MANIFOLDS WITH MINIMAL HOROSPHERES
AND BOUNDED DERIVATIVE STABILITY VECTOR FIELDS

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Abstract

Let $(M,g)$ be a geodesically complete and an infinite volume Riemannian manifold with minimal horospheres and bounded derivative stability vector field. We prove that bounded functions on $M$ having the mean-value property are constant. We extend thus a result of the authors in [4] where they proved a similar result for bounded harmonic functions on harmonic manifolds with minimal horospheres.
1 Introduction

Let \((M, g)\) be a non-compact and geodesically complete Riemannian manifold. For \(r > 0\), we define the stability vector field \(H(., r)\) by:

\[
H(p, r) =: \int_{B(p, r)} \exp_{p}^{-1}(q) \, d\mu(q), \quad \forall \, p \in M,
\]

where \(\exp_{p}^{-1}\) denotes the inverse of the exponential map, \(d\mu\) the Riemannian volume element and \(B(p, r)\) the closed ball with centre \(p\) and radius \(r\).

For special manifolds like harmonic manifolds, d’Atri spaces (see [3] for definition and more information) or compact locally symmetric spaces, the stability vector field vanishes identically for any \(r > 0\) (see [1]).

The vanishing of the stability vector field for any radius \(r > 0\) means that any geodesic ball in \((M, g)\) has its Riemannian center of mass (or center of gravity) at the centre of the ball.

By replacing \(\exp_{p}^{-1}\) by \(\exp_{p}^{-1} \circ f\) and integrating on a measurable set \(A\), the author in [2] characterized the center of mass of a measurable map \(f : A \rightarrow M\) as the points of \(M\) where the corresponding stability vector field vanishes.

In [1] the authors studied the class of manifolds for which the stability vector vanishes identically for any radius and they proved that such manifolds are ball-homogeneous and, are locally and weakly harmonic under some restriction on the Ricci tensor.

We say that a function \(u\) defined on \((M, g)\) possesses the mean-value property if:

\[
\forall \, r > 0 \text{ and } \forall \, p \in M, \quad u(p) = \frac{1}{V(p, r)} \int_{B(p, r)} u(q) \, d\mu(q),
\]

where \(V(p, r)\) denotes the volume of the ball \(B(p, r)\).

Well known examples of functions possessing the mean-value property are harmonic functions defined on harmonic manifolds.

In [4] the authors proved that on non-compact harmonic manifolds with minimal horospheres, bounded harmonic functions are constant.

Our aim is to extend this result on bounded functions having the mean-value property and defined on non-compact manifolds with minimal horospheres and bounded derivative (w.r.t. \(r\)) stability vector fields.

2 Volume functions and stability vector fields

Let \(V : (p, r) \in M \times [0, +\infty] \mapsto V(p, r)\) be the function associating to each couple \((p, r) \in M \times [0, +\infty]\) the volume \(V(p, r)\) of the ball \(B(p, r)\).

The volume function \(V\) and the stability vector field \(H\) are related by the following differential equation of which there is an equivalent version in [1, p. 215-217]:

\[
2 \]
Lemma 1.1

Let $\nabla$ denote the gradient operator on $(M, g)$. For any $r > 0$ and $p \in M$, there exists a vector $U_p \in T_pM$ independent of $r$ such that:

$$\nabla V(p, r) + \frac{1}{r} \frac{\partial}{\partial r} H(p, r) = U_p,$$

or equivalently:

$$\frac{\partial}{\partial r} \left( \nabla V(p, r) + \frac{1}{r} \frac{\partial}{\partial r} H(p, r) \right) = 0.$$

Proof

We have to compute the variations of $H(p, \cdot)$ w.r.t. $r$.

Let $K$ be a compact subset of $M$ and $\delta(K)$ the diameter of $K$.

Consider $\psi \in C^\infty(M)$ and $h \in C^\infty([0, +\infty[)$ with compact support strictly included in $[0, \delta(K)]$. We have:

$$\int_K h(d^2(p, q)) \psi(q) d\mu(q) = \int_0^{\delta(K)} h(t^2) \int_{S(p, t)} \psi_t(q) d\sigma(q) dt,$$

where $d\sigma$ denotes the induced Riemannian measure on the sphere $S(p, t)$ with centre $p$ and radius $t$, and $\psi_t$ the restriction of $\psi$ to $S(p, t)$.

Let us take the gradient (w.r.t. $p$) of both sides of the previous equality.

We have:

$$\int_K \nabla (h(d^2(p, q))) \psi(q) d\mu(q) = \int_0^{\delta(K)} h(t^2) \nabla \left( \int_{S(p, t)} \psi_t(q) d\sigma(q) \right) dt.$$

Equivalently:

$$\int_K \nabla (h(d^2(p, q))) \psi(q) d\mu(q) = \int_0^{\delta(K)} h(t^2) \nabla (M_p[t, \psi]) dt \quad (i),$$

where $M_p[t, \psi]$ denotes the non-normalized mean-value of $\psi$ on $S(p, t)$.

We now compute the left-hand side of $(i)$. We get:

$$\int_K \nabla (h(d^2(p, q))) \psi(q) d\mu(q) = \int_K h'(d^2(p, q)) \nabla d^2(p, q) \psi(q) d\mu(q)$$

$$= \int_0^{\delta(K)} h'(t^2) \left( \int_{S(p, r)} \nabla d^2(p, q) \psi_t(q) d\sigma(q) \right) dt$$

$$= \int_0^{\delta(K)} h'(t^2) M_p[t, \nabla d^2(p, \cdot)\psi] dt$$

$$= \int_0^{\delta(K)} \frac{1}{2\sqrt{t}} h(t) M_p[\sqrt{t}, \nabla d^2(p, \cdot)\psi] dt$$

$$= -\int_0^{\delta(K)} \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_p[\sqrt{t}, \nabla d^2(p, \cdot)\psi] \right) dt,$$
by parts integration and since \(h\) has compact support strictly included in \([0, \delta(K)]\) and, where \(M_p[t, \nabla d^2(p, .)\psi]\) denotes the vector-valued mean-value (non-normalized) of the map \(\nabla d^2(p, .)\psi : q \in M \mapsto \nabla d^2(p, q)\psi(q) = 2\psi(q)\exp_p^{-1} q \in T_p M\). The right-hand side of \((i)\) can be written as:

\[
\int_K \nabla \left( h(d^2(p, q)) \right) \psi(q) \, d\mu(q) = \int_0^{\delta(K)} \frac{1}{2\sqrt{t}} h(t) \nabla \left( M_p[\sqrt{t}, \psi] \right) \, dt.
\]

The relation \((i)\) becomes then:

\[
\int_0^{\delta(K)} \frac{1}{2\sqrt{t}} h(t) \nabla \left( M_p[\sqrt{t}, \psi] \right) \, dt = - \int_0^{\delta(K)} h(t) \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_p[\sqrt{t}, \nabla d^2(p, .)\psi] \right) \, dt,
\]

for any function \(h \in C^\infty([0, +\infty[)\) with compact support strictly included in \([0, \delta(K)]\). It follows that:

\[
\frac{1}{2\sqrt{t}} \nabla \left( M_p[\sqrt{t}, \psi] \right) = - \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_p[\sqrt{t}, \nabla d^2(p, .)\psi] \right), \quad \forall t \in [0, \delta(K)] .
\]

And since the previous equality holds for any compact subset \(K\) of \(M\), it holds then on any interval \([0, \alpha[, \, \alpha \in \mathbb{R}\), by considering a compact \(K \subset M\) of diameter \(\delta(K) \geq \alpha\); for example a closed ball of \(M\) of radius \(r \geq 2\alpha\) that is compact since \(M\) is geodesically complete. Hence we have:

\[
\frac{1}{2\sqrt{t}} \nabla \left( M_p[\sqrt{t}, \psi] \right) = - \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_p[\sqrt{t}, \nabla d^2(p, .)\psi] \right), \quad \forall t \in [0, +\infty[ .
\]

Equivalently:

\[
\nabla (M_p[t, \psi]) = \frac{1}{2t^2} M_p[t, \nabla d^2(p, .)\psi] - \frac{1}{2t} \frac{\partial}{\partial t} (M_p[t, \nabla d^2(p, .)\psi]) \quad (ii) .
\]

By evaluating the relation \((ii)\) for \(\psi \equiv 1\) and using the fact that:

\(M_p[t, \nabla d^2(p, .)] = 2\frac{\partial}{\partial t} H(p, t)\), we obtain:

\[
\nabla (M_p[t, 1]) = \frac{1}{t^2} \frac{\partial}{\partial t} H(p, t) - \frac{1}{t} \frac{\partial^2}{\partial t^2} H(p, t) .
\]

But:

\[
\nabla (M_p[t, 1]) = \nabla \left( \frac{\partial}{\partial t} V(p, t) \right) = \frac{\partial}{\partial t} (\nabla V(p, t))
\]

and

\[
\frac{1}{t^2} \frac{\partial}{\partial t} H(p, t) - \frac{1}{t} \frac{\partial^2}{\partial t^2} H(p, t) = - \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} H(p, t) \right) .
\]

Thus we get:

\[
\frac{\partial}{\partial t} (\nabla V(p, t)) = - \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} H(p, t) \right) .
\]

Hence

\[
\frac{\partial}{\partial t} \left( \nabla V(p, t) + \frac{1}{t} \frac{\partial}{\partial t} H(p, t) \right) = 0 .
\]

Thus we get the result. \(\square\)
3 A derivative formula

Let \( p \in M \) and \( X \) a unit vector in \( T_pM \). Consider as in [4] the function:

\[
\theta_X : M \setminus \{ p \} \longrightarrow \mathbb{R}
\]

\( q \longmapsto \theta_X(q) =: \angle_p(X, \dot{\gamma}_q(0)) \),

where \( \gamma_q \) denote the unique geodesic from \( p \) to \( q \) and \( \angle_p(X, \dot{\gamma}_q(0)) \) the angle at \( p \) between the vectors \( X \) and \( \dot{\gamma}_q(0) \). For the geodesic \( c \) with \( c(0) = p \) and \( \dot{c}(0) = X \), let \( P_t \) be the parallel translation along \( c \) and \( f_t \) the one parameter family of diffeomorphisms of \( M \) given by \( f_t = \exp_{c(t)} \circ P_t \circ \exp^{-1}_p \).

Let \( u \) be a differentiable function on \( M \) possessing the mean-value property.

It holds:

**Proposition 3.1**

\[
X.u(p) = \frac{1}{V(p, r)} \int_{S(p, r)} u \cos \theta_X d\sigma + \frac{1}{V(p, r)} \int_{B(p, r)} u \frac{\partial}{\partial t} D(f_t)_{t=0} d\mu
\]

\[
+ \frac{1}{r V(p, r)} \langle \frac{\partial}{\partial r} H(p, r), X \rangle - \frac{u(p)}{V(p, r)} < U_p, X >
\]

where \( U_p \) is the vector given by the lemma 2.1 and \( D(f_t) \) the Jacobian determinant of the diffeomorphism \( f_t \).

**Proof**

Since the function \( u \) possesses the mean-value property, we have:

\[
u(c(t)) = \frac{1}{V(c(t), r)} \int_{B(c(t), r)} u(q) d\mu(q) .
\]

And then

\[
X.u(p) = \frac{d}{dt} u(c(t))_{t=0}
\]

\[
= \frac{d}{dt} \left( \frac{1}{V(c(t), r)} \int_{B(c(t), r)} u d\mu \right)_{t=0}
\]

\[
= -\frac{1}{V(p, r)^2} \langle \nabla V(p, r), X \rangle \int_{B(p, r)} u d\mu
\]

\[
+ \frac{1}{V(p, r)} \frac{d}{dt} \left( \int_{B(c(t), r)} u d\mu \right)_{t=0} (i).
\]

From lemma 2.1 we have:

\[
\nabla V(p, r) = U_p - \frac{1}{r} \frac{\partial}{\partial r} H(p, r).
\]
It then follows:
\[
\frac{1}{V(p, r)^2} < \nabla V(p, r), X > \int_{B(p, r)} u \, d\mu = \frac{1}{V(p, r)^2} < U_p, X > \int_{B(p, r)} u \, d\mu \\
- \frac{1}{V(p, r)^2} < \frac{1}{r} \frac{\partial}{\partial r} H(p, r), X > \int_{B(p, r)} u \, d\mu \\
= \frac{u(p)}{V(p, r)} < U_p, X > - \frac{1}{r} \frac{u(p)}{V(p, r)} < \frac{\partial}{\partial r} H(p, r), X > \quad (ii),
\]

since \( u(p) = \frac{1}{V(p, r)} \int_{B(p, r)} u(q) \, d\mu(q) \).

In addition it holds:
\[
\frac{d}{dt} \left( \int_{B(c(t), r)} u \, d\mu \right) |_{t=0} = \frac{d}{dt} \left( \int_{B(p, r)} (f_t^* u) D(f_t) \, d\mu \right) |_{t=0} = \int_{B(p, r)} \frac{d}{dt} (f_t^* u) |_{t=0} \, d\mu + \int_{B(p, r)} u \frac{d}{dt} D(f_t) |_{t=0} \, d\mu, \quad \text{since} \quad f_0 = \text{id}_{M}.
\]

By the theorem 2.1 in [4] we get:
\[
\int_{B(p, r)} \frac{d}{dt} (f_t^* u) |_{t=0} \, d\mu = \int_{S(p, r)} u \cos \theta_X \, d\sigma.
\]

Thus:
\[
\frac{d}{dt} \left( \int_{B(c(t), r)} u \, d\mu \right) |_{t=0} = \int_{S(p, r)} u \cos \theta_X \, d\sigma + \int_{B(p, r)} u \frac{d}{dt} D(f_t) |_{t=0} \, d\mu \quad (iii).
\]

By replacing (\(ii\)) and (\(iii\)) in the relation (\(i\)) we obtain the result. \(\Box\) For each radius \(r > 0\), the function \(\frac{d}{dt} D(f_t) |_{t=0}\) is bounded on the ball \(B(p, r)\) by a constant \(\lambda(r)\) depending on \(r\).

Let \((M, g)\) be a geodesically complete and an infinite volume manifold with minimal horospheres.

As an immediate consequence of proposition 3.1 we have the following result:

**Corollary 3.1**

Assume that the vector field \(\frac{\partial}{\partial r} H(., r)\) is bounded (independently of \(r\)) and \(\lim_{r \to +\infty} \lambda(r) = 0\).

Then any bounded function on \(M\) having the mean-value property is constant.

**Proof**

Notations are as in proposition 3.1. Let \(\alpha \geq 0\) and \(\beta \geq 0\) such that
\[
|u| \leq \alpha \quad \text{and} \quad \| \frac{\partial}{\partial r} H(p, r) \| \leq \beta, \quad \forall \ r > 0 \quad \text{and} \quad \forall \ p \in M.
\]

From the derivative formula given in proposition 3.1, we get:
\[
|X.u(p)| \leq \alpha \frac{A(S(p, r))}{V(p, r)} + \alpha \lambda(r) + \frac{\alpha \beta}{r} \frac{\lambda(r)}{V(p, r)} - \frac{\alpha}{V(p, r)} \| U_p \|,
\]

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where $A(S(p, r))$ is the area of the sphere $S(p, r)$.

We have:

$$\lim_{r \to +\infty} V(p, r) = \infty,$$

since $M$ is assumed to be of infinite volume;

and due to the minimality of horospheres

$$\lim_{r \to +\infty} \frac{A(S(p, r))}{V(p, r)} = K_\infty = 0.$$

By the assumption on the constants $\lambda(r)$ and by taking the limit of the previous inequality as $r \to \infty$, it follows then:

$$|X.u(p)| = 0,$$

for any $p \in M$ and any unit vector $X \in T_p M$.

Hence $u$ is a constant function. \hfill \Box

**Remark**

For harmonic manifolds the diffeomorphisms $f_t = \exp_{c(t)} \circ P_t \circ \exp_p^{-1}$ are volume preserving; i.e. $f_t^*(d\mu) = d\mu$. Then $\forall \ t, \ D(f_t) \equiv 1$ and in particular $\frac{d}{dt} D(f_t)|_{t=0} = 0$. Also as we already observed in the introduction the stability vector field vanishes identically for harmonic manifolds. Thus the assumptions on the constants $\lambda(r)$ and on the derivative of the stability vector fields are trivially satisfied in the case of harmonic manifolds.

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**References**


