HERMITE-PADÉ APPROXIMATION APPROACH TO HYDROMAGNETIC FLOWS IN CONVERGENT-DIVERGENT CHANNELS

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Abstract

The problem of two-dimensional, steady, nonlinear flow of an incompressible conducting viscous fluid in convergent-divergent channels under the influence of an externally applied homogeneous magnetic field is studied using a special type of Hermite-Padé approximation approach. This semi-numerical scheme offers some advantages over solutions obtained by using traditional methods such as finite differences, spectral method, shooting method, etc. It reveals the analytical structure of the solution function and the important properties of overall flow structure including velocity field, flow reversal control and bifurcations are discussed.

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1. Introduction

In modern times the theory of flow through convergent-divergent channels has many applications in aerospace, chemical, civil, environmental, mechanical and bio-mechanical engineering as well as in understanding rivers and canals. The mathematical investigations of this type of problem were pioneered by Jeffery (1915) and Hamel (1916), and have been extensively studied by several authors and discussed in many textbooks e.g., Fraenkel (1962), Batchelor (1967), Sobey and Drazin (1986), Banks et al. (1988), Hamadiche et al. (1994), Makinde (1995), Makinde (1997), Makinde (1999), Khan et al. (2003), etc. Jeffery-Hamel flows are interesting models of the phenomenon of separation of boundary layers in divergent channels. These flows have revealed a multiplicity of solutions, richer perhaps than other similarity solutions of the Navier-Stokes equations, no doubt because of the dependence on two non-dimensional parameters i.e. the flow Reynolds number and channel angular widths.

Meanwhile, the study of electrically conducting viscous fluid flowing through convergent-divergent channels under the influence of an external magnetic field is not only fascinating theoretically, but also finds applications in mathematical modelling of several industrial and biological systems. A possible practical application of the theory we envisage is in the field of industrial metal casting, the control of molten metal flows. Another area in which the theoretical study may be of interest is in the motion of liquid metals or alloys in the cooling systems of advanced nuclear reactors. Clearly, the motion in the region with intersecting walls may represent a local transition between two parallel channels with different cross-sections, a widening or a contraction of the flow. A survey of Magneto-Hydrodynamics (MHD) studies in the mentioned technological field can be found in Moreau (1990). The problem is basically an extension of classical Jeffery-Hamel flows of ordinary fluid mechanics to MHD. In the MHD solution an external magnetic field acts as a control parameter for both convergent and divergent channel flows. Here, beside the flow Reynolds number and the channel angular widths, at least an additional dimensionless parameter appears e.g. the Hartman number ($Ha$). Hence, a much larger variety of solutions than in the classical problem are expected.
In this paper we exploit the Hermite-Pade approximation technique to study how the flows evolve, and bifurcating as the flow parameters vary. The main result is that for \( Ha \) larger than 2, the flow reversal near the walls, which is a typical feature of the classical Jefrey-Hamel flows without magnetic field and moderately large Reynolds number, tends to disappear. Moderate values of \( Ha = 4 \) are sufficient to suppress flow reversal up to \( Re = 7 \) for channel semi-angle as large as 45°. We also observed that turning points, whose magnitude depends on various flow parameters, exist in the flow field. In Sections 2 & 3, we establish the mathematical formulation for the problem. Computer extension of the resulting perturbation series solution and the bifurcation study are conducted using a special type of Hermite-Pade approximation technique in Section 4. In Section 5, we discuss the entire findings.

2. Mathematical formulation
Consider the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at the intersection between two rigid plane walls under the influence of an externally applied homogeneous magnetic field as shown in figure (1) below. It is assumed that the fluid has small electrical conductivity and the electromagnetic force produced is very small. Let \((r, \theta)\) be polar coordinate, with \( r = 0 \) as the sink or source. Let \( \alpha \) be the semi-angle and let the domain of the flow be \(-|\alpha|<\theta<|\alpha|\). Denote the velocity components in the radial and tangential direction by \( u \) and \( v \) respectively. Then, the governing equations in terms of the vorticity (\( \omega \)) and stream-function (\( \Psi \)) formulation are given as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi}{\partial r} \right) = \frac{1}{r \rho} \frac{\partial}{\partial \theta} \left( r \sigma \frac{\partial \psi}{\partial \theta} \right) - \frac{\sigma e B_0^2}{\rho r^2}, \quad \omega = -\nabla^2 \Psi, \quad (1)
\]

\[
\omega = \nabla^2 \Psi.
\]
\[ \Psi = \frac{Q}{2} , \quad \frac{\partial \Psi}{\partial \theta} = 0 , \quad \text{at} \quad \theta = \pm \alpha , \]  
(2)

where

\[ Q = \int_{-\alpha}^{\alpha} wr d\theta , \]  
(3)

is the volumetric flow rate, \( B_0 = (\mu, H_0) \) the electromagnetic induction, \( \mu_e \) the magnetic permeability, \( H_0 \) the intensity of magnetic field, \( \sigma_e \) the conductivity of the fluid, \( \rho \) the fluid density, \( V^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \) and \( \nu \) is the kinematic viscosity coefficient. For Jeffery-Hamel flow of conducting fluid, we assume a purely symmetric radial flow (Banks et al. 1988), so that the tangential velocity \( v = 0 \) and as a consequence of the mass conservation, we have the stream-function given by \( \Psi = QG(\theta) / 2 \). If we require \( Q \geq 0 \) then for \( \alpha < 0 \) the flow is converging to a sink at \( r = 0 \).

The dimensionless form of equations (1)-(2) now become

\[ \frac{d^4 G}{dy^4} + 2 \text{Re} \alpha \frac{dG}{dy} \frac{d^2 G}{dy^2} + (4 - Ha) \alpha^2 \frac{d^2 G}{dy^2} = 0 \]  
(4)

with

\[ G = 1 , \quad \frac{dG}{dy} = 0 , \quad \text{at} \quad y = \pm 1 \]  
(5)

where \( y = \theta / \alpha \) and \( Ha = \sqrt{\sigma B_0^2 / \rho \nu} \), \( \text{Re} = Q / 2 \nu \) are the Hartmann number and the flow Reynolds number respectively.

3. Perturbation expansion

The problem posed by equations (3)-(4) is non-linear, for small channel angular width, we shall seek asymptotic expansion of the form

\[ G(y) = \sum_{i=0}^{\infty} \alpha^i G_i \]  
(6)

Substituting the above expressions (5) into (3) - (4) and collecting the coefficients of like powers of \( \alpha \) we obtained and solved the equations governing \( G \). Since it seems cumbersome to obtain many terms of the solution series manually, we have written a MAPLE programme that calculates successively the coefficients of the solution series. Some of the solutions for stream-function and radial velocity obtained are given as follows;
\[
G(y; \alpha, \text{Re}, \text{Ha}) = \frac{1}{2}(3y - y^3) - \frac{3}{280} y(y^2 - 5)(y^2 - 1)^2 \text{Re} \alpha
- \frac{1}{43120} y(y^2 - 1)^2(98 \text{Re}^2 y^6 - 959 \text{Re}^2 y^4 + 247 y^2 \text{Re}^2 - 2875 \text{Re}^2 + 10780 \text{Ha}^2 - 43120)\alpha^2 + ...
\]

\[
u(y; \alpha, \text{Re}, \text{Ha}) = \frac{3}{2}(1 - y^2) - \frac{3}{280}(y^2 - 1)(7y^4 - 28y^2 + 5) \text{Re} \alpha
- \frac{1}{43120}(y^2 - 1)(1078 \text{Re}^2 y^6 - 9317 \text{Re}^2 y^4 + 22099y^2 \text{Re}^2 - 21791y^2 \text{Re}^2 - 215600y^2 + 53900y^2 \text{Ha}^2 - 1078\text{Ha}^2 + 2875 \text{Re}^2 + 43120)\alpha^2 + ...
\]

Using the computer symbolic algebra package (MAPLE), we obtained the first 24 terms of the above solution series (7)-(8) as well as the series for the centerline velocity profile \(\nu(0; \alpha, \text{Re}, \text{Ha})\).

4. Bifurcation study

The main tool of this paper is a simple technique of series summation based on the generalization of the Padé approximation technique (Baker and Graves-Morris, 1996) and may be described as follows. Let us suppose that the partial sum

\[
U_{N-1}(\lambda) = \sum_{i=0}^{N-1} a_i\lambda^i = U(\lambda) + O(\lambda^N) \quad \text{as } \lambda \to 0,
\]

is given. We are concerned with the bifurcation study by analytic continuation as well as the dominant behaviour of the solution by using partial sum (9). We expect that the accuracy of the critical parameters will ensure the accuracy of the solution. It is well known that the dominant behaviour of a solution of a differential equation can often be written as Guttmann (1989),

\[
U(\lambda) \approx \left\{
\begin{array}{ll}
K(\lambda_c - \lambda)^m & \text{for } m \neq 0, 1, 2, ...
\end{array}
\right.
\]

\[
\text{as } \lambda \to \lambda_c \quad \text{as } m \neq 0, 1, 2, ...
\]

where \(K\) is some constant and \(\lambda_c\) is the critical point with the exponent \(m\). However, we shall make the simplest hypothesis in the contest of nonlinear problems by assuming that \(U(\lambda)\) is the local representation of an algebraic function of \(\lambda\). Therefore, we seek an expression of the form

\[
F_d(\lambda, U_{N-1}) = A_{0N}(\lambda) + A_{1N}(\lambda)U^{(1)} + A_{2N}(\lambda)U^{(2)} + A_{3N}(\lambda)U^{(3)},
\]

such that

\[
A_{0N}(\lambda) = 1, \quad A_{1N}(\lambda) = \sum_{j=1}^{d+i} b_j \lambda^{j-1},
\]

and

\[
F_d(\lambda, U) = O(\lambda^{N+1}) \quad \text{as } \lambda \to 0,
\]
where $d \geq 1$, $i=1, 2, 3$. The condition (12) normalizes the $F_d$ and ensures that the order of series $A_{iN}$ increases as $i$ and $d$ increase in value. There are thus $3(2+d)$ undetermined coefficients $b_{ij}$ in the expression (12). The requirement (13) reduces the problem to a system of $N$ linear equations for the unknown coefficients of $F_d$. The entries of the underlying matrix depend only on the $N$ given coefficients $a_i$. Henceforth, we shall take

$$N=3(2+d),$$

(14)

so that the number of equations equals the number of unknowns. Equation (13) is a new special type of Hermite-Padé approximants. Both the algebraic and differential approximants form of equation (13) are considered. For instance, we let

$$U^{(1)}=U, \quad U^{(2)}=U^2, \quad U^{(3)}=U^3,$$

(15)

and obtain a cubic Padé approximant. This enables us to obtain solution branches of the underlying problem in addition to the one represented by the original series. In the same manner, we let

$$U^{(1)}=U, \quad U^{(2)}=DU, \quad U^{(3)}=D^2U,$$

(16)

in equation (12), where $D$ is the differential operator given by $D=d/d\lambda$. This leads to a second order differential approximants. It is an extension of the integral approximants idea by Hunter and Baker (1979) and enables us to obtain the dominant singularity in the flow field i.e. by equating the coefficient $A_{3N}(\lambda)$ in equation (13) to zero. Meanwhile, it is very important to know that the rationale for choosing the degrees of $A_{iN}$ in equation (12) in this particular application is based on the simple technique of singularity determination in second order linear ordinary differential equation with polynomial coefficients as well as the possibility of multiple solution branches for the nonlinear problem, Vainberg and Trenogin (1974). In practice, one usually finds that the dominant singularities are located at zeroes of the leading polynomial $A_{3N}^{(d)}$ coefficients of the second order linear ordinary differential equation. Hence, some of the zeroes of $A_{3N}^{(d)}$ may provide approximations of the singularities of the series $U$ and we expect that the accuracy of the singularities will ensure the accuracy of the approximants.

The critical exponent $m_N$ can easily be found by using Newton’s polygon algorithm. However, it is well known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains $m_N = 1/2$. If we assume a singularity of algebraic type as in equation (10), then the exponent may be approximated by
$$m_N = 1 - \frac{A_N(a_{CN})}{DA_N(\lambda_{CN})}.$$  \hspace{1cm} (17)

For details on the above procedure, interested readers can see Vainberg and Trenogin (1974), Common (1982), Makinde (2005), etc.

5. Results and discussion
The bifurcation procedure above is applied on the first twenty-four terms of the solution series and we obtained the results as shown in tables (1) -(3) below:

Table 1: Computations showing the procedure rapid convergence for $Ha = 1.0$, $Re=20.0$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$N$</th>
<th>$\alpha_c$</th>
<th>$m_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>0.26791645</td>
<td>0.4798161</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>0.26920722</td>
<td>0.4978656</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>0.26915666</td>
<td>0.4998759</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>0.26916217</td>
<td>0.4999999</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
<td>0.26916246</td>
<td>0.500000</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>0.26916246</td>
<td>0.500000</td>
</tr>
</tbody>
</table>

Table 2: Computation showing the divergent channel flow critical Reynolds number for bifurcation at $\alpha=0.1$.

<table>
<thead>
<tr>
<th>$Ha$</th>
<th>$Re_c$</th>
<th>$m_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>54.4389</td>
<td>0.5000</td>
</tr>
<tr>
<td>1</td>
<td>54.47179</td>
<td>0.5000</td>
</tr>
<tr>
<td>2</td>
<td>54.58087</td>
<td>0.5000</td>
</tr>
<tr>
<td>3</td>
<td>54.66510</td>
<td>0.5000</td>
</tr>
<tr>
<td>4</td>
<td>55.22071</td>
<td>0.5000</td>
</tr>
<tr>
<td>5</td>
<td>55.52727</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Table 3: Computation showing the divergent channel critical semi-angles for bifurcation at $Re =20.0$

<table>
<thead>
<tr>
<th>$Ha$</th>
<th>$\alpha_c$</th>
<th>$m_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.267960</td>
<td>0.5000</td>
</tr>
<tr>
<td>1</td>
<td>0.269162</td>
<td>0.5000</td>
</tr>
<tr>
<td>2</td>
<td>0.272906</td>
<td>0.5000</td>
</tr>
<tr>
<td>3</td>
<td>0.279878</td>
<td>0.5000</td>
</tr>
<tr>
<td>4</td>
<td>0.290431</td>
<td>0.5000</td>
</tr>
<tr>
<td>5</td>
<td>0.307406</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Table (1) shows the rapid convergence of the dominant singularity $\alpha_c$, i.e. the divergent channel semi-angle at bifurcating point in the flow field for $Re=20.0$ and $Ha=1.0$ together with its corresponding critical exponent $m_c$ with gradual increase in the number of series coefficients utilized in the approximants. In figure (2) we show the fluid radial velocity profile for a symmetrically divergent...
channel at semi-angle $\alpha = 45^\circ$. It is interesting to note that an increase in the magnetic field intensity causes a general decrease in the fluid velocity around the centerline region of the channel. We can see that the Jeffery-Hamel flow corresponding to $Ha=0$ (no magnetic field) shows flow reversal near both walls i.e. internal boundary layer separation. As the Hartmann number ($Ha$) increases the flow reversal disappears, for $Ha = 2$, it has been suppressed already. The profile becomes distinctively convex for $Ha=4$. We see that moderate values of $Ha$ are sufficient to avoid the flow reversal that is typical of ordinary fluids in divergent channels. The case of convergent channel (i.e. $\alpha = -45^\circ$) is shown in figure (3). No reversal in the flow field near the walls occur in convergent channel, however, an increase in the magnetic field intensity also causes a general decrease in the fluid velocity around the channel centerline region. In figure (4), we show a slice of the bifurcation diagram at $\alpha=0.1$. The diagram suggests that, for this value of $\alpha$, there is a turning point at $Re_c(Ha)$. It is interesting to note that the magnitude of $Re_c$ increases with an increase in the magnetic field intensity (i.e. $Ha$) as shown in table (2). For the case of classical Jeffery-Hamel flow (i.e. $Ha=0$), we obtained $Re_c(Ha=0) \approx 54.4389$. This is in good agreement with Fraenkel’s result, namely $Re_c(Ha=0) \approx 5.461/\alpha$ as $\alpha \to 0$. We remark that, as $\alpha \to 0$, the flow tends to plane Poiseuille flow. Consequently, as the flow tends to hydromagnetic plane Poiseuille flow, we observe that turning point in the flow field varies depending on the magnitude of magnetic field intensity (table 2). Another slice of the bifurcation diagram at $Re=20$ is shown in figure (5). The diagram suggests that, for negative values of $\alpha$ (i.e. convergent channel), the solution is unique. It also suggests that the solution bifurcates at $\alpha_c(Ha)$ as shown in table (3). For the case of classical Jeffery-Hamel flow (i.e. $Ha = 0$), we obtained $\alpha_c(Ha=0) \approx 0.26796$. This is in good agreement with Khan et al. (2003) result, namely $\alpha_c(Re=20) \approx 0.27$. However, we remark here that this particular result of Khan et al. (2003) contains round up error. These bifurcation diagrams (figures 4 and 5) illustrate how the flow changes and bifurcates as the angle of inclination and magnetic field intensity vary or the Reynolds number increases. In particular, for every $\alpha$, there is a critical value $Re_c(Ha)$ such that, for $0 \leq Re(Ha) < Re_c(Ha)$ there are two solutions (labelled I and II) and the solution II diverges to infinity as $Re \to 0$. 
Figure (2): Fluid radial velocity profile for a divergent channel at semi-angle $\alpha = \pi/4$; $Re = 7.0$; $Ha = 0$; $Ha = 2.0$; $Ha = 4.0$

Figure (3): Fluid radial velocity profile for a convergent channel at semi-angle $\alpha = -\pi/4$; $Re = 7.0$; $Ha = 0$; $Ha = 2.0$; $Ha = 4.0$
Figure (4): A slice of approximate bifurcation diagram in the \((Re, u(0; \alpha=0.1, Re, Ha))\) plane.

Figure (5): A slice of approximate bifurcation diagram in the \((\alpha, u(0; \alpha,Re=20, Ha))\) plane.
Conclusion
The nonlinear problem of hydromagnetic Jeffery-Hamel flow is studied using a new perturbation series summation and improvement technique. A bifurcation study by analytic continuation of a power series in the bifurcation parameter for a particular solution branch is performed. The procedure reveals accurately the analytical structure of the solution function and pertinent results for velocity field, flow reversal control and bifurcations are discussed quantitatively. Finally, the above series summation procedure can be used as an effective tool to investigate several other parameter dependent nonlinear boundary-value problems mathematical sciences.

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References


