In this paper we introduce the (warped) product of maps defined between Riemannian (warped) product spaces and we give necessary and sufficient conditions for (warped) product maps to be (bi)-harmonic. We obtain from these results good characterizations of non trivial harmonic metrics and nonharmonic biharmonic metrics on warped product spaces.
1 Introduction

Let \( f : (M^m, g) \rightarrow (N^n, h) \) be a map between the \( m \)-dimensional Riemannian manifold \((M, g)\) and the \( n \)-dimensional Riemannian manifold \((N, h)\).

The energy of the map \( f \) is given by \( E(f) = \int_M e(f) d\text{vol}_g \), where \( d\text{vol}_g \) is the volume form on \((M, g)\) and \( e(f)(x) := \frac{1}{2} \| df(x) \|^2_{\nabla^* M \otimes f^{-1} TN} \) the energy density of \( f \) at the point \( x \in M \). In local coordinates \((x^i)_m=1\) on \( M \) and \((y^a)_n=1\) on \( N \), the energy density is given by \( e(f)(x) = \frac{1}{2} g^{ij}(x) h_{ab}(f(x)) \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} \).

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map \( f \) is harmonic if and only if its tension field \( \tau(f) \) vanishes identically, where the tension field is given by

\[
\tau(f) := \text{trace}_g \nabla f = g^{ij} \left( \frac{\partial^2 f^a}{\partial x^i \partial x^j} - \nabla^i f^a - \nabla_j f^a \right) + N^k \Gamma^a_{ij} \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} - \nabla^a f^b \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j},
\]

where \( \Gamma^a_{ij} \) and \( N^k \Gamma^a_{ij} \) are the Christoffel symbols of the metrics \( g \) and \( h \) respectively.

The theory of harmonic maps was introduced by J. Eells and J. H. Sampson [7] and has been developed by many authors.

The bienergy \( E_2(f) \) of the map \( f \) is defined by \( E_2(f) := \frac{1}{2} \int_M \| \tau(f) \|^2 d\text{vol}_g \).

As it was observed in [1] the bienergy is the first variation of the energy w.r.t. a particular variation of the map; precisely that with \(-\tau(f)\) as variational vector field.

The map \( f \) is said to be biharmonic if it is a critical point of the bienergy functional. The Euler-Lagrange equation associated to the bienergy functional is given by:

\[
\tau_2(f) := -\Delta^f \tau(f) - \text{trace}_g R^N(df, \tau(f)) df = 0,
\]

where \( \Delta^f = -\text{trace}_g \nabla^f \nabla^f - \nabla^f \tau \) is the Laplacian on the sections of \( f^{-1}(TN) \), and \( R^N \) the Riemannian curvature operator of \((N, h)\).

Note that \( \tau_2(f) = -J_f(\tau(f)) \), where \( J_f \) is the Jacobi operator of \( f \), which gives the second variation of the energy functional at its critical (harmonic) points.

Harmonic maps are obviously biharmonic and are absolute minimum of the bienergy. In [10] the author proved that harmonicity and biharmonicity are equivalent if \( M \) is compact and \( R^N \leq 0 \), or if \( \| \tau(f) \| \) is constant and \( R^N \leq 0 \).

Examples of nonharmonic biharmonic maps are given in [3] and also in [9], where it is proved that the generalized Clifford torus \( S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1} \) with \( p+q = m \), \( p \neq q \), is a nonharmonic biharmonic submanifold of \( S^{m+1} \). A classification of nonharmonic biharmonic submanifolds of \( S^3 \) is given in [4] and [5].

The method of conformal deformation of metrics has been used by several authors to study the existence or properties of (bi)-harmonic maps. In the case of warped or twisted product spaces the same techniques can be used, not by deforming conformally the whole metric but by acting only on the warping or twisting functions. In the present paper we consider a particular class of
maps defined between (warped) product spaces, the so-called (warped) product maps, and we examine (bi)-harmonic properties of these maps in relation to that of the component maps. We apply the results obtained to characterize (bi)-harmonic metrics on (warped) product spaces. In the next section we recall some general facts about the Riemannian product of manifolds and we study the (bi)-harmonic properties of product maps defined between Riemannian product spaces. The third section will be devoted to the case of warped product maps with nontrivial warping functions.

Throughout the paper, manifolds, metrics, maps are assumed to be smooth.

2 (Bi)-harmonic properties of product maps

2.1 Riemannian product of manifolds

Let \((M_1, g_1)\) and \((M_2, g_2)\) be two Riemannian manifolds of dimension \(m_1\) and \(m_2\) respectively, \(M_1 \times M_2\) the usual cartesian product of \(M_1\) and \(M_2\), \(\pi\) and \(\sigma\) the projection maps on \(M_1\) and \(M_2\) respectively. The Riemannian product of \((M_1, g_1)\) and \((M_2, g_2)\) is the \((m_1 \times m_2)\)-dimensional manifold \(M_1 \times M_2\) endowed with the metric tensor \(g = \pi^* g_1 + \sigma^* g_2\), where \(\pi^*\) and \(\sigma^*\) are the pull-back of \(\pi\) and \(\sigma\) respectively.

The tangent space \(T(P_1, P_2) (M_1 \times M_2)\) at a point \((P_1, P_2) \in M_1 \times M_2\) is isomorphic to the direct sum \(T_{P_1} M_1 \oplus T_{P_2} M_2\), and thus a vector field \(X\) on \(M_1 \times M_2\) can be written as

\[ X = X_1 + X_2, \text{ with } X_1 \in \mathcal{L}_H(M_1) \text{ and } X_2 \in \mathcal{L}_V(M_2). \]

Obviously we have

\[ \pi_*(\mathcal{L}_H(M_1)) = TM_1 \text{ and } \sigma_*(\mathcal{L}_V(M_2)) = TM_2, \]

where \(\pi_*\) and \(\sigma_*\) are the differential maps of \(\pi\) and \(\sigma\) respectively.

For simplicity we use, as in [11], the same notations \(X_1\) and \(X_2\) for \(\pi_*(X_1)\) and \(\sigma_*(X_2)\) respectively. For \(X = X_1 + X_2 \in T(M_1 \times M_2)\) and \(Y = Y_1 + Y_2 \in T(M_1 \times M_2)\), it holds:

\[ g(X, Y) = g_1(X_1, Y_1) + g_2(X_2, Y_2). \]

Let \(\^1\nabla, \^2\nabla\) and \(\nabla\) the Levi-Civita connections on \(M_1, M_2\) and \(M\) respectively. We have (see [11]):
Lemma 2.1.1
Let $X_1, Y_1 \in \mathcal{L}_H(M_1)$ and $X_2, Y_2 \in \mathcal{L}_V(M_2)$. It holds:

(i) $\pi_* (\nabla_{X_1} Y_1) = 1 \nabla_{X_1} Y_1$ and $\sigma_* (\nabla_{X_1} Y_1) = 0$; i.e. $\nabla_{X_1} Y_1 \in \mathcal{L}_H(M_1)$ is the horizontal lift of $1 \nabla_{X_1} Y_1$.

(ii) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = 0$

(iii) $\sigma_* (\nabla_{X_2} Y_2) = 2 \nabla_{X_2} Y_2$ and $\pi_* (\nabla_{X_2} Y_2) = 0$; i.e. $\nabla_{X_2} Y_2 \in \mathcal{L}_V(M_2)$ is the vertical lift of $2 \nabla_{X_2} Y_2$.

Let $(X_1, X_2, \cdots, X_{m_1+m_2}) = (x_1, x_2, \cdots, x_{m_1}, u_1, u_2, \cdots, u_{m_2})$ be a local coordinates system in $M_1 \times M_2$, where $(x_1, x_2, \cdots, x_{m_1})$ and $(u_1, u_2, \cdots, u_{m_2})$ are local coordinates systems in $M_1$ and $M_2$ respectively. We use the indices notations $\hat{i}, \hat{j}, \hat{k}, \cdots \in \{1, \cdots, m_1+m_2\}$ on $M_1 \times M_2$; $i, j, k, \cdots \in \{1, \cdots, m_1\}$ on $M_1$, and $\bar{i}, \bar{j}, \bar{k}, \cdots \in \{1, \cdots, m_2\}$ on $M_2$ and for $\hat{I}, \hat{J}, \hat{K}, \cdots \in \{1, \cdots, m_1\}$ we put $\hat{I} = i$, $\hat{J} = j$, $\hat{K} = k$, \cdots; for $\bar{I}, \bar{J}, \bar{K}, \cdots \in \{m_1, \cdots, m_1+m_2\}$ we put $\bar{I} = m_1 + \bar{i} \equiv \bar{i}$, $\bar{J} = m_1 + \bar{j} \equiv \bar{j}$, $\bar{K} = m_1 + \bar{k} \equiv \bar{k}$. We have:

$$g = (g_{\hat{i}\hat{j}}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{2ij} \end{pmatrix} \quad \text{and} \quad g^{-1} = (G^{\hat{I}\hat{J}}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{2ij} \end{pmatrix}.$$  

Let $\Gamma^k_{ij}, \hat{1}\Gamma^k_{ij}$ and $\hat{2}\Gamma^k_{ij}$ the Christoffel’s symbols on $(M_1 \times M_2, g)$, $(M_1, g_1)$ and $(M_2, g_2)$ respectively. From Lemma 2.1.1 we have:

Lemma 2.1.2

(i) $\forall i, j, k \in \{1, \cdots, m_1\}$, $\Gamma^k_{ij} = \hat{1}\Gamma^k_{ij}$.

(ii) $\forall i, j, k \in \{1, \cdots, m_1\}$ and $\forall \bar{i}, \bar{j}, \bar{k} \in \{1, \cdots, m_2\}$, $\Gamma^k_{ij} = \hat{1}\Gamma^k_{ij} = \hat{2}\Gamma^k_{ij} = 0$.

(iii) $\forall \bar{i}, \bar{j}, \bar{k} \in \{1, \cdots, m_2\}$, $\Gamma^k_{ij} = \hat{2}\Gamma^k_{ij}$.

2.2 The product maps between Riemannian product spaces

Let $(M_1, g_1)$, $(M_2, g_2)$, $(N_1, h_1)$ and $(N_2, h_2)$ be Riemannian manifolds of dimension $m_1, m_2$, $n_1, n_2$ respectively, $(M, g) := (M_1, g_1) \times (M_2, g_2)$ and $(N, h) := (N_1, h_1) \times (N_2, h_2)$.

We denote by $\Gamma^k_{ij}, \hat{1}\Gamma^k_{ij}$ and $\hat{2}\Gamma^k_{ij}$ the Christoffel’s symbols on $(M, g)$, $(M_1, g_1)$ and $(M_2, g_2)$ respectively, and by $\hat{\Gamma}^c_{\hat{A}\hat{B}}, \hat{1}\Gamma^c_{\hat{A}\hat{B}}$ and $\hat{2}\Gamma^c_{\hat{A}\hat{B}}$ the Christoffel’s symbols on $(N, h)$, $(N_1, h_1)$ and $(N_2, h_2)$ respectively, where we use for indices notations $\hat{A}, \hat{B}, \hat{C}$ on $N_1 \times N_2$, $a, b, c$ on $N_1$ and $\bar{a}, \bar{b}, \bar{c}$ on $N_2$.\n
\hfill 4
with the corresponding conventions between the indices as in the previous paragraph.
We introduce in the following the definition of product maps:

**Definition 2.2.1**

Let \( \phi_1 : (M_1, g_1) \to (N_1, h_1) \) and \( \phi_2 : (M_2, g_2) \to (N_2, h_2) \) be two maps.
We define the product map of \( \phi_1 \) and \( \phi_2 \) as the map \( \phi \) defined by:

\[
\phi : (M, g) \to (N, h) \\
(P_1, P_2) \mapsto (\phi_1(P_1), \phi_2(P_2)),
\]

where \( (M, g) \) and \( (N, h) \) are the Riemannian product spaces defined above. The maps \( \phi_1 \) and \( \phi_2 \) are then called the components of \( \phi \).

Denoting by \( \pi', \sigma' \) the projection maps of \( N_1 \times N_2 \) on \( N_1 \) and \( N_2 \) respectively, we have:

**Lemma 2.2.1**

\[
\pi'_* (\nabla d\phi) = 1\nabla d\phi_1 \text{ and } \sigma'_* (\nabla d\phi) = 2\nabla d\phi_2,
\]

where \( \nabla \), \( 1\nabla \) and \( 2\nabla \) are the Levi-Civita connections of \( (M, g) \), \( (M_1, g_1) \) and \( (M_2, g_2) \) respectively.

**Proof**

Let \( (X_1, X_2, \ldots, X_{m_1+m_2}) = (x_1, x_2, \ldots, x_{m_1}, u_1, u_2, \ldots, u_{m_2}) \) be a local coordinates system in \( M_1 \times M_2 \), with \( (x_1, x_2, \ldots, x_{m_1}) \) and \( (u_1, u_2, \ldots, u_{m_2}) \) local coordinates systems in \( M_1 \) and \( M_2 \) respectively, and

\( (Y_1, Y_2, \ldots, Y_{n_1+n_2}) = (y_1, y_2, \ldots, y_{n_1}, v_1, v_2, \ldots, v_{n_2}) \) be a local coordinates system in \( N_1 \times N_2 \), where \( (y_1, y_2, \ldots, y_{n_1}) \) and \( (v_1, v_2, \ldots, v_{n_2}) \) are local coordinates systems in \( N_1 \) and \( N_2 \) respectively.

We have to compute the components \( (\nabla d\phi)^a_{ij} \) of the second fundamental form of the product map \( \phi \). From Lemma 2.1.2 we have:

\[
(\nabla d\phi)^a_{ij} = \frac{\partial^2 \phi^a}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial x_k} + \Gamma^a_{bc} \frac{\partial \phi^b}{\partial x_i} \frac{\partial \phi^c}{\partial x_j}
\]

\[
= \frac{\partial^2 \phi^a}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial x_k} + \Gamma^a_{bc} \frac{\partial \phi^b}{\partial x_i} \frac{\partial \phi^c}{\partial x_j}
\]

\[
= (1\nabla d\phi_1)^a_{ij}, \forall i, j \in \{1, \ldots, m_1\} \text{ and } \forall a \in \{1, \ldots, n_1\}.
\]
\begin{align*}
(\nabla d\phi)^a_{ij} &= \frac{\partial^2 \phi^a}{\partial X_i \partial X_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial X_K} + \Gamma^a_{BC} \frac{\partial \phi^B}{\partial X_i} \frac{\partial \phi^C}{\partial X_j} \\
&= \frac{\partial^2 \phi^a}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial x_k} + \Gamma^a_{BC} \frac{\partial \phi^b}{\partial x_i} \frac{\partial \phi^c}{\partial x_j} + \Gamma^a_{bc} \frac{\partial \phi^b_i}{\partial x_i} \frac{\partial \phi^b_j}{\partial x_j} \\
&= 0, \; \forall \, i, j \in \{1, \ldots, m_1\} \text{ and } \forall \, a \in \{1, \ldots, n_2\}.
\end{align*}

\begin{align*}
(\nabla d\phi)^a_{ij} &= \frac{\partial^2 \phi^a}{\partial X_i \partial X_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial X_K} + \Gamma^a_{BC} \frac{\partial \phi^b}{\partial X_i} \frac{\partial \phi^c}{\partial X_j} \\
&= \frac{\partial^2 \phi^a}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial x_k} + \Gamma^a_{bc} \frac{\partial \phi^b_i}{\partial x_i} \frac{\partial \phi^c_j}{\partial x_j} \\
&= 0, \; \forall \, i \in \{1, \ldots, m_1\}, \; \forall \, j \in \{1, \ldots, m_2\} \text{ and } \forall \, a \in \{1, \ldots, n_1\}.
\end{align*}

\begin{align*}
(\nabla d\phi)^a_{ij} &= \frac{\partial^2 \phi^a}{\partial X_i \partial X_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial X_K} + \Gamma^a_{BC} \frac{\partial \phi^b}{\partial X_i} \frac{\partial \phi^c}{\partial X_j} \\
&= \frac{\partial^2 \phi^a}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi^a}{\partial x_k} + \Gamma^a_{bc} \frac{\partial \phi^b_i}{\partial x_i} \frac{\partial \phi^c_j}{\partial x_j} \\
&= 0, \; \forall \, a \in \{1, \ldots, n_1\} \text{ and } \forall \, i, j \in \{1, \ldots, m_2\}.
\end{align*}

From the components of $\nabla d\phi$ follows the result.

Let $\tau(\phi)$ be the tension field of the product map $\phi$. It holds:

**Proposition 2.2.1**

$$
\tau(\phi) = \tau(\phi_1) + \tau(\phi_2),
$$

where $\tau(\phi_1)$ and $\tau(\phi_2)$ are respectively the tension fields of the components maps $\phi_1$ and $\phi_2$.

**Proof**

From Lemma 2.2.1 we have:

\begin{align*}
\tau(\phi) &= \text{trace}_g(\nabla d\phi) \\
&= \langle g, (\nabla d\phi) \rangle \\
&= \langle g_1, \tau_1(\nabla d\phi) \rangle + \langle g_2, \tau_2(\nabla d\phi) \rangle \\
&= \langle g_1, \nabla d\phi_1 \rangle + \langle g_2, \nabla d\phi_2 \rangle \quad \text{by Lemma 2.2.1} \\
&= \text{trace}_{g_1}(\nabla d\phi_1) + \text{trace}_{g_2}(\nabla d\phi_2) \\
&= \tau(\phi_1) + \tau(\phi_2).
\end{align*}
As an immediate consequence of proposition 2.2.1, we get:

**Corollary 2.2.1**
The product map $\phi$ is harmonic if and only if its components $\phi_1$ and $\phi_2$ are harmonic.

**Proof**
If $\phi_1$ and $\phi_2$ are harmonic, then $\phi$ is evidently harmonic.
Conversely if $\phi$ is harmonic, we have

$$\tau(\phi) = \tau(\phi_1) + \tau(\phi_2) = 0$$

It follows then

$$0 = \pi'_x(\tau(\phi)) = \pi'_x(\tau(\phi_1)) = \tau(\phi_1)$$

and

$$0 = \sigma'_x(\tau(\phi)) = \sigma'_x(\tau(\phi_2)) = \tau(\phi_2),$$

since $\pi \circ \pi = \pi$, $\sigma \circ \sigma = \sigma$, $\pi \circ \sigma = \sigma \circ \pi = O$.
Thus $\phi_1$ and $\phi_2$ are harmonic.

In the following we give an application of the result above to a special case.
Let $(M, g), (N_1, h_1)$ and $(N_2, h_2)$ be three Riemannian manifolds,
$\phi_1 : (M, g) \longrightarrow (N_1, h_1)$ and $\phi_2 : (M, g) \longrightarrow (N_2, h_2)$ two maps.
We consider the map $\psi$ defined from $\phi_1$ and $\phi_2$ as follows:

$$\psi : (M, g) \longrightarrow (N_1, h_1) \times (N_2, h_2)$$

$$x \longmapsto (\phi_1(x), \phi_2(x)).$$

**Proposition 2.2.2**
The map $\psi$ is harmonic if and only if $\phi_1$ and $\phi_2$ are totally geodesic.

**Proof**
We have

$$\psi = \phi \circ i,$$

where $\phi$ is the product map of $\phi_1$ and $\phi_2$ and $i : M \longrightarrow M \times M$ is defined by $i(x) = (x, x)$.
Let us recall that $\phi \circ i$ is harmonic if $i$ is harmonic and $\phi$ is totally geodesic (see for example [7], second corollary on the page 131).
And from lemma 2.2.1 $\phi$ is totally geodesic if and only if $\phi_1$ and $\phi_2$ are totally geodesic.
Since $i$ is an harmonic map (by a direct computation) we get the result.
Examples 2.2.1

1 - Let \( T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \) be the two-dimensional torus, \( \psi \) the Veronese’s map from the sphere \( S^2(\sqrt{3}) \) with radius \( \sqrt{3} \) to the unit sphere \( S^4 \) and \( \phi : T \rightarrow S^3 \) defined by \( \phi(\theta_1, \theta_2) = (\cos \theta_1 \cos \theta_2, \sin \theta_1 \cos \theta_2, \cos \theta_1 \sin \theta_2, \sin \theta_1 \sin \theta_2) \).

The product map

\[
\Phi : T \times S^2(\sqrt{3}) \rightarrow S^3 \times S^4
\]

is harmonic, since its components \( \phi \) and \( \psi \) are harmonic (see [8]).

2 - The product map of the Hopf’s map \( \eta : S^3 \rightarrow S^2 \) and the Veronese’s map \( \psi \) from \( S^2(\sqrt{3}) \) to \( S^4 \) gives an harmonic map from \( S^3 \times S^2(\sqrt{3}) \) to \( S^2 \times S^4 \).

In the examples above the manifolds are endowed with their standard metrics.

Assume the manifolds \( M_1 \) and \( M_2 \) of finite volumes and the maps \( \phi_1 \) and \( \phi_2 \) of finite energies. Then the energy of the product map \( \phi \) is related to the energies of its components \( \phi_1 \) and \( \phi_2 \) by the following relation:

**Lemma 2.2.2**

\[
E(\phi) = Vol(M_2).E(\phi_1) + Vol(M_1).E(\phi_2),
\]

where \( Vol(M_1) \) and \( Vol(M_2) \) are the volume of \( M_1 \) and \( M_2 \) respectively.

**Proof**

We have:

\[
\|d\phi\|^2 = g^{ij} \frac{\partial \phi^A}{\partial x^i} \frac{\partial \phi^B}{\partial x^j} h_{AB} + g^{\tilde{a}\tilde{b}} \frac{\partial \phi_{\tilde{a}}}{\partial u^\tilde{a}} \frac{\partial \phi_{\tilde{b}}}{\partial u^\tilde{b}} h_{\tilde{a}\tilde{b}}
\]

Thus:

\[
e(\phi) = e(\phi_1) + e(\phi_2),
\]

where \( e(\phi) \), \( e(\phi_1) \), \( e(\phi_2) \) are the energy densities of \( \phi_1 \), \( \phi_1 \) and \( \phi_2 \) respectively.
It follows then:

\[ E(\phi) = \frac{1}{2} \int_{M_1 \times M_2} \|d\phi\|^2 d\mu_1 d\mu_2 \]
\[ = \frac{1}{2} \int_{M_1 \times M_2} \|d\phi_1\|^2 d\mu_1 d\mu_2 + \frac{1}{2} \int_{M_1 \times M_2} \|d\phi_2\|^2 d\mu_1 d\mu_2 \]
\[ = \text{Vol}(M_2) \int_{M_1} e(\phi_1) d\mu_1 + \text{Vol}(M_1) \int_{M_2} e(\phi_2) d\mu_2 \]
\[ = \text{Vol}(M_2).E(\phi_1) + \text{Vol}(M_1).E(\phi_2) , \]

where \( d\mu_1 \) and \( d\mu_2 \) are the volume elements on \((M_1, g_1)\) and \((M_2, g_2)\) respectively. \hfill \Box

Let \( C \) be the class of product maps between the Riemannian product spaces \((M, g)\) and \((N, h)\) defined from \((M_1, g_1)\), \((M_2, g_2)\), \((N_1, h_1)\) and \((N_2, h_2)\) as above. From Lemma 2.2.2 we get:

**Corollary 2.2.2**

The product map \( \phi \in C \) is energy minimizing in \( C \) if and only if its components \( \phi_1 \) and \( \phi_2 \) are energy minimizing in \( C^\infty(M_1, N_1) \) and in \( C^\infty(M_2, N_2) \) respectively.

**Proof**

If \( \phi_1 \) and \( \phi_2 \) are energy minimizing, it follows directly from lemma 2.2.2 that \( \phi \) is energy minimizing in \( C \).

Conversely let \( \phi \) be energy minimizing in \( C \).

If for example \( \phi_1 \) is not energy minimizing in \( C^\infty(M_1, N_1) \), then there exists \( \phi_1' \in C^\infty(M_1, N_1) \) such that \( E(\phi_1') < E(\phi_1) \).

Let \( \phi' \in C \) be the product map with components maps \( \phi_1' \) and \( \phi_2 \). We have

\[ E(\phi') = \text{Vol}(M_2).E(\phi_1') + \text{Vol}(M_1).E(\phi_2) \] by Lemma 2.2.2
\[ < \text{Vol}(M_2).E(\phi_1) + \text{Vol}(M_1).E(\phi_2) \] since \( E(\phi_1') < E(\phi_1) \)
\[ = E(\phi) . \]

So we get a contradiction. Thus \( \phi_1 \) and \( \phi_2 \) are energy minimizing. \hfill \Box

### 2.3 Biharmonicity of the product map

The notations being as at the beginning of the paragraph 2.2, we assume the manifolds \( M_1 \) and \( M_2 \) of finite volumes and the maps \( \phi_1 \) and \( \phi_2 \) of finite bienergies. Then the bienergy of the product map \( \phi \) is related to the bienergies of its components maps \( \phi_1 \) and \( \phi_2 \) as follow:

**Lemma 2.3.1**

\[ E_2(\phi) = \text{Vol}(M_2).E_2(\phi_1) + \text{Vol}(M_1).E_2(\phi_2) . \]
Proof
By Proposition 2.2.1, we have
\[ \tau(\phi) = \tau(\phi_1) + \tau(\phi_2) \]
And then
\[ ||\tau(\phi)||^2 = h(\tau(\phi), \tau(\phi)) = h_1(\tau(\phi_1), \tau(\phi_1)) + h_2(\tau(\phi_2), \tau(\phi_2)) = ||\tau(\phi_1)||^2 + ||\tau(\phi_2)||^2. \]
Integrating the previous relation on \( M_1 \times M_2 \), we get the result. \( \square \)

From Lemma 2.3.1 we get:

**Proposition 2.3.1**
The product map \( \phi \) is biharmonic if and only if its components \( \phi_1 \) and \( \phi_2 \) are biharmonic.

**Proof**
Let \( \{\phi_t\}_t \) be a \( C^\infty \)-variation of \( \phi \) with \( \frac{\partial \phi_t}{\partial t} |_{t=0} = V \in T(M_1 \times M_2) \). This variation of \( \phi \) induces \( C^\infty \)-variations \( \{\phi_{1t}\}_t \) of \( \phi_1 \) and \( \{\phi_{2t}\}_t \) of \( \phi_2 \) with \( V = V_1 + V_2 \), where \( \frac{\partial \phi_{1t}}{\partial t} |_{t=0} = V_1 \in TM_1 \) and \( \frac{\partial \phi_{2t}}{\partial t} |_{t=0} = V_2 \in TM_2 \).

By Lemma 2.3.1 \( E_2(\phi_t) = Vol(M_2) E_2(\phi_{1t}) + Vol(M_1) E_2(\phi_{2t}) \).
And then:
\[
\frac{dE_2(\phi_t)}{dt} |_{t=0} = Vol(M_2) \frac{dE_2(\phi_{1t})}{dt} |_{t=0} + Vol(M_1) \frac{dE_2(\phi_{2t})}{dt} |_{t=0} \\
\quad = Vol(M_2) \int_{M_1} h_1(J_{\phi_1}(\tau(\phi_1)), V_1) \, d\mu_1 + Vol(M_1) \int_{M_2} h_2(J_{\phi_2}(\tau(\phi_2)), V_2) \, d\mu_2 \\
\quad = \int_{M_1 \times M_2} h_1(J_{\phi_1}(\tau(\phi_1)), V_1) \, d\mu_1 d\mu_2 + \int_{M_1 \times M_2} h_2(J_{\phi_2}(\tau(\phi_2)), V_2) \, d\mu_1 d\mu_2 \\
\quad = \int_{M_1 \times M_2} [h_1(J_{\phi_1}(\tau(\phi_1)), V_1) + h_2(J_{\phi_2}(\tau(\phi_2)), V_2)] \, d\mu_1 d\mu_2 \\
\quad = \int_{M_1 \times M_2} h(J_{\phi_1}(\tau(\phi_1)) + J_{\phi_2}(\tau(\phi_2)), V) \, d\mu_1 d\mu_2. \\
\]
Since \( \frac{dE_2(\phi_1)}{dt} |_{t=0} = h(J_\phi(\tau(\phi)) , V) \), it follows:
\[ J_\phi(\tau(\phi)) = J_{\phi_1}(\tau(\phi_1)) + J_{\phi_2}(\tau(\phi_2)) \).

By the same arguments as in the proof of Corollary 2.2.1, \( J_\phi(\tau(\phi)) = 0 \) if and only if \( J_{\phi_1}(\tau(\phi_1)) = 0 \) and \( J_{\phi_2}(\tau(\phi_2)) = 0 \).
Thus we get the result. □

In an analogous manner to the case of the energy, the product map $\phi$ is bienergy minimizing in the class $C$ of product maps if and only if its components $\phi_1$ and $\phi_2$ are bienergy minimizing in $C^\infty(M_1, N_1)$ and in $C^\infty(M_2, N_2)$ respectively.

3 Harmonicity of warped product maps, harmonic metrics and nonharmonic biharmonic metrics on warped product spaces.

3.1 Warped product of manifolds

Let $(M,g)$ and $(N,h)$ be two Riemannian manifolds of dimension $m$ and $n$ respectively, and $\lambda \in C^\infty(M)$ a strictly positive function on $M$.

The warped product of $(M,g)$ and $(N,h)$, with warping function $\lambda$, is the $(m \times n)$-dimensional manifold endowed with the metric $G_\lambda$ defined by:

$$G_\lambda =: \pi^* g + (\lambda \circ \pi)^2 \sigma^* h,$$

where $\pi$ and $\sigma$ are the projections of $M \times N$ on $M$ and $N$ respectively.

The metric $G_\lambda$ is also called the warped product of the metrics $g$ and $h$ with warping function $\lambda$.

For $X,Y \in \mathcal{L}_H(M))$ and $U,V \in \mathcal{L}_V(N))$,

$$G_\lambda(X + U, Y + V) = g(X,Y) + (\lambda \circ \pi)^2 h(U,V),$$

where we use the same convention as in section 2 for the notations of vector fields.

Let $^g\nabla$, $^h\nabla$ and $\nabla$ the Levi-Civita connections of $(M,g)$, $(N,h)$ and $(M \times N, G_\lambda)$ respectively.

We have (see [11]):

**Lemma 3.1.1**

For $X,Y \in \mathcal{L}_H(M)$ and $U,V \in \mathcal{L}_V(N)$, it holds:

(i) $\pi_*(\nabla_X Y) = ^g\nabla_X Y$ and $\sigma_*(\nabla_X Y) = 0$

(ii) $\nabla_X U = \nabla_U X = \frac{d\lambda(X)}{\lambda} U$

(iii) $\pi_*(\nabla_U V) = -\frac{<U,V>}{\lambda} \text{grad} \lambda = -\lambda h(U,V) \text{grad} \lambda$

(iv) $\sigma_*(\nabla_U V) = ^h\nabla_U V$.

Let $(X_1, X_2, \ldots, X_{m+n}) = (x_1, x_2, \ldots, x_m, u_1, u_2, \ldots, u_n)$ be a local coordinates system in
$M \times N$ defined as in the previous section with the same notations of indices and relation’s conventions between them.

We have:

$$G_\lambda = (G_{\lambda ij}) = \left( \begin{array}{cc} g_{ij} & 0 \\ 0 & \lambda^2 h_{ij} \end{array} \right) \quad \text{and} \quad G_\lambda^{-1} = (G_{\lambda j}^i) = \left( \begin{array}{cc} g_{ij} & 0 \\ 0 & \lambda^{-2} h_{ij} \end{array} \right).$$

Let $^1\Gamma_{ij}^k$, $^2\Gamma_{ij}^k$ and $^\phi\Gamma_{ij}^k$ be the Christoffel’s symbols on $(M, g)$ and $(N, h)$ and $(M \times N, G_\lambda)$ respectively.

A direct computation from Lemma 3.1.1 leads to (see for example [12, p. 111]):

**Lemma 3.1.2**

(i) $\Gamma_{ij}^k = \Gamma_{ij}^k$, $\forall i, j, k \in \{1, \ldots, m\}$.

(ii) $\Gamma_{ij}^k = \Gamma_{ij}^k = 0$, $\forall i, j, k \in \{1, \ldots, m\}$ and $\forall \bar{j}, \bar{k} \in \{1, \ldots, n\}$.

(iii) $\Gamma_{ij}^k = \frac{\partial \log \lambda}{\partial x_i} \delta_j^k \quad \forall i \in \{1, \ldots, m\}$ and $\forall \bar{j}, \bar{k} \in \{1, \ldots, n\}$.

(iv) $\Gamma_{ij}^k = -\frac{1}{2}(\text{grad} \lambda^2)_{ij}^k \quad \forall k \in \{1, \ldots, m\}$ and $\forall \bar{i}, \bar{j} \in \{1, \ldots, n\}$.

(v) $\Gamma_{ij}^k = 2\Gamma_{ij}^k$, $\forall \bar{i}, \bar{j}, \bar{k} \in \{1, \ldots, n\}$.

The relations (i) – (v) of lemma 3.1.2 can also be obtained directly from the formula:

$$\Gamma_{ij}^k = \frac{1}{2} G_{\lambda ij}^L \left( \frac{\partial G_{\lambda j}^i}{\partial X_j} + \frac{\partial G_{\lambda i}^j}{\partial X_j} - \frac{\partial G_{\lambda i}^j}{\partial X_L} \right).$$

### 3.2 Warped product maps

Let $(M, G_\lambda)$ be the warped product of two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ with warping function $\lambda \in C^\infty(M_1)$, and $(N, h)$ be the warped product of the Riemannian manifolds $(N_1, h_1)$ and $(N_2, h_2)$ with warping function $\varphi \in C^\infty(N_1)$.

Let $\phi_1 : (M_1, g_1) \rightarrow (N_1, h_1)$ and $\phi_2 : (M_2, g_2) \rightarrow (N_2, h_2)$ two maps.

**Definition 3.2.1**

We define the warped product of $\phi_1$ and $\phi_2$, with warping functions $\lambda$ and $\varphi$, as the map $\Phi_{\lambda, \varphi}$ defined by:

$$\Phi_{\lambda, \varphi} : (M, G_\lambda) \rightarrow (N, H_\varphi)$$

$$(P_1, P_2) \mapsto (\phi_1(P_1), \phi_2(P_2))$$.

The maps $\phi_1$ and $\phi_2$ are then called the components of $\Phi_{\lambda, \varphi}$. 

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Let \((X_1, X_2, \cdots, X_{m_1+m_2}) = (x_1, x_2, \cdots, x_{m_1}, u_1, u_2, \cdots, u_{m_2})\) and \\
\((Y_1, Y_2, \cdots, Y_{n_1+n_2}) = (y_1, y_2, \cdots, y_{n_1}, v_1, v_2, \cdots, v_{n_2})\) be local coordinates systems in \(M_1 \times M_2\) \\
and \(N_1 \times N_2\) respectively as in paragraph 2.2 with the same notation’s conventions for the indices.

The second fundamental form of the warped product map \(\Phi_{\lambda, \Theta}\) of \(\phi_1\) and \(\phi_2\) is given by:

**Lemma 3.2.1**

(i) For \(i, j \in \{1, \cdots, m_1\}\) and \(a \in \{1, \cdots, n_1\}\),

\[
(\nabla d \phi_{\lambda, \Theta})^a_{ij} = (1 \nabla d \phi_1)^a_{ij} .
\]

(ii) For \(i, j \in \{1, \cdots, m_1\}, a \in \{1, \cdots, n_1\}, \tilde{j} \in \{1, \cdots, m_2\}\) and \(\tilde{a} \in \{1, \cdots, n_2\}\),

\[
(\nabla d \phi_{\lambda, \Theta})^\tilde{a}_{ij} = (\nabla d \phi_2)^\tilde{a}_{ij} = 0 .
\]

(iii) For \(i \in \{1, \cdots, m_1\}, \tilde{j} \in \{1, \cdots, m_2\}\) and \(\tilde{a} \in \{1, \cdots, n_2\}\),

\[
(\nabla d \phi_{\lambda, \Theta})^\tilde{a}_{ij} = - \frac{\partial \log \lambda}{\partial x_i} \frac{\partial \phi_1^{\tilde{a}}}{\partial u_j} + \frac{\partial \log \theta}{\partial y_b} \frac{\partial \phi_2^{\tilde{a}}}{\partial x_i} \frac{\partial \phi_1^{\tilde{a}}}{\partial u_j} .
\]

(iv) For \(\tilde{i}, \tilde{j} \in \{1, \cdots, m_2\}\) and \(a \in \{1, \cdots, n_1\}\),

\[
(\nabla d \phi_{\lambda, \Theta})^a_{\tilde{i} \tilde{j}} = \frac{1}{2} (\text{grad} \lambda^2)^k \frac{\partial \phi_1^a}{\partial x_k} g_{2 \tilde{j}} - \frac{1}{2} (\text{grad} \theta^2)^a \frac{\partial \phi_2^a}{\partial u_i} \frac{\partial \phi_2^a}{\partial u_j} h_{2 \tilde{b} \tilde{c}} .
\]

(v) For \(\tilde{i}, \tilde{j} \in \{1, \cdots, m_2\}\) and \(\tilde{a} \in \{1, \cdots, n_2\}\)

\[
(\nabla d \phi_{\lambda, \Theta})^\tilde{a}_{\tilde{i} \tilde{j}} = (2 \nabla d \phi_2)^\tilde{a}_{\tilde{i} \tilde{j}} .
\]

**Proof**

\[
(\nabla d \phi_{\lambda, \Theta})^a_{ij} = \frac{\partial^2 \phi_1^a}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi_1^a}{\partial x_k} + \Gamma^a_{bc} \frac{\partial \phi_1^b}{\partial x_i} \frac{\partial \phi_1^c}{\partial x_j} .
\]

\[
(\nabla d \phi_{\lambda, \Theta})^\tilde{a}_{ij} = \frac{\partial^2 \phi_2^\tilde{a}}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \phi_2^\tilde{a}}{\partial x_k} + \Gamma^a_{bc} \frac{\partial \phi_2^b}{\partial x_i} \frac{\partial \phi_2^c}{\partial x_j} .
\]

by (i) of lemma 3.1.2

\[
(\nabla d \phi_{\lambda, \Theta})^a_{\tilde{i} \tilde{j}} = (1 \nabla d \phi_1)^a_{\tilde{i} \tilde{j}} .
\]

\[
(\nabla d \phi_{\lambda, \Theta})^\tilde{a}_{\tilde{i} \tilde{j}} = 0 , \text{ since by lemma 3.1.2 } \Gamma^k_{ij} = \Gamma^a_{bc} = 0 .
\]
\[
(\nabla d\Phi_{\lambda,\theta})_{ij}^a = \frac{\partial^2 \Phi_{\lambda,\theta}^a}{\partial X_i \partial X_j} - \Gamma_{ij}^k \frac{\partial \Phi_{\lambda,\theta}^a}{\partial X_k} + \Gamma^a_{bc} \frac{\partial \Phi_{\lambda,\theta}^B}{\partial X_i} \frac{\partial \Phi_{\lambda,\theta}^C}{\partial X_j}
\]

\[
= \frac{\partial^2 \phi_2^a}{\partial u_i \partial u_j} - \Gamma_{ij}^k \frac{\partial \phi_2^a}{\partial u_k} + \Gamma^a_{bc} \frac{\partial \phi_2^b}{\partial u_i} \frac{\partial \phi_2^c}{\partial u_j}
\]

\[
= \frac{1}{2} \lambda^2 \nabla \log \lambda \frac{\partial \phi_2^a}{\partial u_i} \frac{\partial \phi_2^a}{\partial u_j} - \frac{1}{2} \mathcal{E}^a \frac{\partial \phi_2^b}{\partial u_i} \frac{\partial \phi_2^c}{\partial u_j}
\]

by (iii) of lemma 3.1.2.

\[
(\nabla d\Phi_{\lambda,\theta})_{ij}^a = \frac{\partial^2 \Phi_{\lambda,\theta}^a}{\partial X_i \partial X_j} - \Gamma_{ij}^k \frac{\partial \Phi_{\lambda,\theta}^a}{\partial X_k} + \Gamma^a_{bc} \frac{\partial \Phi_{\lambda,\theta}^B}{\partial X_i} \frac{\partial \Phi_{\lambda,\theta}^C}{\partial X_j}
\]

\[
= \frac{\partial^2 \phi_2^a}{\partial u_i \partial u_j} - \Gamma_{ij}^k \frac{\partial \phi_2^a}{\partial u_k} + \Gamma^a_{bc} \frac{\partial \phi_2^b}{\partial u_i} \frac{\partial \phi_2^c}{\partial u_j}
\]

\[
= \frac{1}{2} \lambda^2 \nabla \log \lambda \frac{\partial \phi_2^a}{\partial u_i} \frac{\partial \phi_2^a}{\partial u_j} - \frac{1}{2} \mathcal{E}^a \frac{\partial \phi_2^b}{\partial u_i} \frac{\partial \phi_2^c}{\partial u_j}
\]

by (iv) of lemma 3.1.2.

Let \(\tau(\Phi_{\lambda,\theta})\), \(\tau(\phi_1)\) and \(\tau(\phi_2)\) be the tension fields of the warped product map \(\Phi_{\lambda,\theta}\) and of its components \(\phi_1\) and \(\phi_2\). From Lemma 3.2.1 it follows:

**Proposition 3.2.1**

\[
\tau(\Phi_{\lambda,\theta}) = \tau(\phi_1) + \lambda^{-2} \tau(\phi_2)
\]

\[
+ \lambda^{-2} \frac{m_2}{2} \left( \nabla_{M_1 \lambda^2} \phi_1 - \mathcal{E}(\phi_2) \nabla_{N_1} \phi_2 \right)
\]

with \((\nabla_{M_1 \lambda^2}) \phi_1 = [\nabla_{M_1 \lambda^2} \phi_1^a] \frac{\partial}{\partial x^a}\), where \(\mathcal{E}(\phi_2)\) is the energy density of \(\phi_2\) and, \(\nabla_{M_1}\) and \(\nabla_{N_1}\) denote the gradient operators on \(M_1\) and \(N_1\) respectively.
Proof

We have:

\[
\tau(\Phi_{\lambda, \varphi}) = G^i_j (\nabla d\Phi_{\lambda, \varphi})^{-1} \frac{\partial}{\partial y_i} = G^i_j (\nabla d\Phi_{\lambda, \varphi})^{\alpha} \frac{\partial}{\partial y_\alpha} + G^i_j (\nabla d\Phi_{\lambda, \varphi})^{\bar{\alpha}} \frac{\partial}{\partial \bar{v}_\bar{\alpha}} \frac{1}{\lambda^2} - 2 g^{ij}_i \frac{\partial}{\partial y_i} \frac{\partial}{\partial \bar{v}_j} \frac{\partial}{\partial \bar{v}_j} + \lambda^{-2} g^{ij}_i \frac{\partial}{\partial \bar{v}_i} \frac{\partial}{\partial \bar{v}_j} \frac{\partial}{\partial \bar{v}_j}.
\]

By evaluating each term of the second hand member of the last equation, we obtain:

\[
\begin{align*}
\frac{1}{2} \lambda^{-2} g^{ij}_i (\text{grad}_{M_1} \varphi^2) a \frac{\partial}{\partial \bar{v}_a} \frac{\partial}{\partial \bar{v}_j} h_{2\bar{a} \bar{c}} \frac{\partial}{\partial \bar{v}_a} + \lambda^{-2} g^{ij}_i (\nabla d\varphi) \frac{\partial}{\partial \bar{v}_i} \frac{\partial}{\partial \bar{v}_j} \frac{\partial}{\partial \bar{v}_j} = \lambda^{-2} \tau(\varphi_2).
\end{align*}
\]

Thus we get the result. \(\square\)

The following result is a consequence of the previous proposition:

**Corollary 3.2.1**

Let \(\varphi_1 : (M_1, g_1) \rightarrow (N_1, h_1)\) and \(\varphi_2 : (M_2, g_2) \rightarrow (N_2, h_2)\) be two harmonic maps. The warped product \(\Phi_{\lambda, \varphi}\) of \(\varphi_1\) and \(\varphi_2\), with warping functions \(\lambda \in C^\infty(M_1)\) and \(\varphi \in C^\infty(N_1)\), is harmonic if and only if

\[
\frac{m_2}{2} (\text{grad}_{M_1} \varphi^2) \varphi_1 = \varphi(\varphi_2) \text{grad}_{N_1} \varphi^2.
\]

### 3.3 Harmonic metrics on warped product spaces

Let us first recall the definition of harmonic metrics.

**Definition 3.3.1**

Let \((M, g)\) be a Riemannian manifold. A metric \(G\) on \(M\) is said to be harmonic w.r.t. \(g\) if the identity \(\text{id} : (M, g) \rightarrow (M, G)\) is an harmonic map (see [6] for more information about harmonic metrics).

Let \((M \times N, G_\lambda)\) be the warped product of two Riemannian manifolds \((M, g)\) and \((N, h)\) with warping function \(\lambda \in C^\infty(M), \lambda > 0\).

Let \(G_\varphi\) be the metric defined on \(M \times N\) by: 

\[G_\varphi = \pi^* g + \left(\varphi \circ \pi\right)^2 \sigma^* h,\]

with \(\varphi \in C^\infty(M), \varphi > 0\), where \(\pi\) and \(\sigma\) are as usual the projections of \(M \times N\) on \(M\) and \(N\) respectively. It holds:
**Theorem 3.3.1**

We assume $M$ connected.

The metric $G_\varrho$ is harmonic w.r.t. $G_\lambda$ if and only if

$$\lambda^2 - \varrho^2$$

is a constant function on $M$.

**Proof**

Let $\mathcal{I}_{\lambda,\varrho} : (M \times N, G_\lambda) \rightarrow (M \times N, G_\varrho)$ be the warped product of the identity maps $\text{id}_M : (M, g) \rightarrow (M, g)$ and $\text{id}_N : (N, h) \rightarrow (N, h)$, with warping functions $\lambda$ and $\varrho$.

The metric $G_\varrho$ is harmonic w.r.t. $G_\lambda$ if and only if $\mathcal{I}_{\lambda,\varrho}$ is an harmonic map.

Since $\text{id}_M$ and $\text{id}_N$ are harmonic, it follows from Corollary 3.2.1 that $\mathcal{I}_{\lambda,\varrho}$ is an harmonic map if and only if

$$\frac{n}{2}(\text{grad}_M \lambda^2).\text{id}_M = e(\text{id}_N) \text{grad}_M \varrho^2.$$

But

$$e(\text{id}_N) = \frac{n}{2} \text{ and } (\text{grad}_M \lambda^2).\text{id}_M = \text{grad}_M \lambda^2.$$

Thus $\mathcal{I}_{\lambda,\varrho}$ is harmonic if and only if

$$\text{grad}_M \lambda^2 - \text{grad}_M \varrho^2 = 0.$$

That is:

$$\text{grad}_M (\lambda^2 - \varrho^2) = 0.$$

Since $M$ is assumed to be connected, we get the result. □

**Examples 3.3.1**

1. Consider $S^3 \setminus \{(+1, 0, 0, 0)\}$ with the metric $G$ defined in local coordinates $(t, \theta, \phi)$ by:

$$ds_G^2 = dt^2 + \sin^2 t \ d\theta^2 + \sin^2 t \sin^2 \theta d\phi^2.$$

For any real $c < 0$, the metric $G'$ defined by:

$$ds_{G'}^2 = dt^2 + (\sin^2 t - c)d\theta^2 + (\sin^2 t - c) \sin^2 \theta d\phi^2,$$

is harmonic w.r.t. $G$.

Indeed $S^3 \setminus \{(+1, 0, 0, 0)\}$ can be viewed as the warped product of the open interval $]0, \pi[$ and of $S^2$ with the warping function $\lambda$ defined on $]0, \pi[$ by $\lambda(t) = \sin t$.

2. On the space $H^3 \setminus \{(0)\}$ with the metric $G$ defined in local coordinates $(t, \theta, \phi)$ by:

$$ds_G^2 = dt^2 + \sinh^2 t \ d\theta^2 + \sinh^2 t \sin^2 \theta d\phi^2,$$

the metric $G'$ defined by:

$$ds_{G'}^2 = dt^2 + (\sinh^2 t - c)d\theta^2 + (\sinh^2 t - c) \sin^2 \theta d\phi^2$$

with $c < 0$,

is harmonic w.r.t. $G$.

Indeed $H^3 \setminus \{(0)\}$ can be viewed as the warped product of the open interval $]0, \infty[$ and of $S^2$ with the warping function $\lambda$ defined on $]0, \infty[$ by $\lambda(t) = \sinh t$. 17
3.4 Nonharmonic biharmonic metrics on product spaces

In this paragraph we study the biharmonicity of the warped product of two identity maps with warping functions of which one is constant. We obtain then necessary and sufficient conditions for warped metrics to be nonharmonic biharmonic on product spaces.

Before giving the definition of biharmonic metrics, let us point out that, due to the fact that we are interested by nonharmonic biharmonic metrics, we conserve the same places of the domain and the codomain of the identity maps as in the definition of harmonic metrics (see [6]), contrary to the authors in [1] where the places of the domain and codomain are reversed.

Definition 3.4.1

Let $(M, g)$ be a Riemannian manifold. A metric $G$ on $M$ is said to be biharmonic w.r.t. $g$ if the identity $\text{id} : (M, g) \rightarrow (M, G)$ is a biharmonic map.

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds of dimensions $m$ and $n$ respectively, and $(M \times N, G)$ the Riemannian product of $(M, g)$ and $(N, h)$ respectively; i.e. $(M \times N, G) = (M, g) \times (N, h)$.

Let $\varphi \in C^\infty(M)$ be a strictly positive function on $M$ and $G_\varphi = \pi^*g + (\varphi \circ \pi)^2\sigma^*h$ be the warped product metric of $g$ and $h$ with warping function $\varphi$.

We have the following result:

Theorem 3.4.1

The warped product metric $G_\varphi$ is nonharmonic biharmonic w.r.t. $G$, if and only if

$$\text{grad} \varphi \neq 0 \quad \text{and} \quad \Delta_g \omega + \frac{n}{4} d (||\omega||^2) - 2(Ric_g(\omega))^b = 0,$$

with $\omega = dg^2$ and $(Ric_g(\omega))^b(X) = Ric_g(\omega^2, X)$, $\forall X \in TM$, where $\Delta_g = dd^* + d^*d$ and $Ric_g$ are respectively the gradient, the Laplacian and the Ricci tensor on $(M, g)$.

Proof

By Theorem 3.3.1 applied to $\lambda \equiv 1$ and the function $\varphi$, the metric $G_\varphi$ is nonharmonic w.r.t. $G$ if and only if $\text{grad} \varphi^2 = 2\varphi \text{grad} \varphi \neq 0$.

Since $\varphi$ is a strictly positive function, $G_\varphi$ is then nonharmonic w.r.t $G$ if and only if $\text{grad} \varphi \neq 0$ (i).

Let $\tau$ be the tension field of the identity map $\text{id} : (M \times N, G) \rightarrow (M \times N, G_\varphi)$, and $J$ be the Jacobi field along $\text{id}$. We have:

$$\tau = -\frac{n}{2} \text{grad} \varphi^2 \quad \text{and} \quad J(\tau) = \Delta^{\text{id}} \tau + \text{trace}_G R_\varphi(\text{, , } \tau),$$
with $\Delta^{id}\tau = -\text{trace}_{G}(\nabla\nabla\tau - \nabla\nabla\tau)$ the Laplacian on the sections of $T(M \times N)$ and where $R_q$ is the Riemannian curvature operator on $(M \times N, G_q)$.

Let $(E_1, \cdots, E_{m+n})$ be a geodesic orthonormal basis on $T(M \times N)$.

$$\Delta^{id}\tau = - \sum_{l=1}^{m+n} (\nabla E_l \nabla E_l \tau - \nabla\nabla E_l E_l \tau) = - \left( \sum_{l=1}^{m} \nabla E_l \nabla E_l \tau + \sum_{l=m+1}^{m+n} \nabla E_l \nabla E_l \tau \right).$$

By Lemma 3.1.1 we have

$$\sum_{l=1}^{m} \nabla E_l \nabla E_l \tau = \sum_{l=1}^{m} \nabla^g E_l \nabla^g E_l \tau ,$$

where $\nabla^g$ is the Levi-Civita connection on $(M, g)$.

Otherwise for $\hat{l} \in \{ m+1, \cdots, m+n \},$

$$\nabla E_{\hat{l}} \nabla E_{\hat{l}} \tau = \frac{\tau.\dot{\sigma}}{\sigma} E_{\hat{l}}$$

by Lemma 3.1.1

and then

$$\nabla E_{\hat{l}} (\nabla E_{\hat{l}} \tau) = \nabla E_{\hat{l}} \left( \frac{\tau.\dot{\sigma}}{\sigma} E_{\hat{l}} \right)$$

$$= \left( \frac{\tau.\dot{\sigma}}{\sigma} \right) \nabla E_{\hat{l}} E_{\hat{l}} + (E_{\hat{l}}, \frac{\tau.\sigma}{\sigma}) E_{\hat{l}}$$

$$= \frac{\tau.\dot{\sigma}}{\sigma} \left[ -<E_{\hat{l}}, E_{\hat{l}}>|\nabla \sigma + \nabla^h E_{\hat{l}} \right],$$

since $E_{\hat{l}}, \frac{\tau.\dot{\sigma}}{\sigma} = 0$ and by Lemma 3.1.1

$$= n \| \nabla \sigma \|^2 \sigma h_{\hat{l}} \nabla \sigma ,$$

since $\nabla^h E_{\hat{l}} = \sigma_{\dot{\sigma}}(\nabla E_{\hat{l}} E_{\hat{l}}) = 0,$

where $\nabla^h$ is the Levi-Civita connection on $(N, h)$.

Thus

$$\sum_{l=m+1}^{m+n} \nabla E_l \nabla E_l \tau = \frac{n^2}{2} \| \nabla \sigma \|^2 \sigma \| \nabla \sigma \|^2 .$$

We obtain then:

$$\Delta^{id}\tau = - \sum_{l=1}^{m} \nabla^g E_l \nabla^g E_l \tau - \frac{n^2}{2} \| \nabla \sigma \|^2 \sigma \| \nabla \sigma \|^2 .$$

On the other hand:

$$\sum_{l=1}^{m+n} R_q(E_l, \tau)E_l = \sum_{l=1}^{m} R_q(E_l, \tau)E_l + \sum_{l=m+1}^{m+n} R_q(E_l, \tau)E_l$$

$$= \sum_{l=1}^{m} R_q(E_l, \tau)E_l + \sum_{l=m+1}^{m+n} \frac{<E_l, E_l>}{\sigma} \nabla \sigma (\nabla \sigma) ,$$


$$= \sum_{l=1}^{m} R_q(E_l, \tau)E_l + n q \nabla \sigma (\nabla \sigma)$$

$$= - \sum_{l=1}^{m} R_q(\tau, E_l)E_l - \frac{n^2 q^2}{2} \nabla \sigma (\| \nabla \sigma \|^2) ,$$

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since
\[ \nabla_\tau (\text{grad } \varrho) = -\frac{n}{2} \nabla_{\text{grad } \varrho}^2 (\text{grad } \varrho) = -n \varrho \nabla_{\text{grad } \varrho} (\text{grad } \varrho) \]
and
\[ \nabla_{\text{grad } \varrho} (\text{grad } \varrho) = \frac{1}{2} \text{grad}(\| \text{grad } \varrho \|^2) . \]

It follows:
\[ J(\tau) = -\frac{n}{2} \left( - \sum_{i=1}^{m} \nabla_{E_i}^g \nabla_{E_i}^g \text{grad } \varrho^2 - \sum_{i=1}^{m} R_g (\text{grad } \varrho^2 , E_i) E_i \right. \\
+ n\| \text{grad } \varrho \|^2 \text{grad } \varrho^2 + n\varrho^2 \text{grad}(\| \text{grad } \varrho \|^2) \} \\
= -\frac{n}{2} \left( - \sum_{i=1}^{m} \nabla_{E_i}^g \nabla_{E_i}^g \text{grad } \varrho^2 - \sum_{i=1}^{m} R_g (\text{grad } \varrho^2 , E_i) E_i \right. \\
+ \left. \frac{n}{4} \text{grad}(\| \text{grad } \varrho^2 \|^2) \} , \right.
\]
since
\[ \| \text{grad } \varrho \|^2 \text{grad } \varrho^2 + \varrho^2 \text{grad}(\| \text{grad } \varrho \|^2) = \frac{1}{4} \text{grad}(\| \text{grad } \varrho^2 \|^2) . \]
Thus \( G_\varrho \) is biharmonic w.r.t. \( G \) if and only if
\[ - \sum_{i=1}^{m} \nabla_{E_i}^g \nabla_{E_i}^g \text{grad } \varrho^2 - \sum_{i=1}^{m} R_g (\text{grad } \varrho^2 , E_i) E_i + \frac{n}{4} \text{grad}(\| \text{grad } \varrho^2 \|^2) = 0 \ (\ast) . \]

By the Weitzenböck formula
\[ \Delta_g \omega = - \text{trace } \nabla_g^2 \omega + (\text{Ric}_g (\omega^b))^b , \]
where \( ^b \) and \( ^a \) are the usual musical isomorphisms.

Hence the relation (\( \ast \)) is equivalent to
\[ \Delta_g \omega + \frac{n}{4} d (\| \omega \|^2) - 2(\text{Ric}_g (\omega^b))^b = 0 \ (\ast i) . \]

From (\( i \)) and (\( ii \)) we get the result. \( \square \)

When \( (M, g) \) is Einstein, the 1-form \( (\text{Ric}_g (\omega^b))^b \) is proportional to \( \omega \). Thus as an immediate consequence of the previous theorem, we have:

**Corollary 3.4.1**
Let \( \varrho \in C^\infty(M) \), \( \varrho > 0 \), with \( \text{grad } \varrho \neq 0 \). Assume that \( (M, g) \) is an Einstein manifold and that \( \text{grad}(\| \text{grad } \varrho \|) \) is parallel to \( \text{grad } \varrho \).

If \( G_\varrho \) is harmonic w.r.t. \( G \), then \( \varrho \) is an isoparametric function on \( M \).

We recall that a function \( f \) on \( M \) is said to be isoparametric if there exist real functions \( \alpha \)
and \( \beta \) such that \( \|df\|^2 = \alpha \circ f \) and \( \Delta f = \beta \circ f \) or equivalently if \( \text{grad}(\|\text{grad} f\|) \) and \( \text{grad}(\Delta f) \) are parallel to \( \text{grad} f \), with \( \text{grad} f \neq 0 \) (see [1] and [2] for more information about isoparametric functions).

**Acknowledgments.** This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

**References**


