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GRAVITATIONAL WAVE EMISSION FROM A BOUNDED SOURCE:
A TREATMENT IN THE FULL NONLINEAR REGIME

H.P. de Oliveira
Universidade do Estado do Rio de Janeiro, Instituto de Física, Departamento de Física Teórica, CEP 20550-013, Rio de Janeiro, RJ, Brazil
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

I. Damião Soares
Centro Brasileiro de Pesquisas Físicas, R. Dr. Xavier Sigaud 150, CEP 22290-180, Rio de Janeiro, RJ, Brazil.

Abstract

The dynamics of a bounded gravitational collapsing configuration emitting gravitational waves is studied. The exterior spacetime is described by Robinson-Trautman geometries and have the Schwarzschild black hole as its final gravitational configuration, when the gravitational wave emission ceases. The full nonlinear regime is examined by using the Galerkin method that allows us to reduce the equations governing the dynamics to a finite-dimensional dynamical system, after a proper truncation procedure. Gravitational wave emission patterns from given initial configurations are exhibited for several phases of the collapse and the mass-loss ratio that characterizes the amount of mass extracted by the gravitational wave emission is evaluated. We obtain that the smaller initial mass $M_{\text{init}}$ of the configuration, the more rapidly the Schwarzschild solution is attained and a larger fraction of $M_{\text{init}}$ is lost in the process of gravitational wave emission. Within all our numerical experiments, the distribution of the mass fraction extracted by gravitational wave emission is shown to satisfy the distribution law of nonextensive statistics and this result is independent of the initial configurations considered.

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1Regular Associate of the Abdus Salam ICTP. oliveira@dft.if.uerj.br
2ivano@cbpf.br
1 Introduction

In the study of the dynamics of formation of black holes, the final state of the collapsing configuration is fixed by Wheeler’s lemma *A black hole has no hair*[1]. For a Schwarzschild black hole this lemma is sustained by the complete analysis of scalar, electromagnetic and gravitational perturbations on the background geometry performed in Refs. [2]-[7] and extended for non-classical fields in Refs. [8]-[10]. The analysis carried out in these references treat the perturbations as test fields and do not consider possible back-reactions of these perturbations on the full dynamics of the gravitational configuration. In the particular case of gravitational perturbations, the approach does not consider some important issues as how do the gravitational perturbations of radiative character extract mass of the configuration, what is the amount of mass carried out by a particular gravitational radiation pole emitted, and what is the dynamical pattern in the evolution to the final gravitational configuration if nonlinear effects are taken into account. In this work, our aim is to examine these issues by considering a simple class of radiative spacetimes, which is the family of Robinson-Trautman (RT) metrics[11] that have the Schwarzschild geometry as a particular limit. This approach is basically distinct from the perturbative one since the total curvature of the spacetime, which satisfies the full Einstein’s equation, contains already all the information on the radiative dynamics.

The paper begins with a presentation of the basic equations that govern the dynamics of the RT spacetimes (Section II). The characterization of the gravitational wave zone through the Peeling theorem is discussed in Section III. We apply the Galerkin method[12] to study the nonlinear evolution of RT spacetimes from a dynamical system perspective in Section IV. In Sections V and VI we discuss the relevant numerical results connected to the dynamics of RT spacetimes until the Schwarzschild configuration is established. Finally, Section VII is devoted to the summary of results and conclusion. Throughout the paper we use units such that $8\pi G = c = 1$.

2 Robinson-Trautman spacetimes: basic aspects

Robinson-Trautman metrics are the simplest known solutions of vacuum Einstein’s equations which may be interpreted as representing an isolated gravitational radiating system[11]. By construction, RT spacetimes are assumed to admit a shear-free null congruence of geodesics that are surface orthogonal[13]. This family of null hypersurfaces foliates the spacetime globally and, in a coordinate system where they are labeled by $u = const$, the metric can be expressed as

$$
\begin{align*}
 ds^2 &= \alpha^2(u, r, \theta)du^2 + 2dudr - \\
 & \quad \quad - r^2K^2(u, \theta)(d\theta^2 + \sin^2 \theta d\varphi^2),
\end{align*}
$$

(1)
where $r$ is an affine parameter defined along the shear-free null geodesics determined by the
vector field $\partial/\partial r$. We use the angular coordinates $(\theta, \varphi)$ to label the points of the spacelike
surfaces $u = \text{const}, r = \text{const}$, and we assume that these two-dimensional manifolds are compact
and orientable. The geometry is nonstationary and axially symmetric, admitting the obvious
Killing vector $\partial/\partial \varphi$. Einstein equations result in

$$\alpha^2(u, r, \theta) = \lambda(u, \theta) + \frac{B(u)}{r} + 2r \frac{K(u, \theta)}{K(u, \theta)}$$

(2)

where $B(u)$ is an arbitrary function of $u$, and $\lambda(u, \theta)$ is the Gaussian curvature of the surfaces
$(u = \text{const}, r = \text{const})$ defined by

$$\lambda(u, \theta) = \frac{1}{K^2} - \frac{K_{\theta \theta}}{K^3} + \frac{K_{\theta}^2}{K^4} - \frac{K_{\theta}}{K^4} \cot \theta.$$  

(3)

If $B(u)$ is non-zero, it can always be reduced to a constant by a proper coordinate transformation.
For future reference, we will fix $B = -2m_0$. The remaining Einstein equations yield

$$-6m_0 \frac{K}{K} + \frac{(\lambda_\theta \sin \theta)_{\theta}}{2K^2 \sin \theta} = 0.$$  

(4)

In the above, a dot and a subscript $\theta$ denotes derivatives with respect to $u$ and $\theta$, respectively.

Eq. (4), denoted the RT equation, governs the dynamics of the gravitational field and will be the
basis of our analysis of the gravitational wave emission processes in RT spacetimes. Formally
speaking, it allows to evolve initial data $K(u, \theta)$ prescribed on a given null surface $u = u_0$
(except in the case $m_0 = 0$). A particular and important solution of Eqs. (3) and (4) is the
Schwarzschild metric obtained when $K(u, \theta) = K_0 = \text{const}$. In this instance, $\lambda(u, \theta) = K_0^{-2}$ and
the corresponding Schwarzschild mass is

$$M_{\text{Schw}} = m_0 K_0^3$$

(5)

The above expression will be of particular importance in our characterization of the mass function
of the configuration. Concerning this point and for future reference, we note that in general the
function $M(u, \theta) = B(u)K^3(u, \theta)$ is invariant under the coordinate transformation that reduces
$B(u)$ to a constant $-2m_0$; also this transformation induces that the type-D $O(1/r^3)$ curvature
scalar associated with the mass aspect of the spacetime has a $(u, \theta)$ dependence given exactly
by $BK^3$, when made invariant in value under this transformation.

A lot of work has been realized on the evolution of RT spacetimes and on the existence
of solutions for the full nonlinear equation. The most general analysis of the existence and
asymptotic behavior of the RT equation (4) was given by Chrusciel and Singleton[14]. The
established result is that the RT spacetimes exist globally for all positive times $u$ and converge
asymptotically to the Schwarzschild metric, this global time extension being realized for arbitrary
smooth initial data. The relevant aspect we intend to explore here is pattern of emission of
gravitational waves and the associated processes of radiative transfer until the final Schwarzschild configuration has settled down.

3 The Structure of the Curvature Tensor and the Characterization of the Gravitational Wave Zone

In order to define distinct classes of RT spacetimes and to characterize their radiative nature, we now proceed to discuss the algebraic structure of the curvature tensor of the geometry (1). Let us introduce the semi-null tetrad basis determined by the 1-forms

\[ \Theta^0 = du, \quad \Theta^1 = (\alpha^2/2)du + dr, \]
\[ \Theta^2 = rKd\theta, \quad \Theta^3 = rK\sin\theta d\varphi \]  

(6)

In this basis, the non-zero curvature tensor components are given by

\[ R_{2323} = -R_{0101} = 2R_{0212} = -\frac{2m_0}{r^3}, \quad R_{0323} = \frac{\lambda_\theta}{2Kr^2} \]
\[ R_{0303} = -R_{0202} = -\frac{A(u, \theta)}{r^2} - \frac{D(u, \theta)}{r}, \]  

(7)

where the functions \( A \) and \( D \) are

\[ A(u, \theta) = \frac{1}{4K^2} \left( \lambda_\theta - 2\frac{\lambda_\theta K_\theta}{K} - \lambda_\theta \cot \theta \right) \]  

(8)

\[ D(u, \theta) = \frac{1}{2K^2} \frac{\partial}{\partial u} \left[ \frac{K_{\theta\theta}}{K} - \frac{K_\theta}{K} \cot \theta - 2 \left( \frac{K_\theta}{K} \right)^2 \right]. \]  

(9)

We note that the \( r \)-dependence of the curvature components (7) is respectively \( 1/r^3 \), \( 1/r^2 \) and \( 1/r \), and we may express

\[ R_{ABCD} = \frac{D_{ABCD}}{r^3} + \frac{III_{ABCD}}{r^2} + \frac{N_{ABCD}}{r} \]  

(10)

where \( D_{ABCD}, III_{ABCD} \) and \( N_{ABCD} \) are of algebraic type D, type III and type N in the Petrov classification[15] and have the vector field \( k = \partial/\partial r \) as a principal null direction. In the coordinate basis they have the property of being covariantly constant along the null direction \( k \)[11]. From (10) we can now establish an invariant characterization of the radiative character of the spacetime and of a corresponding gravitational radiation wave zone. This is based on two pillars: (i) the Peeling Theorem (for the linearized Riemann tensor of retarded multipole fields see Refs. [16, 18]; for the general case, see Ref. [19]; for a review, including peeling properties of the Maxwell tensor, see Ref. [20]); (ii) the analysis of the spacetime of gravitational
wave solutions of Einstein's equations, and their relation to electromagnetic waves in Maxwell theory\cite{1, 11, 20, 21, 22}. The Peeling Theorem states that the Weyl tensor (or the vacuum Riemann tensor) of a radiative gravitational bounded source, expanded in powers of $1/r$, has the general form

$$R_{ABCD} = \frac{N_{ABCD}}{r} + \frac{III_{ABCD}}{r^2} + \frac{II_{ABCD}}{r^3} + \frac{I_{ABCD}}{r^4} + \ldots \quad (11)$$

where $r$ is the parameter distance defined along the null geodesics determined by the null vector field $k = \partial/\partial r$. The quantities $N_{ABCD}$, $III_{ABCD}$, $II_{ABCD}$ and $I_{ABCD}$, when expressed in the coordinate basis, have vanishing covariant derivatives along the null vector field $k$. They are of algebraic type $N$, $III$, $II$ and $I$, respectively in the Petrov classification. The direction of propagation $k$ is a repeated principal null direction\cite{23} of the curvature tensor to order $r^{-4}$, and satisfies

$$N_{ABCD}k^D = 0$$
$$III_{ABC}[Dk^Ck_E] = 0$$
$$II_{ABC}[Dk_Bk^C] = 0$$
$$k_{[E}I_{A]}BC[Dk_F]k^Bk^C = 0 \quad (12)$$

If the spacetime is such that $N_{ABCD}$ is non-zero then, for large values of the distance parameter $r$, the curvature tensor has the approximate asymptotic expansion $R_{ABCD} \simeq N_{ABCD}/r$, that is, it is of Petrov type $N$, with the degenerate principal null direction given by $k$ (cf. (12)). In other words, the field looks like a gravitational wave at large distances. The wave fronts are the $u =$const. surfaces with $k$ its propagation vector. The non-vanishing of the scalars $N_{ABCD}$ is then taken as an invariant criterion for the presence of gravitational waves, and the asymptotic region (where the $O(1/r)$ term in (11) is dominant) defined as the wave zone. For the RT spacetimes the non-vanishing of the function $D(u, \theta)$ will therefore characterize the wave zone and will allow to determine the angular pattern of the amplitudes of the waves emitted. Also from expressions (7) we see that the wave zone of the RT spacetime corresponds to a gravitational wave polarized in the mode $T2+$, in the terminology of Eardley et al\cite{24}.

4 Towards a full nonlinear approach through the Galerkin method.

Before presenting our full nonlinear approach for the evolution of RT spacetimes in connection with the gravitational waves, it will be useful to discuss briefly the linearized solution exhibited by Foster and Newman\cite{25}. Basically, they assumed that $K(u, \theta)$ is approximated as
where $A_0$ is a constant, $|\epsilon| \ll 1$ and $P_n(\cos \theta)$ is the Legendre polynomial of order $n$. Without loss of generality they have chosen $n \geq 2$, since $n = 0$ and $n = 1$ both represent spheres (considering, of course, the submanifold $r^2 K^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)$), that in the latter case has its center displaced at a distance $\epsilon A_0$ along the axis of revolution). After linearizing the field equation, an immediate solution is found to be

$$\epsilon_n(u) = \epsilon_n(0) e^{-k_n u}$$  \tag{14}$$

where $\epsilon_n(0)$ is the initial value on the hypersurface $u = 0$ and $k_n = \frac{1}{12 m_0 A_0} (n - 1)n(n + 1)(n + 2) > 0$ for $n \geq 0$. Since $\epsilon \to 0$ as $u \to \infty$, the Schwarzschild metric is the asymptotic state for generic initial data consistent with the linearization procedure. Foster and Newman were able to interpret this solution as representing radiation from a bounded source after carrying a coordinate transformation to a system in which the boundary conditions suggested by Bondi[17] and Sachs[16] are satisfied. In this system a consistent expression for the mass lost during the evolution of the spacetime towards the final static state is established, and it becomes evident the change of mass due the emission of gravitational waves; this aspect will be discussed in the next Section.

It is possible to generalize the approach of Foster–Newman in a rather distinct way in order to take into account the nonlinearities of the field equations. For this proposal, we shall consider the Galerkin projection method[12] in which, as the first step, we adopted the following decomposition

$$K^2(u, x) = A_0^2 e^{Q(u, x)} = A_0^2 \exp \left( \sum_{k=0}^{N} b_k(u) P_k(x) \right),$$  \tag{15}$$

where we have introduced a new variable $x = \cos \theta$ such that $-1 \leq x \leq 1$; $N$ is the order of the truncation, $b_k(u)$ are the modal coefficients, and we have adopted the Legendre polynomials $P_k(x)$ as the basis functions of the projective space. The internal product defined in this projection space is

$$\langle P_j(x), P_k(x) \rangle = \int_{-1}^{1} P_j(x)P_k(x) \, dx = \frac{2 \delta_{kj}}{2k + 1},$$  \tag{16}$$

Combining the decomposition of $K(u, x)$ with Eq. (3) results in a expression for $\lambda$ given by

$$\lambda(u, x) = \frac{e^{-Q(u, x)}}{A_0^2} \left( 1 + \sum_{k=0}^{N} \frac{1}{2} k(k + 1)b_k(u)P_k(x) \right).$$  \tag{17}$$

The next step is to substitute the decomposition provided by Eq. (15) into Eq. (4) along with the change $\theta \to x$. We then obtain

$$K^2(u, \theta) \simeq A_0^2 \left[ 1 + \epsilon P_n(\cos \theta) \right],$$  \tag{13}$$
\[ 6m_0 \sum_{k=0}^{N} \dot{b}_k(u)P_k(x) - \frac{1}{A_0^2} e^{-Q(u,x)} [(1 - x^2) \lambda']' = 0. \]

where a prime denotes derivative with respect to \( x \), and \( \lambda(u,x) \) is given by Eq. (17). At this point we use the Galerkin projection in order to derive the evolution equations for the modal coefficients. This means to project the above equation onto each basis function \( P_n(x) \), \( n = 0, 1, 2..., N \), resulting

\[ \dot{b}_n(u) = \frac{(2n + 1)}{12m_0 A_0^2} \left\langle e^{-Q(u,x)} [(1 - x^2) \lambda']', P_n(x) \right\rangle. \]

with \( n = 0, 1, 2..., N \). Therefore, the dynamics of RT spacetimes is reduced to this system of \((N + 1)\) equations. As an instructive illustration of the above system, we shall exhibit the linear and quadratic (nonlinear) terms with respect to the modal coefficients. After some calculation, it follows that

\[ \dot{b}_n + \frac{1}{12m_0 A_0^2} (n - 1)n(n + 1)(n + 2)b_n - \frac{(1 + 2n)}{24m_0 A_0^2} \sum_{k,j=0}^{N} \langle P_k P_j, P_n \rangle \gamma_{kj} - \frac{(1 + 2n)}{12m_0 A_0^2} \sum_{k,j=0}^{N} \langle (1 - x^2) P'_k P'_j, P_n \rangle \times \alpha_{kj} + ... = 0, \]

where \( \alpha_{kj} \) and \( \gamma_{kj} \) are quadratic combinations of \( b_k, k = 0, 1, 2..., N \). Note that the Foster–Newman solution is naturally recovered by considering only the linear terms on the modal coefficients, indicating that the approach provided by the Galerkin method generalizes the mentioned linear treatment.

Before presenting the results of the numerical study of the system (20), it is necessary to establish the appropriate initial conditions \( b_k(0), k = 0, 1, 2..., N \), or in other words, we need to specify initial data family \( K(u = 0, x) \equiv k(x) \). For instance, \( k(x) = k_0 \) corresponds to a sphere and is identified as the Schwarzschild spacetime; on the other hand, the following initial data family

\[ k(x) = \frac{k_0 \sqrt{1 - \epsilon^2}}{\sqrt{1 - \epsilon^2 x^2}} \]

actually corresponds to an ellipse with eccentricity \( \epsilon \) (0 < \( \epsilon < 1 \)). In any case, once the initial data \( k(x) \) is given, it is always possible to obtain the initial values for modal coefficients \( b_j(0) \) recognizing that the following decomposition can be performed, \( K(0, x) = k(x) = A_0 \exp \left( 1/2 \sum_{k=0}^{N} b_k(0) P_k(x) \right) \), which renders, as a consequence,
\begin{equation}
\begin{aligned}
b_j(0) = \frac{2 \langle \ln k(x), P_j \rangle}{\langle P_j, P_j \rangle}
\end{aligned}
\end{equation}

Usually, for most problems the Galerkin method guarantees that an expansion with few modal coefficients provides a good approximation in the sense of reproducing quite well the actual behavior of the system under consideration. The second family of initial data we are going to deal with has the same form of Eq. (21), but the eccentricity is no longer constant and given by 
\[\epsilon = \epsilon_0 + \epsilon_1 x^3,\]
where \(\epsilon_0\) and \(\epsilon_1\) are the parameters. Finally, the third family is assumed to have the following form

\begin{equation}
\begin{aligned}
k(x) = \exp \left[ C_1 e^{-\left(\tanh^{-1} x - a_1\right)^2} + C_2 e^{-\left(\tanh^{-1} x - a_2\right)^2} \right]
\end{aligned}
\end{equation}

where \(C_{1,2}, a_{1,2}\) are the parameters of the distribution. In order to illustrate the effect of choosing a particular value of \(N\), we exhibit the plot of the initial distribution \(k(x)\) corresponding to the ellipse, and \(1/2 \sum_{j=1}^{N} b_j(u)P_j(x)\) as shown in Fig. 1. We note that the exact curve is rapidly attained for increasing values of \(N\).

5 Numerical results

In this Section we shall exhibit some important aspects of the nonlinear dynamics of RT spacetimes. We begin with the behavior of the modal coefficients, as shown in Fig. 2, for the initial data corresponding to an ellipse.

For this task it was assumed \(\epsilon = 0.7\), \(m_0 = 20\) and \(A_0 = k_0 = 1\). For arbitrary values of the constants \(k_0\) and \(\epsilon\), all modal coefficients tend to zero asymptotically; the exception is \(b_0(u)\) that approaches to a constant value which we denote by \(b_0(\infty)\). According to the Galerkin decomposition (15) this asymptotic configuration actually corresponds to the Schwarzschild solution with mass

\[M_\infty = m_0 A_0^3 \exp(3b_0(\infty)/2).\]

As we have mentioned in Section II, this feature is in agreement with previous analytical studies on RT spacetimes. In particular for an elliptical initial data, only the even-order modes will be excited. In the case of the initial data corresponding to the deformed ellipse and the one given by Eq. (23), all modes are excited and, as expected, evolve towards the Schwarzschild spacetime. This feature does not represent a new result, but the details of the dynamics of RT spacetimes until the Schwarzschild configuration has been settled down is of interest for the problem of emission of gravitational waves, as we are going to discuss later. In Figs. 3, we exhibit the evolution of some modal coefficients for the second initial distribution. As an instructive illustration of the dynamics of all modes together, a sequence of polar plots of \(K(u, x) = A_0 \exp \left(1/2 \sum_{k=0}^{N} b_k(u) P_k(x)\right)\) is shown in Fig. 4 for times \(u\) until the Schwarzschild configuration is settled down (corresponding to the final sphere).
Figure 1: In the graphs on the left, we depict the error $E = k(x) - A_0 \exp \left( \frac{1}{2} \sum_{k=0}^{N} b_k(0) P_k(x) \right)$ between each initial data and the respective Galerkin decompositions for $N = 4, 7$ and 11. Also the plot the initial data (solid line) versus the Galerkin decomposition (diamonds) for $N = 7$ are shown on the right. The following choices are assumed: (a) $A_0 = k_0 = 1, \epsilon = 0.7$ for the ellipse; (b) $A_0 = k_0 = 1, \epsilon_0 = 0.8, \epsilon_1 = 0.1$ for the deformed ellipse, and (c) $a_1 = 0.1, a_2 = -0.8, C_1 = C_2/2 = 1.362$ for the third initial distribution given by Eq. (23). It is worth mentioning that the convergence of the Galerkin decomposition is achieved satisfactorily as $N$ increases, producing as a consequence $b_k \to 0$.

An interesting physical feature also valid for any type of smooth initial data, $K(0, x)$, is the relation between the time necessary for the system to reach the Schwarzschild configuration and the magnitude of the mass of the initial distribution, hereafter denoted by $M_{\text{init}}$. The idea of small and large values of $M_{\text{init}}$ are associated with a given initial distribution. For instance, in the case of distribution (21), an analytical expression for $M_{\text{init}}$ can be consistently defined,

$$M_{\text{init}} = \frac{1}{2} m_0 \int_{-1}^{1} K(0, x)^3 \, dx = m_0 k_0^3 \left( 1 - \epsilon^2 \right),$$

(25)

where $m_0$, $k_0$ and $\epsilon$ are three parameters that regulate the magnitude of the initial mass of the distribution. By assuming fixed values of $m_0$, $k_0$, the eccentricity $\epsilon$ turns to be our control parameter such that $0 < M_{\text{init}} < m_0 k_0^3$. With respect to the deformed ellipse we fix $\epsilon_1 = 0.1$ and $\epsilon_0$ is the control parameter; for the third initial data family the choices are $a_1 = 0.1, a_2 = -0.8$.
Figure 2: Evolution of all modal coefficients for the initial data distribution corresponding to an ellipse \((m_0 = 20, A_0 = 1, \epsilon = 0.7)\). Only the even modes are excited, exhibiting a decay to the Schwarzschild solution.

Figure 3: Behavior of some modal coefficients for the second distribution corresponding to a deformed ellipse with \(\epsilon = 0.8\) and \(\epsilon_1 = 0.1\). The first graph accounts for \(b_0(u)\), the second for \(b_4(u), b_6(u)\), and the last for the odd modes \(b_3(u)\) and \(b_5(u)\).

and \(C_2 = 2C_1 = -\ln 7 a_0\), \(a_0\) being the control parameter. In both situations by increasing \(\epsilon_0, a_0\) the initial mass as evaluated by Eq. (25) decreases. Basically, the numerical simulations indicated that for small values of \(M_{\text{init}}\), more rapidly the Schwarzschild solution is settled down; in addition a large fraction of \(M_{\text{init}}\) is lost in the process. On the other hand, if \(M_{\text{init}}\) is large, then the evolution of RT spacetime towards the asymptotic Schwarzschild final state is very slow. Correspondingly, the fraction of \(M_{\text{init}}\) lost is very small. The mass loss of the system is discussed in detail in Section VI. In some sense, we may understand this characteristic of our system as a kind of classical analog of the Hawking process, since the smaller is the initial mass of the black hole, it evaporates faster than in the case of a larger initial data. In Fig. 5 we illustrate this feature by showing the behavior of the first and fourth modal coefficients for the third distribution and corresponding to \(a_0 = 0.9, 0.7, 0.5\), labelled, respectively, by \(A, B\) and \(C\), where \(M_{\text{init}}(A) < M_{\text{init}}(B) < M_{\text{init}}(C)\).

Another important aspect is related to the emission of gravitational waves, whose invariant characterization was established on the basis of the Peeling theorem. In this case the function \(D(u, \theta)\) defined by Eq. (9) plays a crucial role in characterizing the radiation zone and the pattern of gravitational wave emission. In order to express \(D(u, \theta)\) conveniently will be useful to introduce \(x = \cos \theta\) and substitute the Galerkin decomposition (15) into Eq. (9). A direct calculation yields
Figure 4: Sequence of some $K(u, x) = A_0 \exp(1/2 \sum_k b_k(u) P_k(u))$ starting from the initial configuration for several times $u$ (from left to right, from top to bottom). The asymmetries are expelled out by gravitational wave emission and the continuous deformations stop when the sphere is formed (Schwarzschild final configuration) is formed.

Figure 5: Decay of the modes $b_1(u)$ and $b_4(u)$ for distinct values of the initial mass labelled by $A$, $B$ and $C$, for which $M_{\text{init}}(A) < M_{\text{init}}(B) < M_{\text{init}}(C)$. As indicated the greater is the initial mass more time is needed to reach the asymptotic Schwarzschild solution.

$$D(u, x) = \frac{1}{4A_0^2} e^{-Q(u, x)} (1 - x^2) \left(Q'' - \frac{1}{2} Q^2\right).$$  

(26)

where $Q(u, x) = \sum_{k=1}^{N} b_k(u) P_k(x)$. Taking $|D(u, x)|$ as the amplitude of the gravitational waves at the radiation zone, we use this expression to construct Figs. 6 and 7, where the evolution of the angular pattern of the gravitational waves emitted is displayed for the first and third initial data families, respectively. The families of curves corresponds to increasing values of $u$ with associated decreasing of the amplitudes. For the initial configuration (21) the pattern has symmetry of reflexion by the plane $\theta = \pi/2$ ($x = 0$), whereas for the third and more arbitrary initial distribution (Eq. (23)) we note that the pattern does not display any symmetry. The projections of $|D(u, x)|$ in the plane $(|D|, x)$ evaluated in distinct instants reveal an intense emission, or a burst of gravitational waves in the region $-1 < x < 0$ ($\pi < \theta < \pi/2$), characterized by high values of $|D(u, x)|$. The reason for this peculiar behavior can be understood from Fig.
Figure 6: Evolution of the angular pattern of gravitational waves emitted for the first initial distribution (ellipse). The pattern has a symmetry of reflection by the plane $x = 0$ ($\theta = \pi/2$). The left figure corresponds to early values of $u$, when the pattern exhibits three peaks about $\theta = \pi/6, \pi/2$ and $5\pi/6$. The right figure corresponds to later values of $u$, with the small amplitudes peaked about $\theta = \pi/2$.

Figure 7: Evolution of the angular emission pattern for the third (asymmetric) initial data. At early times the pattern shows an intense asymmetric emission in the southern hemisphere ($-1 < x < 0$), connected to the accentuated asymmetry of the initial data in this hemisphere, as shown in the first Fig. 4. As this asymmetry is expelled out by the emission of gravitational waves (cf. the successive Figs. 4) the emission pattern becomes more symmetric and for late times it recovers the symmetry about $\theta = \pi/2$.

4, in which the deformation of the inferior part is more accentuated and evolves faster in the first instants than in other part of the same distribution. As a consequence of the emission of gravitational waves, the initial distribution loses mass until the asymptotic Schwarzschild static configuration has been settled down. As our next step we make an quantitative evaluation of the process of mass extraction from the initial configuration due to the emission of gravitational waves.

6 The distribution of the mass fraction extracted by gravitational wave emission

The gravitational wave emission until the Schwarzschild black hole has been formed has a striking consequence, which is the extraction of mass from the collapsing configuration. Let us explore further this phenomenon by posing the following question. Consider given initial data represented, say, by a one parameter function that regulates the magnitude of $M_{init}$. We have found that the fraction of mass extracted,
depends, roughly speaking, on the inverse of the initial mass. Therefore, what is the precise relation between $M_{\text{init}}$ and the above fraction of mass extracted by gravitational radiation? To answer this question, we have examined three families of initial data: the ellipse defined by Eq. (21), the deformed ellipse and the third distribution given by Eq. (23). In all cases we have selected one control parameter by fixing the remaining parameters, such that for each value of this control parameter corresponds a value of $M_{\text{init}}$. In Figs. 8(a) and 8(b) we present the plot of $M_{\text{init}}$ versus $\Delta$ for the first and the last initial data (21), (23), respectively; the points represent the values determined by numerical integration of the system (19) with the continuous lines fitting these points account for the distribution provided by the nonextensive statistics[26] given by

$$\Delta = C_0 (y_0 - y)^\alpha [1 + (q - 1)\lambda_2 (y_0 - y)]^{1/1-q}, \quad (28)$$

Figure 8: Log-linear plot of $\Delta$ versus $M_{\text{init}}$. The points were generated after integrating the dynamical system (19) using (a) the first and (b) the third initial data families. The continuous curve are the best fit corresponding to the nonextensive distribution law (28).

where $y$ is associated to the initial mass (for instance for the case of the ellipse (21), $y = M_{\text{init}}/m_0k_0^3$); $C_0$, $\alpha$ and $q$ are the parameters of the distribution to be determined and we have set $y_0 = 1$. As it can be seen from Fig. 8, the above distribution fits beautifully the numerically generated points. The parameters corresponding the best fit for the first and third initial data, respectively, are given by $C_0 \simeq 4.46, \alpha \simeq 1.99, \lambda_2 \simeq -1.22, q \simeq 1.73$, and $C_0 \simeq 3.02, \alpha \simeq 1.89, \lambda_2 \simeq -1.93, q \simeq 1.45$.

7 Final remarks

In this paper we have exhibited some interesting features of the nonlinear dynamics of the collapse of a bounded system, the exterior of which is described by RT spacetimes. This is the case of the simplest axisymmetric collapse followed by gravitational wave emission. As a
consequence, a fraction of the initial mass of the system is lost in the form of gravitational waves. To deal with the nonlinear dynamics we have applied the Galerkin projection method which approximates the partial differential equation (4) as a finite dynamical system of ordinary differential equations.

Despite the apparent simplicity of the evolution equation (4) together with the the fact that the Schwarzschild spacetime is the asymptotic configuration for reasonable smooth families of initial data, the details of the dynamics in the course of the establishment of the final configuration are quite rich. Such details include: the structure of gravitational radiation, which we have properly characterized using the Peeling theorem; an classical analogy with the Hawking process; and the relation between the fraction of mass extracted by gravitational waves and the initial mass of the system. Probably, the most interesting result is that the distribution of mass fraction extracted by gravitational wave emission satisfies with good precision the distribution law of the nonextensive statistics. An important question is related to the generality of the above results, in particular the validity of the statistical distribution law, if we consider less restricted exterior spacetimes as it is the present case of RT spacetime.

We have a final comment about the definition we adopted for the mass function $M(u) = \frac{1}{2} m_0 \int_{-1}^{1} K(u, x)^3 dx$. This mass function has the following properties: (i) it is invariant in value under coordinate transformations that alter $B(u)$; (ii) it is positive definite for all $u$; (iii) its $u$-derivative is negative definite for all $u$, as expected since gravitational waves are extracting mass from the system and (iv) it gives the correct Schwarzschild mass in the asymptotic limit $u \to \infty$ or in the static case. Therefore it satisfies all demands for a satisfactory mass function. It would be interesting to have its relation with the Bondi mass but this is not a trivial task since we should implement a coordinate transformation from RT coordinates to Bondi's asymptotic coordinates. However, as shown by Janis and Newman[27] no closed form is known for such transformation. Nevertheless we were able to check that they are approximately equal, up to second order in a perturbative scheme as done in [25].

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