(u,\rho)^T = -\theta \text{grad} H_\theta^{(N)} = K^{(N)}[u,\rho], \tag{35}
\]

where the Hamiltonian function \(H_\theta^{(N)} \in D(M(u,\rho))\) has the form

\[
H_\theta^{(N)} = \frac{1}{2} \int_0^{2\pi} dx \left[ \sum_{n=1}^{N} \rho_n u_n^2 + \sum_{m,n=1}^{N} (\beta \rho_m \rho_n + 2\alpha \rho_m \rho_n u_n) \right]. \tag{36}
\]
The system (34) is obtained from the quasiclassical limit [1] of a manycomponent generalization of the Schrödinger type nonlinear dynamical system (23), written in the form

\[ \psi_{n,t} = \frac{i}{2} \psi_{n,xx} - \sum_{k=1}^{N} (i \beta \psi_{n} \psi_{k}^{*} \psi_{k} - 2 \alpha \psi_{n,x} \psi_{k}^{*} \psi_{k}) , \]

\[ \psi_{n,t}^{*} = -\frac{i}{2} \psi_{n,xx}^{*} + \sum_{k=1}^{N} (i \beta \psi_{n}^{*} \psi_{k} \psi_{k}^{*} + 2 \alpha \psi_{n,x} \psi_{k}^{*} \psi_{k}) , \]

\( 1 \leq n \leq N \). Namely, using the expressions

\[ \psi_{n}(x) = \sqrt{\rho_{n}(x)} \exp \left( -i \int_{x}^{x_{0}} u_{n}(y) dy \right) , \]

\[ \psi_{n}^{*}(x) = \sqrt{\rho_{n}(x)} \exp \left( i \int_{x_{0}}^{x} u_{n}(y) dy \right) , \]

\( 1 \leq n \leq N \), in (37) and passing to the quasiclassical limit, we obtain the manycomponent hydrodynamical system (34). As (37) is a bi-Hamiltonian completely integrable flow on the manifold \( M_{(\psi, \psi^{*})} \), we infer that the one-dimensional Benney type system (34) is also bi-Hamiltonian with an infinite hierarchy of polynomial conservation laws. We shall develop this further in a forthcoming article.

3. COMPLETE INTEGRABILITY OF THE BENNEY TYPE SYSTEM ASSOCIATED WITH THE BOLTZMANN-VLASOV EQUATION

3.1 As stated in the article [2] and in [9] for a more general multicomponent case, the Boltzmann-Vlasov equation for a manyparticle system on an axis with a special singular interaction is equivalent to the equation for a flow of an incompressible liquid with a free surface subject to gravity. These equations, as was first indicated in [10], are completely Lax integrable via the Adler-Kostant scheme [8, 16, 17, 18, 19]. Hence there is great interest in solving the analogous problem for the Boltzmann-Vlasov system (12) and also for its two-dimensional analogue - the Benney type hydrodynamic equations.

To ward these goals, let us rewrite the generalized Benney equations in exact form by introducing the new variable

\( \mathbb{R}_{+} \ni y = \int_{-\infty}^{u(x,y;t)} df(x,p,t) , \)

which makes it possible to invert the above integral for \( u \in C^{\infty}(\mathbb{R}/2\pi; \mathbb{R}) \) for each value of the parameter \( t \in \mathbb{R} \). Owing to (12), we obtain the two-dimensional Benney type system of equations

\[ \left( \begin{array}{c}
\frac{u}{p} \\
\frac{\rho}{p}
\end{array} \right) = K[u; \rho] := \left( \begin{array}{c}
-uu_{x} + u_{y} \int_{0}^{y} dy(u + \alpha \rho)_{x} - \alpha \frac{\partial}{\partial x} \int_{0}^{\rho} dy - [(\alpha u + \beta)\rho]_{x} \\
-\frac{\partial}{\partial x} \int_{0}^{\rho} dy(u + 2\alpha y)
\end{array} \right) . \]

Here \( u(x,y;t) \) is the horizontal velocity of a particle in the fluid at the vertical level \( y \in [0, \rho(x;t)] \subset \mathbb{R}_{+} \) and \( \rho(x;t) \) is the height of the liquid surface at time \( t \).
Obviously the dynamical system (39) for the case of static flow, i.e., when \( u \equiv 0 \), is the hydrodynamic system (18) which also has applications to gas dynamics when convection is taken into account.

If one introduces, in analogy to (13), the dynamical moment functions \( a_n \in M_{(Z_+)} \) defined as

\[
a_n(x) := \int_0^{\rho(x,t)} dy u^n(x,y,t),
\]

then it is easy to see that (39) is equivalent to (16) in these new coordinates. Therefore we have full equivalence [2, 9] of the Boltzmann-Vlasov equation (12) to both the moment Benney type system of equations and the hydrodynamic system (39) describing incompressible fluid flow with a free surface in convective terms.

3.2 To verify the complete integrability of the Benney type system (39), we shall first study its Lax type integrability using algebraic methods [8, 9, 15, 21]. Toward this end it is useful to describe certain algebraic concepts.

Let \( G \) be an arbitrary metrizable Lie algebra [22] over \( \mathbb{C} \), that is, \( G \) has an invariant symmetric nondegenerate scalar product \( (\cdot,\cdot) \) which satisfies

\[
(l_1,l_2) = (l_2,l_1), \quad (l_1,[l_2,l_3]) = ([l_1,l_2],l_3)
\]

for all \( l_1, l_2, l_3 \in G \), where \([\cdot,\cdot]\) is the Lie bracket on \( G \). If, in addition, \( G \) is reflexive, i.e., \( G^{**} \cong G \), then in view of (41), \( G^* \) can be identified with \( G \) as the modulus \( \mathcal{G} \) with respect to \( \mathbb{C} \).

Definition. A Lie algebra \( G \) together with a linear map \( \mathcal{R} : \mathcal{G} \to \mathcal{G} \) is called an \( \mathcal{R} \)-structure if the bracket defined as

\[
[l_1,l_2]_{\mathcal{R}} := [\mathcal{R}l_1,l_2] + [l_1,\mathcal{R}l_2]
\]

coincides with the second Lie bracket on \( \mathcal{G} \) for all \( l_1, l_2 \in G \).

It is easy to show that

\[
-[[\mathcal{R}l_1,\mathcal{R}l_2] + \mathcal{R}[l_1,\mathcal{R}l_2]]_{\mathcal{R}} = \nu[l_1,l_2]
\]

for \( l_1, l_2 \in G \) and some complex number \( \nu \) is a necessary condition for \( [\cdot,\cdot]_{\mathcal{R}} \) to be the second Lie bracket on \( \mathcal{G} \). The relationship (43) is called the classical Yang-Baxter (YB(\( \nu \))) equation [22], and it plays an important role in the theory of Lax integrability.

To find a solution of (43), let us suppose that \( G \) is a direct sum of a pair of its subalgebras, i.e., \( G = G_+ \oplus G_- \). Denoting the projection operators (projectors) on \( G_+ \) and \( G_- \) by \( P_+ \) and \( P_- \), respectively, it is easy to verify that

\[
\mathcal{R} = P_+ - P_-
\]

is a solution of the Yang-Baxter equation for \( \nu = 1 \). Hence (44) provides an \( \mathcal{R} \)-structure on \( G \). Given any solution of the Yang-Baxter equation (43), additional \( \mathcal{R} \)-structures can be obtained from (44) via composition \( \mathcal{R} \circ \mathcal{R} \circ \mathcal{R} \circ \cdots \); however, there are only three independent such
structures [21] given by

\begin{align*}
(1) \ [l_1, l_2], \\
(2) \ [l_1, l_2]_R &= 2[l_{1+}, l_{2+}] - 2[l_{1-}, l_{2-}], \\
(3) \ [l_1, l_2]_{R^*} &= 4[l_{1+}, l_{2+}] + 4[l_{1-}, l_{2-}],
\end{align*}

where \( l_j \in G \) and \( l_{j\pm} \in P_{\pm}l_j \in G_{\pm}, j = 1, 2. \)

When \( \nu = 0 \) it is easy to show that any symplectic 2-cocycle \( \omega : G \times G \to \mathbb{C} \), where

\[ \omega(l_1, l_2) = (R^{-1}l_1, l_2), \]
\[ \omega(l_1, [l_2, l_3]) + \omega(l_2, [l_3, l_1]) + \omega(l_3, [l_1, l_2]) = 0, \]

for all \( l_j \in G, 1 \leq j \leq 3 \), defines the “unitary” solution of \( \text{YB}(0) \).

Let us now consider the space \( D(G^*) \) of all Fréchet smooth functionals on the adjoint \( G^* \), on which there are two canonical Lie-Poisson structures \( \{\cdot, \cdot\}_L \) and \( \{\cdot, \cdot\}_\theta \); namely, defined for \( f, g \in D(G^*) \) and \( l \in G^* \) by

\[ \{f, g\}_L(l) := (l, [\nabla f(l), \nabla g(l)]), \]
\[ \{f, g\}_\theta(l) := (l, [\nabla f(l), \nabla g(l)]_R). \]

Here the “gradient” \( \nabla f(l) \in G \) for a given \( f \in D(G^*) \) is defined as

\[ (m, \nabla f(l)) := \frac{d}{d\varepsilon} f(l + \varepsilon m) \bigg|_{\varepsilon=0}, \]

for all \( m \in G^* \).

**Definition.** An element \( f \in D(G^*) \) is called a Casimir functional with respect to the Lie-Poisson bracket (47) (denoted \( f \in I(G^*) \)) if for all \( g \in D(G^*) \)

\[ \{f, g\} = (l, [\nabla f(l), \nabla g(l)]) = 0. \]

One has the following theorem [8, 11].

**Theorem 1.**

Let \( \mu, \gamma \in D(G^*) \) be Casimir functionals on \( G^* \) with respect to the Lie-Poisson bracket \( \{\cdot, \cdot\}_L \). Then:

1) \( \mu, \gamma \in D(G^*) \) are involutive on \( G^* \) with respect to \( \{\cdot, \cdot\}_\theta \);

2) The orbit \( \text{Orb}(K) \) of the vector field \( K : G^* \to T(G^*) \), defined by

\[ \frac{dl}{dt} = ad^*_R_{\nabla\gamma(l)}l = K(l), \]

where \( l \in G^* \) and \( t \in \mathbb{R} \) is a real evolution parameter, lies in a symplectic leaf of \( G^* \) with respect to \( \{\cdot, \cdot\}_\theta \).

The statement of Theorem 1 means that if for all \( l \in G^* \)

\[ ad^*_R_{\nabla\gamma(l)}l = 0, \quad ad^*_R_{\nabla\mu(l)}l = 0, \]

(50)
then \( \{ \gamma, \mu \}_\theta = 0 \) and the vector field (49) on \( G^* \) is Hamiltonian for the Poisson bracket \( \{ \cdot, \cdot \}_\theta = (\text{grad}(\cdot), \theta \text{grad}(\cdot)) \), where \( \theta = R^* L + L R \) in view of (47). In addition, because of the identification \( G^* \cong G \),

\[
dl/dt = [l, R \nabla \gamma(l)], \quad [l, \nabla \gamma(l)] = 0.
\] (51)

Thereby the vector field \( K \) of (49) can be written in the canonical form of the Lax type representation [8]. For the \( R \)-structure (44) the Lie Poisson bracket has the following form:

\[
\{ \gamma, \mu \}_\theta = 2(l, [\nabla \gamma(l)_+, \nabla \mu(l)_+]) - 2(l, [\nabla \gamma(l)_-, \nabla \mu(l)_-]).
\] (52)

In particular the Lax type equation (51) may be rewritten as

\[
dl_+ / dt = -2[l_+, \nabla \gamma(l)_-],
\] (53)

and

\[
dl_- / dt = 2[l_-, \nabla \gamma(l)_+],
\] (54)

where \( l = l_+ \oplus l_- \in G^* \cong G \) and \( \gamma \) is a Casimir functional on \( G^* \) with respect to the canonical Lie-Poisson bracket \( \{ \cdot, \cdot \}_\mathcal{L} \). Since obviously (53) and (54) leave the subspaces \( G^*_\pm \) invariant, (51) on \( G^* \cong G \) is the natural augmentation of the dynamical systems constructed by Adler, Symes and Kostant [8, 16, 17, 18, 19, 23] on the invariant subalgebras \( G_\pm \subset G \). This augmentation makes it possible for both the Lax type representation and the regular Hamiltonian formalism to be constructed for the hydrodynamic Benney type system (39).

We have shown above that for the dynamical system \( K : G \to T(G^*) \) given by (51), there exists the Lie-Poisson bracket on \( D(G^*) \) given by (47) for which it is Hamiltonian. In [21] it was observed that for the case of a metrizable Lie algebra \( G \) with a unitary \( R \)-structure and a commutator Lie bracket (i.e, \( [l_1, l_2] = l_1 \circ l_2 - l_2 \circ l_1, l_1, l_2 \in G \), on the functional space \( D(G^*) \)), there exists another Lie-Poisson bracket \( \{ \cdot, \cdot \}_\eta \) defined by

\[
\{ \gamma, \mu \}_\eta := (l \circ \nabla \gamma(l), R l \circ \nabla \mu(l)) - (\nabla \gamma(l) \circ l, R \nabla \mu(l) \circ l).
\] (55)

Here \( \gamma, \mu \in D(G^*) \) and “\( \circ \)” is an associative product operation in the Lie algebra \( G \) which turns it into the ring \( \bar{G} \) over \( \mathbb{C} \). As is stated in [21], the bracket (55) is compatible via Magri [8, 24] with the Lie-Poisson bracket \( \{ \cdot, \cdot \}_\theta \) if \( \gamma \) is a Casimir functional on \( G^* \) with respect to \( \{ \cdot, \cdot \}_\eta \). Therefore, if the above conditions on the \( R \) -structure on \( G \) are satisfied, the dynamical system (51) on \( G^* \) is bi-Hamiltonian with an infinite involutive hierarchy of conservation laws.

In order to use the above algebraic scheme for the Benney type system being studied, let us consider the Lie algebra realization of \( G \) as a ring of pseudodifferential operators on \( \mathbb{R} \) with symbol \( \xi \):

\[\mathcal{G} = \bigcup_{m \in \mathbb{Z}_+} \left\{ a(x; \xi) = \sum_{j=-\infty}^{m} a_j(x) \xi^j : a_j \in \mathcal{G}(\mathbb{R}, \mathbb{C}) \right\},\] (56)
where $\mathcal{G}(\mathbb{R}, \mathbb{C})$ is a ring of functions on $\mathbb{R}$. The product operation on $\mathcal{G}$ is defined as follows:

$$a \circ b := \sum_{j \in \mathbb{Z}_+} \frac{1}{j!} \frac{\partial^j a(x; \xi)}{\partial \xi^j} \frac{\partial^j b(x; \xi)}{\partial x^j},$$

(57)

for $a, b \in \mathcal{G}$. There is a further useful decomposition of $\mathcal{G}$ into two Lie subalgebras with respect to the Lie bracket $[\cdot, \cdot]$, namely $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, where

$$\mathcal{G}_+ := \bigcup_{m \in \mathbb{Z}_+} \left\{ \sum_{j=0}^{m} u_j(x) \xi^j : u_j \in \mathcal{G}(\mathbb{R}; \mathbb{C}) \right\},$$

$$\mathcal{G}_- := \left\{ \sum_{j \in \mathbb{Z}_+} \xi^{(j+1)} \circ a_j(x) : a_j \in \mathcal{G}(\mathbb{R}; \mathbb{C}) \right\}.$$

(58)

Define the “trace” functional $Tr : \mathcal{G} \to \mathbb{C}$ by means of the formula

$$Trb(\xi) = Tr \left( \sum_{j=0}^{m(l)} u_j \xi^j + \sum_{j \in \mathbb{Z}_+} \xi^{-(j+1)} \circ a_j \right) := \int_{\mathbb{R}} dx \text{res}_{\xi=0} l(\xi) = \int_{\mathbb{R}} dx a_0(x).$$

(59)

Hence

$$(a, b) = Tr(a \circ b),$$

(60)

is a nondegenerate, symmetric invariant scalar product satisfying

$$(l_+, l_-) = Tr(l_+ \circ l_-) = \int_{\mathbb{R}} dx \langle a, b \rangle,$$

(61)

where

$$l_+ = \sum_{j=0}^{m(l)} b_j \xi^j \in \mathcal{G}_+, \quad l_- = \sum_{j \in \mathbb{Z}_+} \xi^{(j+1)} \circ a_j \in \mathcal{G}_-$$

and $\langle \cdot, \cdot \rangle$ is the usual inner product in $l_2(\mathbb{Z}; \mathbb{C})$.

The ring of functions $\mathcal{G}(\mathbb{R}; \mathbb{C})$ carries the usual $L_2$ scalar product and the subalgebra $\mathcal{G}_-$ is endowed with the topology of $l_2(\mathbb{Z}; \mathcal{G})$. It follows from (61) that $\mathcal{G}_+$ is densely embedded in $\mathcal{G}_-^*$. Therefore the identifications $\mathcal{G}_-^* \cong \mathcal{G}_+$ and $\mathcal{G}_+^* \cong \mathcal{G}_-$ are valid, modulo the closure operation in $l_2(\mathbb{Z}; \mathcal{G})$.

3.3 Let us now consider the following quasiclassical limit of the Lie bracket of $\mathcal{G}$:

$$[a, b]_0 := \lim_{\varepsilon \to 0} [a, b]_\varepsilon = \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial x} - \frac{\partial b}{\partial \xi} \frac{\partial a}{\partial x},$$

(62)

where

$$[a, b]_\varepsilon := \frac{1}{\varepsilon} (a_0 b - b_0 a) \bigg|_{\frac{\partial}{\partial \xi} \to \frac{\partial}{\partial x}},$$

$$a, b \in \mathcal{G}_0 := \lim_{\varepsilon \to 0} \bigg|_{\frac{\partial}{\partial \xi} \to \frac{\partial}{\partial x}}.$$

Define an element $l(\xi) \in \mathcal{G}_0$ by the rule

$$\mathcal{G}_+^* \ni l(\xi) := l[a(\xi)] \in C^\infty(M(\mathbb{Z}_+); \mathcal{G}_+^*),$$

(63)
where, by definition,
\[
\{a_j : j \in \mathbb{Z}_+\} \in \mathcal{M}(\mathbb{Z}_+), \quad a(\xi) := \sum_{j \in \mathbb{Z}_+} a_j(x)\xi^{-(j+1)} \in \mathcal{G}_0 \cong \mathcal{G}_0^+.
\]
and \(l : M(\mathbb{Z}_+) \to \mathcal{G}_0^+\) is a standard moment map [8] with the equivalent coadjoint action of \(\mathcal{G}\) on \(M(\mathbb{Z}_+)\).

According to the representation (58), the Lie algebra \(\mathcal{G}_0\) also possesses a decomposition analogous to that of \(\mathcal{G}\), i.e. \(\mathcal{G}_0 = \mathcal{G}_{0+} \oplus \mathcal{G}_{0-}\). Whence by (47), for functionals \(\gamma, \mu \in D(\mathcal{G}_{0+})\), the Lie-Poisson bracket \(\{\cdot, \cdot\}_\theta\) may be written as
\[
\{\gamma, \mu\}_\theta = \int_\mathbb{R} dx \langle \text{grad}_\gamma, \theta(a)\text{grad}\mu \rangle,
\]
where the cosymplectic operator \(\theta : T^*(M(\mathbb{Z}_+)) \to T(M(\mathbb{Z}_+))\) is defined as
\[
\theta(a) = [\theta_{mn}(a)], \quad m, n \in \mathbb{Z}_+,
\]
\[
\theta_{mn}(a) := ma_{m+n-1} \frac{d}{dx} + n \frac{d}{dx}a_{m+n-1}.
\]

Thereby the Lie-Poisson bracket (64) is seen to coincide with the bracket (15) which is obtained from the Lie-Poisson bracket (8) via the canonical [9] map (13) of the manifold \(M(\mathcal{f})\) into the manifold \(M(\mathbb{Z}_+)\).

Suppose now that the expression (63) is representable in \(\mathcal{G}\) as follows:
\[
l[a(\xi)] = \xi + \frac{\beta}{2\alpha} + b(\xi),
\]
where \(\alpha \neq 0\) and \(\beta\) are real parameters and \(b(\xi) := \sum_{j \in \mathbb{Z}_+} b_j(x)\xi^{-(j+1)} \in \mathcal{G}_0^+ \cong \mathcal{G}_0^-.\) Then we have a map \(\hat{l} : \mathcal{G}_0^- \to \mathcal{G}_0^-\) defined by
\[
\hat{l}[a(\xi)] := b(\xi) = l[a(\xi)] - \xi - \frac{\beta}{2\alpha}.
\]

Let \(\gamma \in D(\mathcal{G}_0^0)\) be a Casimir functional, i.e for all \(b(\xi) \in \mathcal{G}_0^0, \quad ad^\gamma v_{\gamma(b)}(b) = 0,\) or equivalently in view of (41) and \(\mathcal{G}^*_0 \cong \mathcal{G}_0, [b(\xi), \nabla \gamma(b)] = 0.\) Suppose further that there exists an \(\omega \in \mathcal{G}_0\) such that \((\omega, [\mathcal{G}_{0\pm}, \mathcal{G}_{0\pm}]) = 0.\) Then we have the following result [8, 15]:

**Theorem 2.**

The dynamical system \(db/dt = ad^\gamma \nabla \gamma(b)(\xi)\) on \(\mathcal{G}_0^+ \cong \mathcal{G}_0^-\), where \(\nabla \gamma(b) := \nabla \gamma(b + \omega),\) is the standard completely integrable Hamiltonian flow on \(\mathcal{G}_0^+\) having the Lax type representation
\[
\frac{d}{dt} (b(\xi) + \omega) = [b(\xi) + \omega, \nabla \gamma(b)]_+.
\]

**Sketch of Proof:** As \(\gamma\) is a Casimir functional on \(\mathcal{G}_0^0, [\nabla \gamma(\omega + b), \omega + b] = 0\) for all \(b \in \mathcal{G}_0^0\) and also \(\nabla \gamma(\omega + b) = \nabla \gamma_\omega(b)_+ \oplus \nabla \gamma_\omega(b)_-,\) so we immediately obtain from (67) the Lax type representation (68) taking into account the invariance of \(\omega, i.e d\omega/dt = 0\) on \(\mathbb{R}.\)
Define the functionals \( \gamma_j := Tr(b + \omega)^j, j \in \mathbb{Z}_+, b \in G_{0-} \). As the functionals \( \tilde{\gamma}_j = Tr b^j \) are Casimir on \( G_0^* \), any \( \gamma_j \in D(G_0^*) \) generates, due to the equality
\[
\nabla \gamma_j(b) = \nabla \tilde{\gamma}_j(b + \omega) = j(b + \omega)(j-1),
\]
the completely integrable dynamical system
\[
\frac{d}{dt} \gamma_j(b) = j[b + \omega, (b + \omega)(j-1)],
\]
where the \( t_j \) are evolution parameters and \( \{\gamma_j, \gamma_k\}_\theta = 0 \) for all \( j, k \in \mathbb{Z}_+ \). Hence, if the Casimir system of invariants \( \{\tilde{\gamma}_j\} \subset D(G_0^*) \) is complete, i.e. for any \( \gamma \in D(G_0^*) \) we have
\[
\nabla \gamma(b) \in \operatorname{span}_\mathbb{C} \{\nabla \tilde{\gamma}_j(b) \in G_0 : j \in \mathbb{Z}_+ \}
\]
then the dynamical system (68) is a completely Lax integrable system on \( M(\mathbb{Z}_+) \), as was to be proved. □

Note that in our case of the Lie algebra \( G_0 \), the hypotheses of Theorem 2 are satisfied if \( \omega = \xi + \frac{\beta}{2\alpha} \in G_0 \). Therefore, it follows from (66) that \( l[a(\xi)] \in G_0, a(\xi) \in G_{0-} \), satisfies the Lax type representation
\[
dl/dt = [l, \nabla \gamma(l)_+].
\]
On the other hand taking into account the map (67), we can consider the Hamiltonian system (68) as that induced from a Hamiltonian system of the form
\[
da(\xi)/dt = \{\gamma, a(\xi)\}_\theta(a), \quad a(\xi) \in G_{0-}.
\]
Indeed, the map \( \hat{l} : G_0^* \rightarrow G_0^* \cong G_0 \) can be considered as the canonical [8] map of \( (M(\mathbb{Z}_+), \theta) \) onto itself when (70) obtains, which by (67) is equivalent to the Lax representation (70) for the vector field
\[
dl/dt = K(l), \quad t \in \mathbb{R}.
\]
Moreover, since under the canonical map an involutive system of functionals on \( M(\mathbb{Z}_+) \) transforms into an involutive system, the induced dynamical system for \( a(\xi) \) will be a completely integrable Lax type bi-Hamiltonian flow on \( G \).

The Lax type representation (70) has an equivalent interpretation for vector fields owing to (62), namely, if to an element \( l(\xi) \in G_0 \) one assigns a corresponding vector field \( \tilde{l}(\xi) := \frac{\partial l}{\partial \xi} \frac{\partial}{\partial x} - \frac{\partial l}{\partial x} \frac{\partial}{\partial \xi} \), then
\[
d\tilde{l}(\xi)/dt = [\tilde{l}(\xi), \nabla \gamma(l)_+],
\]
where the bracket is the usual commutator of vector fields on \( \mathbb{R}^2 \).

3.4 In aid of applying the above results to an investigation of the Benney type system (39), let us set in (63) the element \( l[a(\xi)] \in G_0 \) equal to
\[
l[a(\xi)] = \xi - 2\alpha a(\xi) + 2\alpha^{-1} \beta \frac{1 - 2\alpha a(\xi)}{1 - 2\alpha a(\xi)},
\]
where res denotes the residue. Setting $\gamma_j = T r l^j, j \in \mathbb{Z}_+$, we immediately obtain the following theorem.

**Theorem 3.**

The Hamiltonian system

$$\frac{d a(\xi)}{d t} = \{H, a(\xi)\}_\theta,$$

where $H = (\alpha \gamma_2 - 4 \alpha \beta \gamma_1 + 3 \beta^2 \gamma_0)/8 \alpha^2$, is equivalent to the Hamiltonian moment Benney type system (16) and, owing to the map $\hat{\xi}: M_{(u, \rho)} \rightarrow M_{(\mathbb{Z}^+)}$ defined by

$$(u, \rho) \xrightarrow{\hat{\xi}} a(\xi) := \int_0^{\rho(x,t)} \frac{dy}{\xi - u(x, y; t)} \in \mathcal{G}_0,$$

also to the Hamiltonian Benney type system (39). The equivalence for (39) holds if

$$\gamma(l) = \frac{T r l^2}{8 \alpha} - \frac{\beta T r l}{2 \alpha} + \left(\frac{\beta^2}{\alpha} \frac{d}{d \beta} T r l\right)_{|\beta=0} \in \mathcal{D}(\mathcal{G}_0^\ast).$$

This theorem can easily be proved by straightforward computations using

$$\gamma_0 = \int_{\mathbb{R}} dx a_0, \quad \gamma_1 = 2 \alpha \int_{\mathbb{R}} dx a_1 + \beta \int_{\mathbb{R}} dx a_0,$$

$$\gamma_2 = 8 \alpha H + 8 \beta \int_{\mathbb{R}} dx a_1 + \alpha^{-1} \beta^2 \int_{\mathbb{R}} dx a_0,$$

$$H = \frac{1}{2} \int_{\mathbb{R}} dx \left( a_2 + \beta a_0^2 + 2 \alpha a_0 a_1 \right),$$

and Theorem 2.

Note also the following interesting fact: the Lax representation (70) has a singularity at $\alpha = 0$, while the Benney type system (39) and its Hamiltonian $H$ given in (76) are regular at $\alpha = 0$. Obviously, this can be interpreted as follows: the singularity at $\alpha = 0$ in (70) corresponds to an instability at the bottom of the fluid motion when the convective parameter $\alpha$ changes from positive to negative values for fixed positive $\beta$.

In addition to the statement of Theorem 3, one can infer from (55) that the Benney type system (73) on the manifold $M_{(\mathbb{Z}^+)}$ is bi-Hamiltonian with respect to two compatible cosymplectic structures $\theta$ and $\eta$. As the map (74) is canonical, the same statement applies to the hydrodynamical Benney type system (39) on the manifold $M_{(u, \rho)}$.

3.5 In the article [14] the Riemann invariants [15, 20] are studied for the dynamical system (18) on the functional manifold for the case $\alpha = 0$. As is well known, the Riemann invariants determine on the $x, t$-plane a characteristic velocity $\xi \in T(M)$ as a local functional on $M_{(u, \rho)}$, that is $dx/dt = \xi[u, \rho]$, where at $\alpha \neq 0$

$$\det \begin{bmatrix} u + 2 \alpha \rho - \xi & 2 \alpha \beta u \\ \rho & u + 2 \alpha \rho - \xi \end{bmatrix} = 0.$$  

Whence we obtain the two values for the characteristic velocity

$$\xi_{\pm}[u, \rho] = (u + 2 \alpha \beta) \pm \sqrt{\rho(\beta + 2 \alpha u)},$$
for which the corresponding Riemann invariants on the characteristic curves $dx/dt = \xi_{\pm}[u, \rho]$ are

$$\lambda_{\pm} = \frac{\sqrt{\beta}}{2} \left( \sqrt{1 + 2\alpha\beta^{-1}u - 1} \right) \pm 2\sqrt{\rho}. \quad (79)$$

It is easy to see that the local functionals (79) satisfy $d\lambda_{\pm}/dt = 0$ on the characteristic curves. Indeed, for the Lax type representation (70) there is a corresponding curve

$$\lambda = \lambda(\xi) = \frac{\xi - 2\alpha a_0 - (2\alpha)^{-1}\beta}{1 - 2\alpha a(\xi)}, \quad (80)$$

where $\lambda, \xi \in \mathbb{C}$, which for the case of the Hamiltonian system (18) takes the form

$$\lambda = \frac{\xi - 2\alpha \rho + (2\alpha)^{-1}\beta}{\xi - u - 2\alpha \rho} (\xi - u). \quad (81)$$

The turning points of the algebraic curve (81) are determined by the relationship $d\lambda/d\xi = 0, \xi \in \mathbb{C}$. As $\lambda \in \mathbb{C}$ is an invariant of the Lax representation (70), the parameter $\xi$ at a turning point of (81) depends only on the variables $x, t \in \mathbb{R}$ and coincides with the expression (78). Thus (79) yields the values of the invariant $\lambda$ of (81) at the turning points (78). From the invariant expression (79) on the characteristics $dx/dt = \xi_{\pm}[u, \rho]$, we also obtain the equations

$$\frac{\partial \lambda_{\pm}}{\partial t} + \xi_{\pm}[u, \rho] \frac{\partial \lambda_{\pm}}{\partial x} = 0. \quad (82)$$

Fixing one of the Riemann invariants, for example putting $\lambda_+ = 0$, we find by direct calculation that the dynamical system (18) admits solutions that become discontinuous in finite time. As noted in the article [2], this phenomenon is not a defect of the Boltzmann-Vlasov equation (12), but stems from the substitution $f(x,p,t) = \rho(x,t)\delta(p-u(x,t)), x, p \in \mathbb{R}$, resulting in the nonlinear dynamical system (18). The question of whether the Benney type hydrodynamical system (39) has singular solutions of the blow-up type requires a deeper investigation of the turning points of the curve (80) and the corresponding equations (82).

3.6 Let $W_2^{(\infty)}(\mathbb{R}; \mathbb{C})$ be a functional space on which the action of the symbol $\xi : W_2^{(\infty)} \to W_2^{(\infty)}$ is “diagonal”, that is equivalent to the product operation on the function $\xi \in C^{\infty}(\mathbb{R}; \mathbb{R})$. In other words, for any $\phi \in W_2^{(\infty)}$ the expression $\xi \circ \phi(x) = \xi(x)\phi(x)$ obtains. This action is in agreement with the action of the operator symbol $\xi : W_2^{(\infty)} \to W_2^{(\infty)}$ on the Lie algebra $\mathfrak{g}_0$ studied above. It is easy to see that the Lax representation (71) is equivalent to the compatibility condition of the following system of eigenvalues:

$$l(\xi(x))\phi(x) = \lambda \phi(x),$$
$$d\phi(x)/dx = -\nabla \gamma(l) \phi(x), \quad (83)$$

where $\phi \in W_2^{(\infty)}$ and $\xi \in C^{\infty}(\mathbb{R}, \mathbb{R})$ is a solution of the transcendental equation $l(\xi(x)) = \lambda \in \mathbb{R}, x \in \mathbb{R}$, with $d\lambda/dt = 0$ for all $t \in \mathbb{R}$. Choosing $l(\xi) \in \mathfrak{g}_0$ in the form (72), we obtain an algebraic scheme for describing the integrability of the nonlinear Benney type system (39). As the expression (72) is singular at $\alpha = 0$, it does not admit a regular limit transition of the
hydrodynamic system (39) at \( \alpha = 0 \), but it does at \( \beta = 0 \). In connection with this, let us consider an a priori expression of \( l(\xi) \) in the following form:

\[
\hat{l}[a(\xi)] = \frac{\xi + 2\beta - \text{res}(\xi a(\xi))}{1 - \xi a(\xi) + \text{res}(\xi)} + \frac{\xi + 2\beta}{1 - (\xi a(\xi) - \text{res}(\xi))^2},
\]

(84)

where \( a(\xi) = \sum_{j \in \mathbb{Z}_+} a_j(x)\xi^{-j+1} \in G_{0-} \) and \( \beta \) is an arbitrary real parameter. The expression (84), analogously to (67), determines the canonical map \( \hat{l} : G_{0+} \to G_{0+} \cong G_{0-} \), where

\[
\hat{l}[a(\xi)] = b(\xi) = l[a(\xi)] - 2(\xi + 2\beta) \in G_{0-}.
\]

(85)

Therefore, if \( \gamma \in D(G_0^0) \) is a Casimir functional, then \( \gamma \) restricted to \( G_{0+}^0 \cong G_{0-}^0 \) together with the shift operator \( w \in 2(\xi + 2\beta) \in G_0^0 \) leads to the integrable Hamiltonian system on \( G_{0+}^0 \) possessing the Lax representation (70). In particular, if we put \( \gamma = \text{Tr}l(\xi) \in D(G_0^0) \) and define the Hamiltonian function on the manifold \( G_{0+}^0 \) by \( H = \frac{1}{2}\text{Tr}l[a(\xi)] \), where \( a(\xi) \in G_{0+}^0 \cong G_{0-}^0 \), then the Hamiltonian system (73) can be written in component form as

\[
da_n/dt = \{H, a_n\} = -a_{n+1,x} - (a_0a_1)_x - \beta a_{0,x},
\]

(86)

\[
da_1/dt = \{H, a_1\} = -a_{2,x} - 3(a_1a_1)_x - \beta a_{1,x},
\]

\[
da_n/dt = \{h, a_n\} = -a_{n+1,x} - (a_1a_n)_x - na_na_1x - \beta a_{n,x},
\]

where the Hamiltonian function is

\[
H = \frac{1}{2} \int_\mathbb{R} dx(a_2 + a_1^2 + 2\beta a_1) \in D(M_{(\mathbb{Z}_+)}).
\]

Using the canonical transformation \( M(f) \ni f \to a_n \in M_{(\mathbb{Z}_+)}) \), where for each \( n \in \mathbb{Z}_+ \)

\[
a_n := \int_\mathbb{R} df(x,p)p^n,
\]

(87)

we immediately obtain

\[
df /dt = \{\{H, f\}\} = -(p + a_1)f_x - pa_1xf_p + \beta f_x,
\]

(88)

which is the Boltzmann-Vlasov kinetic equation for the many-particle system with Hamiltonian \( H \in D(\mathbb{R}^{2N}; \mathbb{R}) \) defined by

\[
H := \frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{1}{2} \sum_{j \neq k=1}^N p_jp_k\delta(x_j - x_k) + \beta \sum_{j=1}^N p_j,
\]

(89)

where \( N \) is the number of particles on the axis \( \mathbb{R} \).

The Hamiltonian (89) has a rather specific physical meaning which we will not go into here. Note only that a quantization of the Hamiltonian (89) transforms it into the operator \( \hat{H} \) on \( L_2(\mathbb{R}^n; \mathbb{C}) \) given by

\[
\hat{H} = -\sum_{j=1}^N \left( \frac{\hbar^2}{2} \frac{\partial^2}{\partial x_j^2} - \beta \frac{\hbar}{i} \frac{\partial}{\partial x_j} \right) - \frac{\hbar^2}{2} \sum_{j \neq k=1}^N \frac{\partial}{\partial x_j} \delta(x_j - x_k),
\]

(90)
and the second quantized expression via the scheme (19) has the form (for $\hbar = 1$)

$$H = \frac{1}{2} \int_{\mathbb{R}} dx \left[ \psi_{x}^2 + \psi_{x}^4 \psi^\dagger \psi \psi - i \beta (\psi^\dagger \psi - \psi_{x}^2) \right],$$  \hspace{1cm} \text{(91)}

acting on the Fock Hilbert space $\Phi$. The classical field expression for the Hamiltonian operator (91) is a functional $H \in D(M_{(\psi, \psi^\dagger)})$, where

$$H = \frac{1}{2} \int_{\mathbb{R}} dx \left[ \psi_{x}^2 \psi_x + \psi_{x}^4 \psi^\dagger \psi \psi - i \beta (\psi^\dagger \psi - \psi_{x}^2 \psi) \right].$$  \hspace{1cm} \text{(92)}

The corresponding Poisson bracket on $M_{(\psi, \psi^\dagger)}$ is determined for all $x, y \in \mathbb{R}$ by the following canonical relationship:

$$\{ \psi(x), \psi(y) \}_\theta = \{ \psi^\ast(x), \psi^\ast(y) \}_\theta = 0, \quad \{ \psi(x), \psi^\ast(y) \}_\theta = -i \delta(x - y).$$  \hspace{1cm} \text{(93)}

Using now the functional (92) and the expression (93), we obtain the Schrödinger type field equation for a point $(\psi, \psi^\ast) \in M_{(\psi, \psi^\dagger)} \subset C^\infty(\mathbb{R}; \mathbb{C}^2)$; namely

$$\psi_t = \frac{i}{2} \psi_{xx} (1 + \psi^\ast \psi) + \beta \psi_x + \frac{i}{2} \psi^\ast \psi^2, \quad \psi_t^\ast = \frac{i}{2} \psi_{xx} (1 + \psi^\ast \psi) + \beta \psi_x^\ast - \frac{i}{2} (\psi^\ast)^2 \psi.$$

\hspace{1cm} \text{(94)}

Using the change of variables (23) and (24) and then applying the quasiclassical approximation $d/dt \rightarrow \varepsilon d/dt, d/dx \rightarrow \varepsilon d/dx$ and taking the limit as $\varepsilon \rightarrow 0$, we obtain the following hydrodynamic system of equations on the functional manifold $M_{(u, \rho)}$:

$$\begin{pmatrix} u_t \\ \rho_t \end{pmatrix} = K[u, \rho] := \begin{pmatrix} -uu_x - (u^2 \rho + \beta u_x) \\ -\rho u + \rho^2 u + \beta \rho_x \end{pmatrix}. \hspace{1cm} \text{(95)}$$

The dynamical system (95) is Hamiltonian on $M_{(u, \rho)}$ with respect to the Poisson bracket (29), i.e., we have

$$(u, \rho)^T = -\theta \text{grad} H_\theta = K[u, \rho], \hspace{1cm} \text{(96)}$$

where $H_\theta = \frac{1}{\varepsilon} \int_{\mathbb{R}} dx (\rho u^2 + \rho^2 u^2 + 2 \beta \rho u) \in D(M_{(u, \rho)})$ because of (92).

Taking into account the bi-Hamiltonicity of the moment dynamical system (86), we infer that the dynamical system (96) on $M_{(u, \rho)}$ is also bi-Hamiltonian; that is, there exists an implicit operator $\eta : T^\ast(M_{(u, \rho)}) \rightarrow T(M_{(u, \rho)})$, compatible with (29), such that

$$(u_t, \rho_t)^T = -\eta \text{grad} H_\eta \hspace{1cm} \text{(97)}$$

for some $H_\eta \in D(M_{(u, \rho)})$.

Note. The existence of a completely integrable quasiclassical approximation for (23) and its moment analogue (16) demonstrated above, shows that some statements in [2] concerning such systems are false. It is also stated in [2] that a field analogue of (95) is the nonlinear Schrödinger equation (at $\beta = 0$)

$$\begin{align*}
\psi_t &= \frac{i}{2} \psi_{xx} - \frac{i}{2} (\psi^\ast \psi \psi)_x, \\
\psi_t^\ast &= -\frac{i}{2} \psi_{xx}^\ast - \frac{i}{2} (\psi^\ast \psi^\ast \psi)_x.
\end{align*}$$

\hspace{1cm} \text{(98)}
on the manifold $M(\psi, \psi^*)$ with Hamiltonian $H = \frac{1}{4} \int_{\mathbb{R}} dx \left[ |\psi^* \psi|^2 - i(\psi \psi_x^* - \psi_x \psi^*) \right]$ and the following noncanonical Poisson bracket on $M(\psi, \psi^*)$:

\[
\{\psi(x), \psi(y)\}_\theta = 0 = \{\psi^*(x), \psi^*(y)\}_\theta,
\]

\[
\{\psi(x), \psi^*(y)\}_\theta = \delta'(x - y).
\] (99)

In fact, as we showed earlier, the correct field analogue of (95) in the framework of the article [1] is the nonlinear Schrödinger type equation (94).

3.7 The Benney type hydrodynamic system (17) and (86) are associated with quasiclassical approximations of the canonically Hamiltonian completely integrable Schrödinger equations (23) and (94), which are in a natural one-to-one relation to the kinetic Boltzmann-Vlasov completely integrable equations for one-dimensional particle flows with respect to the Lie-Poisson bracket (8).

The nonlinear Schrödinger type modified equation (98) discussed in [2] does not have a canonical (Fock) field Hamiltonian structure (see (99)), therefore there is no way to construct an associated Boltzmann-Vlasov equation that is physically meaningful.

To investigate the problem of constructing the associated Boltzmann-Vlasov equations for nonlinear dynamical systems with nonlocal Hamiltonian structure, let us consider in more detail the generalized modified Schrödinger equation (98) with $\beta \neq 0$:

\[
\begin{align*}
\psi_t &= i \frac{1}{2} \psi_{xx} - \frac{1}{2} (\psi^* \psi)_x + i \beta \psi^* \psi, \\
\psi^*_t &= -i \frac{1}{2} \psi_{xx} - \frac{1}{2} (\psi^* \psi^*)_x + i \beta \psi^* \psi, 
\end{align*}
\] (100)

on the manifold $M(\psi, \psi^*) \in C^\infty(\mathbb{R}; \mathbb{C}^2)$ which has the symplectic structure determined as follows [6]:

\[
\theta(\psi, \psi^*) = \begin{bmatrix}
2 \beta \psi \partial^{-1} \psi & \partial - 2 \beta \psi \partial^{-1} \psi^* \\
\partial - 2 \beta \psi \partial^{-1} \psi^* & -2 \beta \psi^* \partial^{-1} \psi^* 
\end{bmatrix}.
\] (101)

Then the dynamical system (100) has a Hamiltonian representation in the form $(\psi, \psi^*)^T = -\theta \text{grad} H_\theta$, where the Hamiltonian functional $H_\theta \in D(M(\psi, \psi^*))$ is

\[
H_\theta = \frac{1}{4} \int_{\mathbb{R}} dx \left[ (\psi^* \psi)^2 + i(\psi \psi^* - \psi_x \psi^*) + 2 \beta \psi^* \psi \right].
\] (102)

The quasiclassical substitution (24) into formulas (100)-(102) leads to the Hamiltonian hydrodynamic system

\[
\left( \begin{array}{c}
\dot{u} \\
\rho_t 
\end{array} \right) = K[u, \rho] := \left( \begin{array}{c}
-uu_x - \frac{1}{2}(up)_x - \beta \rho_x \\
-(up)_x - \frac{3}{2} \rho \rho_x 
\end{array} \right),
\] (103)

where because of (102),

\[
\begin{align*}
H_\theta &= \frac{1}{4} \int_{\mathbb{R}} dx (\rho^2 + 2u \rho + 2 \beta \rho), \\
\theta(u, \rho) &= \begin{bmatrix}
3 \beta \partial & \partial u - \beta \partial \\
\partial u - \beta \partial & \rho \partial + \partial \rho 
\end{bmatrix}.
\end{align*}
\] (104)
If we introduce the moment functions \( a_n(x) = \int_0^x dyu^n(x, y) \in M(\mathbb{Z}_+), n \in \mathbb{Z}_+ \), the Hamiltonian in (103) can be expressed in the form \( H_\theta = \frac{1}{4} \int_\mathbb{R} dx (a_0^2 + 2a_1 + 2\beta a_0) \). If, further, we a priori use the Hamiltonian structure (64) on \( M(\mathbb{Z}_+) \), then the corresponding evolution equations \( da_n/dt = \{ H_\theta, a_n \}, n \in \mathbb{Z}_+ \), do not coincide with the moment equations obtained from the relationship (103). This again shows that the correct field analogue of the hydrodynamic system (95), discussed in [2], is the nonlinear modified Schrödinger equation (100).

The dynamical system (103) closely resembles the Benney type system (18), but it cannot be reduced to it. In fact, it is not probable that a natural analogue of the kinetic Boltzmann-Vlasov equation for the hydrodynamic system (103) exists, as the Hamiltonian structure (104) on the manifold \( M_{(u, \rho)} \) does not correspond canonically to the Lie-Poisson structure (15) on the moment manifold \( M(\mathbb{Z}_+) \). This statement obviously applies for all noncanonically Hamiltonian hydrodynamic systems.

3.8 As was shown in [25], when describing high-temperature superconductivity, a model of quasi-one-dimensional crystals is used that admits soliton-like states that are stable with respect to time. The Hamiltonian operator in a representation of the second quantization is determined by the expression (at \( \hbar = 1 \))

\[
H = \frac{1}{2} \int_\mathbb{R} dx \left[ \psi_x^+ \psi_x + 2v(x)\psi^+ \psi + i(\psi_x^+ \psi - \psi^+ \psi_x) + v^2(x) \right],
\]

(105)

where the scalar field \( v \in C^\infty(\mathbb{R}; \mathbb{R}) \) corresponds to the deformation energy of crystal perturbations. Its quasiclassical field analogue is given by the Hamiltonian functional

\[
H_\theta = \frac{1}{2} \int_\mathbb{R} dx \left[ \psi_x^* \psi_x + 2v(x)\psi^* \psi + i(\psi_x^* \psi - \psi^* \psi_x) + v^2(x) \right]
\]

(106)

on the canonical symplectic manifold \( M_{(\psi, v, \psi^*)} \subset C^\infty(\mathbb{R}; \mathbb{C} \times \mathbb{R} \times \mathbb{C}) \) with the symplectic structure

\[
\theta(\psi, v, \psi^*) = \begin{bmatrix} 0 & 0 & i \\ 0 & \partial & 0 \\ -i & 0 & 0 \end{bmatrix}.
\]

(107)

The dynamical system corresponding to the Hamiltonian

\[
\begin{align*}
\psi_t &= \frac{1}{2} \psi_{xx} - iv \psi - \psi_x, \\
v_t &= -v_x - (\psi^* \psi)_x, \\
\psi_t^* &= -\frac{1}{2} \psi_{xx}^* - iv \psi^* - \psi_x^*,
\end{align*}
\]

(108)

where \( t \in \mathbb{R} \) is an evolution parameter and \( x \in \mathbb{R} \).

Carrying out the quasiclassical approximation (24) in (108), we obtain the following hydrodynamic system set on the functional manifold \( M_{(u, v, \rho)} \subset C^\infty(\mathbb{R}; \mathbb{R}^2 \times \mathbb{R}^+) \):

\[
\begin{pmatrix}
u_t \\
u_t \\
\rho_t
\end{pmatrix} = \begin{pmatrix}
\{ H_\theta, u \}_\theta \\
\{ H_\theta, v \}_\theta \\
\{ H_\theta, \rho \}_\theta
\end{pmatrix} = K[u, v, \rho] := \begin{pmatrix}
-uw_x - u_x - v_x \\
-v_x - \rho_x \\
-\rho u_x - \rho_x
\end{pmatrix},
\]

(109)

where the Hamiltonian functional for (109) on \( M_{(u, v, \rho)} \) and the Hamiltonian structure are given, respectively, by

\[
H_\theta = \frac{1}{2} \int_\mathbb{R} dx \left[ \rho u^2 + v^2 + 2\rho(u + v) \right], \quad \theta(u, v, \rho) = \text{antidiag}(\partial, \partial, \partial).
\]

(110)
Introducing the standard map
\[ a_n(x) := \int_0^{\rho(x)} du^n(x, y) \]
for all positive integers \( n \), we find that on the manifold \( M_{(Z^+)} \),
\[ H_0 = \frac{1}{2} \int_\mathbb{R} dx (a_2 + v^2 + v a_0 + a_1) \in D(M_{(Z^+)}). \]

4. SWEPt VOLUME DYNAMICAL SYSTEMS AND THEIR KINETIC MODELS

4.1 It is well known [28, 29] that motion planning, numerically controlled machining and robotics are just a few of the many areas of manufacturing automation in which the analysis and representation of swept volumes plays a crucial role. Swept volume modeling is also an important part of task-oriented robot motion planning. In these problems a robot carrying some objects is moved through a domain in space containing obstacles (phase space constraints) for the purpose of reaching a desired goal. A typical motion planning problem consists of moving a collection of solid objects around obstacles from an initial to a final position. This may include, in particular, solving collision detection problems and obtaining optimal solution paths.

When an object undergoes a rigid motion, the totality of points through which it passes constitutes a region in space called the swept volume generated by the (rigid) sweep. In order to have a mathematical framework for these sweeps in real 3-space \( \mathbb{R}^3 \), we introduce some definitions that will prove useful in the sequel.

A Euclidean motion in \((n+1)\)-dimensional Euclidean space \( E^{n+1} \) is a mapping \( \sigma: E^{n+1} \rightarrow E^{n+1} \) such that \( \|\sigma(x) - \sigma(y)\| = \|x - y\| \) for all \( x, y \in E^{n+1} \), where \( \|\cdot\| \) is the standard Euclidean norm in \( E^{n+1} \). The simplest cases of these sweeps or motions are translations and rotations of the form
\[ \alpha(x) := x + a, \quad \beta(x) := gx, \] (111)
where \( x, a \in E^{n+1} (= \mathbb{R}^{n+1}) \) and \( g \in O(n+1) \). The following result characterizes Euclidean motions.

**Theorem 4.**

Let \( \sigma \) be a Euclidean motion in \( E^{n+1} \). Then there exist a unique orthogonal mapping \( \beta \in O(n+1) \) and a unique translation \( \alpha \) such that \( \sigma = \alpha \circ \beta \).

**Sketch of Proof:** Let \( a := \sigma(0) \) and \( \alpha \) be as in (111). Define \( \beta := \alpha^{-1} \circ \beta \). It is easy to show that \( \beta \) is orthogonal. Indeed, \( \beta \) is a motion since
\[ \|\beta(x) - \beta(y)\| = \|\sigma(x) - \sigma(y)\| = \|x - y\|, \] (112)
and \( \|\beta\| = 1 \) because
\[ \|\beta\| = \|\beta(x) - \beta(0)\| = \|x - 0\| = \|x\| \]
for all \(x, y \in E^{n+1}\). We need to show that \(\beta\) is linear. Note first that the standard inner product \(\langle \cdot, \cdot \rangle\) in \(E^{n+1}\) is preserved by the mapping \(\beta : E^{n+1} \to E^{n+1}\), viz.

\[
\langle \beta(x), \beta(y) \rangle = \frac{1}{2} \left( \| \beta(x) \|^2 + \| \beta(y) \|^2 - \| \beta(x) - \beta(y) \|^2 \right) = \frac{1}{2} \left( \| x \|^2 + \| y \|^2 - \| x - y \|^2 \right) = \langle x, y \rangle
\]

(113)

for all \(x, y \in E^{n+1}\). To prove the linearity of \(\beta\), it suffices to show that

\[
\beta(k_1 x + k_2 y) = k_1 \beta(x) + k_2 \beta(y),
\]

(114)

or equivalently that the form

\[
E(x, y) := \beta(k_1 x + k_2 y) - k_1 \beta(x) - k_2 \beta(y) \equiv 0
\]

(115)

for all \(x, y \in E^{n+1}\), \(k_1, k_2 \in \mathbb{R}\). To prove (115) it is necessary that \(E(x, y)\) in (115) be orthogonal to each vector of some basis in \(E^{n+1}\). If \(\{e_1, ..., e_{n+1}\}\) is an orthonormal basis for \(E^{n+1}\), obviously so is \(\{\beta(e_1), ..., \beta(e_{n+1})\}\) since \(\beta\) preserves the scalar product in view of (113). Therefore we find that for all \(1 \leq j \leq n + 1\),

\[
\langle E(x, y), \beta(e_j) \rangle = \langle \beta(k_1 x + k_2 y), \beta(e_j) \rangle - k_1 \langle \beta(x), e_j \rangle - k_2 \langle \beta(y), e_j \rangle = k_1 \langle x, e_j \rangle + k_2 \langle y, e_j \rangle - k_1 \langle x, e_j \rangle - k_2 \langle x, e_j \rangle = 0.
\]

(116)

This proves the linearity of \(\beta\), and the uniqueness of \(\alpha\) and \(\beta\) are easily verified. \(\square\)

The following is a direct consequence of the above theorem:

Corollary 5.

Let \(\sigma\) be a Euclidean motion in \(E^{n+1}\). Then: (i) \(\sigma\) is a smooth mapping; (ii) \(\sigma\) maps \(E^{n+1}\) orthogonally onto itself; and (iii) \(\langle \sigma'(x), \sigma'(y) \rangle = \langle x, y \rangle\) for all \(x, y \in E^{n+1}_p \cong T_p(E^{n+1}), p \in E^{n+1}\).

Indeed, for each point \((p, x) \in E^{n+1}_p, p \in E^{n+1}\) we have

\[
\sigma'(p)x = \left. \frac{d}{dt} \sigma(p + tx) \right|_{t=0} = \left. \frac{d}{dt} \sigma \circ \beta(p + t\beta(x)) \right|_{t=0} = \left. \frac{d}{dt} \beta(p + t\beta(x) + \sigma(0)) \right|_{t=0} = \beta(x).
\]

(117)

Whence for \((p, x), (p, y) \in E^{n+1}_p\), we obtain

\[
\langle \sigma'(p)x, \sigma'(p)y \rangle = \langle \beta(x), \beta(y) \rangle = \langle x, y \rangle.
\]

(118)

from which the required properties follow.

4.2 Let us consider a simply-connected solid body \(V\) embedded in \(E^3\) having boundary surface \(S = \partial V\) parametrized as follows:

\[
S = \bigcup_{\tau \in [0, h]} \{ x(s, \tau) \in \mathbb{R}^3 : x(s + 2\pi, \tau) = x(s, \tau), s \in \mathbb{R}/2\pi \mathbb{Z} \}.
\]

(119)

Here \(\tau \in \mathbb{R}/2\pi \mathbb{Z}\) is the usual parametrization via the identity \(\langle dx, dx \rangle = ds^2\) of curve \(x(\cdot, \tau), \tau \in [0, h]\), obtained by cutting \(V\) straight across its diameter through a point \(\tau \in [0, h]\). This means
that the diameter of $V$ is parametrized by $\tau$ and the set of curves $\{x(\cdot, \tau) : \tau \in [0, h]\}$ covers $S$ in a unique fashion.

To proceed further in our description of the motion of a solid body $V$ in $E^3$, we reformulate it in terms of manifolds swept out by the surface $S$ in a time interval $[0, t_0]$. We denote these surfaces by $S_{t_0}(\tau)$; they can be represented by

$$S_{t_0}(\tau) := \bigcup_{t \in [0, t_0]} \{x(t, \tau)\}.$$  \hspace{1cm} (120)

Therefore, the swept volume manifold $S_{t_0}(V)$ of the solid body defined in [29] can be written as

$$S_{t_0}(V) = \bigcup_{\tau \in [0, h]} S_{t_0}(\tau).$$  \hspace{1cm} (121)

It is obvious that $S_{t_0}(V)$ is equal to the compact three-dimensional submanifold with boundary comprised of points swept by the set of curves $\{x(\cdot, \tau)\} \subset S$ corresponding to the diameter points $\tau \in [0, h]$. This leads naturally to the problem of constructing special dynamical systems - called swept volume dynamical systems - intimately associated with the Euclidean motion of a solid body in space and studying their differential-geometric and differential-topological properties [28]] which are useful for applications in manufacturing automation.

Let us assume that a sweep of a solid body $V$ in 3-space is generated by a family of Euclidean motions $\sigma(t), t \in [0, t_0]$, giving rise to a swept volume such that each of the curves $x(\cdot, \tau)$ maintains its planarity and arc-length for all $t \in [0, t_0]$. This means, in particular, that the Gaussian curvature of each curve $x(\cdot, \tau), \tau \in [0, h]$, is time independent while its torsion $\xi$ is zero for all $t \in [0, h]$. The invariance of planarity of the family of Euclidean motions $\sigma(t), t \in [0, t_0]$, completely characterizes the motion of $V$ in $E^3$. To the above properties one needs only to add length invariance, which can be expressed as

$$\langle dx/ds, K'(t, x)dx/ds \rangle = 0$$  \hspace{1cm} (122)

for all $t \in [0, t_0]$. Here we have postulated the evolution of curves $x(\cdot, \tau, t)$ due to the Euclidean motion as follows:

$$dx/dt = K(t, x),$$  \hspace{1cm} (123)

where $K(t, \cdot) : E^3 \to T(E^3)$ is a parametric family of vector fields on $E^3$. Vector fields satisfying (122) are called Killing vector fields. A rather complete description of such fields associated with infinitesimal rigid body motions can be found in [30, 31]. Our next goal is to obtain an analogous theory for swept volumes using modern differential-geometric and algebraic-topological tools.

Now suppose we are given a swept volume manifold $S_{t_0}(V)$ as defined in (121). Over this manifold we can define a connection $\Gamma$ [32] together with the following representation of parallel transport with respect to local spatial parameters $\tau \in [0, h], s \in \mathbb{R}/2\pi\mathbb{Z}$, and the temporal parameter $t \in [0, t_0]$ via the covariant derivative

$$\nabla_{\frac{\partial}{\partial y^j}} f := \frac{\partial f}{\partial y^j} + \Gamma_j f,$$  \hspace{1cm} (124)

24
where \( y = (y^1, y^2, y^3)^T := (s, \tau, t)^T \in S_{t_0}(V) \) and \( \Gamma_j, 1 \leq j \leq 3 \), are Christoffel matrices acting in the adjoint vector bundle over the swept volume manifold. The Christoffel matrices can be determined uniquely by requiring that \( \nabla_{\frac{\partial}{\partial y^i}} f^s = 0 \) for all horizontal \([33, 34]\) vector fields \( f^s \in T(S_{t_0}(V)) \). This means that parallel transport along the manifold \( S_{t_0}(V) \) must be generated by some Euclidean motion \( \sigma(t) : E^3 \rightarrow E^3, t \in [0, t_0] \). We first consider Cartan’s main structure equations in a differential-geometric setting \([33]\):

\[
d\theta = -\frac{1}{2}[\omega, \theta] + \Theta, \quad d\omega = -\frac{1}{2}[\omega, \omega] + \Omega, \tag{125}\]

where \( \omega : P(S_{t_0}(V); GL(3; \mathbb{R})) \rightarrow gl(3; \mathbb{R}) \) is the connection form,

\[
\theta : P(S_{t_0}(V); GL(3; \mathbb{R})) \rightarrow E^3
\]

is the canonical affine form, \( \Omega \) is the corresponding curvature form and \( \Theta \) is the torsion form, all on the principal fiber bundle \( P(S_{t_0}(V); GL(3; \mathbb{R})) \) of frames over the manifold \( S_{t_0}(V) \). In component form the structure equations (125) are as follows:

\[
d\theta^i = -\omega^i_j \wedge \theta^j + \Theta^i, \quad d\omega^i = -\omega^i_k \wedge \omega^k + \Omega^i_j, \tag{126}\]

where \( \theta = \sum_{i=1}^3 \theta^i e_i, \omega = \sum_{i,j=1}^3 \omega^i_j A^j_i, \Theta = \sum_{i=1}^3 \Theta^i e_i, \Omega = \sum_{i,j=1}^3 \Omega^i_j A^j_i, \{e_i : 1 \leq i \leq 3\} \) is a basis for \( E^3 \) and \( \{A^j_i \in gl(3; \mathbb{R}) : 1 \leq i, j \leq 3\} \) are in the Lie algebra \( gl(3; \mathbb{R}) \) and satisfy the condition \( A^i_j e_k = e_j \delta^i_k \) for all \( i, j, k = 1, 2, 3 \). The structure equations (126) completely describe the swept volume dynamical system generated by a rigid sweep \( \sigma(t) : E^3 \rightarrow E^3 \) of a solid body in 3-space. The motion \( \sigma(t) \) considered as an affine motion in \( \mathbb{R}^3 \) must satisfy the main defining conditions on the canonical and connection one-forms:

\[
\mathbb{R} a^* \omega = Ad_{a^{-1}} \omega, \quad \mathbb{R} a^* \theta = a^{-1} \theta
\]

for all \( a \in GL(3; \mathbb{R}) \). These conditions are obviously satisfied if the following canonical conditions obtain:

\[
\theta = X^{-1}dy, \quad \omega = X^{-1}(dX + (dy, \Gamma(y)) X), \tag{127}\]

where \( y = (s, \tau, t)^T \in S_{t_0}(V) \) and \( X = [X_1, X_2, X_3] \in GL(3; \mathbb{R}) \) is an arbitrary basis for the tangent space \( T_y(S_{t_0}) \). To determine the Riemannian connection matrix \( \Gamma(y), y \in S_{t_0} \), we write the first fundamental form as follows:

\[
\delta l^2 := \langle \delta x, \delta x \rangle = \sum_{i,j=1}^3 g_{ij}(y) dy^i dy^j. \tag{128}\]

If the embedding \( x : S_{t_0}(V) \rightarrow E^3 \) is generated by a sweep, the above Riemannian connection matrices must have the form

\[
\Gamma^j_{k,s} = \frac{1}{2} \sum_{s=1}^3 g^j_{is} \left( \frac{\partial g_{is}}{\partial y^{k'}} + \frac{\partial g_{sk}}{\partial y^{i'}} - \frac{\partial g_{ki}}{\partial y^{s'}} \right), \tag{129}\]
where \( \sum_{s=1}^{3} g^{ks} a_{kj} = \delta^i_j \), \( i, j, k = 1, 2, 3 \); and as is well known [33], (129) is uniquely determined by the condition \( \nabla \frac{\partial}{\partial y} g_{ks} = 0 \) for all \( 1 \leq j, k, s \leq 3 \). From (128) we see that the above swept volume manifold (with boundary) is not Euclidean but actually a Riemannian space with non-trivial curvature and torsion. Thus we have arrived at the following important classification problem: to describe an effective differential-geometric procedure for determining the manifolds with boundary that are generated by a Euclidean sweep of a solid object. We can simplify this problem by employing the geometric theory of Cartan. For the case under consideration we assume that a Lie group \( G \) acts on \( S_{t_0}(V) \) in a manner described by

\[
dy^j + \sum_{i=1}^{n} \xi^i_j(y)\omega^i(a, da) = 0,
\]

(130)

where \( y \in S_{t_0}(V) \), \( \omega^i(a, da), 1 \leq i \leq n = \dim G \), are the Maurer-Cartan left-invariant forms of the Lie group \( G \), \( a \in G \) is an arbitrary element and the \( \xi^i_j(y) \) are characteristic functions on the manifold \( S_{t_0}(V) \). The following theorem of Cartan is useful in describing a geometric object that is invariant with respect to the group action \( G \times S_{t_0}(V) \to S_{t_0}(V) \).

**Theorem 6.**

The differential system (130), with characteristic left-invariant one-forms \( \omega^i(a, da), 1 \leq i \leq n = \dim G \) on the Lie group \( G \), is tantamount to invariance of the group action on \( S_{t_0}(V) \) if and only if the following conditions obtain:

(i) The coefficients \( \xi^i_j \) are analytic functions of \( y \in S_{t_0}(V) \).

(ii) The system (130) is completely integrable in the Frobenius-Cartan sense.

We intend to investigate the above geometric aspects of the problem more thoroughly in a paper that is now in preparation. Here we are just going to formulate several hydrodynamic models for swept volume dynamical systems in \( E^3 \) having very rich symmetry groups and discuss some of their interesting and useful properties.

4.3 If \( S_{t_0}(V) \) is a swept volume manifold generated by a Euclidean motion \( \sigma(t), t \in [0, t_0] \), then there exists a set of tangent vector fields

\[
ds/da = u(s, \tau, t), \quad d\tau/da = v(s, \tau, t), \quad dt/da = w(s, \tau, t),
\]

(131)

where \( (s, \tau, t) \in S_{t_0}(V), \alpha \in \mathbb{R} \) is an evolution parameter and \( (u, v, w) \) is a smooth vector field on \( S_{t_0}(V) \). To more effectively describe the vector field (131), let us assume that the surface \( S = \partial V \) of the solid body satisfies the invariant equation

\[
\hat{\gamma}(\bar{x}) = 0,
\]

(132)

where \( \hat{\gamma} : \mathbb{R}^3 \to \mathbb{R} \) is a smooth function. This means that (132) is identically satisfied for the parametrized surface \( \bar{x} = \bar{x}(s, \tau) \) for all \( \tau \in [0, h] \) and \( s \in \mathbb{R}/2\pi \mathbb{Z} \):

\[
\hat{\gamma}(\bar{x}(s, \tau)) \equiv 0.
\]

(133)
When \( t \neq 0 \), Theorem 4 implies that the Euclidean motion can be written in the form
\[
x(s, \tau, t) = \xi(t) + a(t)\bar{x}(s, \tau)
\] (134)
for all \((s, \tau, t)^T \in S_{t_0}(V)\). Solving (134) for \( \bar{x} \) and substituting this in (133), we find that
\[
\bar{\gamma}(a^*(t)(x - \xi(t)) := \gamma(t, x) = 0
\] (135)
for \( t \in [0, h] \) and \( x \in S_{t_0}(V)\). Recalling now the general form (131) of a vector field on the manifold \( S_{t_0}(V) \), it follows directly from (135) that
\[
\frac{dw}{dt}(\gamma, x', (u, v, w)^T) = 0,
\] (136)
where the prime denotes the Fréchet derivative of the mapping (134). Thus (136) gives a necessary condition for the set of vector fields (131) to belong to \( T(S_{t_0}(V)) \).

If (131) satisfies (136) for the parameter \( \alpha := t \in [0, t_0] \), then we obtain
\[
\frac{ds}{dt} = u(s, \tau, t), \quad \frac{d\tau}{dt} = v(s, \tau, t),
\] (137)
\( w(s, \tau, t) \equiv 1 \) for all \((s, \tau, t)^T \in S_{t_0}(V)\). Using once again the representation (134), a simple but tedious calculation shows that equations (137) admit the prolongation
\[
\frac{Du}{Dt} = -\frac{\partial p}{\partial s}, \quad \frac{Dv}{Dt} = -\frac{\partial p}{\partial \tau},
\] (138)
where \( D/Dt \) is the Eulerian total or material derivative and \( p(s, \tau, t) := q(s, t) + \mathbb{R}(\tau, t) \).

Moreover, if we use the equation \( Dv/Dt = -\partial\mathbb{R}/\partial \tau \), the dynamical system (138) takes the form of the well known Navier-Stokes equations for the virtual flow of an ideal incompressible two-dimensional fluid under external pressure \( p(s, \tau) \), namely
\[
\frac{Du}{Dt} = -\rho^{-1}\frac{\partial p}{\partial s}, \quad \frac{Dv}{Dt} = -\rho^{-1}\frac{\partial p}{\partial \tau}, \quad \frac{Dp}{Dt} = -\rho(\partial u/\partial s + \partial v/\partial \tau),
\] (139)
where \((u, v)^T\) is the velocity vector in the \( s, \tau \)-plane and \( \rho > 0 \) is the density of the liquid. As the liquid is incompressible, i.e., \( D\rho/Dt = \partial \rho/\partial t + u\partial \rho/\partial s + v\partial \rho/\partial \tau \equiv 0 \), we easily establish the condition \( \partial u/\partial s + \partial v/\partial \tau = 0 \) for all \( s \in \mathbb{R}/2\pi \mathbb{Z} \) and \( \tau \in [0, h] \). We may assume that the density is normalized to unity, i.e., \( \rho \equiv 1 \).

Consider the Navier-Stokes equations (139) with a free surface given by the equation \( \tau = h(s, t) \), where \( h(\cdot, t), t \in [0, t_0], \) is the height of the fluid above the bottom (the \( s \)-axis) at time \( t \). Then (139) reduces to the system
\[
\begin{pmatrix}
\frac{du}{d\tau} \\
\frac{dh}{d\tau}
\end{pmatrix} = \mathcal{K}[u, q, h] := \begin{pmatrix}
-uw_s - u_s \int_0^\tau u_s d\tau - q_s \\
-h_s + \frac{\partial}{\partial s} \int_0^h u d\tau
\end{pmatrix},
\] (140)
which is similar to (109) subject to the condition \( u_\tau = 0 \). Here we have also assumed that \( Dh/Dt = v \mid \tau = h \), which ensures that the virtual fluid does not pass through the free surface \( \tau = h(s,t) \), and \( dq/dt = -\partial h/\partial s \), which stems from a wind pressure of unity along the \( s \)-axis.

The system (140) generates a nonlinear integro-differential dynamical system on an infinite-dimensional functional manifold \( M_{(u,q,h)} \subset C^\infty(\mathbb{R}^2;\mathbb{R} \times \mathbb{R}_+^2) \). In order to better understand this system, we will further investigate its Lax integrability, i.e., we study the existence of an infinite hierarchy of involutive conservation laws (with respect to some Poisson bracket) and a special operator representation of Lax type. In addition, we shall show that (140) has a natural connection with the nonlinear kinetic Boltzmann-Vlasov equation for a one-dimensional particle flow with a pointwise interaction potential between particles. This property of (140) enables us to establish a physical analogy between turbulence in kinetic multiparticle systems connected with stochastization of particle trajectories and instability and shocks in the flow of an ideal incompressible fluid flowing over a horizontal bottom and having a free boundary.

4.4 The Boltzmann-Vlasov equation can be obtained from (140) by use of the representation

\[
\tau = \int_{-\infty}^{u(s,\tau)} dpf(s,p,t),
\]

where \( \tau \in [0,h] \) and \( f \in C^2(\mathbb{R}^2;\mathbb{R}_+) \) is the Boltzmann distribution function for the kinetics of a one-dimensional system of particles. The equation of the free boundary in (140) is determined by the compatibility condition for the distribution function:

\[
h(s) = \int_{-\infty}^{u(s,h)} dpf(s,p,t), \quad s \in \mathbb{R}/2\pi\mathbb{Z}.
\]

Note also that the above transformation of the dynamical system (140) is canonical, i.e., the Boltzmann-Vlasov equation obtained is Hamiltonian and has a special symplectic structure on the functional manifold \( M(f) \subset C^2(\mathbb{R}^2;\mathbb{R}_+) \), and the same is true of (140).

In order to better understand the dynamical system (140), we introduce the moment functionals

\[
a_n(s) := \int_0^{h(s)} d\tau u^n(s,\tau), \quad s \in \mathbb{R}/2\pi\mathbb{Z}, n \in \mathbb{Z}_+.
\]

Then by direct calculation we find that (140) is equivalent to the following infinite-dimensional system of moment equations on the functional manifold \( M(\mathbb{Z}_+) := \{a_n \in C^2(\mathbb{R};\mathbb{R}) : n \in \mathbb{Z}_+, \sup_n n^k |a_n| < \infty, k \in \mathbb{Z}_+ \} \):

\[
\begin{pmatrix}
da_n/dt \\
dq/dt
\end{pmatrix} = K[a,q] :=
\begin{pmatrix}
-na_n-1q_x - a_{n+1,x} \\
-a_{0,x}
\end{pmatrix}.
\]

(141)

We first establish the complete integrability of the dynamical system (141) on \( M(\mathbb{Z}_+) \). For this purpose we consider the Lie algebra \( G_0 \) of symbols

\[
l(\xi) := \sum_{j \gg \infty} a_j(s)\xi^{-(j+1)},
\]

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where \( \{a_j(s)\} \in M(\mathbb{Z}_+), \) with bracket defined by (62):

\[
[l_1(\xi), l_2(\xi)]_0 = \frac{\partial l_1}{\partial \xi^0} \frac{\partial l_2}{\partial x} - \frac{\partial l_2}{\partial \xi^0} \frac{\partial l_1}{\partial x} \quad (142)
\]

As shown in [3], the bracket \([\cdot, \cdot]_0\) is a natural hydrodynamic limit of the standard bracket \([\cdot, \cdot]\) on the Lie algebra \(G\) (56) of symbols of pseudodifferential operators on \(\mathbb{R}\).

The Lie algebra \(G_0\) admits a natural direct sum decomposition (58): \(G_0 = G_{0+} \oplus G_{0-}\); moreover, the identifications \(G_{0+}^* \cong G_{0-}, G_{0-}^* \cong G_{0+}\) hold for the space \(G_0^*\) dual to \(G_0\) with respect to the standard invariant inner product having the form \(\langle l_1, l_2 \rangle := Tr(l_1 \circ l_2)\), where \(Tr(l(\xi)) := \int_R ds \text{res}_l(\xi)\) for \(l_1, l_2, l \in G_0\). We define the gradient as in (48) and \(R\) with its corresponding projectors \(P_\pm\) as in (44).

Consider the vector field \(\hat{K}\) on \(G_0^*\) defined by

\[
dl(\xi)/dt = K[l(\xi)] := ad_{R^\gamma(l)}^*l(\xi),
\]

which is a coadjoint action of \(R^\gamma(l) \in G_0^*\) on \(G_0\), where \(\gamma \in D(G_0^*)\) a Casimir functional. The isomorphism \(G_0^* \cong G_0\) obtained from the inner product on \(G_0\) implies that (141) can be represented on \(G_0^*\) in the form

\[
dl/dt = K[l] := [l, R^\gamma(l)]. \quad (143)
\]

Let us show that (143) is Hamiltonian with respect to the standard symplectic Lie-Poisson structure on \(G_0^*\). Indeed, the Lie-Poisson bracket

\[
\{\gamma, \mu\}_L := \langle l, [\nabla \gamma, \nabla \mu]_0 \rangle
\]

is defined naturally on \(G_0^*\) via (47). Then it clearly follows from the properties of the scalar product \(\langle \cdot, \cdot \rangle\) that (143) is equivalent to the Hamiltonian system

\[
dl/dt = \{\gamma, l\}_\theta,
\]

where \(\theta := LR + R^*L\). Define

\[
l(a(\xi)) = \frac{\xi^2}{2} + q + A(\xi), \quad (144)
\]

where \(A(\xi) := \sum_{j \in \mathbb{Z}_+} a_j(s)\xi^{-(j+1)} \in G_{0-}\) is constructed from the values of the moment functions on \(M(\mathbb{Z}_+).\) According to the Kostant-Symes theorem [17, 23], all of the functionals \(\gamma_j = Tr^j/2\) are Casimir and in involution with respect to the Lie-Poisson bracket \(\{\cdot, \cdot\}_\theta\) on \(G_0^*\). Consequently, (143) with \(\gamma = H = Tr l^2 \in D(M(\mathbb{Z}_+))\) is a completely integrable Hamiltonian flow on \(M(\mathbb{Z}_+) \cong M(u, q, h)\), where the Lie-Poisson bracket is given by

\[
\{\gamma, \mu\}_\theta := \int_R ds \langle \text{grad}\gamma, \theta(a, q)\text{grad}\mu \rangle. \quad (145)
\]

Here \(\langle \cdot, \cdot \rangle\) is the standard scalar product on the space of sequences \(l_2(\mathbb{R})\) and

\[
\theta(a, q) := [\theta_{mn}(a)] \otimes \theta(q), m, n \in \mathbb{Z}_+,
\]

\[
\theta_{mn}(a) := ma_{m+n-1} + n \frac{d}{dx} a_{m+n-1},
\]

\[
\theta(q) := \frac{d}{dx}. \quad (146)
\]

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With the above we have proved the following result.

**Theorem 7.**

The moment dynamical system (141) on the functional manifold \( M_{(\mathbb{Z}_+)} \) is a completely Lax integrable Hamiltonian flow with respect to the Lie-Poisson bracket (145)-(146); the Hamiltonian functional is \( H := T r l^2 \) and the Lax representation has the form

\[
dl/dt = [l, \mathcal{R} \nabla H(l)],
\]

where \( l \in \mathcal{G}_0^s \) is given by (144).

We now prove that the above mappings of \( M_{(\mathbb{Z}_+)} \) onto \( M_{(u,q,h)} \) and \( M_{(f,q)} \) are canonical. In the first case it is easy to verify that the mapping

\[
a_n(s) = \int_0^{h(s)} d\tau u^n(s, \tau) \in M_{(\mathbb{Z}_+)}, \ n \in \mathbb{Z}_+, \ \sim \in \mathbb{R}/\pi\mathbb{Z},
\]

transforms the Hamiltonian structure \( \{\cdot, \cdot\}_{\theta(u,q,h)} \) into \( \{\cdot, \cdot\}_{\theta(u,q,h)} \); moreover

\[
d(u, q, h)^T / dt = -\theta(u, q, h) \text{grad} H = K[u, q, h],\]

where \( \theta(u, q, h) := \text{antidiag}(d/ds, d/ds, d/ds) \) is the canonical structure on \( M_{(u,q,h)} \). To prove that the mapping of \( M_{(u,q,h)} \) onto \( M_{(f)} \) is canonical, we use the mapping \( \tau = \int_{-\infty}^{u(s,t)} dp f(s, p; t) \) assuming that the free surface is given by \( h(s, t) = \int_{-\infty}^{u(s,h)} dp f(s, p; t), (s, t) \in \mathbb{R}/\pi\mathbb{Z} \times [0, T] \), and consider the space \( F \) of smooth functions on \( T^*(\mathbb{R}) \) with canonical Poisson bracket \( \{f, g\}(s, p) := \partial_q \partial_p f - \partial_p \partial_q f, (s, p) \in \mathbb{R}/\pi\mathbb{Z} \times \mathbb{R} \), which transforms it into a Lie algebra. If we introduce Hilbert space structure on \( F \) via the inner product \( (\cdot, \cdot) \), then by the Riesz theorem we can identify \( F^* \) with \( F \). Note that \( (f, g) = (g, f) = \int_\mathbb{R} ds \int_\mathbb{R} dp f(s, p) g(s, p) \) so that the scalar product is invariant with respect to the Lie-Poisson bracket on \( F \), i.e., \( (f, \{g, h\}) = (\{f, g\}, h) \) for all \( f, g, h \in F \). We now define the gradient mapping \( \nabla : D(F^*) \to F \) for a functional \( \gamma \in D(F^*) \) by \( (\nabla \gamma)(f, g) := \frac{d}{d\varepsilon} \gamma(f + \varepsilon g) \mid_{\varepsilon=0} \). Then \( \nabla \gamma(f) = \gamma/\tau f \) is an ordinary Euler variational derivative of \( \gamma \) at \( f \in F^* \). Clearly \( \{\gamma, \mu\} := (f, [\nabla \gamma, \nabla \mu]) \) defines a canonical Hamiltonian structure on \( F^* \).

Consider a functional \( H \in D(F^*) \) and its gradient \( \nabla H(f) \in F \). Then the vector field \( df/dt = ad_{\nabla H(f)} f \) on \( F^* \) is generated by the coadjoint action of the Lie algebra \( F \) on \( F^* \).

It follows from the properties of the inner product on \( F \) that this vector field is equivalent to the following Lax type equation:

\[
df/dt = \{f, \nabla H(f)\},
\]

which coincides with the Hamiltonian equation \( df/dt = \{H, f\} \) on the manifold \( F^* \).

We now make the following identification: \( M_{(\mathbb{Z}_+)} \ni a_n(s) = \int_\mathbb{R} dp p^n f(s, p), s \in \mathbb{R}/2\pi\mathbb{Z} \), consistent with the mapping of \( M_{(\mathbb{Z}_+)} \) into \( M_{(f)} \) introduced above. As a result, the Lie-Poisson bracket \( \{\cdot, \cdot\} \) on \( F^* \) transforms into the Lie-Poisson bracket (145) on \( \mathcal{G}_0^s \), i.e., the mapping of
$M(\mathbb{Z}_+, \nu)$ into $M(f, q)$ is canonical. Using now the Hamiltonian $H = \int_\mathbb{R} ds(a_2 + 2qa_0)$, we find the kinetic Boltzmann-Vlasov equation for the distribution function $f \in M(f)$ and the field function $q \in M(q)$:

\[
\begin{align*}
\frac{df}{dt} &= -pf_s + q_sf_p, \\
\frac{dq}{dt} &= -\int_\mathbb{R} dpf_s(s, p, t).
\end{align*}
\] (150)

The consistent system of evolution equations (150) is important and has interesting applications to the theory of kinetic processes. Consider a system of interacting particles on an axis $\mathbb{R}$, and assume that its density $\rho$ is a constant ($=1$) and the potential of particle interaction has the form $\Phi(s - s')$, $s, s' \in \mathbb{R}/2\pi\mathbb{Z}$. Then the distribution function satisfies the kinetic Boltzmann-Vlasov equation

\[
\frac{df}{dt} = \int_0^{2\pi} ds' \int_\mathbb{R} dp' \Phi(s - s') f(s', p'; t),
\] (151)

provided that there is no multiparticle correlation. Comparing (150) with (151), we see that the following identification obtains:

\[
q(s, t) = \int_0^{2\pi} ds' \int_\mathbb{R} dp' \Phi(s - s') f(s', p'; t),
\] (152)

which can be used to reduce the second equation in (150) to the linear integral equation

\[
\int_0^{2\pi} ds' \int_\mathbb{R} dp' \Phi(s - s') pf(s', p') = \int_\mathbb{R} dpf(s, p) - \text{const},
\] (153)

valid for all $f \in M(f)$ satisfying (150). But owing to the definition of the distribution function $f$, (139) and the fact that the density $\rho(s, t) = \int_\mathbb{R} dpf(s, p; t) \equiv 1$ for all $t \in [0, T]$, we obtain that

\[
\int_0^{2\pi} ds' \int_\mathbb{R} dp\Phi(s - s') pf(s', p) = 1 - \text{const}
\] (154)

for all $s \in \mathbb{R}/2\pi\mathbb{Z}$. Since, in general, $\Phi(s - s') \rightarrow 0$ as $|s - s'| \rightarrow \infty$, it follows from (153) and (154) that const. = 1; hence, $\int_\mathbb{R} dppf(s, p) = 0$ for all $s \in \mathbb{R}/2\pi\mathbb{Z}$. Thus, $f(s, p) = f(s, -p)$ for all $(s, p) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, and the solution (152) is consistent with the initial dynamical system (140). A complete description of all possible solutions to (140) and (150) will be presented elsewhere.

4.3 Here we consider some differential-geometric aspects of a swept volume manifold $S_0(V)$ generated by a Euclidean motion in $\mathbb{R}^3$. We shall use the basic notation from Theorem 6 of Subsection 4.2.

We define the following system of one-forms (130) generated by a group action $G \times Y \rightarrow Y$ on a manifold $Y$:

\[
\beta^i := dy^i + \sum_{i=1}^n \xi^i(a, da),
\] (155)

For this system to be completely Frobenius integrable, the canonical one-forms $\bar{\omega}^i : 1 \leq i \leq n$ must satisfy the Maurer-Cartan equations

\[
d\bar{\omega}^j + \frac{1}{2} \sum_{i,k=1}^n C^j_{ik} \bar{\omega}^i \wedge \bar{\omega}^k := \bar{\Omega}^j = 0
\] (156)
for all $1 \leq j \leq n$, where the $C_{ik}^j$ are the structure constants. If the canonical one-forms $\{\bar{\omega}^i\}$ are defined via the scheme

$$
\begin{align*}
T^*(M) & \xrightarrow{n} T^*(G) \\
M & \xrightarrow{n} G,
\end{align*}
$$

where $M$ is a smooth finite-dimensional manifold and $\eta$ is a smooth mapping, then (156) takes the simple form

$$
\eta^*\bar{\Omega}^j |_{\tilde{M}} = 0, \quad 1 \leq j \leq n
$$

on some integral submanifold $\tilde{M} \subset M$ that is diffeomorphic to $S_{t_0}(V)$.

Let $\{\alpha^j \in \Lambda^2(M) : 1 \leq j \leq m\}$ be a basis of two-forms generating the ideal $I(\alpha)$ over the two-forms (156). Using this basis we can formulate the Cartan criterion for the system of one-forms (155) to define a group action of $G$ on $S_{t_0}(V)$. The ideal $I(\alpha, \beta)$ generated by both (155) and $\{\alpha^j\}$ over the prolonged locally defined manifold $M \times Y$ must be completely integrable; therefore

$$
d\beta^j = m \sum_{k=1}^{m} f_{kj} \alpha^k + \sum_{i=1}^{n} g_{ij}^i \wedge \beta^i,
$$

where $f_{kj} \in \Lambda^0(M \times Y)$, $g_{ij}^i \in \Lambda^1(M \times Y)$, $1 \leq i, j \leq n$, $1 \leq k \leq m$. We note that the ideal $I(\alpha, \beta)$ over the two-forms (156) should also be completely integrable via the Cartan criterion, because it follows from the equations $d\bar{\Omega}^j = 0$, $1 \leq j \leq n$, that $dI(\alpha) \subset I(\alpha)$ and $I(\alpha) |_{\tilde{M}} = 0$.

The above enables us to interpret the locally defined manifold $M \times S_{t_0}(V)$ as the adjoint of a principal fiber bundle $P(M; G)$ over the base manifold $M$ with structure group $G$ acting on the fibered manifold $P$. By representing a point locally as $(Z, a) \in P(M; G)$, we obtain the local representation of the connection one-form as

$$
\omega(Z, a) := \bar{\omega}(a, da) + Ad_{a^{-1}} \langle \Gamma(Z), dZ \rangle,
$$

where $\Gamma(Z) \in \mathcal{G}$ is the Christoffel elements, $Z \in M$, $\bar{\omega}(a, da) := \sum_{i=1}^{n} \bar{\omega}^i(a, da) A^i$ and $\{A^i\}$ is a basis for $\mathcal{G}$. Hence, the curvature two-form (defined just on $M$) is

$$
\Omega := d\omega + \omega \wedge \omega = \frac{1}{2} Ad_{a^{-1}} \sum_{i,j=1}^{m} F_{ij}(Z) dZ^i \wedge dZ^j,
$$

where for all $Z \in M$, $i, j = 1, \ldots, m = \dim M$,

$$
F_{ij}(Z) := \frac{\partial \Gamma_j}{\partial Z^i} - \frac{\partial \Gamma_i}{\partial Z^j} + [\Gamma_i, \Gamma_j].
$$

The results obtained above for the one-forms (155) imply that

$$
\Omega |_{\tilde{M}} = 0 \iff F_{ij} |_{\tilde{M}} = 0
$$

for all $1 \leq i, j \leq m$, and this is equivalent to

$$
\frac{1}{2} \sum_{i,j=1}^{m} F_{ij}(Z) dZ^i \wedge dZ^j \in I(\alpha).
$$
Thus we can find the structure constants of the Lie group $G$ by a simple yet tedious computation of the holonomy algebra of the connection (160), which is generated by all linear combinations of elements $F_{ij}, \nabla_k F_{ij}, \nabla_k \nabla_l F_{ij}, \ldots, i, j, k, l = 1, \ldots, m$, where $\nabla_k := \frac{\partial}{\partial Z^k} - \Gamma^k$, are the appropriate covariant derivatives in $T^*(M)$. Whence, we obtain the following:

**Theorem 8.**

Suppose that the holonomy Lie algebra $G$ associated with a Euclidean motion of the solid object $V$ in $E^3$ is generated by parallel transport along a two-dimensional integral submanifold $\tilde{M} \subset M$ with local coordinates $(s, t) \in \mathbb{R}/2\pi \mathbb{Z} \times [0, T)$. Then owing to (158), the following system is compatible on $\tilde{M}$:

$$\frac{\partial f}{\partial s} = -\Gamma(s)f, \quad \frac{\partial f}{\partial t} = -\Gamma(t)f,$$

for all $f$ spanning a linear representation space for the Lie algebra $G$, where $\Gamma(s), \Gamma(t)$ are the nontrivial Christoffel matrices that define the curvature form $\Omega$ over $M$.

The above theorem gives rise to a new way of constructing exact forms of a group actions $G \times Y \to Y$ which produce Euclidean motions in $\mathbb{R}^3$. Some interesting applications of this new approach are presented in a forthcoming paper.

5. CONCLUDING REMARKS

The investigation of hydrodynamic equations based upon the kinetic theory of many-particle systems goes back to Bogoliubov [4] who formulated the first really effective equations for the flow of an ideal incompressible fluid including heat transfer. These equations are quite complicated, so one- and two-dimensional approximations are often used to solve practical problems associated with such phenomena as vorticity and convective and turbulent flows. When viscosity is neglected, Hamiltonian methods [26] are well-suited to these equations and lead to the construction of conservation laws and their integration in quadratures.

In the present study it has been shown that for many Lax integrable nonlinear dynamical systems with canonical symplectic structure there exists an associated pair of completely integrable equations: the kinetic Boltzmann-Vlasov equation and a corresponding "two-dimensionalized" Benney type hydrodynamic system describing the flow of an ideal incompressible fluid with a free surface over a horizontal bottom together with appropriate boundary conditions (accounting for gravity, wind pressure, surface tension and other physical phenomena). A new class of these equations was studied from the point of view of Lax integrable dynamical systems.

The algebraic techniques developed in this paper for the hydrodynamic systems investigated makes it possible to classify their Lax type representations by means of a choice of the initial "hydrodynamical" Lie algebra $G_0$ of pseudodifferential operators on an appropriate Schwartz space of functionals. There are a number of possible generalizations of $G_0$; for example, the Lie
algebra $G_0 [G]$ with values in a finite-dimensional Lie algebra $G$ endowed with a non-degenerate symmetric Killing form.

As we have shown here, all kinetic Boltzmann-Vlasov equations for many-particle dynamical systems are Hamiltonian with respect to the standard symplectic Lie-Poisson structure. These equations reduce to the moment equations on the functional manifold $M_{(Z_+)}$ only when the initial many-particle system in a quasiclassical approximation is the canonically Lax integrable system. In view of the fact that the Vlasov approximation [3] for the Boltzmann kinetic equations is rather restrictive for many interparticle interaction potentials, there is great practical interest in constructing the coupled kinetic Boltzmann-Vlasov equations for one and two particle distribution functions using the induced Lie-Poisson bracket as in [7].

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References


