PHASE TRANSITIONS FOR ISING MODEL
WITH FOUR COMPETING INTERACTIONS

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Abstract

In this paper we consider an Ising model with four competing interactions (external field, nearest neighbor, second neighbors and triples of neighbors) on the Cayley tree of order two. We show that for some parameter values of the model there is phase transition. Our second result gives a complete description of periodic Gibbs measures for the model. We also construct uncountably many non-periodic extreme Gibbs measures.

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1 Introduction

Lattice spin systems is a large class of systems considered in statistical mechanics. Some of them have a real physical meaning, others are studied as suitable simplified models of more complicated systems. The structure of the lattice plays an important role in investigations of spin systems. For example, in order to study a phase transition problem for a system on $\mathbb{Z}^d$ and on a Cayley tree there are two different methods: Pirogov-Sinai theory on $\mathbb{Z}^d$, Markov random field theory and recurrent equations of this theory on Cayley tree. In [2-6] for several models on a Cayley tree, using the Markov random field theory, Gibbs measures are described.

In the paper we investigate a model with four competing interactions on the Cayley tree.

The paper is organized as follows.

In section 2 we give definitions of the model, Cayley tree and Gibbs measures.

In section 3 we reduce the problem of describing limit Gibbs measures to the problem of solving a nonlinear functional equations.

Section 4 is devoted to describe translation-invariant Gibbs measures. We show that two (minimal and maximal) of translation-invariant Gibbs measures are extreme in the set of all Gibbs measures.

In section 5 we study periodic Gibbs measures and show that our model admits only translation-invariant and periodic with period two (chess-board) Gibbs measures.

In the last section we construct uncountably many non-periodic extreme Gibbs measures.

2 Definitions

Cayley tree. The Cayley tree $\Gamma^k$ (see [1]) of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Let $\Gamma^k = (V, L, i)$ where $V$ is the set of vertices of $\Gamma^k$, $L$ is the set of edges of $\Gamma^k$ and $i$ is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then $x$ and $y$ are called nearest neighboring vertices and we write $l = < x, y >$. The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d|x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ such that the pairs}$$

$$< x_0, x_1 >, ..., < x_{d-1}, x_d > \text{ are neighboring vertices}\}.$$

For the fixed $x^0 \in V$ we set

$$W_n = \{x \in V|d(x, x^0) = n\}, V_n = \{x \in V|d(x, x^0) \leq n\},$$

$$L_n = \{l = < x, y > \in L|x, y \in V_n\}.$$

A collection of the pairs $< x, x_1 >, ..., < x_{d-1}, y >$ is called a path from $x$ to $y$. We write $x < y$ if the path from $x^0$ to $y$ goes through $x$. We call the vertex $y$ a direct successor of $x$, if
y > x and x, y are nearest neighbors. The set of the direct successors of x is denoted by \( S(x) \), i.e.

\[
S(x) = \{ y \in W_{n+1} | d(x,y) = 1 \}, \quad x \in W_n.
\]

We observe that for any vertex \( x \neq x^0 \), x has \( k \) direct successors and \( x^0 \) has \( k+1 \).

The vertices x and y are called second neighbor which is denoted by \( >x;y< \), if there exists a vertex \( z \in V \) such that \( x, z \) and \( y, z \) are nearest neighbors. We will consider only second neighbors \( >x,y< \), for which there exists \( n \) such that \( x, y \in W_n \). Three vertices \( x, y \) and \( z \) are called a triple of neighbors and they are denoted by \( <x,y,z> \), if \( <x,y>, <y,z> \) are nearest neighbors and \( x, z \in W_n, y \in W_{n-1} \), for some \( n \in N \). The fixed vertex \( x^0 \) is called the 0-th level and the vertices in \( W_n \) are called the \( n \)-th level.

It is known [5] that there exists a one-to-one correspondence between the set \( V \) of the vertices of the Cayley tree of order \( k \geq 1 \) and the group \( G_k \) of the free products of \( k+1 \) cyclic groups of the second order with generators \( a_1, a_2, ..., a_{k+1} \).

Let us define a group structure on the \( \Gamma^k \) as follows. Vertices which correspond to the “words” \( g, h \in G_k \) are called nearest neighbors and are connected by an edge if either \( g = ha_i \) or \( h = ga_j \) for some \( i \) or \( j \). The graph thus defined is a Cayley tree of order \( k \). Consider a left (resp. right) transformation shift on \( G_k \) defined as: for \( g_0 \in G_k \) we put

\[
T_{g_0}h = g_0h (\text{resp.} \ T_{g_0}h = g_0h) \quad \forall h \in G_k.
\]

Then the set of all left (resp. right) shifts on \( G_k \) is isomorphic to the group \( G_k \).

**The model.** The Ising model, which was originally regarded as a ferromagnetic model, has found some applications in many other physical, biological and chemical systems, and even in sociology. The model that is considered in [8] is a natural generalization of the Ising model, and a model of the similar form has recently been investigated by Monroe [15, 16] to understand the physical aspects associated with the Husimi tree or the Kagome lattice. On a similar note, the topic of statistical mechanics on non amenable graphs is a modern growing field [2, 14]. In the same paper [8], we have presented the exact solution of an Ising model with competing restricted interactions and zero external magnetic field on the Cayley tree \( \Gamma^2 \) for order 2.

In this paper we consider the Ising Model with four competing interactions on the Cayley tree which is defined by the following Hamiltonian

\[
H(\sigma) = -J_3 \sum_{<x,y,z>} \sigma(x)\sigma(y)\sigma(z) - J \sum_{>x,y<} \sigma(x)\sigma(y) - J_1 \sum_{<x,y>} \sigma(x)\sigma(y) - \alpha \sum_{x \in V} \sigma(x) \quad (1)
\]

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and the spin variables \( \sigma(x) \) assume the values \( \pm 1 \). (See [11] for models with competing interactions, and see [2], [14]-[16] for the physical motivation underlying the study of these models.)
Remark. If one considers the Hamiltonian with all possible triple (without condition \(x, z \in W_n\)) and second neighbors (without condition \(x, y \in W_n\)) then the problem of describing limit Gibbs measures becomes a difficult problem.

The various partial cases of this model have been investigated in numerous works, for example, the case \(J_3 = \alpha = 0\) was considered in [15], [16] and [8]. In [8], the exact solution of an Ising model with competing restricted interactions with zero external field was presented. The case \(J = \alpha = 0\) was considered in [7], [16], and [17]. In [7], the exact solution was found for the problem of phase transitions. In [17] it is proven that there are two translation - invariant and uncountable number of distinct non-translation - invariant extreme Gibbs measures. In [9] the phase transition problem was solved for \(\beta = 0\), \(J_1, J_3 \neq 0\) and for \(J_3 = 0\), \(\alpha \cdot J_1 \neq 0\) as well.

In the paper we will consider the case \(J_1, J_2, J_3 \neq 0\).

Gibbs measures. Let \(\Lambda\) be a finite subset of \(V\). Assume \(\Omega(\Lambda)\) is the set of all configurations on \(\Lambda\), that is the functions \(\{\sigma(x), x \in \Lambda\}\). Let \(\vec{\sigma}(V \setminus \Lambda)\) be a fixed boundary configuration. The total energy of configuration \(\sigma(\Lambda) \in \Omega(\Lambda)\) under condition \(\vec{\sigma}(V \setminus \Lambda)\) is defined as

\[
H(\sigma(\Lambda)|\vec{\sigma}(V \setminus \Lambda)) = -J_3 \sum_{<x,y,z> \atop x,y,z \in \Lambda} \sigma(x)\sigma(y)\sigma(z) - J \sum_{>x,y< \atop x,y \in \Lambda} \sigma(x)\sigma(y) - \\
J_1 \sum_{<x,y> \atop x,y \in \Lambda} \sigma(x)\sigma(y) - \alpha \sum_{x \in \Lambda} \sigma(x) - J_3 \sum_{<x,y,z> \atop x \in \Lambda, y \in \Lambda, z \in \Lambda or \atop x \notin \Lambda} \sigma(x)\sigma(y)\sigma(z) - \\
J \sum_{>x,y< \atop x \in \Lambda, y \notin \Lambda} \sigma(x) \vec{\sigma}(y) - J_1 \sum_{<x,y> \atop x \in \Lambda, y \notin \Lambda} \sigma(x) \vec{\sigma}(y).
\]  
(2)

When all boundary points \(\{\vec{\sigma}(y), y \in V \setminus \Lambda\}\) are fixed as \(+1\), we have the positive boundary condition and when they are fixed as \(-1\), we have negative boundary condition. The free boundary condition corresponds to the case when the last three sums in the above are absent, that is formally all boundary points are fixed as 0.

The partition function \(Z_\Lambda(\vec{\sigma}(V \setminus \Lambda))\) in volume \(\Lambda\) under boundary condition \(\vec{\sigma}(V \setminus \Lambda)\) is defined as

\[
Z_\Lambda = \sum_{\sigma(\Lambda) \in \Omega(\Lambda)} \exp(-\beta H(\sigma(\Lambda))|\vec{\sigma}(V \setminus \Lambda)),
\]

where \(\beta = \frac{1}{kT}\) is the inverse temperature. Then the conditional Gibbs measure \(\mu_\Lambda\) in volume \(\Lambda\) under boundary condition \(\vec{\sigma}(V \setminus \Lambda)\) is defined as

\[
\mu_\Lambda(\sigma(\Lambda)) = \frac{\exp(-\beta H(\sigma(\Lambda))|\vec{\sigma}(V \setminus \Lambda))}{Z_\Lambda}.
\]  
(3)

3 The functional equation

There are several approaches to derive the equation solutions of which describes the limit Gibbs measures for lattice models on the Cayley tree. One approach is based on properties of Markov
random fields on Cayley tree [23] and [18]. Another approach is based on recurrent equations for partition functions [7], [12].

Here we shall use the Markov random field method.

Let $h : x \rightarrow R$ be a real valued function of $x \in V$. Given $n = 1, 2, \ldots$, consider the probability measure $\mu^{(n)}$ on $\{-1, +1\}^{V_n}$ which is defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left\{ -\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x) \right\}.$$ 

Here, as before, $\beta = \frac{1}{kT}$ and $\sigma_n : x \in V_n \rightarrow \sigma_n(x)$ and $Z_n$ is the corresponding partition function

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega(V_n)} \exp \left\{ -\beta \tilde{H}(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x) \right\}.$$ 

The consistency condition for $\mu^{(n)}(\sigma_n)$, $n \geq 1$ is

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}),$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

Let $V_1 \subset V_2 \subset \ldots, \cup_{n=1}^{\infty} V_n = V$ and $\mu_1, \mu_2, \ldots$ be a sequence of the probability measures on $\Phi^{V_1}, \Phi^{V_2}, \ldots$ satisfying the consistency condition, where $\Phi = \{-1, +1\}$. Then, according to the Kolmogorov theorem, (see, e.g. [21]), there is a unique limit Gibbs measure $\mu_h$ on $\Omega$ such that for every $n = 1, 2, \ldots$ and $\sigma_n \in \Phi^{V_n}$ the following equality holds

$$\mu(\{\sigma|V_n = \sigma_n\}) = \mu^{(n)}(\sigma_n).$$

The following statement describes the conditions on $h_x$ which guarantee the consistency condition of measures $\mu^{(n)}(\sigma_n)$.

**Proposition 1.** The measure $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfies the consistency condition (4) if and only if for any $x \in V$ the following equation holds:

$$h_x = \frac{1}{2} \log \left( \frac{\theta_1^2 \theta_2 \theta_3 e^{2(h_y + h_z)} + \theta_1 e^{2h_y} + e^{2h_z} + \theta_2 \theta_3 e^{2h_y + 2h_z}}{\theta_1 \theta_2^2 \theta_3 + \theta_1 \theta_2 (e^{2h_y} + e^{2h_z}) + \theta_2 e^{2h_y + 2h_z}} \right),$$

where $S(x) = \{y, z\}$, $< y, x, z >$ is a ternary neighbor and $\theta_1 = e^{2\beta J_1}, \theta_2 = e^{2\beta J_2}, \theta_3 = e^{2\beta J_3}, \theta_4 = e^{2\beta J_4}$.

**Proof.** Necessity. According to the consistency condition (4) we have

$$\frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)}} \exp \left\{ -\beta H_{n-1}(\sigma_{n-1}) + \beta J_1 \sum_{x \in W_{n-1}, y, z \in S(x)} \sigma(x)(\sigma(y) + \sigma(z)) \right\}$$

$$+ \beta J \sum_{x \in W_{n-1}, y, z \in S(x)} \sigma(x)\sigma(y) + \beta J_3 \sum_{x \in W_{n-1}, y, z \in S(x)} \sigma(x)\sigma(y)\sigma(z)$$
\[ + \sum_{x \in W_{n-1}} \beta \alpha \sigma(x) + \sum_{x \in W_{n-1}} h_x \sigma(x) \right\) \\
= \exp \left\{ -\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_x \sigma(x) \right\} \\
\]

Consequently we have
\[
\frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)} \in W_{n-1}} \prod_{x \in W_{n-1}} \exp\{\beta J_1 \sigma(x)(\sigma(y) + \sigma(z)) + \beta J_3 \sigma(y)\sigma(z) \\
+ \beta J_3 \sigma(x)\sigma(y)\sigma(z) + \beta \alpha \sigma(x) + h_y \sigma(y) + h_z \sigma(z)\} \\
= \prod_{x \in W_n} \exp\{h_x \sigma(x)\}. \\
\]

Assume \( x \in W_{n-1} \) and \( S(x) = \{y, z\}, \sigma_x^{(n)} = \{\sigma(y), \sigma(z)\} \). Since \( \sigma^{(n)} = \cup_{x \in W_{n-1}} \sigma_x^{(n)} \), we get
\[
\frac{Z_{n-1}}{Z_n} \sum_{x \in W_{n-1}} \prod_{\sigma_x^{(n)} \in W_{n-1}} \exp\{\beta J_1 \sigma(x)(\sigma(y) + \sigma(z)) + \beta J_3 \sigma(y)\sigma(z) \\
+ \beta J_3 \sigma(y)\sigma(z) + \beta \alpha \sigma(x) + h_y \sigma(y) + h_z \sigma(z)\} = \prod_{x \in W_n} \exp\{h_x \sigma(x)\}. \tag{6} \\
\]

Now fix \( x \in W_{n-1} \) and rewrite (6) for the cases \( \sigma(x) = 1 \) and \( \sigma(x) = -1 \). If \( \sigma(x) = 1 \), we have
\[
N = \sum_{\sigma_x^{(n)} = \{\sigma(y), \sigma(z)\}} \exp\{\beta J_1 (\sigma(y) + \sigma(z)) + \beta J_3 \sigma(y)\sigma(z) \\
+ \beta J_3 \sigma(y)\sigma(z) + \beta \alpha + h_y \sigma(y) + h_z \sigma(z)\} \\
= \exp\{h_x\}; \\
\]
and if \( \sigma(x) = -1 \), then
\[
D = \sum_{\sigma_x^{(n)} = \{\sigma(y), \sigma(z)\}} \exp\{-\beta J_1 (\sigma(y) + \sigma(z)) + \beta J_3 \sigma(y)\sigma(z)\} \\
+ \beta J_3 \sigma(y)\sigma(z) - \beta \alpha + h_y \sigma(y) + h_z \sigma(z)\} \\
= \exp\{-h_x\}. \\
\]

So that
\[
\frac{N}{D} = \exp \{2h_x\}. \tag{7} \\
\]

The numerator \( N \) of the left-hand side is equal to
\[
N = \exp(2\beta J_1 + \beta J + \beta J_3 + \beta \alpha + h_y + h_z) + \exp(-\beta J - \beta J_3 - \beta \alpha - h_y + h_z) \\
+ \exp(-\beta J - \beta J_3 + \beta \alpha + h_y - h_z) + \exp(-2\beta J_1 + \beta J + \beta J_3 - \beta \alpha - h_y - h_z) \\
\]
while the denominator \( D \) is equal to
\[
D = \exp(-2\beta J_1 + \beta J + \beta J_3 - \beta \alpha + h_y + h_z) + \exp(-\beta J - \beta J_3 - \beta \alpha - h_y + h_z) \\
+ \exp(-\beta J - \beta J_3 - \beta \alpha + h_y - h_z) + \exp(2\beta J_1 + \beta J + \beta J_3 - \beta \alpha - h_y - h_z). \\
\]
Then the equality \( \frac{N}{D} = \exp\{2h_x\} \) implies (5).

**Sufficiency.** Assume that (5) is valid. From (7) we get

\[
\sum_{\sigma_x^{(n)} = \{\sigma(y), \sigma(z)\}} \exp\{\beta J_1 \sigma(x)(\sigma(y) + \sigma(z)) + \beta J_3 \sigma(x)\sigma(y)\sigma(z) + \beta \alpha \sigma(x) + h_y \sigma(y) + h_z \sigma(z)\} = a(x) \exp\{\sigma(x)h_x\},
\]

where \( \sigma(x) = \pm 1 \). This equality implies

\[
\prod_{x \in W_{n-1}} \sum_{\sigma_x^{(n)} = \{\sigma(y), \sigma(z)\}} \exp\{\beta J_1 \sigma(x)(\sigma(y) + \sigma(z)) + \beta J_3 \sigma(x)\sigma(y)\sigma(z) + \beta \alpha \sigma(x) + h_y \sigma(y) + h_z \sigma(z)\} = \prod_{x \in W_{n-1}} a(x) \exp\{\sigma(x)h_x\}.
\]

Denoting \( A_n(x) = \prod_{x \in W_{n-1}} a(x) \), we have from (8)

\[
Z_{n-1} A_{n-1} \mu^{(n-1)}(\sigma_{n-1}) = Z_n \sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}).
\]

As \( \mu^{(n)} \), \( n \geq 1 \) is a probability, we have

\[
\sum_{\sigma_{n-1}, \sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \sum_{\sigma_{n-1}} \mu^{(n-1)}(\sigma_{n-1}) = 1.
\]

From these equalities we get \( Z_{n-1} A_{n-1} = Z_{n} \), which means that (4) holds.

According to Proposition 1 the problem of describing the Gibbs measures is reduced to the description of the solutions of the functional equation (5).

Denote \( \Omega = \{-1, +1\}^V \). Note that any transformation \( S \) of the group \( G_k \) induces a shift automorphism \( \tilde{S} : \Omega \to \Omega \) by

\[
(\tilde{S}\sigma)(g) = \sigma(Sg), \ g \in G_k, \ \sigma \in \Omega.
\]

By \( G_k \) we denote the set of all shifts on \( \Omega \). We say that a Gibbs measure \( \mu \) on \( \Omega \) is translation - invariant if for any \( T \in G_k \) the equality \( \mu(T(A)) = \mu(A) \) is valid for all \( A \in \mathcal{F} \), where \( \mathcal{F} \) is a standard \( \sigma \)-algebra of subsets of \( \Omega \) generated by cylinder subsets.

### 4 Translation-invariant Gibbs measures: phase transition

The analysis of the solution of (5) is rather tricky. It is natural to begin with the translation-invariant solutions where \( h_x = h \) is constant for all \( x \in V \). In this case from (5), we have

\[
u = \theta_1 \theta_2 \theta_3 u^2 + 2\theta_1 u + \theta_2 \theta_3
\]

(9)
where \( u = e^{2h} \).

Note that if there is more than one positive solution for equation (9), then there is more than one translation-invariant Gibbs measure corresponding to these solutions. We say that a phase transition occurs for model (1), if equation (9) has more than one positive solution. The number of the solutions of equation (9) depends on the parameter \( \beta = \frac{1}{kT} \). The phase transition usually occurs for low temperature. It is possible to find an exact value of temperature \( T^* \) such that a phase transition occurs for all \( T < T^* \) where \( T^* \) is called a critical value of temperature.

Finding the exact value of the critical temperature for some models means to exactly solve the models.

**Proposition 2.** If \( \theta_1^2 > 3 \), \( \theta_2 > \frac{2\theta_1}{\theta_1^2 - 3} \), and

\[
\frac{\sqrt{\theta_1^2(\theta_1^2 + 2\theta_1^2 - 3) - 4\theta_1^2} - 8\theta_1}{2\theta_1} - \frac{\sqrt{\theta_1^2(\theta_1^2 + 2\theta_1^2 - 3) - 4\theta_1^2} - 8\theta_1}{2\theta_1} = \theta_3 < 0
\]

then equation (9) has three positive roots \( u_1^* < u_2^* < u_3^* \). Here

\[
\eta_i(\theta_1, \theta_2, \theta_3) = \frac{1}{u_i} \frac{\theta_1^2 \theta_2 \theta_3 u_i^2 + 2\theta_1 u_i + \theta_2 \theta_3}{\theta_1^2 \theta_3 u_i + 2\theta_1 \theta_3 u_i + \theta_2 u_i^2}
\]

where \( u_i, i = 1, 2 \) are the solutions of

\[
\theta_1^2 \theta_2 \theta_3 u_i^4 + 4\theta_1 \theta_2 u_i^3 + \theta_3(3\theta_2^2 - \theta_1^2 \theta_3^2 + 4\theta_1^2)u_i^2 + 4\theta_1 \theta_2 \theta_3 u_i^2 + \theta_1^2 \theta_3^2 = 0. \tag{10}
\]

**Proof.** Denote

\[
f(u) = \frac{\theta_1^2 \theta_2 \theta_3 u^2 + 2\theta_1 u + \theta_2 \theta_3}{\theta_1^2 \theta_2 + 2\theta_1 \theta_3 u + \theta_2 u^2}.
\]

We have

\[
f'(u) = 2\theta_2 \frac{\theta_1 \theta_1 \theta_3^2 - 1)u^2 + \theta_2 \theta_3(\theta_1^4 - 1)u + \theta_1(\theta_2^4 - \theta_3^4)}{(\theta_1^2 \theta_2 + 2\theta_1 \theta_3 u + \theta_2 u^2)^2},
\]

\[
f''(u) = 2\theta_2 \frac{\theta_2 \theta_2^2 + 2\theta_1 \theta_3 u + \theta_2 \theta_3^2)^2}{(\theta_1^2 \theta_2 + 2\theta_1 \theta_3 u + \theta_2 u^2)^3}
\]

\[
(-2\theta_1 \theta_2 (\theta_1^2 \theta_3^2 - 1)u^3 + \theta_1 \theta_3(\theta_1^4 - 1)u^2 + 6\theta_1 \theta_2 \theta_2 \theta_3^2 - \theta_1 \theta_2 \theta_3^2)u + \theta_1 \theta_3(\theta_1^2 \theta_3^2 - 1)u + \theta_1 \theta_3(\theta_1^2 \theta_3^2 - 1 - 4\theta_1^2 + 4\theta_3^2).
\]

Denote

\[
A = 2\theta_1 \theta_2 (\theta_1^2 \theta_3^2 - 1); \quad B = 3\theta_2 \theta_3(\theta_1^4 - 1);
\]

\[
C = 6\theta_1 \theta_2 (\theta_3^2 - \theta_1^2); \quad D = \theta_1 \theta_3(\theta_2^4 - 1) - 4\theta_1^2 + 4\theta_3^2).
\]

It is easy to see that under conditions of the proposition we have \( A > 0, B > 0, C > 0, D > 0 \). If equation \( f''(u) = 0 \) is equivalent to \( Au^3 + Bu^2 - Cu - D = 0 \), one can easily prove that the
last equation has unique positive solution, say $u^*$. Thus $f$ is convex for $u < u^*$ and concave for $u > u^*$. Consequently there are at most three solutions. On the other hand, it is easy to see that (9) has more than one solution if and only if there is more than one solution of the equation $uf'(u) = f(u)$ which is equivalent to equation (10). Now consider (10), which can be rewritten as

$$\theta_1^2 \theta_2^2 (u + \frac{1}{u})^2 + 4 \theta_1 \theta_2 (\frac{u}{\theta_3} + \frac{\theta_3}{u}) + 3 \theta_2^2 - \theta_1^2 \theta_2^2 + 4 \theta_1^2 - 2 \theta_1^2 \theta_2^2 = 0.$$ 

Denote

$$\varphi(u) = 4 \theta_1 \theta_2 (\frac{u}{\theta_3} + \frac{\theta_3}{u}),$$

$$\psi(u) = \theta_2^2 (\theta_1^2 + 2 \theta_1^2 - 3) - \theta_1^2 \theta_2^2 (u + \frac{1}{u})^2 - 4 \theta_1^2.$$ 

A simple analysis of these functions shows that under conditions of the proposition equation (9) has three positive solutions. This completes the proof.

Thus by Propositions 1 and 2 we can formulate the following

**Theorem 3.** Assume the conditions of Proposition 2 are satisfied then for the model (1) there are three translation-invariant Gibbs measures $\mu_1, \mu_2, \mu_3$ i.e. there is phase transition.

Note that $\mu_1 (\mu_3)$ corresponds to positive (resp. negative) boundary condition. The boundary condition corresponding to $\mu_2$ is unclear.

The following Proposition 4 describes a useful property of general (non translation-invariant) solutions $h_x$ to (5)

**Proposition 4.** Assume the conditions of Proposition 2 are satisfied and $h_x$ is a solution of (5), with $u_x = e^{2h_x}$, then

$$u_1^* \leq u_x \leq u_3^*, \quad x \in V$$

(11)

where $u_1^* < u_3^*$ are solutions of (9).

**Proof.** It is clear that $u_x > 0$, for any $x \in V$. For $u, v > 0$ denote

$$F(u, v) = \theta_1^2 \theta_2 \theta_4 u v + \theta_1 (u + v) + \theta_2 \theta_3.$$ 

Equation (5) can be rewritten as $u_x = F(u_y, u_z)$.

Observe that under conditions of Proposition 2 the function $F(u, v)$ is increasing with respect to $u$ and $v$ on $(0, \infty)$. Hence we conclude that

$$\frac{\theta_3 \theta_4}{\theta_1} < F(u, v) < \theta_1^2 \theta_3 \theta_4,$$

for all $u, v > 0$. Now we consider the function with $u, v \in (\frac{\theta_3 \theta_4}{\theta_1}, \theta_1^2 \theta_3 \theta_4)$. By similar reason as above we get

$$f(\frac{\theta_3 \theta_4}{\theta_1}) < F(u, v) < f(\theta_1^2 \theta_3 \theta_4),$$


where \( f(u) = F(u, u) \). Repeating this argument one gets

\[
f^{(n)}(\frac{\theta_2 \theta_4}{\theta_1}) < F(u, v) < f^{(n)}(\frac{\theta_2^2 \theta_3 \theta_4}{\theta_1^2}),
\]

for all \( n \geq 1 \). Here \( f^{(n)} \) is \( n \)-th iteration of the map \( x \rightarrow f(x) \). The sequence \( f^{(n)}(\frac{\theta_2^2 \theta_3 \theta_4}{\theta_1^2}) \) is decreasing and bounded below by \( u_3^* \). Its limit is a fixed point of \( f \) and thus equal to \( u_3^* \). This proves that \( u_x \leq u_3^* \). The lower bound for \( u_x \) is similar and gives \( u_1^* \).

Using Proposition 4 by similar argument as in the proof of Theorem 12.31 of [10] one can prove the following

**Theorem 5.** Assume conditions of Proposition 2 are satisfied then translation-invariant measures \( \mu_1, \mu_3 \) (see Theorem 3) are extreme.

**Remark.** The problem of extremity for measure \( \mu_2 \) is a difficult problem. Usually (see [3], [24]) such measure which corresponds to unordered phase is extreme for the temperature which is lower than the critical temperature of phase transition.

### 5 Periodic Gibbs Measures

In this section we study a periodic (see Definition 6) solutions of (5).

**Definition 6.** Let \( K \) be a subgroup of \( G_k, k \geq 1 \). We say that a collection (of functions) \( h = \{h_x \in \mathbb{R}^k : x \in G_k\} \) is \( K \)-periodic if \( h_{yx} = h_x \) for all \( x \in G_k \) and \( y \in K \).

**Definition 7.** A Gibbs measure is called \( K \)-periodic if it corresponds to \( K \)-periodic collection \( h \).

Observe that a translation-invariant Gibbs measure is \( G_k \)-periodic.

We give a complete description of periodic Gibbs measures i.e. a characterization of such measures with respect to any normal subgroup of finite index in \( G_k \).

Let \( K \) be a subgroup of index \( r \) in \( G_k \), and let \( G_k/K = \{K_0, K_1, ..., K_{r-1}\} \) be the quotient group, with the coset \( K_0 = K \). Let \( q_i(x) = |S_i(x) \cap K_i|, \ i = 0, 1, ..., r - 1 \); \( N(x) = |\{ j : q_j(x) \neq 0\}| \), where \( S_i(x) = \{y \in G_k : (x, y)\}, x \in G_k \) and \( |\cdot| \) is the number of elements in the set. Denote \( Q(x) = (q_0(x), q_1(x), ..., q_{r-1}(x)) \).

We note (see [19]) that for every \( x \in G_k \) there is a permutation \( \pi_x \) of the coordinates of the vector \( Q(e) \) (where \( e \) is the identity of \( G_k \)) such that

\[
\pi_x Q(e) = Q(x).
\]

It follows from (12) that \( N(x) = N(e) \) for all \( x \in G_k \).

Each \( K \)-periodic collection is given by

\[
\{h_x = h_i \ \text{for} \ x \in K_i, \ i = 0, 1, ..., r - 1\}.
\]
For $k = 2$ by Proposition 1 and (12), $h_n, \ n = 0, 1, ..., r - 1,$ satisfies

$$h_n = \frac{1}{2} \log F(e^{2h \pi_n(i)}, e^{2h \pi_n(j)}),$$

where $F(u, v)$ is defined in the proof of Proposition 4 and $\pi_n$ is permutation of $Q(e)$ for $x \in K_n, i, j \in Q(e)$.

**Proposition 8.** Suppose the conditions of Proposition 2 are satisfied then $F(u, v) = F(h, v)$ if and only if $u = h$ \ (F(u, v) = F(u, h) if and only if $v = h$)

**Proof.** Follows from monotonity of $F$ with respect to $u$ (resp. $v$).

Let $G^*_2$ be the subgroup in $G_2$ consisting of all words of even length. Clearly, $G^*_2$ is a subgroup of index 2.

**Theorem 9.** Let $K$ be a normal subgroup of finite index in $G_2$. Then each $K$–periodic Gibbs measure for model (1) is either translation-invariant or $G^*_2$–periodic.

**Proof.** We see from (13) that

$$F(e^{h \pi_n(i)}, e^{h \pi_n(j)}) = F(e^{h \pi_n(i')}, e^{h \pi_n(j')}),$$

For any $i, j, i', j' \in Q(e), n = 0, 1, ..., r - 1$. Hence from Proposition 8 we have

$$h_{\pi_n(i_1)} = h_{\pi_n(i_2)} = ... = h_{\pi_n(i_{N(e)})}.$$ 

Therefore,

$$h_x = h_y = h, \ \text{if} \ x, y \in S_1(z), \ z \in G^*_2;$$

$$h_x = h_y = l, \ \text{if} \ x, y \in S_1(z), \ z \in G_2 \setminus G^*_2.$$ 

Thus the measures are translation-invariant (if $h = l$) or $G^*_2$–periodic (if $h \neq l$). This completes the proof of the theorem.

Let $K$ be a normal subgroup of finite index in $G_2$. What condition on $K$ will guarantee that each $K$–periodic Gibbs measure is translation-invariant? We put $I(K) = K \cap \{a_1, a_2, a_3\}$, where $a_i, \ i = 1, 2, 3$ are generators of $G_2$.

**Theorem 10.** If $I(K) \neq \emptyset$, then each $K$–periodic Gibbs measure for model (1) is translation-invariant.

**Proof.** Take $x \in K$. We note that the inclusion $xa_i \in K$ holds if and only if $a_i \in K$. Since $I(K) \neq \emptyset$, there is an element $a_i \in K$. Therefore $K$ contains the subset $Ka_i = \{xa_i : x \in K\}$. By Theorem 9 we have $h_x = h$ and $h_{xa_i} = l$. Since $x$ and $xa_i$ belong to $K$, it follows that
Thus each $K$-periodic Gibbs measure is translation-invariant. This proves Theorem 10.

Theorems 9 and 10 reduce the problem of describing $K$-periodic Gibbs measure with $I(K) \neq \emptyset$ to describing the fixed points of $f(u) = F(u, u)$ (see (9)) which describes translation-invariant Gibbs measures. If $I(K) = \emptyset$, this problem is reduced to describing the solutions of the system:

\[
\begin{cases}
  u = f(v), \\
  v = f(u)
\end{cases}
\]

with

\[
f(u) = \frac{\theta_1^2 \theta_2 \theta_3 u^2 + 2 \theta_1 u + \theta_2 \theta_3}{\theta_1^2 \theta_2 + 2 \theta_1 \theta_3 u + \theta_2 u^2}.
\]

Evidently the positive roots of the equation

\[
\frac{f(f(u)) - u}{f(u) - u} = 0
\]

describe the periodic (non translation-invariant) Gibbs measures.

As we are looking for positive roots (16) has the following form:

\[
\begin{align*}
\theta_1^2 \theta_2 (\theta_1^2 \theta_2 \theta_3 u^2 + 2 \theta_1 \theta_3 u + \theta_2) u^2 + \theta_3 (\theta_1^2 \theta_2^2 u + 2 \theta_1 \theta_2 \theta_3 + 2 \theta_1^2 \theta_2 + 4 \theta_1^2 \theta_4 - \theta_2^2 \theta_4) u + \\
\theta_1^2 \theta_2 (\theta_1 \theta_2 \theta_3^2 + 2 \theta_1 \theta_4 + \theta_2) = 0,
\end{align*}
\]

The discriminant $\Delta$ of (17) is equal to

\[
\Delta = -4 \theta_1^2 \theta_2^3 \theta_3 (\theta_1 \theta_2 \theta_4 + 2) \theta_4^3 + A \theta_3^2 - 4 \theta_1^2 \theta_2^3 (\theta_1 \theta_2 + 2 \theta_4),
\]

where

\[
A = -\theta_2^3 (3 \theta_1^2 + 6 \theta_1^4 - 1) \theta_4^2 - 4 \theta_1^2 \theta_4 (1 + \theta_2^2) (1 + \theta_4^2) \theta_1^2 + \\
4 \theta_1^2 \theta_2 (\theta_1 \theta_4^2 + \theta_1^4 - 2 \theta_2 \theta_4^2) + 16 \theta_1^2 \theta_2 (1 + \theta_2^2) \theta_2 + 16 \theta_1^4 \theta_4^2.
\]

Using simple analysis one can see that (17) has two positive solutions if

\[
\theta_1 < 1, \quad \theta_2 > \frac{2 \theta_1}{1 - \theta_1^2}, \quad \frac{1}{\theta_4} < \theta_4 < \theta_4^*,
\]

where

\[
\theta_4^* = \frac{\theta_2^2 - 4 \theta_1^2 - \theta_4^2 \theta_2^2 + \sqrt{(4 \theta_1^2 + \theta_4^2 \theta_2^2 - \theta_2^2)^2 - 16 \theta_1^4 \theta_2^2}}{4 \theta_1^2 \theta_2}
\]

and

\[
A^2 > 64 \theta_1^{10} \theta_2^6 \theta_4 (\theta_1 \theta_2 \theta_4 + 2) (\theta_1 \theta_2 + 2 \theta_4),
\]

\[
\theta_3^< < \theta_2^< < \theta_3^<,
\]

where $\theta_3^<$ are solutions of $\Delta = 0$.

Therefore, the following theorem is proved:
Theorem 11. Assume \((\theta_1, \theta_2, \theta_3, \theta_4)\) satisfied conditions (18)-(20) then for the model (1) there are two \(G_2^+\) periodic Gibbs measures \(\mu_{1, \text{per}}, \mu_{2, \text{per}}\).

Remarks 1. By construction measures \(\mu_{1, \text{per}}, \mu_{2, \text{per}}\) are non translation-invariant, but periodic with period 2 (= index of normal subgroup).

2. For \(\theta_4 = 1\) the condition (19) can be rewritten as (see [9])

\[
(1 - 3\theta_1^2)(\theta_1^2 + 1)(\theta_2^2 - 2\theta_1)(\theta_2^2 - 2\theta_1 + 1) > 0.
\]

This factorization gives a more simple formulation of the conditions (18)-(20) i.e. for \(\theta_4 = 1\) conditions (18)-(20) can be reduced to

\[
0 < \theta_1 < \frac{1}{\sqrt{3}}, \quad \theta_2 > \frac{2\theta_1}{1 - 3\theta_1^2}, \quad \theta_3^- < \theta_3 < \theta_3^+.
\]

6 Non periodic Gibbs measures

In this section we consider the case of phase transition (i.e. assume that the conditions of Proposition 2 are satisfied). We show that functional equation (5) admits uncountably many non periodic solutions.

Take an arbitrary infinite path \(\pi = \{x^0 = x_0, x_1, \ldots\}\) on the Cayley tree of order 2. There is (see [4], [20]) one-to-one correspondence between such paths and real numbers \(t \in [0; 1]\). We will map the path \(\pi\) to a function \(h_\pi : x \in V \rightarrow h_\pi^x\) satisfying (5). Path \(\pi\) splits Cayley tree \(\Gamma^2\) into two parts \(\Gamma_1^2\) and \(\Gamma_2^2\).

Function \(h_\pi\) is defined by

\[
h_\pi^x = \begin{cases} 
\log u_1^t, & \text{if } x \in \Gamma_1^2 \\
\log u_3^t, & \text{if } x \in \Gamma_2^2 
\end{cases}
\]

(21)

Denote

\[
\Phi(x, y) = \frac{1}{2} \log \left( \frac{\theta_1 \theta_2 \theta_3 e^{2(x+y)} + \theta_4 e^{2x} + e^{2y} + \theta_2 \theta_3}{\theta_1 \theta_2 + \theta_1 \theta_3 e^{2x} + e^{2y} + \theta_2 e^{2(x+y)}} \right)
\]

Proposition 12. The following inequality holds:

\[
|\Phi(x_1, y) - \Phi(x_2, y)| \leq \gamma(\theta_1, \theta_2, \theta_3)|x_1 - x_2|,
\]

where

\[
\gamma(\theta_1, \theta_2, \theta_3) = \max_{t \in [u_1^t, u_3^t]} \frac{\sqrt{\theta_1 t + \theta_2 t + \theta_3 t}}{\theta_1 t + \theta_2 t + \theta_3 t + 1} < 1
\]

Proof. The function \(\Phi(x, y)\) can be rewritten as

\[
\Phi(x, y) = \frac{1}{2} \log \theta_4 + \frac{1}{2} \log \frac{A e^{2x} + B}{C e^{2x} + D}.
\]
where $A, B, C, D$ depends on $\theta_1, \theta_2, \theta_3$ and $y$. It is easy to see that

$$|\Phi'(x, y)| \leq \frac{\sqrt{AD} - \sqrt{BC}}{\sqrt{AD} + \sqrt{BC}}.$$ 

This completes the proof.

With the help of Proposition 12 it is easy to prove the following Theorem 13, similar to Theorem 3 of [20]:

**Theorem 13.** For any infinite path $\pi$, there exists a unique function $h^\pi$ satisfying (5) and (21).

In the standard way (see e.g. [4], [20]) one can prove that functions $h^{\pi(t)}$ are different for different $t \in [0; 1]$.

Now let $\mu(t)$ denote the Gibbs measure corresponding to function $h^{\pi(t)}$, $t \in [0; 1]$.

Using Theorem 5, similar to the analogous theorem of [4] we obtain the following

**Theorem 14.** For any $t \in [0; 1]$, there exists a unique extreme Gibbs measure $\mu(t)$. Moreover, the above Gibbs measures $\mu_i$, $i = 1, 3$, are specified as $\mu(0) = \mu_3, \mu(1) = \mu_1$.

Because measures $\mu(t)$ are different for different $t \in [0; 1]$ we obtain a continuum of distinct extreme Gibbs measures.

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