HERMITIAN-EINSTEIN METRICS ON HOLOMORPHIC VECTOR BUNDLES
OVER HERMITIAN MANIFOLDS

Xi Zhang
Department of Mathematics, Zhejiang University,
Hangzhou 310027, Zhejiang, People's Republic of China
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this paper, we prove the long-time existence of the Hermitian-Einstein flow on a holomorphic vector bundle over a compact Hermitian (non-kähler) manifold, and solve the Dirichlet problem for the Hermitian-Einstein equations. We also prove the existence of Hermitian-Einstein metrics for holomorphic vector bundles on a class of complete noncompact Hermitian manifolds.

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1mzhangxi@ictp.trieste.it; xizhang@zju.edu.cn
1 Introduction

Let \((M, g)\) be a Hermitian manifold with Hermitian metric \(g\), and \(E\) be a rank \(r\) holomorphic vector bundle over \(M\). Given any Hermitian metric \(H\) on the holomorphic vector bundle \(E\) there exists one and only one complex metric connection \(A_H\). If the curvature form \(F_H\) of \(A_H\) satisfies

\[
\sqrt{-1} \Lambda F_H = \lambda Id
\]

where \(\lambda\) is a real number, then \(H\) will be called a Hermitian-Einstein metric. After the pioneering work of Kobayashi ([8], [9]), the relation between the existence of Hermitian-Einstein metrics and stable holomorphic vector bundles over closed Kähler manifolds is by now well understood due to the work of Narasimhan-Seshadri, Donaldson, Siu, Uhlenbeck-Yau and others ([16], [3], [19], [21], [22]). Later, in [4] the Dirichlet boundary value problem was solved for Hermitian-Einstein metrics over compact Kähler manifolds with non-empty boundary. In this paper, we study the existence of Hermitian-Einstein metrics for holomorphic vector bundles over Hermitian (non-Kähler) manifolds. We should point out that if \((M, g)\) is non-Kähler then the basic Kähler identities

\[
\bar{\partial} A^* = -\sqrt{-1} \Lambda \partial A; \partial A^* = \sqrt{-1} \Lambda \bar{\partial} A.
\]

(1.2)

do not hold. So the non-Kähler case is analytically more difficult than the Kähler case.

We first investigate the associated parabolic system i.e. Hermitian-Einstein flow over compact Hermitian manifolds, and we prove the long-time existence of the Hermitian-Einstein flow. In general, the Hermitian-Einstein flow does not converge to a Hermitian-Einstein metric when \(M\) is a closed Hermitian manifold without boundary (In this case, the stability of holomorphic vector bundle may ensure the convergence of the Hermitian-Einstein flow under some conditions ([2], [12], [13], [19]).) However we prove the solvability of the Dirichlet problem for Hermitian-Einstein metric over compact Hermitian manifolds with smooth boundary.

Main Theorem Let \(E\) be a holomorphic vector bundle over the compact Hermitian manifold \(\bar{M}\) with non-empty smooth boundary \(\partial M\). For any Hermitian metric \(\varphi\) on the restriction of \(E\) to \(\partial M\) there is a unique Hermitian-Einstein metric \(H\) on \(E\) such that \(H = \varphi\) over \(\partial M\).

In the second part of this paper, we study the Hermitian-Einstein equation on holomorphic vector bundles over complete Hermitian manifolds, here complete means complete, noncompact and without boundary. In section 5, we prove the long-time existence of the Hermitian-Einstein flow on any complete Hermitian manifold under the assumption that the initial metric has bounded mean curvature. It is reasonable that the long-time solution will converge to a Hermitian-Einstein metric under some assumptions on manifold and initial metric. But, in section 6, we adapt the direct elliptic method, using theorem 4.1 and compact exhaustion to prove the existence of Hermitian-Einstein metric on some complete Hermitian manifolds.
2 Preliminary Results

Let \((M, g)\) be a compact Hermitian manifold, and \(E\) be a rank \(r\) holomorphic vector bundle over \(M\). Denote by \(\omega\) the Kähler form, and define the operator \(\Lambda\) as the contraction with \(\omega\), i.e. for \(\alpha \in \Omega^{1,1}(M, E)\), then
\[
\Lambda \alpha = \langle \alpha, \omega \rangle.
\] (2.1)

A connection \(A\) on the vector bundle \(E\) is called Hermitian-Einstein if it is integrable and the corresponding curvature form \(F_A\) satisfies the following Einstein condition:
\[
\sqrt{-1} \Lambda F_A = \lambda Id,
\]
where \(\lambda\) is some real constant. When \((M, g)\) is a Kähler manifold, we know that the connection \(A\) must be Yang-Mill connection. So in this case, \(A\) is also called Hermitian-Yang-Mills.

Let \(H\) be a Hermitian metric on holomorphic vector bundle \(E\), and denote the holomorphic structure by \(\bar{\partial}E\), then there exists a canonical metric connection which is denoted by \(A_H\). Taking a local holomorphic basis \(e_\alpha (1 \leq \alpha \leq r)\), the Hermitian metric \(H\) is a positive Hermitian matrix \((H_{\alpha\bar{\beta}})_{1 \leq \alpha, \bar{\beta} \leq r}\) which will also be denoted by \(H\) for simplicity, here \(H_{\alpha\bar{\beta}} = H(e_\alpha, e_\bar{\beta})\). In fact, the complex metric connection can be written as follows
\[
A_H = H^{-1} \bar{\partial}H,
\] (2.2)
and the curvature form
\[
F_H = \bar{\partial}A_H = \bar{\partial}(H^{-1} \bar{\partial}H).\] (2.3)

In the literature sometimes the connection is written as \((\bar{\partial}H)H^{-1}\) because of the reversal of the roles of the row and column indices.

**Definition 2.1** If for a Hermitian metric \(H\) on \(E\), the corresponding canonical metric connection \(A_H\) is Hermitian-Einstein, then the metric \(H\) is called a Hermitian-Einstein metric.

It is well known that any two Hermitian metrics \(H\) and \(K\) on bundle \(E\) are related by \(H = Kh\), where \(h = K^{-1}H \in \Omega^0(M, \text{End}(E))\) is positive and self adjoint with respect to \(K\). It is easy to check that
\[
A_H - A_K = h^{-1} \bar{\partial}_K h,
\] (2.4)
\[
F_H - F_K = \bar{\partial}(h^{-1} \bar{\partial}_K h),\] (2.5)

Let \(H_0\) be a Hermitian metric on \(E\). Consider a family of Hermitian metrics \(H(t)\) on \(E\) with initial metric \(H(0) = H_0\). Denote by \(A_{H(t)}\) and \(F_{H(t)}\) the corresponding connections and curvature forms, denote \(h(t) = H_0^{-1}H(t)\). When there is no confusion, we will omit the
parameter \( t \) and simply write \( H, A_H, F_H, H \) for \( H(t), A_{H(t)}, F_{H(t)}, h(t) \) respectively. The Hermit-Einstein evolution equation is

\[
H^{-1} \frac{\partial H}{\partial t} = -2(\sqrt{-1} \Lambda F_H - \lambda d).
\]

(2.6)

We also call it the Hermitian-Einstein flow. Choosing local complex coordinates \( \{ z^i \}_{i=1}^m \) on \( M \), as in [10], we define the holomorphic Laplace operator for functions

\[
\tilde{\Delta}f = -2\sqrt{-1} \bar{\partial} \partial f = 2g^{i\bar{j}} \partial^2 f/\partial z^i \partial \bar{z}^j,
\]

(2.7)

where \((g^{i\bar{j}})\) is the inverse matrix of the metric matrix \((g_{i\bar{j}})\). As usual, we denote the Beltrami-Laplacian operator by \( \Delta \). The difference of the two Laplacians is given by a first order differential operator as follows

\[
(\tilde{\Delta} - \Delta)f = \langle V, \nabla f \rangle_g,
\]

(2.7')

where \( V \) is a well-defined vector field on \( M \). The holomorphic Laplace operator \( \tilde{\Delta} \) coincides with the usual Laplace operator if and only if the Hermitian manifold \((M, g)\) is Kähler. By taking a local holomorphic basis \( e_{\alpha} \) \((1 \leq \alpha \leq r)\) on bundle \( E \) and local, complex coordinates \( \{ z^i \}_{i=1}^m \) on \( M \), then the Hermitian-Einstein flow equation (2.6) can be written as following:

\[
\frac{\partial H}{\partial t} = -2\sqrt{-1} \bar{\partial} \partial H + 2\sqrt{-1} \Lambda \bar{\partial} H H^{-1} \partial H + 2\lambda H = \tilde{\Delta} H + 2\sqrt{-1} \bar{\partial} H H^{-1} \partial H + 2\lambda H.
\]

(2.6')

Where \( H \) denote the Hermitian matrix \((H_{\alpha\beta})_{1 \leq \alpha, \beta \leq r}\). From the above formula, we see that the Hermit-Einstein evolution equation is a nonlinear strictly parabolic equation.

**Proposition 2.2** Let \( H(t) \) be a solution of Hermitian-Einstein flow (2.6), then

\[
(\frac{\partial}{\partial t} - \tilde{\Delta})|\sqrt{-1} \Lambda F_H - \lambda Id|_H^2 \leq 0.
\]

(2.8)

**Proof.** For simplicity, we denote \( \sqrt{-1} \Lambda F_H - \lambda Id = \theta \). By calculating directly, we have

\[
\tilde{\Delta}\theta|_H^2 = -2\sqrt{-1} \bar{\partial} \partial \{ \text{tr} \theta H^{-1} \bar{\theta} H \}
\]

\[
= -2\sqrt{-1} \Lambda \text{tr} \{ \bar{\partial} \partial \theta H^{-1} \bar{\theta} H \} + \bar{\partial} \partial \theta H^{-1} \bar{\theta} H + 2\lambda \theta H^{-1} \bar{\theta} H
\]

\[
+ 2\text{Re}(-2\sqrt{-1} \bar{\partial} \partial \theta H + 2|\partial H \theta|_H^2 + 2|\bar{\partial} \theta|_H^2)
\]

and

\[
\frac{\partial}{\partial t}(\Lambda F_H) = \frac{\partial}{\partial t}(\Lambda \bar{\partial}(h^{-1} \partial_0 h)) = \Lambda \bar{\partial} \{ \frac{\partial}{\partial t}(h^{-1} \partial h + h^{-1} H_0^{-1} \partial H_0 h) \}
\]

\[
= \Lambda \bar{\partial} \{ h^{-1} \partial_0 h - h^{-1} \frac{\partial h}{\partial t} H^{-1} \partial H + H^{-1} \partial H h^{-1} \frac{\partial h}{\partial t} \}
\]

\[
= \Lambda \bar{\partial}(\partial_H(\sqrt{-1} \Lambda F_H - \lambda Id))
\]

(2.10)
where \( h = H_0^{-1}H \) and \( D_H = \partial_H + \overline{\partial} \). Using above formulas, we have

\[
(\bar{\Delta} - \frac{\partial}{\partial t})(\sqrt{-1}\Lambda F_H - \lambda Id_H) = 2|\partial_H \theta_H|^2_H + 2|\overline{\partial} \theta_H|^2_H \geq 0. \tag{2.11}
\]

For further discussion, we will introduce the Donaldson’s ”distance” on the space of Hermitian metrics as follows.

**Definition 2.3** For any two Hermitian metrics \( H, K \) on bundle \( E \) set

\[
\sigma(H, K) = trH^{-1}K + trK^{-1}H - 2\text{rank}E. \tag{2.12}
\]

It is obvious that \( \sigma(H, K) \geq 0 \), with equality if and only if \( H = K \). The function \( \sigma \) is not quite a metric but it serves almost equally well in our problem, moreover the function \( \sigma \) compares uniformly with \( d(\cdot, \cdot) \), where \( d \) is the Riemannian distance function on the metric space, in that \( f_1(d) \leq \sigma \leq f_2(d) \) for monotone functions \( f_1, f_2 \). In particular, a sequence of metrics \( H_i \) converges to \( H \) in the usual \( C^0 \) topology if and only if \( \text{Sup}_M \sigma(H_i, H) \rightarrow 0 \).

Let \( h = K^{-1}H \), and apply \( -\sqrt{-1}\Lambda \) to (2.5) and taking the trace in the bundle \( E \), we have

\[
tr(\sqrt{-1}h(\Lambda F_H - \Lambda F_K)) = \frac{1}{2} \Delta tr h + tr(-\sqrt{-1}\Lambda \overline{\partial} hh^{-1}\overline{\partial}_K h). \tag{2.13}
\]

Similarly, we have

\[
tr(\sqrt{-1}h^{-1}(\Lambda F_K - \Lambda F_H)) = \frac{1}{2} \Delta tr h^{-1} + tr(-\sqrt{-1}\Lambda \overline{\partial} hh^{-1}\overline{\partial}_H h^{-1}). \tag{2.14}
\]

Since \( h \) is a positive Hermitian endomorphism, by choosing local normal coordinates of \( M \) at the point under consideration and a local trivialization of bundle \( E \), it is easy to check ([3], [19]) that \( tr(-\sqrt{-1}\Lambda \overline{\partial} hh^{-1}\overline{\partial}_K h) \) is non-negative, so we have the following proposition.

**Proposition 2.4** Let \( H \) and \( K \) be two Hermitian-Einstein metrics, then \( \sigma(H, K) \) is sub-harmonic with respect to the holomorphic Laplace operator, i.e. :

\[
\bar{\Delta} \sigma(H, K) \geq 0. \tag{2.15}
\]

Let \( H(t), K(t) \) be two solutions of the Hermitian-Einstein flow (2.6), and denote \( h(t) = K(t)^{-1}H(t) \). Using formula (2.13) and (2.14) again, we have

\[
(\bar{\Delta} - \frac{\partial}{\partial t})(tr h(t) + tr h^{-1}(t)) \geq 0.
\]

\[
2tr(-\sqrt{-1}\Lambda \overline{\partial} E h h^{-1}\overline{\partial}_K h) + 2tr(-\sqrt{-1}\Lambda \overline{\partial} E h^{-1} h \overline{\partial}_H h^{-1}) \geq 0.
\]
So we have proved the following proposition.

**Proposition 2.5** Let $H(t)$, $K(t)$ be two solutions of the Hermitian-Einstein flow (2.6), then
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\sigma(H(t), K(t)) \geq 0.
\] (2.16)

**Proposition 2.6** Let $H(x, t)$ be a solution of the Hermitian-Einstein flow (2.6) with the initial metric $H_0$, then
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\lg\{tr(H_0^{-1}H) + tr(H^{-1}H_0)\} \geq -2|\sqrt{-1}\Lambda F_{H_0} - \lambda Id|_{H_0}.
\] (2.17)

**Proof.** Let $h = H_0^{-1}H$, direct calculation shows that
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\trh = 2tr(\sqrt{-1}h\Lambda F_{H_0} - \lambda h) + 2tr(-\sqrt{-1}\Lambda \bar{\partial}hh^{-1}\bar{\partial}_0h).
\] (2.18)
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\trh^{-1} = -2tr(\sqrt{-1}h^{-1}\Lambda F_{H_0} - \lambda h^{-1}) + 2tr(-\sqrt{-1}\Lambda \bar{\partial}_E h^{-1}h\partial_H h^{-1}).
\] (2.19)

It is easy to check that ([19])
\[
2(trh)^{-1}tr(-\sqrt{-1}\Lambda \bar{\partial}hh^{-1}\bar{\partial}_0h) - (trh)^{-2}|dtrh|^2 \geq 0,
\]
\[
2(trh^{-1})^{-1}tr(-\sqrt{-1}\Lambda \bar{\partial}_E h^{-1}h\partial_H h^{-1}) - (trh^{-1})^{-2}|dtrh^{-1}|^2 \geq 0.
\] (2.20)

From above two inequalities, it is easy to check
\[
(trh + trh^{-1})^{-1}\{ -2\sqrt{-1}\Lambda \bar{\partial}hh^{-1}\bar{\partial}_0h - 2\sqrt{-1}\Lambda \bar{\partial}_E h^{-1}h\partial_H h^{-1} \} \geq (trh + trh^{-1})^{-2}|dtrh + dtrh^{-1}|^2.
\] (2.21)

Then, we have
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\lg\{trh + trh^{-1}\}
= (trh + trh^{-1})^{-1}(\tilde{\Delta} - \frac{\partial}{\partial t})(trh + trh^{-1}) - (trh + trh^{-1})^{-2}|dtrh + dtrh^{-1}|^2
= 2(trh + trh^{-1})^{-1}tr(\sqrt{-1}h\Lambda F_{H_0} - \lambda h)
-2(trh + trh^{-1})^{-1}tr(\sqrt{-1}h^{-1}\Lambda F_{H_0} - \lambda h^{-1})
+2(trh + trh^{-1})^{-1}\{ -2\sqrt{-1}\Lambda \bar{\partial}hh^{-1}\bar{\partial}_0h - 2\sqrt{-1}\Lambda \bar{\partial}_E h^{-1}h\partial_H h^{-1} \}
- (trh + trh^{-1})^{-2}|dtrh + dtrh^{-1}|^2
\geq -2|\sqrt{-1}\Lambda F_{H_0} - \lambda Id|_{H_0}.
\]

\[\square\]

Discussing like that in the above proposition, we have

**Proposition 2.7** Let $H(x)$ and $H_0(x)$ are two Hermitian metric, then
Corollary 2.8  Let $H$ be an Hermitian-Einstein metric, and $H_0$ be the initial Hermitian metric, then
\[
\bar{\Delta} \log \{ tr(H_0^{-1}H + trH^{-1}H_0) \} \geq -2|\sqrt{-1} \Lambda F_{H_0} - \lambda \text{Id}|_{H_0} \nonumber -2|\sqrt{-1} \Lambda F_H - \lambda \text{Id}|_H.
\] (2.22)

3  The Hermitian-Einstein flow on compact Hermitian manifolds

Let $(M, g)$ be a compact Hermitian manifold (with possibly non-empty boundary), and $E$ be a holomorphic vector bundle over $M$. Let $H_0$ be the initial Hermitian metric on $E$. If $M$ is closed then we consider the following evolution equation
\[
\begin{align*}
\frac{\partial H}{\partial t} & = -2(\sqrt{-1} \Lambda F_H - \lambda \text{Id}), \\
H(t)|_{t=0} & = H_0.
\end{align*}
\] (3.1)

If $M$ is a compact manifold with non-empty smooth boundary $\partial M$, and the Hermitian metric $g$ is smooth and non-degenerate on the boundary, for given data $\varphi$ on $\partial M$ we consider the following boundary value problem.
\[
\begin{align*}
\frac{\partial H}{\partial t} & = -2\sqrt{-1}(\Lambda F_H - \lambda \text{Id}), \\
H(t)|_{t=0} & = H_0, \\
H|_{\partial M} & = \varphi.
\end{align*}
\] (3.2)

Here $H_0$ satisfying the boundary condition. From formula (2.6'), we known that the above equations are nonlinear strictly parabolic equations, so standard parabolic theory gives short-time existence:

**Proposition 3.1** For sufficiently small $\epsilon > 0$, the equation (3.1), and (3.2) have a smooth solution defined for $0 \leq t < \epsilon$.

Next we want to prove the long-time existence of the evolution equation (3.1) and (3.2). Let $h = H_0^{-1}H$. By direct calculation, we have
\[
|\frac{\partial}{\partial t}(\log \text{tr} h)| \leq 2|\sqrt{-1} \Lambda F_H - \lambda \text{Id}|_H,
\] (3.3)
and similarly
\[
|\frac{\partial}{\partial t}(\log \text{tr} h^{-1})| \leq 2|\sqrt{-1} \Lambda F_H - \lambda \text{Id}|_H.
\] (3.4)
Theorem 3.2 Suppose that a smooth solution $H_t$ to the evolution equation (3.1) is defined for $0 \leq t < T$. Then $H_t$ converge in $C^0$-topology to some continuous non-degenerate metric $H_T$ as $t \to T$.

Proof: Given $\epsilon > 0$, by continuity at $t = 0$ we can find a $\delta$ such that

$$\sup_M \sigma(H_t, H_{t'}) < \epsilon,$$

for $0 < t, t' < \delta$. Then Proposition 2.5 and the Maximum principle imply that

$$\sup_M \sigma(H_t, H_{t'}) < \epsilon,$$

for all $t, t' > T - \delta$. This implies that $H_t$ are uniformly Cauchy sequence and converge to a continuous limiting metric $H_T$. On the other hand, by proposition 2.2, we known that $|\sqrt{-1}\Lambda F_H - \lambda Id|_H$ are bounded uniformly. Using formula (3.2) and (3.3), one can conclude that $\sigma(H, H_0)$ are bounded uniformly, therefore $H(T)$ is a non-degenerate metric.

We prove the following lemma in the same way as [3; Lemma 19] and [18; Lemma 6.4].

Lemma 3.3 Let $M$ be a compact Hermitian manifold without boundary (with non-empty boundary). Let $H(t), 0 \leq t < T,$ be any one-parameter family of Hermitian metrics on a holomorphic bundle $E$ over $M$ (satisfying Dirichlet boundary condition), and suppose $H_0$ is the initial Hermitian metric. If $H(t)$ converges in the $C^0$ topology to some continuous metric $H_T$ as $t \to T$, and if $\sup_M |\Delta F_H|_{H_0}$ is bounded uniformly in $t$, then $H(t)$ are bounded in $C^1$ and also bounded in $L^p_2$ (for any $1 < p < \infty$) uniformly in $t$.

Proof: Let $h(t) = H_0^{-1}H(t)$. We contend that $h(t)$ are bounded uniformly in $C^1$ topology, and also $H(t)$ are bounded uniformly in $C^1$. If not then for some subsequence $t_j$ there are points $x_j \in M$ with $\sup |\nabla_0 h_j| = l_j$ achieved at $x_j$, and $l_j \to \infty$, here $h_j = h(t_j)$.

(a), First we consider the case that $M$ is a closed manifold. Taking a subsequence we can suppose that the $x_j$ converge to a point $x$ in $M$. Then we choose local coordinates $\{z_\alpha\}_{\alpha=1}^m$ around $x_j$ and rescaled by a factor of $l_j^{-1}$ to a ball of radius $1 \{ |w| < 1 \}$, and pull back the matrixes $h_j$ to matrix $\tilde{h}_j$ via the maps $w_\alpha = l_j z_\alpha$. With respect to the rescaled metrics

$$\sup_{|w| < 1} |\nabla_{\tilde{h}_j}| = 1,$$

is attained at the origin point. By the conditions of the lemma, we known

$$|\Delta \tilde{F}_j - \Delta \tilde{F}_0| = |\tilde{h}_j^{-1} (\Lambda \tilde{h} \partial \partial_{\tilde{h}_j} h_j - \Lambda \tilde{h} \partial \partial_{\tilde{h}_j} h_j^{-1} \partial_{\tilde{h}_j} h_j)|$$

(3.5)
is bounded in \( \{ w \in C^m ||w| < 1 \} \). Since \( \hat{h}_j, \nabla \hat{h}_j \) are bounded, \( |\Lambda \partial \partial_0 \hat{h}_j| \) are bounded independent of \( j \), then \( |\Delta \hat{h}_j| \) is also bounded independent of \( j \). By the properties of the elliptic operator \( \Delta \) on \( L^p \) spaces, \( \hat{h}_j \) are uniformly bounded in \( L^p \) on a small ball. Taking \( p > 2m \), so that \( L^p \rightarrow C^1 \) is compact, thus some subsequence of the \( \hat{h}_j \) converge strongly in \( C^1 \) to \( \hat{h}_\infty \). But on the other hand the the variation of \( \hat{h}_\infty \) is zero, since the original metrics approached a \( C^0 \) limit, which

contradicts

\[
|\nabla \hat{h}_\infty|_{z=0} = \lim_{j \to \infty} |\nabla \hat{h}_j|_{z=0} = 1.
\]

(b), When \( M \) is a compact manifold with non-empty boundary \( \partial M \). Let \( d_j \) denote the distance from \( x_j \) to the boundary \( \partial M \), then there are two cases.

(1), If \( \lim \sup d_j l_j > 0 \), then we can choose balls of radius \( \leq d_j \) around \( x_j \) and rescaled by a factor of \( \frac{l_j}{\epsilon} \) to a ball of radius 1 (where \( \epsilon \leq \lim \sup d_j l_j \), pull back the matrixes \( h_j \) to matrixes \( \hat{h}_j \) defined on \( \{ w \in C^m ||w| < 1 \} \). With respect to the rescaled metrics, we have

\[
\sup |\nabla \hat{h}_j| = \epsilon,
\]

is attained at the origin. By condition of the lemma, and discussing like that in (a), we will deduce contradiction.

(2), On the other hand, if \( \lim \sup d_j l_j = 0 \), we may assume \( x_j \) approach a point \( y \) on the boundary, and let \( \bar{x}_j \in \partial M \) such that \( \text{dist}(\bar{x}_j, x_j) = d_j \), also \( \bar{x}_j \) approach \( y \). Choose half-ball of radius \( \frac{1}{2} \) around \( \bar{x}_j \) and rescale by a factor of \( l_j \) to the unit half-ball. In the rescaled picture the points \( x_j \) approach \( z = 0 \). After rescaling, \( |\Lambda \partial \partial_0 \hat{h}_j| \) is still bounded, \( \hat{h}_j \) is uniformly bounded, and \( \sup |\nabla \hat{h}_j| = 1 \) is attained at point \( x_j \). Since \( \hat{h}_j \) satisfy boundary condition along the face of the half-ball, using elliptic estimates with boundary, and discussing like that in (a), we can also deduce contradiction.

From the above discuss, we known that \( h_t \) are uniformly bounded in \( C^1 \), also \( H(t) \) are uniformly bounded in \( C^1 \) topology. Using formula (2.5) together with the bounds on \( h(t) \), \( |\Lambda F_H| \), and \( \nabla_0 h \) show that \( \Lambda \partial \partial_0 h \) are uniformly bounded. Elliptic estimates with boundary conditions show that \( h(t) \) (also \( H(t) \)) are uniformly bounded in \( L^p \).

\( \square \)

**Theorem 3.4** The evolution equation (3.1) and (3.2) have a unique solution \( H(t) \) which exists for \( 0 \leq t < \infty \).

**Proof.** Proposition 3.1 guarantees that a solution exists for a short time. Suppose that the solution \( H(t) \) exists for \( 0 \leq t < T \). By theorem 3.2, \( H(t) \) converges in \( C^0 \) to a non-degenerate continuous limit metric \( H(T) \) as \( t \to t \). From proposition 2.4 and the maximum principle, we conclude that \( |\sqrt{-1}AF_H - \lambda Id|_H \) are bounded independently of \( t \). Moreover, \( |AF_H|^2_{H_0} \) are bounded independently of \( t \). Hence by lemma 3.3, \( H(t) \) are bounded in \( C^1 \) and also bounded in \( L^p \) (for any \( 1 < p < \infty \)) uniformly in \( t \). Since the evolution equations (3.1) and (3.2) is quadratic
in the first derivative of $H$ we can apply Hamilton’s method [7] to deduce that $H(t) \to H(T)$ in $C^\infty$, and the solution can be continued past $T$. Then the evolution equation (3.1) and (3.2) have a solution $H(t)$ define for all time.

By proposition 2.5 and the maximum principle, it is easy to conclude the uniqueness of the solution.

\[ \square \]

**Remark:** It should be mentioned that the theorem of Li and Yau [13] give the existence of a $\lambda$-Hermitian-Einstein metric in a stable bundle over a closed Gauduchon manifold, where the real constant $\lambda$ depending on the slope of the bundle with respect to the Gauduchon metric; Buchdahl [1] prove the same result for arbitrary surfaces independently; the book which written by Lübke and Teleman [12] is a good reference for this field. When $M$ is a closed Hermitian manifold, the solution of equation (3.1) usually will not convergence to a Hermitian-Einstein metric. However, in the next section, we will show that the solution of equation (3.2) always convergence to a Hermitian-Einstein metric which satisfies the boundary condition.

## 4 The Dirichlet boundary problem for Hermitian-Einstein metric

In this section we will consider the case when $M$ is the interior of a compact Hermitian manifold $\overline{M}$ with non-empty boundary $\partial M$, and the Hermitian metric is smooth and non-degenerate on the boundary, holomorphic vector bundle $E$ is defined over $\overline{M}$. We will discuss the Dirichlet boundary problem for Hermitian-Einstein metric by using the heat equation method to deform an arbitrary initial metric to the desired solution. The main points in the discussion are similar with that in [4] or [18]. For given data $\varphi$ on $\partial M$ we consider the evolution equation (3.2). By theorem 3.4, we known there exists a unique solution $H(t)$ of the equation (3.2). The aim of this section is to prove that $H(t)$ will convergence to the Hermitian-Einstein metric which we want.

By direct calculation, one can check that

$$ |\nabla_H \theta|^2_H \geq |\nabla \theta|^2_H $$

for any section $\theta$ in $End(E)$. Then, using formula (2.11), we have

$$ (\tilde{\Delta} - \frac{\partial}{\partial t})|\sqrt{-1} \Lambda F_H - \lambda Id|_H \geq 0. \quad \text{(4.1)} $$

We first solve the following Dirichlet problem on $M$ ([20; Ch5, proposition 1.8 ]):

\[
\begin{align*}
\Delta v &= -|\sqrt{-1} \Lambda F_{H_0} - \lambda Id|_{H_0}, \\
v|_{\partial M} &= 0.
\end{align*}
\]  

(4.2)
Setting \( w(x, t) = \int_0^t \sqrt{-1} \Lambda F_H - \lambda Id_H(x, s)ds - v(x) \). From (4.1) (4.2), and the boundary condition satisfied by \( H \) implies that, for \( t > 0, |\sqrt{-1} \Lambda F_H - \lambda Id_H(x, t) | \) vanishes on the boundary of \( M \), it is easy to check that \( w(x, t) \) satisfies

\[
\begin{align*}
(\bar{\Delta} - \frac{\partial}{\partial t})w(x, t) &\geq 0, \\
w(x, 0) &\equiv -v(x), \\
w(x, t)|_{\partial M} &\equiv 0.
\end{align*}
\] (4.3)

By the maximum principle, we have

\[
\int_0^t |\sqrt{-1} \Lambda F_H - \lambda Id_H(x, s)ds| \leq \sup_{y \in M} v(y),
\] (4.4)

for any \( x \in M \), and \( 0 < t < \infty \).

Let \( t_1 \leq t \leq t_2 \), and let \( \tilde{h}(x, t) = H^{-1}(x, t_1)H(x, t) \). It is easy to check that

\[
\tilde{h}^{-1} \frac{\partial \tilde{h}}{\partial t} = -2(\sqrt{-1} \Lambda F_H - \lambda Id).
\] (4.5)

Then we have

\[
\frac{\partial}{\partial t} \log tr(\tilde{h}) \leq 2|\sqrt{-1} \Lambda F_H - \lambda Id|_H.
\]

From the above formula, we have

\[
tr(H^{-1}(x, t_1)H(x, t)) \leq r \exp(2 \int_{t_1}^t |\sqrt{-1} \Lambda F_H - \lambda Id|_H ds).
\] (4.6)

We have a similar estimate for \( tr(H^{-1}(x, t)H(x, t_1)) \). Combining them we have

\[
\sigma(H(x, t), H(x, t_1)) \leq 2r(\exp(2 \int_{t_1}^t |\sqrt{-1} \Lambda F_H - \lambda Id|_H ds) - 1).
\] (4.7)

From (4.4), (4.7), we know that \( H(t) \) converge in the \( C^0 \) topology to some continuous metric \( H_\infty \) as \( t \to \infty \). Using lemma 3.3 again, we known that \( H(t) \) are bounded in \( C^1 \) and also bounded in \( L^p_2 \) (for any \( 1 < p < \infty \)) uniformly in \( t \). On the other hand, \( |H^{-1} \frac{\partial H}{\partial t}| \) is bounded uniformly. Then, the standard elliptic regularity implies that there exists a subsequence \( H_t \to H_\infty \) in \( C_\infty \) topology. From formula (4.4), we know that \( H_\infty \) is the desired Hermitian-Einstein metric satisfying the boundary condition. From proposition 2.4 and the maximum principle, it is easy to conclude the uniqueness of solution. So we have proved the following theorem.

**Theorem 4.1** Let \( E \) be a holomorphic vector bundle over the compact Hermitian manifold \( M \) with non-empty boundary \( \partial M \). For any Hermitian metric \( \varphi \) on the restriction of \( E \) to \( \partial M \) there is a unique Hermitian-Einstein metric \( H \) on \( E \) such that \( H = \varphi \) over \( \partial M \)
5 Hermitian-Einstein flow over complete Hermitian manifolds

Let $M$ be a complete, noncompact Hermitian manifold without boundary, in this case, we will simply say $M$ is a complete Hermitian manifold. Let $E$ be a holomorphic vector bundle of rank $r$ over $M$ with a Hermitian metric $H_0$. In this section we are going to prove a long time existence for the Hermitian-Einstein flow over any complete manifold under some conditions on the initial metric $H_0$. As usually, we use the compact exhaustion construction to prove the long time existence.

Let $\{\Omega_i\}_{i=1}^{\infty}$ be a exhausting sequence of compact sub-domains of $M$, i.e. they satisfy $\Omega_i \subset \Omega_{i+1}$ and $\cup_{i=1}^{\infty} \Omega_i = M$. By theorem 3.4 and theorem 4.1, we can find Hermitian metrics $H_i(x,t)$ on $E|_{\Omega_i}$ for each $i$ such that

$$\begin{align*} H_i^{-1} \frac{\partial H_i}{\partial t} &= -2(\sqrt{-1} \Lambda F_{H_i} - \lambda Id), \\
H_i(x,0) &= H_0(x), \\
H_i(x,t)|_{\partial \Omega_i} &= H_0(x), \\
\lim_{t \to -\infty} (\sqrt{-1} \Lambda F_{H_i} - \lambda Id) &= 0. \end{align*}$$ (5.1)

Suppose that there exist a positive number $C_0$ such that $|\sqrt{-1} \Lambda F_{H_0} - \lambda Id|_{H_0} \leq C_0$ on any points of $M$. Denote $h_i = H_0^{-1}H_i$, direct calculation shows that

$$\begin{align*} \frac{\partial}{\partial t} \lg tr h_i &\leq 2|\sqrt{-1} \Lambda F_{H_i} - \lambda Id|_{H_i}, \\
\frac{\partial}{\partial t} \lg tr h_i^{-1} &\leq 2|\sqrt{-1} \Lambda F_{H_i} - \lambda Id|_{H_i}. \end{align*}$$ (5.2)

By Proposition 2.2 and the Maximum principle, we have

$$\sup_{\Omega_i \times [0,\infty)} |\sqrt{-1} \Lambda F_{H_i} - \lambda Id|_{H_i} \leq C_0.$$ (5.3)

Integrating (5.2) along the time direction,

$$|\lg tr h_i(x,t) - \lg r| = \left| \int_0^t \frac{\partial}{\partial s} (\lg tr h_i(x,s)) \right| ds \leq 2C_0t.$$ Then we have

$$\sup_{\Omega_i \times [0,T]} tr h_i \leq r \exp(2C_0T),$$ (5.4)

and

$$\inf_{\Omega_i \times [0,T]} tr h_i \geq r \exp(-2C_0T),$$ (5.5)

These imply that

$$\sup_{\Omega_i \times [0,T]} \sigma(H_0, H_i) \leq 2r(\exp(2C_0T) - 1),$$ (5.6)

and

$$(r \exp(2C_0T))^{-1} Id \leq h_i(x,t) \leq r \exp(2C_0T) Id$$ (5.7)

for any $(x,t) \in \Omega_i \times [0,T]$. In particular, over any compact subset $\Omega$, for $i$ sufficiently large such that $\Omega \subset \Omega_i$, we have the $C^0$-estimate

$$\sup_{\Omega \times [0,T]} \sigma(H_0, H_i) \leq 2r(\exp(2C_0T) - 1).$$ (5.8)
Without loss of generality we can assume that \( \Omega = B_o(R) \), here \( B_o(R) \) denotes the geodesic ball of radius \( R \) with center at a fixed point \( o \in M \). First, we want to show that there exists a subsequence of \( \{ H_i \} \) converging uniformly to a Hermitian metric \( H_{\infty}(x,t) \) on \( B_o(R) \times [0,\frac{T}{2}] \).

Direct calculation as before show that over \( \Omega_i \times [0,T] \)

\[
\bar{\Delta}trh_i = -2\Delta \bar{\partial} \partial trh_i \\
= -2tr(h_i(\sqrt{-1}\Lambda F_{H_i} - \lambda Id)) + 2tr(h_i(\sqrt{-1}\Lambda F_{H_0} - \lambda Id)) \\
-2tr(\sqrt{-1}\Lambda \bar{\partial}trh_i h_i^{-1}\partial H_0 h_i) \\
\geq -C_1 + C_2 e(h_i).
\] (5.9)

Here \( e(h_i) = -2tr(\sqrt{-1}\Lambda \bar{\partial}trh_i \partial H_0 h_i) \), \( C_1 \) and \( C_2 \) are positive constants depending only on \( C_0 \) and \( T \), and we have used formulas (2.13), (5.3), (5.4), and (5.7). Choosing \( i \) sufficiently larger such that \( B_o(4R) \subset \Omega_i \), let \( \psi \) be a cut-off function which equal 1 in \( B_o(2R) \) and is supported in \( B_o(4R) \). Now multiply the above inequality by \( \tau_i \psi^2 \) and integrate it over \( M \). Then

\[
C_2 \int_M tr(h_i) \psi^2 e(h_i) \leq C_1 \int_M trh_i \psi^2 + \int_M trh_i \psi^2 \bar{\Delta}trh_i \\
- C_1 \int_M trh_i \psi^2 + \int_M trh_i \psi^2 \bar{\Delta}trh_i + \int_M trh_i \psi^2 (V, \nabla\nabla trh_i) \\
\leq C_1 \int_M trh_i \psi^2 + 8 \int_M (trh_i)^2 |\nabla \psi|^2 + 8 \int_M (trh_i)^2 \psi^2 |V|^2.
\]

Using (5.4) again, we obtain the following estimate:

\[
\int_0^T \int_{B_o(2R)} e(h_i) \leq C_3.
\] (5.10)

Here \( C_3 \) is a uniform constant depending on \( C_0 \), \( T \), \( R \), and \( V \).

Because \( e(h_i) \) contain all the squares of the first order derivatives (space direction) of \( h_i \), \( h_i \) have uniform \( C^0 \) bound, and also \( \partial \partial h_i \) are uniformly bounded. So, the above inequality imply that \( h_i \) are uniformly bounded in \( L^2(B_o(2R) \times [0,T]) \). Using the fact that \( L^2(B_o(2R) \times [0,T]) \) is compact in \( L^2(B_o(2R) \times [0,T]) \), by passing to a subsequence which we also denoted by \( \{ H_i \} \), we have that \( H_i \) converge in \( L^2(B_o(2R) \times [0,T]) \). Given any positive number \( \epsilon \), we have

\[
\int_0^T \int_{B_o(2R)} \sigma^2(H_j, H_k) \leq \epsilon,
\] (5.11)

for \( j, k \) sufficiently large.

For further discussion, We need the following lemmas. The following Sobolev inequality had been proved by Saloff-Coste (\cite{17; Theorem 3.1}).

**Lemma 5.1** Let \( M^m \) be a \( m \)-dimensional complete noncompact Riemannian manifold, and \( B_x(r) \) be a geodesic ball of radius \( r \) and centered at \( x \). Suppose that \(-K \leq 0 \) is the lower bound of the Ricci curvature of \( B_x(r) \). If \( m > 2 \), there exists \( C \) depending only on \( m \), such that

\[
\left( \int_{B_x(r)} |f|^{2q} \right)^{\frac{1}{q}} \leq \exp \left( C(1 + \sqrt{r}) \right) \text{Vol}(B_x(r))^{-\frac{2}{m} - \frac{2}{m'}} \left( \int_{B_x(r)} (|\nabla f|^2 + r^{-2}|f|^2) \right),
\] (5.12)

for any \( f \in C^\infty_0(B_x(r)) \), where \( q = \frac{m}{m-2} \). For \( m \leq 2 \), the above inequality holds with \( m \) replaced by any fixed \( m' > 2 \), and the constant \( C \) also depending only on \( m' \).
From the above Sobolev inequality (5.12) and the standard Moser iteration argument, it is not hard to conclude the following mean-value type inequality which can be seen as an generalization of the mean value inequality of Li and Tam ([11; Theorem 1.2]) for the nonnegative sub-solution to the heat equation. We should point out that the elliptic case of the following mean value inequality had been discussed in [14; p.344].

**Lemma 5.2** Let \( M^m \) be a \( m \)-dimensional (complex) complete noncompact Hermitian manifold without boundary, and \( B_o(2R) \) be a geodesic ball, centered at \( o \in M \) of radius \( 2R \). Suppose that \( f(x,t) \) be a nonnegative function satisfying

\[
(\Delta - \frac{\partial}{\partial t})f \geq -C_5 f
\]

on \( B_o(2R) \times [0,T] \). If \( -K \leq 0 \) is the lower bound of the Ricci curvature of \( B_o(2R) \), then for \( p > 0 \), there exists positive constants \( C_6 \) and \( C_7 \) depending only on \( C_5, m, R, K, p, T, \) and the difference vector fields \( V \), such that

\[
\sup_{B_o(\frac{1}{4}R) \times [0,\frac{T}{4}]} f^p \leq C_6 \int_0^T \int_{B_o(R)} f^p(y,t)dydt + C_7 \sup_{B_o(R)} f^p(\cdot,0).
\]

**Lemma 5.3** Let \( M^m \) be a \( m \)-dimensional (complex) complete noncompact Hermitian manifold without boundary, and \( B_o(2R) \) be a geodesic ball, centered at \( o \in M \) of radius \( 2R \). Suppose that \( f(x,t) \) be a nonnegative function satisfying (5.13) on \( B_o(2R) \times [0,T] \). If \( -K \leq 0 \) is the lower bound of the Ricci curvature of \( B_o(2R) \), then for \( p > 0 \), there exists a positive constant \( C_8 \) depending only on \( C_5, m, R, K, p, \) \( \delta, \eta, T, \) and \( V \) such that

\[
\sup_{B_o((1-\delta)R) \times [\eta T, (1-\eta)T]} f^p \leq C_8 \int_0^T \int_{B_o(R)} f^p(y,t)dydt,
\]

where \( 0 < \delta, \eta < \frac{1}{2} \).

**Proof.** Setting \( 0 < \eta_1 < \eta_2 \leq \frac{1}{2}, 0 < \delta_1 < \delta_2 \leq 1, \) and let \( \psi_1 \in \mathcal{C}_0^\infty(B_o(2R)) \) be the cut-off function

\[
\psi_1(x) = \begin{cases} 
1; & x \in B_o((1-\delta_2)R) \\
0; & x \in B_o(2R) \setminus B_o((1-\delta_1)R)
\end{cases}
\]

\( 0 \leq \psi_1(x) \leq 1 \) and \( |\nabla \psi_1| \leq 2(\delta_2 - \delta_1)^{-1}R^{-1} \). Let

\[
\psi_2(t) = \begin{cases} 
1; & \eta_2T < t < (1-\eta_2)T \\
0; & t < \eta_1T, \text{ or } t > (1-\eta_1)T.
\end{cases}
\]

\( 0 \leq \psi_2(t) \leq 1 \) and \( |\frac{\partial \psi_2}{\partial t}| \leq 2(\eta_2 - \eta_1)^{-1}T^{-1} \). Multiplying \( f^{q-1}\psi_2^2 \) on both sides of the inequality (5.13) \( (q > 1) \), here \( \psi(x,t) = \psi_1(x)\psi_2(t) \), and integrate it over \( M \). Intergrating by parts, and
using the Schwartz inequality, we have:

$$\frac{2(q - 1)}{q} \int_M \|f \frac{\partial \psi}{\partial t}\|^2 + \int_M \frac{\partial (f^q \psi^2)}{\partial t} \leq \int_M (qC_9 \psi^2 + \frac{4q}{q - 1} \|\nabla \psi\|^2 + 2\psi \frac{\partial \psi}{\partial t}) f^q.$$  

Integrating along the time direction, we have

$$\int_0^T \int_M \|\nabla f \frac{\partial \psi}{\partial t}\|^2 \leq \frac{q}{2(q - 1)} (\int_0^T \int_M (qC_9 \psi^2 + \frac{4q}{q - 1} \|\nabla \psi\|^2 + 2\psi \frac{\partial \psi}{\partial t}) f^q).$$  

(5.16)

Since $M$ is a complete manifold, so, we can use the Sobolev inequality (5.12). Combining with Hölder inequality, and the above inequality, we have:

$$\int_{\eta_2 T}^{(1-m) T} \int_{B_o((1-\delta_2) R)} f^{q(1 + \frac{2}{m})} dx dt \leq C_s \{ (\frac{1}{\eta_2 - \eta_1}) (\frac{1}{\delta_2 - \delta_1})^2 q (\frac{q}{2(q-1)}) \int_{\eta_1 T}^{(1-m) T} \int_{B_o((1-\delta) R)} f^q dx dt \}^{1 + \frac{1}{m}}.$$  

(5.17)

where positive constant $C_s$ depending only on $m, R, K, T$ and the vector field $V$. Using Moser’s iteration, the result follows. The iteration arguments in lemma 5.2 and lemma 5.3 is similar, the only difference is the choice of cut-off function $\psi_2(t)$ in the above.

\[\square\]

On the other hand, from proposition 2.5, we known that $\sigma(H_j, H_k)$ satisfy

$$(\Delta - \frac{\partial}{\partial t}) \sigma(H_j, H_k) \geq 0.$$  

Using (5.11) and (5.14), we have

$$\sup_{B_o(R) \times [0, T]} \sigma^2(H_j, H_k) \leq C_4 \epsilon.$$  

(5.18)

Here $C_4$ is a positive constant depending only on $C_0, R, T$, and the bound of sectional curvature on $B_o(2R)$. From (5.18), we can conclude that, by taking a subsequence, $H_i$ converges uniformly to a continuous Hermitian metric $H_\infty$ on $B_o(R) \times [0, T]$.

Next, we will use the above $C^0$ to obtain the $C^1$-estimate on $B_o(R) \times [0, T]$, the method we used is similar as that in [2; section 2.3]. For any point $x \in B_o(2R)$, choosing a small ball $B_x(r)$ such that the bundle $E$ can be trivialized locally, and let $\{e_a\}$ be the holomorphic frame of $E$. So, a metric $H_i$ can be written as a matrix which also denoted by $H_i$ on $B_x(r)$. The the complex metric connection with respect to $H_i$ can be written as following

$A_i = H_i^{-1} \partial H_i$

and the curvature form

$$F_{H_i} = \bar{\partial} (H_i^{-1} \partial H_i).$$

Choosing a real coordinate $\{y_i\}$ on $B_x(r)$ and centered at $x$. Denote $\rho_i = H_i^{-1} dH_i(\frac{\partial}{\partial y_i})$. It is easy to check that

$$-2\sqrt{-1} \Lambda \bar{\partial} \partial \rho_i - \frac{\partial}{\partial t} \rho_i = -\rho_i H_i^{-1} \frac{\partial H_i}{\partial t} + H_i^{-1} \frac{\partial H_i}{\partial t} \rho_i$$  

(5.19)
on $B_x(r)$. In fact, this follows from (5.1) by considering the one-parameter family of solutions obtained by translating in the direction of $\frac{\partial}{\partial y_l}, H_i^1(y_1, \ldots, y_{2m}) = H_i(y_1, \ldots, y_l + s, \ldots, y_{2m})$. It follows that the square norm $|\rho|^2_{H_i} = \text{tr} \rho H_i^{-1} \rho^* H_i$ satisfy
\[ (\tilde{\Delta} - \frac{\partial}{\partial t})|\rho|^2_{H_i} \geq 0 \] (5.20)
on $B_x(\frac{r}{2})$. On the other hand, there must exist constant $C_9$ and $C_{10}$ such that
\[ C_9 \text{Id} \leq \{g(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_l})\} \leq C_{10} \text{Id} \]
on $B_x(r)$, where $g$ is the Hermitian metric of $M$. So, we have
\[ C_9 \sum_i |H_i^{-1} \frac{\partial H_i}{\partial y_l}|_{H_i} \leq |H_i^{-1} \nabla H_i|^2_{H_i} \leq C_{10} \sum_i |H_i^{-1} \frac{\partial H_i}{\partial y_l}|_{H_i} \]
Using formula (5.10), (5.20), and lemma 5.2, we can conclude that there exists a positive constant $C_{11}$ which is independently of $i$ such that
\[ \sup_{B_o(\frac{r}{4}) \times [0, \frac{T}{4}]} |H_i^{-1} \nabla H_i|^2_{H_0} \leq C_{11}. \] (5.21)

Since $x$ is arbitrary, so we can conclude that the $C^1$-norm of $H_i$ is bounded uniformly on any $B_o(R) \times [0, \frac{T}{4}]$. By the $C^0$-estimate (5.8) and the above $C^1$-estimate, the standard parabolic theory shows that, by passing to a subsequence, $H_i$ converges uniformly over any compact subset of $M \times [0, \infty)$ to a smooth $H_\infty$ which is a solution of the Hermitian-Einstein flow (2.6) on the whole manifold. Therefore we complete the proof of the following theorem.

**Theorem 5.4** Let $M$ be a complete noncompact Hermitian manifold without boundary, let $E$ be a holomorphic vector bundle over $M$ with initial Hermitian metric $H_0$. Suppose that there exists a positive number $C_0$ such that $|\sqrt{-1} \Lambda F_{H_0} - \lambda \text{Id}| \leq C_0$ everywhere, where $\lambda$ is a real number, then the Hermitian-Einstein flow
\[ \begin{cases} H^{-1} \frac{\partial H}{\partial t} = -2(\sqrt{-1} \Lambda F_H - \lambda \text{Id}), \\ H(x, 0) = H_0, \end{cases} \]
has a long time solution on $M \times [0, \infty)$.

6 H-E metrics over complete Hermitian manifolds

In this section, we consider the existence of the Hermitian-Einstein metrics on some complete Hermitian manifolds. As above, complete means complete, noncompact, and without boundary. Since we have established the global existence of the Hermitian-Einstein flow on any complete Hermitian manifold, one could hope that the Hermitian-Einstein flow will converge to a Hermitian-Einstein metric under some assumptions. But, in the following we will adapt the
direct elliptic method, the argument is similar to that Ni and Ran are used in the Kähler case ([15]).

Let \( \{ \Omega_i \}_{i=1}^{\infty} \) be a exhausting sequence of compact sub-domains of \( M \), and \( H_0 \) be a Hermitian metric on the holomorphic vector bundle \( E \). By section 4, we known that the following Dirichlet problem is solvable on \( \Omega_i \), i.e. there exists a Hermitian metric \( H_i(x) \) such that

\[
\begin{align*}
-\Delta F_{H_i} &= 0, \text{for } x \in \Omega_i, \\
H_i(x)|_{\partial \Omega_i} &= H_0(x).
\end{align*}
\] (6.1)

In order to prove that we can pass to limit and eventually obtain a solution on the whole manifold \( M \), we need to establish some estimates. The key is the \( C^0 \)-estimate. From corollary 2.8, we have the following Bochner type inequality.

\[
\begin{align*}
\hat{\Delta} \lg(\text{tr} h_i + \text{tr} h_i^{-1}) &\geq -2|\sqrt{-1} \Delta F_{H_0} - \lambda Id|_{H_0}, \text{ for } x \in \Omega_i \\
\lg(\text{tr} h_i + \text{tr} h_i^{-1})|_{\partial \Omega_i} &= \lg 2r.
\end{align*}
\] (6.2)

Here \( h_i = H_0^{-1} H_i \). Let \( \hat{\sigma}_i = \hat{\sigma}(H_0, H_i) = \lg(\text{tr} h_i + \text{tr} h_i^{-1}) - \lg 2r \), we have

\[
\begin{align*}
\hat{\Delta} \hat{\sigma}_i &\geq -2|\sqrt{-1} \Delta F_{H_0} - \lambda Id|_{H_0}, \text{ for } x \in \Omega_i \\
\hat{\sigma}_i|_{\partial \Omega_i} &= 0.
\end{align*}
\] (6.3)

Next, we impose three invertibility conditions on the holomorphic Laplace operator between suitably chosen function spaces.

**Condition 1:** There exist positive number \( p > 0 \) such that for every nonnegative function \( f \in L^p(M) \cap C^0(M) \), there exists a nonnegative solution \( u \in C^0(M) \) of

\[ \hat{\Delta} u = -f. \]

**Condition 2:** There exist positive number \( \mu > 0 \) such that for every nonnegative function \( f \in C^0_{\mu}(M) \), there exists a nonnegative solution \( u \in C^0(M) \) of

\[ \hat{\Delta} u = -f. \]

**Condition 2’:** There exist positive numbers \( \mu > 0, \mu' > 0 \) such that for every nonnegative function \( f \in C^0_{\mu}(M) \), there exists a nonnegative solution \( u \in C^0_{\mu'}(M) \) of

\[ \hat{\Delta} u = -f. \]

Here \( C^0_{\mu}(M) \) denotes the space of continuous functions \( f \) which satisfy that there exists \( x_0 \in M \) and a constant \( C(f) \) such that \( |f(x)| \leq C(f)(1 + \text{dist}(x, x_0))^{-\mu}. \)

**Theorem 6.1.** Assume that \( M \) is a complete Hermitian manifold such that for the holomorphic Laplace operator \( \hat{\Delta} \) on \( M \), the condition 1 is satisfied with positive number \( p > 0 \)
(or the condition 2 is satisfied with positive number \( \mu \)). Let \((E, H_0)\) be an holomorphic vector bundle with a Hermitian metric \(H_0\). Assume that \(\|\sqrt{-1} \Lambda F_{H_0} - \lambda Id\|_{H_0} \in L^p(M)\) for some real number \( \lambda \) (or \(\|\sqrt{-1} \Lambda F_{H_0} - \lambda Id\|_{H_0} \in C^0_\mu(M)\)), then there exists a Hermitian-Einstein metric \(H\) on \(E\). If \(M\) satisfies the condition \(2'\), and the initial Hermitian metric \(H_0\) satisfies \(\|\sqrt{-1} \Lambda F_{H_0} - \lambda Id\|_{H_0} \in C^0_\mu(M)\), then there exists a unique Hermitian-Einstein metric \(H\) with \(\bar{\sigma}(H_0, H) \in C^0_\mu(M)\), here \(\bar{\sigma}\) is defined in (6.3).

**Proof.** Using the maximum principle, from condition 1 (or condition 2) and formula (6.3), we can conclude that the Donaldson’s distance \(\sigma_i = \sigma(H_0, H_i)\) between \(H_i\) and \(H_0\) must satisfy

\[
\sigma_i \leq 2r \exp u - 2r.
\]

(6.4)

for any \(x \in \Omega_i\). Where \(u\) satisfy \(\tilde{\Delta}u = -2|\sqrt{-1} \Lambda F_{H_0} - \lambda Id|_{H_0}^2\). From the above \(C^0\)-estimates, and discussing like that in the proof of theorem 5.4, we can obtain an uniform \(C^1\)-estimates of \(H_i\). Then standard elliptic theory shows that, by passing a subsequence, \(H_i\) converge uniformly over any compact sub-domain of \(M\) to a smooth Hermitian metric \(H\) satisfying

\[
\sqrt{-1} \Lambda F_{H} - \lambda Id = 0.
\]

When \(M\) satisfies the condition \(2'\), using the maximum principle, the Hermitian-Einstein metric \(H\) which we obtained in the above must satisfy \(\bar{\sigma}(H_0, H) \leq u\). So, \(\bar{\sigma}(H_0, H) \in C^0_\mu(M)\).

Finally, we prove the uniqueness of the Hermitian-Einstein metric with the mentioned properties. Let \(\tilde{H}\) be another Hermitian-Einstein metric for the same real number \(\lambda\) on the bundle \(E\) and satisfies \(\bar{\sigma}(H_0, \tilde{H}) \in C^0_\mu(M)\). Hence for every \(\epsilon\) outside a sufficiently large geodesic ball \(B_\rho(R)\) around an arbitrary \(o \in M\), we have

\[
\sigma(H_0, \tilde{H}) \leq \epsilon, \quad \text{and}, \quad \sigma(H_0, H) \leq \epsilon.
\]

By the definition of the Donaldson distance \(\sigma\), and the above inequalities, it is not hard to conclude that:

\[
\sigma(H, \tilde{H}) \leq 2r (1 + \sqrt{\epsilon - \frac{1}{4} \epsilon^2 + \frac{1}{2} \epsilon})^2 - 2r
\]

outside the geodesic ball \(B_\rho(R)\). On the other hand, from proposition 2.4, we have:

\[
\tilde{\Delta} \sigma(H, \tilde{H}) \geq 0.
\]

By the maximum principle this implies \(\sigma(H, \tilde{H}) \leq 2r (1 + \sqrt{\epsilon - \frac{1}{4} \epsilon^2 + \frac{1}{2} \epsilon})^2 - 2r\) on all of \(M\) for every \(\epsilon > 0\) and hence \(\sigma(H, \tilde{H}) \equiv 0\). This implies \(H \equiv \tilde{H}\).

\(\Box\)

**Remark:** The condition \(2'\) is introduced by Grunau and Kühnel in [5] where they discuss the existence of holomorphic map from complete Hermitian manifold, and they had constructed
some examples which satisfy condition 2’. Next, with the help of the following two definitions, we want to discuss the condition 1 on the holomorphic Laplace operator.

**Definition 6.2. (Positive spectrum)** Let $M$ be a complete Hermitian manifold, we say the holomorphic Laplace operator $\Delta$ has positive first eigenvalue if there exists a positive number $c$ such that for any compactly supported smooth function $\phi$ one has

$$\int_M (-\Delta \phi) \phi \geq c \int_M \phi^2. \tag{6.5}$$

The supremum of these numbers $c$ will be denoted by $\bar{\lambda}_1(M)$.

**Definition 6.3. ($L^2$-Sobolev inequality)** Let $M$ be a $m$-dimensional (complex) complete Hermitian manifold, we say the holomorphic Laplace operator $\Delta$ satisfies $L^2$-Sobolev inequality if there exists a constant $S(M)$ such that for any compact supported smooth function $\phi$ one has

$$\int_M (-\Delta \phi) \phi \geq S(M) \left( \int_M \phi^{\frac{4m}{2m-2}} \right)^{\frac{2m-2}{2m}} \tag{6.6}$$

**Lemma 6.4.** Let $M$ be a complete Hermitian manifold, and the holomorphic Laplace operator $\Delta$ has positive first eigenvalue $\bar{\lambda}_1(M)$. Then for a nonnegative continuous function $f$ the equation

$$\bar{\Delta} u = -f$$

has a nonnegative solution $u \in W^{2,m}_{\text{loc}} \cap C^{1,\alpha}_{\text{loc}}(M)$ $(0 < \alpha < 1)$ if $f \in L^p(M)$ for some $p \geq 2$.

**Proof.** We first solve the following Dirichlet problem on $\Omega_i$ ([20; Ch5, proposition 1.8])

$$\begin{cases}
\bar{\Delta} u_i = -f, \\
u_i|_{\partial \Omega_i} = 0.
\end{cases} \tag{6.7}$$

Here $\Omega_i$ is a exhaustion of $M$. First, by the maximum principle, we know that $u_i \geq 0$. Now multiplying $u_i^{p-1}$ on both sides of the equation and integrating by parts we have that

$$\int_{\Omega_i} f u_i^{p-1} = \int_{\Omega_i} (-\bar{\Delta} u_i) \cdot u_i^{p-1} = (p-1) \int_{\Omega_i} u_i^{p-2} |\nabla u_i|^2 - \int_{\Omega_i} u_i^{p-1} \langle V, \nabla u_i \rangle. \tag{6.8}$$

On the other hand, using the assumption that $\bar{\lambda}_1(M) > 0$, we have

$$\bar{\lambda}_1(M) \int_{\Omega_i} u_i^p \leq \int_{\Omega_i} (-\bar{\Delta} u_i^p) u_i^p = \left( \frac{p}{2} \right) \int_{\Omega_i} u_i^{p-2} |\nabla u_i|^2 - \frac{p}{2} \int_{\Omega_i} u_i^{p-1} \langle V, \nabla u_i \rangle. \tag{6.9}$$

Adding (6.8) and (6.9) we have

$$\frac{p}{2} \int_{\Omega_i} f u_i^{p-1} \geq \frac{p}{2} \left( \frac{p}{2} - 1 \right) \int_{\Omega_i} u_i^{p-2} |\nabla u_i|^2 + \bar{\lambda}_1(M) \int_{\Omega_i} u_i^p. \tag{6.10}$$
From the above inequality, using Hölder inequality, we have
\[
\left( \int_{\Omega_i} u_i^p \right) \frac{1}{p} \leq \frac{p}{2\lambda_1(M)} \left( \int_M f^p \right)^{\frac{1}{p}}.
\] (6.11)

Using the interior \( L^p \) estimates for the linear elliptic equation ([6], Theorem 9.11) we know that, over a compact sub-domain \( \Omega \), there will be a uniform bound for \( \|u_i\|_{W^{2,p}(\Omega)} \). Therefore, using Rellich’s compactness theorem, by passing to a subsequence we know that \( u_i \) will converge to a solution \( u \in W^{2,p}_0(M) \) on the manifold \( M \), and the standard elliptic theory can show that \( u \in C^{1,\alpha}_{loc}(M) \).

□

Replacing the Poincare (6.5) inequality by the Sobolev inequality(6.6) in the proof of above lemma, we can prove the following lemma.

**Lemma 6.5.** Let \( M \) be a \( m \)-dimensional(complex) complete Hermitian manifold, and the holomorphic Laplace operator \( \bar{\Delta} \) satisfy the \( L^2 \)-Sobolev inequality (6.6). Then for a nonnegative continuous function \( f \) the equation
\[
\bar{\Delta} u = -f
\]
has a nonnegative solution \( u \in W^{2,2m}_{loc} \cap C^{1,\alpha}_{loc}(M) (0 < \alpha < 1) \) if \( f \in L^p(M) \) for some \( m > p \geq 2 \).

The above two lemmas show that when the holomorphic Laplace operator \( \bar{\Delta} \) has positive first eigenvalue (or satisfies the \( L^2 \) Sobolev inequality) then the condition 1 must be satisfied for some positive number \( p \).

**Corollary 6.6** Let \( M \) be a complete Hermitian manifold, and the holomorphic Laplace operator \( \bar{\Delta} \) has positive first eigenvalue \( \bar{\lambda}_1(M) \). Let \((E, H_0)\) be a holomorphic vector bundle with Hermitian metric \( H_0 \). Assume that \( \|\sqrt{-1} \Delta F_{H_0} - \lambda Id\| \in L^p(M) \) for some \( p \geq 2 \) and real number \( \lambda \). Then there exists a Hermitian-Einstein metric \( H \) on \( E \).

**Corollary 6.7** Let \( M \) be a \( m \)-dimensional(complex) complete Hermitian manifold, and the holomorphic Laplace operator \( \bar{\Delta} \) satisfy the \( L^2 \)-Sobolev inequality (6.6). Let \((E, H_0)\) be a holomorphic vector bundle with Hermitian metric \( H_0 \). Assume that \( |\sqrt{-1} \Delta F_{H_0} - \lambda Id| \in L^p(M) \) for some \( p \in [2, m) \) and real number \( \lambda \). Then there exists a Hermitian-Einstein metric \( H \) on \( E \).

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