The solution of BBGKY’s chain of kinetic equations that describes the system of particles interacting by pair potential is reduced to the solution of Boltzmann equation.
The present paper is devoted to the definition of the solution of BBGKY’s chain of kinetic equations \cite{Bogoluibov1946, BornGreen1949, Kirkwood1946, Kirkwood1947, Yvon1935, Bogoluibov1975, UhlenbeckFord1963, Klimontovich1982, LiboffPerona1967, Liboff2003} by solving the kinetic equation of Boltzmann \cite{Boltzmann1972}. Since its formulation in 1872, the Boltzmann kinetic equation, is subject to intensive investigation \cite{Cercignani1969, Cercignani1975, CercignaniPulvirenti1993, LebowitzMontroll1983, CercignaniIllnerPulvirenti1994, CercignaniIllner1995, Markowich1990, Noinzert1984}. The interest in Boltzmann kinetic equation increased even more from the second part of the 20th century, because of its relevance in many fields of physics. It is well known that the Boltzmann kinetic equation describes the evolution of one particle interacting with another. However, the use of the Boltzmann kinetic equation for describing the evolution of a system of many interacting particles, should be considered. In this direction, essential results have been obtained, for instance, the derivation of the BBGKY’s chain of kinetic equations and the determination of the solutions of this chain \cite{Petrina1972, PetrinaVidibida1975, Rasulova1980, RasulovaSiddiqi2004}. In previous reports \cite{Bogoluibov1946, Koga1970, BogoluibovBogoluibov1946} it has been shown that the first equation of BBGKY’s chain is similar to Boltzmann kinetic equation.

In the present paper, it is demonstrated that all members of series, which is the solution of the chain, are associated with a solution of Boltzmann kinetic equation under the main order approximation (without terms which contain the gradients of $\psi(t)$ not higher than first order) and for $t > t_{\text{collision}}$.

Suppose we are given a system of monoatomic molecules. Suppose that the molecules interact through a two-body potential $\phi$. In the framework of classical statistical physics, we consider for the given system the problem of solving the hierarchy of BBGKY kinetic equations \cite{Bogoluibov1946}:

$$\frac{\partial}{\partial t} f_n(t) = [H_n, f_n(t)] + \frac{1}{v} \int \sum_{1 \leq i \leq n} [\phi(q_i - q), f_{n+1}(t)] \, dx, \quad (1)$$

where $f_n$ is the probability density of the gas ensemble at time $t \in R_+$ at position $q_1 \in \Lambda$, $q_2 \in \Lambda, \ldots, q_n \in \Lambda$ with the velocities $v_1 \in R^3_+, \ldots; v_n \in R^3_+$ of particles. Therefore, $f : R_+ \times F \to R_+$ with the phase space $F = (\Lambda \times R^3)^n$.

Here, $H_n = \sum_{1 \leq i < j \leq n} \phi(q_i - q_j)$, $T_i = \frac{p_i}{2m}$, $m = 1$ is the mass of a molecule, $p$ the momentum of a molecule, $n \in N$, $N$ is the number of molecules, $V$ - the volume of the system; $N \to \infty, V \to \infty$, $v = \frac{V}{N} = \text{const}$ is volume per molecule, $[,]$ denotes the Poisson brackets.
Introducing the notation

\[(\mathcal{H}f)_n = [H_n, f_n]; \quad (\mathcal{D}_xf)_n(x_1, \cdots, x_n) = f_{n+1}, (x_1, \cdots, x_n, x);\]

\[(\mathcal{A}_xf)_n = \frac{1}{v} \sum_{1 \leq i \leq n} [\phi(q_i - q), f_n];\]

\[f(t) = \{f_1(t_1x_1), \cdots, f_n(t, x_1, \cdots, x_n), \cdots\}, n = 1, 2, \cdots\]

we can cast Eq.(1) in the form

\[\frac{\partial}{\partial t} f(t) = \mathcal{H}f(t) + \int \mathcal{A}_x \mathcal{D}_x f(t) dx. \quad (2)\]

**Derivation of Hierarchy of Kinetic Equations for correlation functions.**

**Proposition 1.** The hierarchy of kinetic equations for the correlation functions has the form

\[\frac{\partial}{\partial t} \varphi(t) = \mathcal{H} \varphi(t) + \frac{1}{2} \mathcal{W} \varphi(t) + \int \mathcal{A}_x \mathcal{D}_x \varphi(t) dx + \int \mathcal{A}_x \mathcal{D}_x \varphi(t) dx, \quad (3)\]


\[f(t) = \Gamma \varphi(t) = I + \varphi(t) + \frac{\varphi(t) \ast \varphi(t)}{2!} + \cdots \left( \ast \varphi(t) \right)^n + \cdots, \quad (4)\]

\[\varphi(t) = \{\varphi_1(t, x_1), \cdots, \varphi(t, x_1, \cdots, x_n), \cdots\};\]

\[(\varphi \ast \varphi)(x) = \sum_{Y \subset X} \varphi(Y) \varphi(X \setminus Y); \quad I \ast \varphi = \varphi; \quad (\ast \varphi)^n = \varphi \ast \varphi \ast \cdots \ast \varphi \text{ n times};\]

\[X = (x_1, \cdots, x_n) = (x_n); \quad Y = (x_n'), \quad n' \in; \quad n \cdot n' = 1, 2, \cdots;\]

\[(\mathcal{U} \varphi_n) = \left[ \sum_{1 \leq i < j \leq n} \phi(q_i - q_j), \varphi_n \right], \mathcal{W}(\varphi, \varphi) = \sum_{Y \subset X} \mathcal{U}(Y, X \setminus Y) \varphi(Y) \varphi(X \setminus Y).\]

**Proof:** To obtain (3), we substitute (4) in (2):

\[\frac{\partial}{\partial t} \Gamma \varphi(t) = \mathcal{H} \Gamma \varphi(t) + \int \mathcal{A}_x \mathcal{D}_x \Gamma \varphi(t) dx. \quad (5)\]
We have

\[ D_x \Gamma \varphi (t) = D_x \varphi (t) \times \Gamma \varphi (t), \quad (6) \]

\[ A_x \Gamma \varphi (t) = A_x \varphi (t) \times \Gamma \varphi (t), \quad (7) \]

\[ A_x D_x \Gamma \varphi (t) = A_x D_x \varphi (t) \times \Gamma \varphi (t) + A_x \varphi (t) \times D_x \varphi (t) \times \Gamma \varphi (t), \quad (8) \]

\[ T \Gamma \varphi (t) = T \varphi (t) \times \Gamma \varphi (t), \quad (9) \]

\[ U \Gamma \varphi (t) = U \varphi (t) \times \Gamma \varphi (t) + \frac{1}{2} W \left( \varphi (t), \varphi (t) \right) \times \Gamma \varphi (t), \quad (10) \]

\[ \frac{\partial}{\partial t} \Gamma \varphi (t) = \frac{\partial}{\partial t} \varphi (t) \times \Gamma \varphi (t). \quad (11) \]

substituting (6) – (11) in (5) and multiplying both sides by \( \Gamma (-\varphi (t)) \) we obtain (3). This proves the proposition.

To investigate our system on the basis of an argument similar to those in [A.I.Akhiezer (ed.), 1974], we can choose as expansion parameter \( v \), setting

\[ \phi (q_i - q_j) = v \theta (q_i - q_j) \quad (12) \]


\[ \varphi_n (t) = v^{n-1} \psi_n (t) \quad (13) \]

On the basis of (12), (13), Eq.(3) for \( n \) molecules takes the form

\[ \frac{\partial}{\partial t} \psi_n (t, X) = \left[ \sum_{1 \leq i \leq n} T_{i}, \psi_n (t, X) \right] + v \left( U \psi (t) \right)_n (X) \]

\[ + \frac{v}{2} (W \psi (t), \psi (t))_n (X) + v^2 \int (A_x D_x \psi (t))_n (X) dx \]

\[ + v \int (A_x \psi (t) \times D_x \psi (t))_n (X) dx \quad (14) \]

To solve Eq.(14), we apply perturbation theory. We shall seek a solution in the form of the series

\[ \psi_n (t, X) = \sum_{\mu} v^\mu \psi_n^\mu (t, X), n = 1, 2, 3, \ldots, \mu = 0, 1, 2, \ldots \quad (15) \]
Substituting the series (15) in Eq. (14) and equating the coefficients of equal powers of \( v \) we obtain

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_1 \right) \psi_1^\alpha(t) = 0, \tag{16}
\]

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_1 + \mathcal{L}_2 \right) \psi_2^\alpha(t) = S_2^\alpha, \tag{17}
\]

\[
\left( \frac{\partial}{\partial t} + \sum_{i=1}^\infty \mathcal{L}_i \right) \psi_n^\mu(t) = S_n^\mu, \tag{18}
\]

\[
\int (A_x \mathcal{D}_x \psi(t))_n(X)dx = 0,
\]

where we have introduced the notation

\[
\mathcal{L}_1 \psi_1^\alpha(t) = v_1 \frac{\partial}{\partial q_1} \psi_1^\alpha(t, x_1) - \int \frac{\partial \theta(q_1 - q)}{\partial q_1} \frac{\partial \psi_1^\alpha(t, x_1)}{\partial p_1} \psi_1^\alpha(t, x)dx,
\]

\[
\mathcal{L}_i \psi_n^\mu(t) = v_i \frac{\partial}{\partial q_1} \psi_n^\mu(t, X) - \int \left( A_x \psi^0(t) \right) (x_i) \left( \mathcal{D}_x \psi^\mu \right)_{n-1}(t, X|X, x_i) dx,
\]

and

\[
S_n^\mu = \left( \mathcal{U} \psi^{\mu-1} \right)_n(X) + \frac{1}{2} \sum_{\delta_1+\delta_2=\mu-1} \left( \mathcal{W} \psi^{\delta_1} \psi^{\delta_2} \right)(X) + v \int \left( A_x \mathcal{D}_x \psi^{\mu-1} \right)_n(X)dx + v \int \sum_{\delta_1+\delta_2=\mu} \left( A_x \psi^{\delta_1} \mathcal{D}_x \psi^{\delta_2} \right)_n(X)dx.
\]

Thus, the solution of Eq. (14) reduces to the solution of the homogeneous (16) and inhomogeneous (17), (18) Vlasov’s [A.A.Vlasov, 1950] equations for \( \psi_1^\alpha(t) \) and \( \psi_n^\mu(t) \), accordingly.

**Proposition 2.** The series (15), \( \psi_n(t, X) = \sum_{\mu} \mu^\mu \psi_n^\mu(t, X) \), where \( \psi_1^\alpha \) is defined in accordance with the solution of Vlasov’s equation and the remaining \( \psi_n^\mu \) on the basis of the formula

\[
\psi_n^\mu(t, X) = \int dx_1 \cdots dx_n \int_{-\infty}^t dt' S_n^\mu(t, x_1', \ldots, x_n') \bigcap_{1 \leq i \leq n} G\left(t-t_i, x_i, x_i'\right), \tag{20}
\]

is a solution of Eq. (14), if \( G \) satisfies equation:

\[
\left( \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial q_1} \right) G\left(t-t_0, x_i, x_i'\right) - \frac{\partial \psi(t, x_i)}{\partial V_i} \int \frac{\partial \theta(q_i - q)}{\partial q_i} G(t-t'; x, x_i') dx - \int \frac{\partial \theta(q_i - q)}{\partial q_i} G(t-t'; x, x_i') dx = 0. \tag{21}
\]
\[- \int \frac{\partial}{\partial q_i} G(t-t'; x_i', x_i) \frac{\partial}{\partial \nu_i} \psi(t,x) dx = 0\]

with the initial condition

\[G(0; x_i, x_i') = \delta(x_i - x_i'). \quad (22)\]

**Proof:** We consider Eqs. (16) and (17) where (16) is the Vlasov equation. This system of coupled equations for the single-molecule and two-molecule can serve for the determination of the successive approximations \(\psi_n^m(t)\), where \(\psi_1^0(t, X)\) is a solution of Vlasov’s equation.


\[\psi_2^o(t, x_1, x_2) = \int dx_1' \int dx_2' \int_{-\infty}^t dt' S_2^o(t'; x_1', x_2'). \quad (23)\]

\[G(t-t'; x_1, x_1') G(t-t'; x_2, x_2') \]

in (17), we see that (21) is a solution of (17) if

\[S_2^o(t, x_1, x_2) = \left[ \theta(q_1 - q_2), \psi_1^0(t; x_1) \psi_1^0(t, x_2) \right] + \int \sum_{1 \leq i \leq 2} \left[ \theta(q_i - q), \psi_1^0(t; x_i) \psi_1^0(t, x) \right] dx \]

and if \(G\) satisfies equation

\[\left( \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial q_1} \right) G(t-t'; x_1, x_1') - \frac{\partial \psi(t, x_1)}{\partial \nu_1} \int \frac{\partial \theta(q_1 - q)}{\partial q_1} G(t-t'; x, x_1') dx - \]

\[- \int \frac{\partial}{\partial q_1} G(t-t'; x_1, x_1') \frac{\partial}{\partial \nu_1} \psi(t,x) dx = 0, \quad (24)\]

with the initial condition

\[G(0; x_1, x_1') = \delta(x_1 - x_1'). \quad (25)\]

The recursive system of Eq.(18) can, with allowance for the established structure of the solutions, serve to determine the successive approximations \(\psi_n^m(t)\) and, therefore, formula (15). Indeed substituting again (20) directly in (18), we can see that (20) is a solution of (18) if \(S_n^m\) is defined in accordance with (19) and if \(G\) satisfies Eq.(21) with the initial condition (22).
Connection between Vlasov and Boltzmann equations.

It is known [J.H.Ferziger, H.G.Kaper, 1976] that on the assumption \( t >> t_{\text{coll}} \) where \( t_{\text{coll}} \) is the time of collision, the integral part of Vlasov’s equation can be represented as

\[
\frac{1}{v} \int \frac{\theta(q_1 - q)}{\partial q_1} \frac{\partial}{\partial p_1} \psi(t, x_1)\psi(t, x)dx = \frac{1}{v} \int \frac{\theta(q_1 - q)}{\partial q_1} \frac{\partial}{\partial p_1} S^{(2)}_\infty(x_1, x)\psi(t, x_1)\psi(t, x)dx.
\]

Here \( S^{(2)}_\infty(x_1, x) \) is \( \lim_{t \to \infty} S^{(2)}_t(x_1, x)S^{(1)}_t(x_1)S^{(1)}_t(x) \) and \( S^{(2)}_t, S^{(1)}_t, S^{(2)}_t \) are evolution operators in the form:

\[
S^{(s)}_t(x_1, ..., x_s) = \exp\left[t\left(\sum_{i=1}^{s} p_i \frac{\partial}{m \partial q_i} - \left( \sum_{1 \leq i < j \leq s} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_j} \right)\right)\right].
\]


\[
S^{(2)}_\infty(X_1, X)x_i = X^{(2)}_i(x_1, x) = (Q^{(2)}_i(x_1, x), P^{(2)}_i(x_1, x)),
\]

and distributing \( Q^{(2)}_1 \) and \( Q^{(2)}_2 \) in the points \( q_i \), restricting by terms which contain the gradients of \( \psi(t) \) not higher than first order, we receive

\[
\frac{1}{v} \int \frac{\theta(q_1 - q)}{\partial q_1} \frac{\partial}{\partial p_1} S^{(2)}_\infty(x_1, x)\psi(t, x_1)\psi(t, x)dx = \frac{1}{v}[J(\psi_1, \psi_1) + J_1(\psi_1, \psi_1)],
\]

where

\[
J(\psi_1, \psi_1) = \int \frac{\theta(q_1 - q)}{\partial q_1} \frac{\partial}{\partial p_1} \psi(t, q_1, P^{(2)}_1)\psi(t, q, P^{(2)}(t))dx,
\]

\[
J_1(\psi_1, \psi_1) = \int \frac{\theta(q_1 - q)}{\partial q_1} \frac{1}{2}(q_1 - q)\nabla q_1 \psi(t, q_1, P^{(2)}_1)\psi(t, q_1, P^{(2)}(t)) + \frac{1}{2}S^{(2)}_\infty(x_1, x)q_2[\psi(t, q_1, P^{(2)}_1)\nabla q_1 \psi(t, q_1, P^{(2)}(t)) - \psi(t, q_1, P^{(2)}_1)\nabla q_1 \psi(t, q_1, P^{(2)}_1)]dx.
\]

In [J.H.Ferziger, H.G.Kaper, 1976] it is shown that \( J(\psi_1, \psi_1) \) may be reduced to the collision part of the Boltzmann equation:

\[
J(\psi_1, \psi_1) = \int \int [\psi(p'_1)\psi(p') - \psi(p_1)\psi(p)]\frac{|p_1 - p|}{m}\partial p \partial d\partial d\partial d^3 p.
\]

In approximation of only main order (without terms which contain the gradients of \( \psi(t) \) not higher than first order) the Vlasov’s equation is reduced to Boltzmann’s equation

\[
\frac{\partial}{\partial t} \psi(t, x_1) = -p_1 \frac{\partial}{\partial q_1} \psi(t, x_1) + \frac{1}{v} \int \frac{\theta(q_1 - q)}{\partial q_1} \frac{\partial}{\partial p_1} \psi(t, x_1)\psi(t, x)dx = \frac{1}{v} \int \int [\psi(p'_1)\psi(p') - \psi(p_1)\psi(p)]\frac{|p_1 - p|}{m}\partial p \partial d\partial d\partial d^3 p.
\]
Using this relation we can reduce the solution of chain of kinetic equations BBGKY to a solution of chain kinetic equations for correlation functions for homogeneous and inhomogeneous Boltzmann equations:

\[ \frac{\partial}{\partial t} + \mathcal{B}_1 \psi_n^0(t) = 0, \]  

(26)

\[ \frac{\partial}{\partial t} + \mathcal{B}_1 + \mathcal{B}_2) \psi_n^0(t) = S_2^0, \]  

(27)

\[ \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mathcal{B}_i \psi_n^\mu(t) = S_n^\mu, \]  

(28)

where we have introduced the notation

\[ \mathcal{B}_1 \psi_n^0(t) = v_1 \frac{\partial}{\partial q_1} \psi_n^0(t, x_1) - \int [\psi_1^0(t, p_1') \psi_1^0(t, p') - \psi_1^0(t, p_1) \psi_1^0(t, p)] \frac{|p_1 - p|}{m} b_i d_p, \]

\[ \mathcal{B}_i \psi_n^\mu(t) = v_i \frac{\partial}{\partial q_i} \psi_n^\mu(t, X) - v \int [\psi(t, p_i'))(D_x \psi_\mu(t) - \psi(t, p_i))(D_x \psi_\mu(t) - \psi(t, p_i)) - \psi(t, p_i))(D_x \psi_\mu(t) - \psi(t, p_i)) \frac{|p_i - p|}{m} b_i d_p, \]

and

\[ S_n^\mu = (U \psi_\mu^{-1}(t))_n(X) + \frac{1}{2} \sum_{\delta_1 + \delta_2 = \mu - 1} (W(\psi_{\delta_1}(t), \psi_{\delta_2}(t))(X) + v \int (A_x D_x \psi_\mu^{-1}(t))_n(X) dx. \]

\[ + v \int \sum_{\delta_1 + \delta_2 = \mu} (A_x \psi_{\delta_1}(t) D_x \psi_{\delta_2}(t))_n(X) dx. \]

Here \( P = (p_1, p_2, ..., p_n) \), and \( \psi_n^\mu(t) \) are the members of series

\[ \psi_n(t, X) = \sum_{\mu} \psi_n^\mu(t, X). \]

So, replacing Eq.(21) by the corresponding "Boltzmann" equation and the members of series (15) defined by the solution of Vlasov equations (16)-(18) and by the solution of Boltzmann equations, (26)-(28) we can express the solution of BBGKY’s chain to the solution of Boltzmann equations.

**Conclusion**

The present paper reports on the method of definition of the solution of BBGKY’s chain of kinetic equations, describing the evolution of a system of many interacting particles. In the approximation \( t > t_{\text{collision}} \) and considering main order (without terms which contain the gradients of \( \psi(t) \) not higher than first order), it is shown, that the solution of chain can be defined through the solution of Boltzmann equation.
Acknowledgments

The author would like to thank the INTAS (grant 20-015, field 2). She would also like to thank the Abdus Salam International Centre for Theoretical Physics (Trieste, Italy), Professor V. Kravtsov and Professor K.R. Sreenivasan for hospitality.

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