Linear Algebraic Groups and K-theory

Notes in collaboration with Hinda Hamraoui, Casablanca

Ulf Rehmann*

Department of Mathematics, University of Bielefeld, Bielefeld, Germany

Lectures given at the
School on Algebraic K-theory and its Applications
Trieste, 14 - 25 May 2007

LNS0823002

*rehmann@math.uni-bielefeld.de

Contents

Introduction

The functors $\mathrm{K}_1, \mathrm{K}_2$ [1] for a commutative field k are closely related to the theory of the general linear group via exact sequences of groups

$$1 \to \mathrm{SL}(k) \to \mathrm{GL}(k) \overset{\det}{\to} K_1(k) \to 1,$$

$$1 \to K_2(k) \to \mathrm{St}(k) \to \mathrm{SL}(k) \to 1.$$

Here the groups $\mathrm{GL}(k)$, $\mathrm{SL}(k)$ are inductive limits of the well known matrix groups of the general and special linear group over k with n rows and columns: $\mathrm{GL}(k) = \varinjlim_{\mathrm{n}} \mathrm{GL}_n(k)$, $\mathrm{SL}(k) = \varinjlim_{\mathrm{n}} \mathrm{SL}_n(k)$, and $K_1(k) \cong k^*$, the latter denoting the multiplicative group of k.

The groups $\mathrm{SL}_n(k)$, $\mathrm{SL}(k)$ are all *perfect*, i.e., equal to their commutator subgroups,[2]. It is known that any perfect group G has a *universal central extension* $\tilde{G}$, i.e., there is an exact sequence of groups

$$1 \to A \to \tilde{G} \to G \to 1$$

such that A embeds into the center of $\tilde{G}$, and such that any such central extension factors from the above sequence. The universal central extension sequence and in particular its kernel A is unique for G up to isomorphism, and the abelian group A is called the fundamental group of G – see Thm. 1.1 ii) and subsequent Remark 1.

The *Steinberg group* $\mathrm{St}(k)$ in the second exact sequence above is the universal central extension of $\mathrm{SL}(k)$ and $K_2(k)$ is its fundamental group.

Similarly, the universal central extension of $\mathrm{SL}_n(k)$ is denoted by $\mathrm{St}_n(k)$ and called the Steinberg group of $\mathrm{SL}_n(k)$. Again there are central extensions

$$1 \to K_2(n,k) \to \mathrm{St}_n(k) \to \mathrm{SL}_n(k) \to 1$$

and the group $K_2(n,k)$ is the fundamental group of $\mathrm{SL}_n(k)$.

One has natural epimorphisms $K_2(n,k) \to K_2(k)$ for $n \geq 2$, which are isomorphisms for $n \geq 3$, but the latter is not true in general for $n = 2$. The groups $K_2(n,k)$ can be described in terms of so-called *symbols*, i.e., generators indexed by pairs of elements from k^* and relations reflecting the

[1] cf. the first lecture by Eric Friedlander

[2] except for (very) small fields k and small n, see Thm. 2.1 below

additive and multiplicative structure of the underlying field k. This is the content of a theorem by Matsumoto (Thm. 2.2).

The proof of this theorem makes heavy use of the internal group structure of $\mathrm{SL}_n(k)$. However, the group $\mathrm{SL}_n(k)$ is a particular example of the so-called "Chevalley groups" over the field k, which are perfect and have internal structures quite similar to those of $\mathrm{SL}_n(k)$.

In full generality, the theorem of Matsumoto describes "k-theoretic" results for all Chevalley groups over arbitrary fields (Thm. 5.2).

Among these groups, there are, e.g., the well known symplectic and orthogonal groups, and their corresponding groups of type K_2 are closely related to each other and to the powers of the fundamental ideal of the "Witt ring" of the underlying field – which play important rôles in the context of quadratic forms (see the lectures given by Alexander Vishik) and in the context of the Milnor conjecture and, more generally, the Bloch-Kato conjecture (see the lectures given by Charles Weibel).

Our lectures are organized as follows:

In section 1, we reveal the internal structure of $\mathrm{SL}_n(k)$ together with a theorem by Dickson and Steinberg which gives a presentation of $\mathrm{SL}_n(k)$ and its universal covering group in terms of generators and relations (Thm. 1.1). We formulate the theorem by Matsumoto for $\mathrm{SL}_n(k)$ (Thm. 1.2). In an appendix, in order to prepare for the general cases of Chevalley groups, we discuss the so-called "root system" of $\mathrm{SL}_n(k)$ in explicit terms.

In section 2, we outline the basic notions for the theory of linear algebraic groups, in particular, we discuss tori, Borel subgroups and parabolic subgroups.

In section 3, the notion of root systems is discussed, in particular, root systems of rank one and two are explicitly given, as they are important ingredients for the formulation of the theorems by Steinberg and Matsumoto for Chevalley groups. The classification of semi-simple algebraic groups over algebraically closed fields is given, Chevalley groups over arbitrary fields are described including their internal structure theorems (Thm. 3.3).

In section 4, the theorems by Steinberg and by Matsumoto for Chevalley groups are given (Thm. 4.1, Thm. 4.2), and relations between their associated "K_2"–type groups are explained.

In section 5, we go beyond Chevalley groups and describe the classification and structure of almost simple algebraic groups (up to their so-called "anisotropic kernel"), in terms of their Bruhat decomposition (relative to their minimal parabolic subgroups) and their Tits index (Thm. 5.2).

In section 6, we describe some K-theoretic results for almost simple algebraic groups which are not Chevalley groups, mostly for the group $\mathrm{SL}_n(D)$, where D is a finite dimensional central division algebra over k (Thms. 6.1, 6.2, 6.3, 6.4, 6.5, 6.6), and finally formulate several open questions.

1 The group structure of SL_n over a field

Let k be any field. By k^*, we denote the multiplicative group of k.

We consider the groups of matrices

$$\begin{array}{rcl} \mathrm{GL}_n(k) & = & \{(a_{kl}) \in \mathrm{M}_n(k) \mid 1 \leq k, l \leq n, \det(a_{kl}) \neq 0\} \\ G := \mathrm{SL}_n(k) & = & \{(a_{kl}) \in \mathrm{GL}_n(k) \mid \det(a_{kl}) = 1\}. \end{array}$$

The determinant map $\det : \mathrm{GL}_n(k) \to k^*$ yields an exact sequence of groups:

$$1 \to \mathrm{SL}_n(k) \to \mathrm{GL}_n(k) \to k^* \to 1.$$

Let $e_{ij} := (a_{kl})$ be the matrix with coefficients in k such that $a_{kl} = 1$ if $(k,l) = (i,j)$ and $a_{kl} = 0$ otherwise, and let $\mathbf{1} = \mathbf{1}_n \in \mathrm{GL}_n(k)$ denote the identity matrix.

We define, for $x \in k$ and $i, j = 1, \ldots, n,\ i \neq j$, the matrices

$$\begin{array}{rcll} u_{ij}(x) & = & \mathbf{1}_n + x e_{ij} & (x \in k) \\ w_{ij}(x) & = & u_{ij}(x)\, u_{ji}(-x^{-1})\, u_{ij}(x) & (x \in k, x \neq 0) \\ h_{ij}(x) & = & w_{ij}(x)\, w_{ij}(-1) & (x \in k, x \neq 0) \end{array}$$

The matrices $w_{ij}(x)$ are monomial, and the matrices $h_{ij}(x)$ are diagonal.

(Recall that a *monomial* matrix is one which has, in each column and in each line, exactly one non-zero entry.)

Example: For $n = 2$, we have:

$$u_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad w_{12}(x) = \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}, \quad h_{12}(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix},$$
$$u_{21}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad w_{21}(x) = \begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix}, \quad h_{21}(x) = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}.$$

For $n \geq 2$ we obtain matrices with entries which look, at positions ii, ij, ji, jj, the same as the matrices above at positions $11, 12, 21, 22$ respectively, and like the unit matrix at all other positions.

The proof of the following is straightforward:

(1) The group $\mathrm{SL}_n(k)$ is generated by the matrices $\{u_{ij}(x) \mid 1 \leq i, j \leq n,\ i \neq j,\ x \in k\}$

(2) The matrices $\{w_{ij}(x) \mid 1 \leq i, j \leq n, i \neq j, x \in k^\star\}$ generate the subgroup N of all monomial matrices of G.

(3) The matrices $\{h_{ij}(x) \mid 1 \leq i, j \leq n, i \neq j, x \in k^\star\}$ generate the subgroup T of all diagonal matrices of G.

The group N is the normalizer of T in G, and the quotient $W := N/T$ is isomorphic to the symmetric group $\mathfrak{S}_n$, i.e., the group of permutations of the n numbers $\{1, \ldots, n\}$.

This isomorphism is induced by the map

$$\sigma : N \to \mathfrak{S}_n,\ w_{ij}(x) \mapsto (ij),$$

where (ij) denotes the permutation which interchanges the numbers i, j, leaving every other number fix.

For the elements $u_{ij}(x),\ w_{ij}(x),\ h_{ij}(x)$ we have the following relations:

(A) $\quad u_{ij}(x + y) = u_{ij}(x)\, u_{ij}(y)$

(B) $$[u_{ij}(x), u_{kl}(y)] = \begin{cases} u_{il}(xy) & \text{if } i \neq l, j = k, \\ u_{jk}(-xy) & \text{if } i = l, j \neq k, \\ \mathbf{1} & \text{otherwise, provided } (i,j) \neq (j,i) \end{cases}$$

(B′) $\quad w_{ij}(t)\, u_{ij}(x)\, w_{ij}(t)^{-1} = u_{ji}(-t^{-2}x),$ for $t \in k^\star, x \in k$.

(C) $\quad h_{ij}(xy) = h_{ij}(x)\, h_{ij}(y)$

Remarks:

- The relation (B) is void if $n = 2$.
- If $n \geq 3$, the relations (A), (B) imply the relation (B′).

We denote by B and U the subgroups of G defined by

$$B = \left\{ \begin{pmatrix} \star & \star & \star \\ & \ddots & \star \\ 0 & & \star \end{pmatrix}, \text{ upper triangular} \right\}$$
$$U = \left\{ \begin{pmatrix} 1 & \star & \star \\ & \ddots & \star \\ 0 & & 1 \end{pmatrix}, \text{ unipotent upper triangular} \right\}$$

Then:

i) The group B is the semi-direct product $T \ltimes U$ with normal subgroup U.

ii) The subgroup B is a maximal connected solvable subgroup of G, it is the stabilizer group of the canonical maximal flag of vector spaces. $0 \subset k \subset k^2 \subset \ldots \subset k^{n-1} \subset k^n$.

iii) The quotient G/B is a projective variety.

iv) Any subgroup P of G containing B is the stabilizer group of a subflag of the canonical maximal flag. $0 \subset k \subset k^2 \subset \ldots \subset k^{n-1} \subset k^n$. In this case G/P is projective variety.

v) Every subgroup of G which is the stabilizer of some flag of vector spaces is conjugate to such a P as in iv).

vi) We have a so-called *Bruhat decomposition*: $G = BNB = \cup_{w\in W} B\tilde{w}B$ into disjoint double cosets $B\tilde{w}B$ where $\tilde{w}$ is a pre-image of w under $N \to W$. Since $T \subset B$, the double coset $B\tilde{w}B$ does not depend on the choice of the pre-image $\tilde{w}$ for $w \in W$, hence we may denote this double coset by BwB. Therefore we may write the Bruhat decomposition as $G = BNB = \cup_{w\in W} BwB$.

Proof: i) As a consequence of the relations (A), (B) resp. (B′), we have the following relations:

$$h_{ij}(t)\; u_{kl}(x)\; h_{ij}(t)^{-1} = \begin{cases} u_{kl}(t^2x) & \text{if } i=k, j=l \\ u_{kl}(t^{-2}x) & \text{if } i=l, j=k \\ u_{kl}(tx) & \text{if } i=k, j\neq l \\ u_{kl}(t^{-1}x) & \text{if } i\neq k, j=l \\ u_{lk}(x) & \text{if } \{i,j\}\cap\{k,l\}=\emptyset \end{cases} \qquad (H)$$

This shows that U is normalized by T, and it is obvious that $U \cap T = \{1\}$

ii), iii) (Cf. [24], 6.2)

iv), v), vi) The proofs here are an easy exercise and recommended as such, if you are a beginner in this subject and want to gain some familiarity with the basics.

Examples:

– If $n = 2$, we have $W \cong \mathfrak{S}_2 \cong \mathbb{Z}/2Z$ and $G = B \cup B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B$

– If $n = 3$, we have $W \cong \mathfrak{S}_3$. For shortness, we write $w_{ij} := w_{ij}(1)$, then we get:

$$w_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\; w_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},\; w_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$w_{12}\; w_{13} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad (w_{12}\; w_{13})^2 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The permutations associated to these elements are given by

$$w_{ij} \mapsto (ij), \quad w_{12}w_{13} \mapsto (132), \quad (w_{12}w_{13})^2 \mapsto (123),$$

and the Bruhat decomposition is given by

$$G = B \cup Bw_{12}B \cup Bw_{12}B \cup Bw_{23}B \cup Bw_{12}\; w_{13}B \cup B(w_{12}\; w_{13})^2B$$

Theorem 1.1 (Dickson, Steinberg, cf. [25], Thm. 3.1 ff., Thm. 4.1 ff.)

i) A presentation of G is given by

(A), (B), (C) *if $n \geq 3$, and*
(A), (B$'$), (C) *if $n = 2$.*

ii) Denote by $\tilde{G}$ the group given by the presentation

(A), (B) *if $n \geq 3$, and*
(A), (B$'$) *if $n = 2$,*

Then the canonical map $\pi : \tilde{G} \to G$ is central, which means that its kernel is contained in the center of $\tilde{G}$.

Assume $|k| > 4$ if $n \geq 3$ and $|k| \neq 4, 9$ if $n = 2$. Then this central extension is universal, which means: Every central extension $\pi_1 : G_1 \to G$ factors from π, i.e., there exists $g : \tilde{G} \to G_1$ such that $\pi = \pi_1 \circ g$.

Remark 1: As G and $\tilde{G}$ are perfect, $\tilde{G}$ as a universal central extension of G is unique up to isomorphism. The group $\mathrm{St}_n(k) := \tilde{G}$ is called the *Steinberg group* of $\mathrm{SL}_n(k)$.

Remark 2: If k is finite field, then G and $\tilde{G}$ equal. (This is proved in [25] as well.)

Remark 3: By the definition of $\tilde{G}$ and G, the elements $h_{ij}(xy)h_{ij}(x)^{-1}h_{ij}(y)^{-1}$ generate the kernel of $\pi : \tilde{G} \to G$.

Theorem 1.2 (Matsumoto, cf. [12], Thm. 5.10, Cor. 5.11, see also Moore [16]*) Let the notations and assumptions be the same as in the preceding theorem.*

The elements $c_{ij}(x, y) := h_{ij}(x)h_{ij}(y)h_{ij}(xy)^{-1}, x, y \in k^*$ yield the relations $c_{ij}(x, y) = c_{ji}(y, x)^{-1}$ for $n \geq 2$, and are independent of i, j for $n \geq 3$.

Let $c(x, y) := c_{12}(x, y)$. These elements fulfill the following relations:

$n = 2:$ (*cf.* [12], Prop. 5.5)

(S1) $c(x, y)\, c(xy, z) = c(x, yz)\, c(y, z)$
(S2) $c(1, 1) = 1,\ c(x, y) = c(x^{-1}, y^{-1})$
(S3) $c(x, y) = c(x, (1 - x)y)$ *for $x \neq 1$*

$n \geq 3:$ (*cf.* [12], Lemme 5.6)

(S°1) $c(x, yz) = c(x, y)\, c(x, z)$
(S°2) $c(xy, z) = c(x, z)\, c(y, z)$
(S°3) $c(x, 1 - x) = 1$ *for $x \neq 1$*

In case $n = 2$, *define* $c^\natural(x,y) := c(x,y^2)$. *Then* $c^\natural(x,y)$ *fulfills the relations* (S°1), (S°2), (S°3),

Moreover, $\ker \pi : \tilde{G} \to G$ *is isomorphic to the abelian group presented by*

(S1), (S2), (S3) *in case* $n = 2$
(S°1), (S°2), (S°3) *in case* $n \geq 3$

Remarks:

i) The relations (S1), (S2), (S3) are consequences of the relations (S°1), (S°2), (S°3).

ii) A map c from $k^* \times k^*$ to some abelian group is called a *Steinberg cocycle* (resp. a *Steinberg symbol*), if it fulfills relations (S1), (S2), (S3) (resp. (S°1), (S°2), (S°3)).

iii) The relations (S°1), (S°2) say that c is bimultiplicative, and hence the group presented by (S°1), (S°2), (S°3) can be described by

$$k^* \otimes_{\mathbb{Z}} k^* / \langle x \otimes (1-x) \mid x \in k^*, x \neq 1 \rangle.$$

iv) $\ker \pi : \mathrm{St}_n(k) \to \mathrm{SL}_n(k)$ is denoted by $K_2(n,k)$. Clearly we have homomorphisms

$$K_2(2,k) \twoheadrightarrow K_2(3,k) \xrightarrow{\sim} K_2(4,k) \xrightarrow{\sim} \ldots \xrightarrow{\sim} K_2(n,k), \ (n \geq 3),$$

hence the inductive limit

$K_2(k) := \varinjlim_{\mathrm{n}} K_2(n,k)$ is defined and isomorphic to $K_2(n,k)$ for $n \geq 3$.

The elements of $K_2(k)$ are generated by the *Steinberg symbols*, very often (in particular in the context of K-theory) denoted by $\{x,y\} := c(x,y)$.

The "Theorem of Matsumoto" very often is just quoted as the fact that, for any field k, the abelian group $K_2(k)$ is presented by the generators $\{x,y\},\ x,y \in k^*$, subject to relations

$$\{x,yz\} = \{x,y\}\,\{x,z\},\ \ \{xy,z\} = \{x,z\}\,\{y,z\},$$

and

$$\{x, 1-x\} = 1 \text{ for } x \neq 1.$$

This does not just hold for $K_2(k)$, but as well for all $K_2(n,k),\ n \geq 3$.

For this (restricted) statement, there is a complete and elementary proof in the book on "Algebraic K-Theory" by Milnor (cf. [15], §§ 11, 12, pages 93–122).

However, this neither does include the case of $K_2(2,k)$, nor the much wider cases of arbitrary simple "Chevalley groups", which were treated by Matsumoto's article as well. We will see more of this in the following sections; in particular we will see that the kernel of the surjective map

$$K_2(2,k) \twoheadrightarrow K_2(n,k) \tilde{\to} K_2(k) \ (n \geq 3),$$

can be described, by a result of Suslin [28], in a satisfying way in terms of other invariants of the underlying field k.

Appendix of section 1: A first digression on root systems

We will exhibit here the *root system* of $G = \mathrm{SL}_n$ as a first example, in spite of the fact that the formal definition of a "root system" will be given in a later talk.

For this we let $\mathrm{Diag}_n(k)$ denote the subgroup of all diagonal matrices in $\mathrm{GL}_n(k)$, and we denote a diagonal matrix just by its components:

$$\mathrm{diag}(d_\nu) = \mathrm{diag}(d_1, \ldots, d_n) := \begin{pmatrix} d_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & d_n \end{pmatrix} \in \mathrm{Diag}_n.$$

Clearly $T \subset \mathrm{Diag}_n(k)$. For $i = 1, \ldots, n$ we define homomorphisms:

$$\varepsilon_i : \mathrm{Diag}_n(k) \to k^* \text{ by } \varepsilon_i(\mathrm{diag}(d_\nu)) = d_i.$$

$\{\varepsilon_i\}$ is a basis of the (additively written) free $\mathbb{Z}$-module $X(\mathrm{Diag}_n)$ of all *characters* (i.e., homomorphisms) $\chi : \mathrm{Diag}_n \to k^*$: In general, such a character is given by

$$\chi(\mathrm{diag}(d_i)) = \prod_{i=1}^{n} d_i^{n_i}, \text{ with } n_i \in \mathbb{Z} \text{ for } i = 1, \ldots, n,$$

hence we have $\chi = \sum_{i=1}^{n} n_i \varepsilon_i$, and χ vanishes on T if and only if $n_1 = \ldots = n_n$. Hence the set $\{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n-1\}$ is a basis of the $\mathbb{Z}$-module $X(T)$ of all characters on T.

The group $W = N/T$ operates on $X(\mathrm{Diag}_n(k))$ by

$$({}^w\chi)(d) := \chi(\tilde{w}d\tilde{w}^{-1}) \text{ for all } d \in \mathrm{Diag}_n(k), w \in N/T,$$

where $\tilde{w} \in N$ is any pre-image of w. (This is unique modulo T, hence $\tilde{w}d\tilde{w}^{-1}$ is independent of its choice.)

Exercise: If $d = \mathrm{diag}(d_\nu)$, then $({}^w\chi)(\mathrm{diag}(d_\nu)) = \mathrm{diag}(d_{w(\nu)})$. (Hint: Use $W = \mathfrak{S}_n$.)

The submodule $X(T)$ (as well as $\mathbb{Z}(\varepsilon_1 + \ldots + \varepsilon_n)$) is invariant under W.

On the $\mathbb{R}$-vector space $X(\mathrm{Diag}_n(k)) \otimes_{\mathbb{Z}} \mathbb{R}$, we consider the scalar product $\langle\ ,\ \rangle$ which has ε_i as an orthonormal basis (we identify the elements $\chi \in X(\mathrm{Diag}_n(k))$ with their images $\chi \otimes 1 \in X(\mathrm{Diag}_n(k)) \otimes_{\mathbb{Z}} \mathbb{R}$). This scalar product is invariant with respect to the operation of W, as ${}^w\varepsilon_i = \varepsilon_{w(i)}$.

For the basis $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ of the $\mathbb{R}$-vector space $V := X(T) \otimes_Z \mathbb{R}$, we obtain the following orthogonality relations:

$$\langle\alpha_i, \alpha_j\rangle = \langle\varepsilon_i, \varepsilon_j\rangle - \langle\varepsilon_i, \varepsilon_{j+1}\rangle - \langle\varepsilon_{i+1}, \varepsilon_j\rangle + \langle\varepsilon_{i+1}, \varepsilon_{j+1}\rangle = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } |j-i| = 1, \\ 0 & \text{if } |j-i| > 1. \end{cases}$$

This means that all α_i are of same length $\sqrt{2}$, and the angle between two distinct elements $\alpha_i, \alpha_j, i \neq j$ of them has cosine

$$\langle\alpha_i, \alpha_j\rangle/2 = \begin{cases} -1/2 & \text{if } |j-i| = 1 \\ 0 & \text{if } |j-i| > 1 \end{cases}$$

which means that the angle between them is either $\pi/3$ or $\pi/2$.

Define $\Sigma = \{\alpha_{i,j} := \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n,\ i \neq j\}$ (hence $\alpha_{i,i+1} = \alpha_i$).

This finite set is, as a subset of $V = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, invariant under W, and symmetric, i.e., $\Sigma = -\Sigma$.

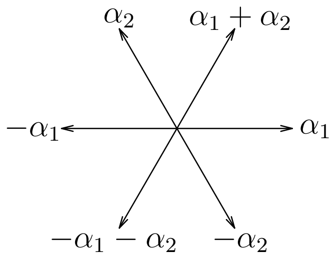
Clearly,

$$\alpha_{ij} = \begin{cases} \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} & \text{if } i < j \\ -\alpha_j - \alpha_{j+1} - \ldots - \alpha_{i-1} & \text{if } i > j \end{cases},$$

hence every element in Σ is an integral linear combination of the α_i with coefficients of the same sign.

For $n = 3$, we obtain $\Sigma = \{\ \pm\alpha_1,\ \pm\alpha_2,\ \pm(\alpha_1 + \alpha_2) = \pm\alpha_{13}\ \}$, and the geometric realization looks as follows:

Root system of $\mathrm{SL}_3(k)$:



In particular, every pair (i,j) of indices with $i \neq j$ (i.e., off-diagonal matrix positions in $\mathrm{SL}_n(k)$) determines a root $\alpha_{ij} = \varepsilon_i - e_j \in \Sigma$.

Using this correspondence, we can write the formulae (H) in a completely uniform way:

$$h_{ij}(t)\; u_{kl}(x)\; h_{ij}(t)^{-1} = u_{kl}(\; t^{\langle \alpha_{ij}, \alpha_{kl} \rangle}\; x\;). \tag{1}$$

We leave the proof of this interesting fact as an exercise to the reader.

2 Linear algebraic groups over fields

The basic language for algebraic groups is that of algebraic geometry. An algebraic k-group G over a field k is an algebraic k-variety, which we won't define here formally. We here will just think of them as the set $G(K)$ of common solutions of one or more polynomials with coefficients in k, as in the easy example of $\mathrm{SL}_n(k)$, which can be considered as the set of zeros $x = (x_{ij}) \in k^{n^2}$ of the polynomial equation $\det(x) - 1 = 0$. Clearly, if k'/k is a field extension (or even a commutative ring extension), then we may as well consider the solutions of the same set of polynomials, but over k' rather than over k, and we will denote those by $G(k')$. Solutions over k' are called the *k'-rational points of G*.

By k-morphisms between such varieties we will mean mappings which are given by polynomials with coefficients in k.

For example, $\det : \mathrm{GL}_n(k) \to k^*$ is such a k-morphism – which at the same time is a homomorphism of groups as well.

For more details concerning the algebraic geometry needed for our topic see [24], chap. 1.

Definition 2.1 *A linear algebraic k-group is an affine k-algebraic variety G, together with k-morphisms $(x, y) \mapsto xy$ of $G \times G$ into G ("multiplication") and $x \mapsto x^{-1}$ of G into G ("inversion") such that the usual group axioms are satisfied, and such that the unit element 1 is k-rational.*

This definition makes $G(k')$ a group for every extension field (or commutative ring) k'/k.

Since every polynomial over k is as well a polynomial over k', an algebraic group G over k becomes an algebraic group over k' in a natural way, this is denoted by $G \times_k k'$ and is said to be obtained from G *by field extension with* k'. Similarly for varieties in general.

The notions "linear algebraic group" and "affine algebraic group" are synonymous: every algebraic group which is an affine variety can be embedded by a k morphism as a closed k-subgroup into some GL_n ([24], 2.3.7).

There is a broader notion of algebraic groups, but here we will restrict to the linear or affine case.

In the following we will make the convention: The notion "k-group" will always mean "linear algebraic k-group"; an "algebraic k-subgroup" of a k-group G will mean an algebraic k-group which is a k-subvariety of G. A homomorphism of k-groups is a k-morphism of algebraic varieties which induces a group homomorphism for the corresponding groups of rational points.

Examples:

- The linear k-group GL_n, defined by

  $$\mathrm{GL}_n(k) = \{ \ (x_{ij}, y) \in k^{n^2+1} \mid \det(x_{ij}) \cdot y = 1 \ \}$$

  Clearly, multiplication and inversion are given by the usual matrix operations, which are "polynomially" defined over any field.

  (As the usual condition $\det(x_{ij}) \neq 0$ is not a polynomial equation, we need the additional coordinate y to describe GL_n as an algebraic variety.)

- A k-subgroup of $\mathrm{GL}_n(k)$ is SL_n, defined by

  $$\mathrm{SL}_n(k) = \{ \ (x_{ij}) \in \mathrm{GL}_n(k) \mid \det(x_{ij}) - 1 = 0 \ \}.$$

- The "additive group" $\mathbf{G}_\mathrm{a}$ is defined by $\mathbf{G}_\mathrm{a}(k) = k$. Here the defining polynomial is just the zero polynomial. Multiplication and inversion are given by the additive group structure of k.

- The "multiplicative group" is given by $\mathbf{G}_\mathrm{m} = \mathrm{GL}_1$. We have $\mathbf{G}_\mathrm{m}(k) = \{ (x,y) \in k^2 \mid x \cdot y = 1 \}$.

- Let q be a regular quadratic form on k-vector space V of finite dimension n. Then we have its associated (or polar) bilinear form by $\langle x, y\rangle = q(x+y) - q(x) - q(y)$ for $x, y \in V$.

  For a basis $e_1, \ldots, e_n$ of V, the form q is represented by the symmetric matrix $A = (\langle e_i, e_j\rangle)_{i,j=1,\ldots,n} \in \mathrm{GL}_n(k)$. The groups defined by

  $$\begin{array}{lcl} \mathrm{O}_q(k) & = & \{ x \in \mathrm{GL}_n(k) \mid xAx^t - A = 0 \ ; \} \\ \mathrm{SO}_q(k) & = & \{ x \in \mathrm{GL}_n(k) \mid xAx^t - A = 0, \ \det x - 1 = 0 \} \end{array}$$

  are k-groups, called the *orthogonal* (resp. *special orthogonal*) group of the quadratic form q.

- The symplectic group $\mathrm{Sp}_{2n}(k)$ is defined by

  $$\mathrm{Sp}_{2n}(k) = \{ x \in \mathrm{GL}_{2n}(k) \mid xJx^t = J \}$$

  where $J = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \in \mathrm{GL}_{2n}(k)$.

Remark: For algebraic groups as varieties, the notions "connectedness" and "irreducibility" coincide. The connected component of an algebraic k-group G containing the unit element is often called its 1-component and denoted by G^0.

The 1-connected component of k-group is a closed normal k-subgroup of finite index, and every closed k-subgroup of finite index contains the 1-component ([24], 2.2.1).

Examples:

- GL_n, SL_n, SO_q, $\mathrm{Sp}_{2n}(k)$, $\mathbf{G}_\mathrm{a}$, $\mathbf{G}_\mathrm{m}$ are connected ([24], 2.2.3).

- The 1-connected component of O_q is SO_q: we have a decomposition into two components $\mathrm{O}_q(k) = \mathrm{SO}_q(k) \cup \mathrm{O}_q^-(k)$ with $\mathrm{O}_q^-(k) = \{x \in O_q(k) \mid \det x{+}1 = 0 \}$. (Clearly, O_q^- is not a group, but a coset in O_q.)

Many concepts of abstract groups (the notion "abstract" used here as opposed to "algebraic" groups) can be transferred to algebraic groups: Very often, if subgroups are concerned, one has to use the notion of "closed subgroups" which are closed as subvarieties.

For example, if H, K are closed k-subgroups of a k-group G, then the subgroup $[H, K]$ generated by the commutators $[x, y] = x\ y\ x^{-1}\ y^{-1}$ with $x \in H(k)$, $y \in K(k)$ (more precisely, the Zariski closure of these) is a k-subgroup of G ([24], 2.2.8).

In this sense, the sequence $G^{(0)} = G$, $G^{(1)} = [G, G], \ldots, G^{(i+1)} := [G^{(i)}, G^{(i)}], \ldots$ of k-subgroups of G is well defined, and G is called *solvable* if this sequence becomes trivial after a finite number of steps.

There is also the concept of quotients: If H is a closed k-subgroup of the linear k-group G, then there is an essentially unique "quotient k-variety" G/H which is an affine k-group in case H is normal, but, as a variety, it is in general not affine (cf. [24], 5.5.5, 5.5.10 for algebraically close fields, and 12.2.1, 12.2.2 in general).

As an example for the latter fact, we look at $G = \mathrm{SL}_2$ and H the k-subgroup of upper triangular matrices. Then G/H is isomorphic to the projective line $\mathbf{P}^1$, since H is the stabilizer subgroup of a line in k^2, and G operates transitively on all lines in k^2.

A linear algebraic k-group T is called a *k-torus*, if, over an algebraic closure $\bar{k}$, the group $T(\bar{k})$ becomes isomorphic to a (necessarily finite) product of copies of groups $\mathbf{G}_{\mathrm{m}}(\bar{k})$.

A k-groups is called *unipotent*, if, after some embedding into some GL_n, every element becomes a unipotent matrix (i.e., a matrix with all eigenvalues equal to 1). It can be shown that this condition is independent of the embedding.

An example is the k-group of strictly upper diagonal matrices in GL_n.

The overall structure for connected linear algebraic groups over a *perfect* field k can be described as follows:

i) G has a unique maximal connected linear solvable normal k-subgroup $G_1 =: \mathrm{rad}\ G$, which is called the *radical of* G. The quotient group G/G_1 is *semisimple*, i.e., its radical is trivial.

ii) G_1 has a unique maximal connected normal unipotent k-subgroup $G_2 =: \mathrm{rad}_{\mathrm{u}}\ G$, which is called the *unipotent radical of* G. The quotient

G_1/G_2 is a k-torus, the quotient G/G_2 is a *reductive* k-group, i.e., it is an almost direct product of a central torus T and a semisimple group G'. (The notion "almost direct product" means that $G = T \cdot G'$ and $T \cap G'$ is finite.)

Let us summarize these facts in a table (recall that k is perfect here):

<table>
<tr><td></td><td>conn.
linear</td><td>(normal)</td><td>conn.
solvable</td><td>(normal)</td><td>conn.
unipotent</td></tr>
<tr><td>Groups:</td><td>G</td><td>$\rhd$</td><td>$G_1 = \mathrm{rad}\ G$</td><td>$\rhd$</td><td>$G_2 = \mathrm{rad_u}\ G$</td></tr>
<tr><td>Quotient:</td><td colspan="3">G/G_1
semisimple</td><td colspan="2">G_1/G_2
torus</td></tr>
<tr><td>Quotient:</td><td colspan="5">$G/G_2 = G' \cdot T$
reductive, i.e.,
almost direct product of semisimple G' with central torus T</td></tr>
</table>

In particular, for a k-group G the following holds:

i) G is reductive if and only if $\mathrm{rad_u}\ G = 1$

ii) G is semisimple if and only if $\mathrm{rad}\ G = 1$

For char $k = 0$ it is known that $\mathrm{rad_u}\ G$ has a reductive complement H such that $G = H\ \mathrm{rad_u}\ G$ is a semidirect product.

Examples:

i) For $G = \mathrm{GL}_n(k)$, $\mathrm{rad}\ G = \mathrm{center}(G)$ is isomorphic to $\mathbf{G}_m$, the group $G/\mathbf{G}_m$ is isomorphic to PGL_n, which is simple, and $\mathrm{rad_u}\ (G) = 1$.

ii) Let $e_1, \ldots, e_n$ be the standard basis of k^n. For $r < n, s = n - r$, take

$$G = \mathrm{Stab}(ke_1 \oplus \ldots \oplus ke_r) = \left\{ \begin{array}{c@{\,}c} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} r \\ s \end{array} & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array} \in \mathrm{GL}_n(k) \right\},$$

where r, s indicate the number of rows and columns of the submatrices denoted by $*$.

Then $\mathrm{rad}\, G$, $\mathrm{rad_u}\, G$ are given, respectively, by the matrices of the shape

$$\begin{pmatrix} \alpha & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & * & \\ 0 & \cdots & \alpha & & & \\ & & & \beta & \cdots & 0 \\ & 0 & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & * & \\ 0 & \cdots & 1 & & & \\ & & & 1 & \cdots & 0 \\ & 0 & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 1 \end{pmatrix}$$

Hence

$$\begin{array}{lcl} G/\mathrm{rad_u}\, G & \cong & \mathrm{GL}_r(k) \times \mathrm{GL}_s(k) \\ G/\mathrm{rad}\, G & \cong & \mathrm{PGL}_r(k) \times \mathrm{PGL}_s(k) \\ \mathrm{rad}\, G/\mathrm{rad_u}\, G & \cong & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \end{array}$$

Remarks about Tori (cf. [24], chap.3, for proofs):

A linear k-group T is a k-torus if, over some algebraic field extension k'/k, it becomes isomorphic to a (necessarily finite) product of multiplicative groups: $T \times_k k' \cong \Pi \mathbf{G}_{\mathrm{m}}$.

If this holds, then in particular $T \times_k k_s \cong \Pi \mathbf{G}_{\mathrm{m}}$ for a separable closure k_s of k.

A k-torus T is said to be *split* if $T \cong \Pi \mathbf{G}_{\mathrm{m}}$ (over k. T is said to be *anisotropic* if it does not contain any split subtorus

Example:

i) The algebraic $\mathbb{R}$-group T defined by

$$T(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \mid x^2 + y^2 = 1 \right\}$$

is an anisotropic torus, since it is compact and hence cannot be isomorphic to $\mathbb{R}^*$.

However, over $\mathbb{C}$ it becomes

$$T \times_{\mathbb{R}} \mathbb{C}(\mathbb{C}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \mid \lambda \neq 0 \right\} \cong \mathbb{C}^*$$

This can be seen by making a suitable coordinate change (Exercise!).

The group $T(\mathbb{R})$ can be considered as the group of norm-1-elements in $\mathbb{C}$.

ii) More generally, for any finite field extension k'/k, the group of norm-1-elements $\{x \in k'^* \mid N_{k'/k}(x) = 1\}$ is an anisotropic k-torus.

If T is a k-torus, k_s a separable closure of k, then the $\mathbb{Z}$-module of characters $X(T) = \mathrm{Hom}_{k_s}(T, \mathbf{G}_m)$ is in fact a module over the Galois group $\Gamma := \mathrm{Gal}(k_s/k)$.

We have:

- The group $X(T)$ is $\mathbb{Z}$-free-module and T splits if and only if Γ operates trivially on $X(T)$.
- The torus T is anisotropic if and only if $X(T)^\Gamma = \{0\}$.
- There exist a unique maximal anisotropic k-subtorus T_a of T, and a unique maximal split k-subtorus T_s of T such that $T_a \cdot T_s = T$ and $T_a \cap T_s$ is finite.

Parabolic subgroups, Borel subgroups, cf.[24], chap. 6

Definition: Let G be connected linear k-group.

- A maximal connected solvable k-subgroup of G is called a *Borel subgroup of G.*
- A k-subgroup of G is called *parabolic* if it contains a Borel subgroup of G.

Theorem 2.2 (Borel 1962)

Let G be a connected k-group and $\bar{k}$ an algebraic closure of k.

- *All maximal tori in G are conjugate over $\bar{k}$. Every semisimple element of G is contained in a torus; the centralizer of a torus in G is connected.*
- *All Borel subgroups in G are conjugate over $\bar{k}$. Every element of G is in such a group.*

- *Let P a closed k-subgroup of G. The quotient G/P is projective if and only if P is parabolic. If P is a parabolic subgroup of G, then it is connected and self-normalizing, i.e., equal to its normalizer subgroup $N_G(P)$ in G. If P, Q are two parabolic subgroups containing the same Borel subgroup B of G and if they are conjugate, then $P = Q$.*

Example: Let $G = \mathrm{GL}_n$.

i) The k-subgroup $T = \mathrm{Diag}_n$ of diagonal matrices is a maximal torus of G.

ii) The k-subgroup $B = \left\{ \begin{pmatrix} \star & \star & \star \\ & \ddots & \star \\ 0 & & \star \end{pmatrix} \in GL_n \right\}$ of upper triangular matrices is a Borel subgroup of G.

iii) $P := \mathrm{Stab}(ke_1) = \left\{ \begin{array}{c c} & \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{array}{c} 1 \\ n-1 \end{array} & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array} \in \mathrm{GL}_n(k) \right\}$ contains the group B of upper triangular matrices and is hence parabolic, and we have $G/P \cong \mathbf{P}^{n-1}$. (Here the numbers $1, n-1$ again denote number of rows and columns.)

3 Root systems, Chevalley groups

Root systems, definitions and basic facts:

Good references for this subsection are [23], chap. V, and [6], chap. VI.

Let V be a finite dimensional $\mathbb{R}$-vector space with scalar product $\langle\ ,\ \rangle$. A set $\Sigma \subset V \setminus \{0\}$ is a *root system* in V if the following statements i), ii), iii) hold:

i) The set Σ is finite, generates V, and $-\Sigma = \Sigma$.

ii) For each $\alpha \in \Sigma$, the linear map $s_\alpha : V \to V$ defined by $s_\alpha(v) = v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha,\alpha\rangle}\alpha$ leaves Σ invariant: $s_\alpha(\Sigma) = \Sigma$.

iii) For each pair $\alpha, \beta \in \Sigma$, the number $n_{\beta,\alpha} = 2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}$ is an integer – a so-called "Cartan integer".

Let Σ be a root system on V.

$\operatorname{rank}(\Sigma) := \dim V$ is called the *rank* of Σ.

Σ is called *reducible* if there exist proper mutually orthogonal sub-spaces V', V'' of V such that $V = V' \perp V''$ and $\Sigma = (V' \cap \Sigma) \cup (V'' \cap \Sigma)$. Otherwise Σ is called *irreducible*.

Σ is called *reduced* if $\mathbb{R}\alpha \cap \Sigma = \{ \pm\alpha \}$ for all $\alpha \in \Sigma$. (In general one has $\mathbb{R}\alpha \cap \Sigma \subset \{ \pm\alpha, \pm\alpha/2, \pm 2\alpha \}$.)

By definition, s_α is the reflection on the hyperplane $H_\alpha = \mathbb{R}\alpha^\perp$ of V orthogonal to α. The *Weyl group* $W(\Sigma)$ *of* Σ is the subgroup of $\operatorname{Aut}(V)$ generated by these reflections $\{s_\alpha \mid \alpha \in \Sigma\}$.

A *Weyl chamber* of Σ is a connected component of $V \setminus \cup_{\alpha\in\Sigma} H_\alpha$. The Weyl group acts simply transitive on all Weyl chambers. Each Weyl chamber C defines an ordering of roots: $\alpha > 0$ if $(\alpha, v) > 0$ for every $v \in C$.

An element in Σ is called a *simple root (with respect to an ordering)* if it is not the sum of two positive roots. Every root is an integral sum of simple roots with coefficients of same sign. The number of simple roots of Σ with respect to any ordering is the same as $\dim V = \operatorname{rank}(\Sigma)$, hence any set of simple roots forms a basis of V.

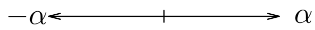
For the Cartan numbers one gets $n_{\beta,\alpha} n_{\alpha,\beta} = 4 \, \frac{\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle} = 4 \, \cos^2\varphi$, where φ denotes the angle between α and β. As this is integral, only 8 angles are possible up to sign, and it is easily concluded that, for reduced root systems, only roots of up to two different lengths can occur. If $\langle\beta,\beta\rangle \geq \langle\alpha,\alpha\rangle$, then $\langle\beta,\beta\rangle = p\langle\alpha,\alpha\rangle$ with $p = 1, 2, 3$.

The *Dynkin diagram* of Σ is a graph whose vertices are the simple roots, two vertices α, β with $p = \frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle} \geq 1$ are combined by p edges, moreover, the direction of the smaller root is indicated by $<$ or $>$ in case α, β have different length.

The Dynkin diagrams classify the root systems uniquely. Below we list all reduced root systems of rank ≤ 2 and their Dynkin diagrams.

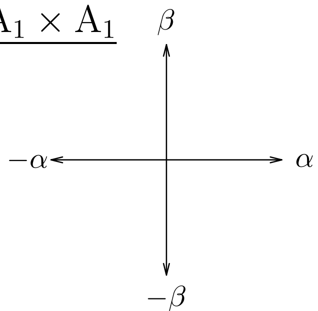
Reduced root systems of rank ≤ 2:

$\underline{\mathrm{A}_1}$



$W(\mathrm{A}_1) = \mathbb{Z}_2$, typical group: SL_2, Dynkin diagram: $\underset{\alpha}{\circ}$

$\underline{\mathrm{A}_1 \times \mathrm{A}_1}$

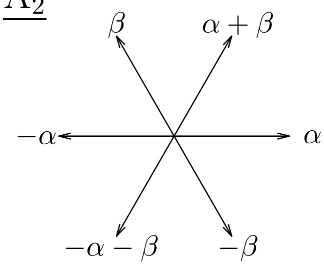


typical group: $\mathrm{SL}_2 \times \mathrm{SL}_2$

$n_{\beta,\alpha} = n_{\alpha,\beta} = 0$

Dynkin diagram: $\underset{\alpha}{\circ} \qquad \underset{\beta}{\circ}$

$\underline{\mathrm{A}_2}$

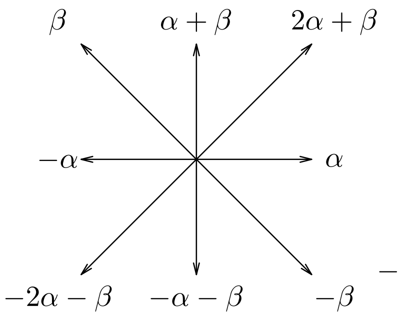


typical group: SL_3

$n_{\beta,\alpha} = n_{\alpha,\beta} = -1$

Dynkin diagram: $\underset{\alpha}{\circ}$——$\underset{\beta}{\circ}$

$\underline{\mathrm{B}_2 \cong \mathrm{C}_2}$

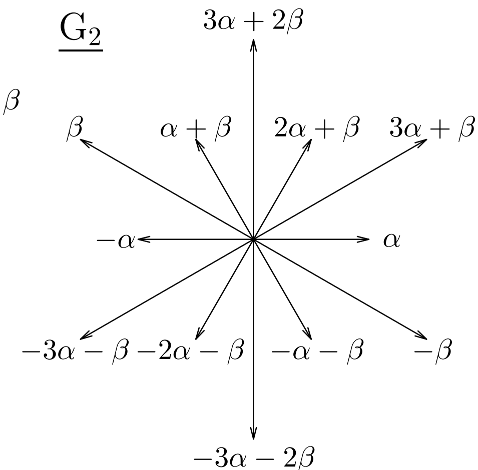


typical groups: SO_5, Sp_4

$n_{\beta,\alpha} = -2, n_{\alpha,\beta} = -1$

Dynkin diagram: $\underset{\alpha}{\circ} \Leftarrow \underset{\beta}{\circ}$

$\underline{\mathrm{G}_2}$



typical group: automorphism group of a Cayley algebra

$n_{\beta,\alpha} = -3, n_{\alpha,\beta} = -1$

Dynkin diagram: $\underset{\alpha}{\circ} \Lleftarrow \underset{\beta}{\circ}$

The Weyl groups of the rank 2 root systems are given by the presentation $W = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = (s_\alpha\ s_\beta)^\kappa = 1 \rangle$ with $\kappa = 2, 3, 4, 6$ respectively.

Root systems are, up to isometry, completely determined by their Dynkin diagrams and completely classified for arbitrary rank. Below we will give a complete list for the irreducible root systems.

Root system of a semisimple k-group G

Proofs for the facts given in the rest of this section can be found in the book by Springer [24], or also in the book by Borel [2].

Let $S \subset G$ be any k-torus.

The k-group G operates on its Lie algebra $\mathfrak{g}$ by the adjoint representation $\mathrm{Ad}_{\mathfrak{g}} : G \to \mathrm{Aut}\ \mathfrak{g}$ (cf. [24], chap. 4.4).

Since S contains only semisimple elements (which all commute with each other), it follows that $\mathrm{Ad}_{\mathfrak{g}}\ (S)$ is diagonalizable. Hence $\mathfrak{g}$ can be decomposed into eigenspaces using characters $\alpha : S \to \mathbf{G}_{\mathrm{m}}, \alpha \in X(S)$:

$$\mathfrak{g} = \mathfrak{g}_0^S \oplus \bigoplus_{\alpha \in X(S)} \mathfrak{g}_\alpha^S, \quad \mathfrak{g}_\alpha^S = \{\ x \in \mathfrak{g} \mid \mathrm{Ad}_{\mathfrak{g}}\ (s)(x) = \alpha(s)x\ \} \text{ for } \alpha \in X(S).$$

We denote the set of characters α which occur in the above decomposition by $\Sigma(G, T)$.

Assume now that k is algebraically closed, i.e., $k = \bar{k}$. Then G has a maximal split torus T. Since all such tori are conjugate in G, the set $\Sigma(G, T)$ is essentially independent of T and is denoted by $\Sigma(G) = \Sigma(G, T)$, and is called “the” *set of roots* of G.

The normalizer $N = \mathrm{Norm}_G T \subset G$ operates on the group of characters $X = X(T) = \mathrm{Hom}(T, \mathbf{G}_{\mathrm{m}})$ (which is a free $\mathbb{Z}$-module of rank $= \dim T$. We choose a scalar product $\langle\ ,\ \rangle$ on the $\mathbb{R}$-vector space $V = X \otimes_{\mathbb{R}} \mathbb{R}$ which is invariant under the (finite) group $W := N/T$.

Then the set $\Sigma := \Sigma(G)$ is a root system.

We choose an ordering (via some Weyl chamber), we also choose a set $\Delta \subset \Sigma$ of simple roots.

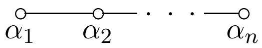
Then, by the preceding, we have $|\Delta| = \dim V = \dim T =:$ rank (G). (By definition, the rank of a semisimple group is the dimension of a maximal torus.)

The *Dynkin diagram of* G is the Dynkin diagram of Σ.

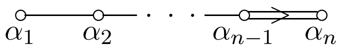
Here we will give a list of all irreducible reduced root systems together with the corresponding groups, insomuch they have special names. The number n will always denote the rank.

We have four infinite series $(\mathrm{A}_n)_{n\geq 1}$, $(\mathrm{B}_n)_{n\geq 2}$, $(\mathrm{C}_n)_{n\geq 3}$, $(\mathrm{D}_n)_{n\geq 4}$, and several "exceptional" root systems: $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$.

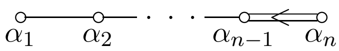
Their Dynkin diagrams look as follows:

A_n:


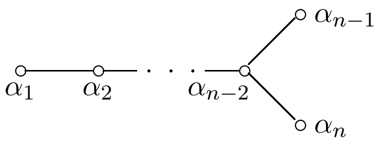
$n \geq 1$ SL_{n+1}

B_n:


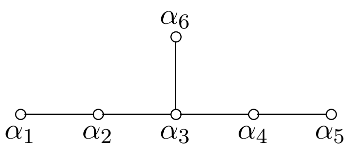
$n \geq 2$ SO_q, $\dim q = 2n+1$

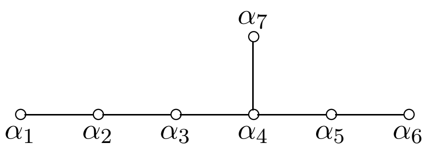
C_n:


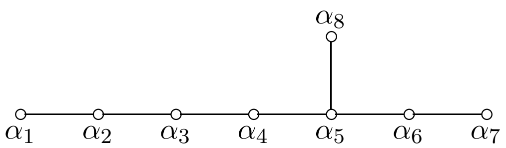
$n\geq 3$, $\mathrm{C}_2 \cong \mathrm{B}_2$ Sp_{2n}

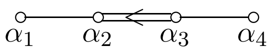
D_n:


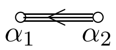
$n\geq 4$, $\mathrm{D}_3 \cong \mathrm{A}_3$ SO_q, $\dim q = 2n$

E_6:


E_7:


E_8:


F_4:


G_2:


Isogenies

An isogeny φ is a k-morphism $H \to G$ of k-groups such that $\ker\varphi$ is finite and φ is surjective over $\bar{k}$.. (This implies that H, G are of the same dimension.)

An isogeny φ is central if $\ker\varphi \subset \mathrm{center} H$.

The k-group G and G' are strictly isogenous if there exists a k-group H and central isogenies $\varphi : H \to G$ and $\varphi' : H \to G'$

A semisimple k-group G is *simply connected*, if there is no proper central isogeny $G' \to G$ with a semisimple k-group G'. It is *adjoint*, if its center is trivial. It is known that for each semisimple k-group there exists a simply connected k-group $\hat{G}$, and an adjoint group $\bar{G}$. Hence that there are central isogenies $\hat{G} \to G \to \bar{G}$. The kernel of the composite (which is a central isogeny as well), is the center of $\hat{G}$ and a finite group, which only depends on the Dynkin diagram of G ([24], §8).

Examples:

i) $\mathrm{SL}_n, \mathrm{PGL}_n$ are simply connected, resp. adjoint, the natural map $\mathrm{SL}_n \to \mathrm{PGL}_n$ is a central isogeny, and its kernel is cyclic of order n.

ii) Let q be a nondegenerate quadratic form over a field k of characteristic $\neq 2$. The homomorphisms $\mathrm{Spin}_q \to \mathrm{SO}_q \to \mathrm{PSO}_q$ are central isogenies as well as their composite, Spin_q is simply connected, PSO_q is adjoint. If $\dim q$ is odd, one has $\mathrm{SO}_q \cong \mathrm{PSO}_q$, otherwise (i.e., if $\dim q$ is even), the kernel of $\mathrm{SO}_q \to \mathrm{PSO}_q$ is of order 2. Moreover

$$\text{center } \mathrm{Spin}_q \cong \begin{cases} \mathbb{Z}_2 & \text{if } \dim q \text{ is odd} \\ \mathbb{Z}_4 & \text{if } \dim q \equiv 2 \bmod 4 \\ \mathbb{Z}_2 \times Z_2 & \text{if } \dim q \equiv 0 \bmod 4 \end{cases}$$

For the definition of Spin_q cf. [10], 7.2A, or any book on quadratic forms.

Main classification theorem: $k = \bar{k}$

The following is the main theorem for semisimple k-groups over algebraically closed fields k. It was proved by W. Killing (1888) for $K = \mathrm{C}$, and for arbitrary algebraically closed fields by C. Chevalley (1958).

Theorem 3.1 *i) A semisimple k-group is characterized, up to strict isogeny, by its Dynkin diagram.*

ii) A semisimple k-group is almost simple if and only if its Dynkin diagram is connected.

iii) Any semisimple k-group G is isogeneous to the direct product of simple groups whose Dynkin diagrams are the connected components of the Dynkin diagram of G.

Chevalley groups (arbitrary k):

Proposition 3.2 *For field k and any Dynkin diagram $\mathfrak{D}$ there exists a semisimple k-group G such that $\mathfrak{D}$ is the Dynkin diagram of G.*

Remark: A group G like this exists even over $\mathbb{Z}$. This was proved by Chevalley, a proof can be found in [27].

Definition A *Chevalley group over k* is a semisimple k-group with a split maximal k torus.

Structural theorem for Chevalley groups:

Let k be any field, and let G be a Chevalley group over k with maximal split torus T and Σ a set of roots for G with respect to T.

Theorem 3.3 *i) For each $\alpha \in \Sigma$, there exists a k-isomorphism $u_\alpha : \mathbf{G}_{\mathrm{a}} \tilde{\to} U_\alpha$ onto a unique closed k-subgroup $U_\alpha \subset G$, such that $t\, u_\alpha(x)\, t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in T, x \in k$. Moreover, $G(k)$ is generated by $T(k)$ and all $U_\alpha(k)$.*

ii) For every ordering of Σ, there is exactly one Borel group B of G with $T \subset B$ such that $\alpha > 0$ if and only if $U_\alpha \subset B$.

Moreover, $B = T \cdot \prod_{\alpha>0} U_\alpha$ and $\mathrm{rad}_{\mathrm{u}}\, B = \prod_{\alpha>0} U_\alpha$.

iii) The subgroup $\langle U_{-\alpha}, U_\alpha \rangle$ is isogeneous to SL_2 for every $\alpha \in \Sigma$.

iv) (Bruhat decomposition) *Let $N = \mathrm{Norm}_C(T)$, then $W = N/T$ is the Weyl group, we have a disjoint decomposition of $G(k)$ into double cosets: $G(k) = B(k)N(k)B(k) = B(k)WB(k) = \cup_{w\in W} B(k)WB(k)$.*

Parabolic subgroups

Let G be a semisimple k-group with a split maximal k-torus (i.e., G is a Chevalley group) with root system Σ, let B be a Borel group containing T, and let $\Delta \subset \Sigma$ be the set of simple roots corresponding to B.

- There is a 1-1 correspondence of parabolic subgroups P_θ containing B and the subsets $\theta \subset \Delta$, given by:

  $$\Theta \mapsto P_\theta = \langle T, U_{\alpha(\alpha\in\Delta)}, U_{-\alpha(\alpha\in\Theta)}\rangle.$$

  In particular one has $B = P_\emptyset$ and $G = P_\Delta$.

- For $\Theta \subset \Delta$, let $W_\theta = \langle s_\alpha \in W \mid \alpha \in \Theta\rangle$.

  There is a Bruhat decomposition into double cosets (disjoint, as usual):

  $$P_\Theta(k) = \bigcup_{w\in W_\Theta} B(k)wB(k).$$

  The structure of P_Θ is as follows: It has a so-called *Levi decomposition*:

  $$P_\Theta = L_\Theta \ltimes \mathrm{rad_u}\, P_\Theta.$$

  L_Θ here is the *Levi subgroup* of P_Θ. This is a reductive k-group, obtained as the centralizer of the k-torus $T_\Theta := \bigcap_{\alpha\in\Theta}(\mathrm{Ker}\ \alpha)^0 \subset T$. Hence

  $$L_\Theta = Z_G(T_\Theta).$$

  The unipotent radical is described by

  $$\mathrm{rad_u}\, P_\Theta = \langle U_\alpha \mid \alpha > 0,\ \alpha \notin \sum_{\gamma\in\Theta} \mathbb{R}\gamma\rangle,$$

  i.e., it is generated by all U_α for which $\alpha > 0$ and α is not a linear combination of elements from Θ.

Remark:

The data $(G(k), B(k), N, S)$ with $S = \{ s_\alpha \mid \alpha \in \Delta \}$ fulfill the axioms of a "BN pair" or a "Tits system" in the sense of [6], chap. IV, §2.

These axioms are the foundation for the setup of the so-called "buildings", which give a geometrical description of the internal groups structure of Chevalley groups.

4 K-theoretic results related to Chevalley groups

Let G be a simply connected Chevalley group over k, (i.e., having no proper algebraic central extension), let T be a maximal split torus and Σ the set of roots of G for T.

For $\alpha \in \Sigma$, recall the embeddings $u_\alpha : \mathbf{G}_{\mathrm{a}} \to U_\alpha \subset G$ from the last section, and define:

$$\begin{array}{rcll} w_\alpha(x) & = & u_\alpha(x)u_{-\alpha}(-x^{-1})u_\alpha(x) & (x \in k, x \neq 0) \\ h_\alpha(x) & = & w_\alpha(x)w_\alpha(-1) & (x \in k, x \neq 0) \end{array}$$

The Steinberg relations for G:

For every $x, y \in K$ and $\alpha, \beta \in \Sigma$:

(A) $u_\alpha(x+y) = u_\alpha(x)u_\alpha(y)$

(B) $[u_\alpha(x), u_\beta(y)] = \prod\limits_{\substack{i,j>0 \\ i\alpha+j\beta\in\Sigma}} u_{i\alpha+j\beta}(c_{\alpha\beta;ij}\; x^i\, y^j)$

Here the product is taken in some lexicographical order, with certain coefficients $c_{\alpha\beta;ij} \in \mathbb{Z}$, called *structure constants*, independent of x, y and only dependent on the Dynkin diagram of G.

(B′) $w_\alpha(t)u_\alpha(x)w_\alpha(t)^{-1} = u_{-\alpha}(-t^{-2}x),$ for $t \in k^\star, x \in k$.

(C) $h_\alpha(xy) = h_\alpha(x)h_\alpha(y)$

Theorem 4.1 (Steinberg, [25], Thms. 3.1, 3.2 ff., Thm. 4.1 ff. Theorem 1.1 in section 1 was just the special case $G = \mathrm{SL}_n$.)

Given Σ, there exists a set of structure constants $c_{\alpha\beta,ij} \in \mathbb{Z}$ for $\alpha, \beta, i\alpha + j\beta \in \Sigma, (i, j \geq 1)$, such that the following holds.

i) Let $\hat{G}$ denote the simply connected covering of G. A presentation of $\hat{G}(k)$ is given by

(A), (B), (C) *if* rank $G \geq 2$, *and*
(A), (B′), (C) *if* rank $G = 1$.

ii) Denote by $\tilde{G}$ the group given by the presentation

(A), (B) *if* rank $G \geq 2$, *and*
(A), (B′) *if* rank $G = 1$,

Then the canonical map $\pi : \tilde{G} \to \hat{G}(k)$ is central, and $\tilde{G}$ is perfect, i.e., $\tilde{G} = [\tilde{G}, \tilde{G}]$.

If, moreover, $|k| > 4$ for rank $G > 1$, *and $|k| \neq 4, 9$ for* rank $G = 1$, *then $\tilde{G}$ is the universal central extension of $\hat{G}(k)$.*

(Note: $\tilde{G}$ is <u>not</u> an algebraic group !)

Remarks:

- Again, $\tilde{G}$, as universal extension of $\hat{G}(k)$, is unique up to isomorphism. For given Σ, the group $\mathrm{St}_\Sigma(k) = \tilde{G}$ is called the Steinberg group for the Chevalley groups defined by Σ or just the Steinberg group of Σ.
- If k is finite, then $\tilde{G}$ and $\hat{G}(k)$ are equal.
- The relation (B′) is a consequence of the relations (A), (B) in case rank $\Sigma) > 1$.
- There are various possible choices for the sets of “structure constants”, the interdependence of the coefficients for a given Chevalley group is delicate, cf. [24], 9.2 for a discussion.
- The roots occurring on the right of (B) can be read off the two dimensional root systems, since $\{\alpha, \beta\}$ generate a sub-root system of rank 2. E.g., for $G = \mathrm{G}_2$, one may have:

  $$[u_\alpha(x), u_\beta(y)] = u_{\alpha+\beta}(x\, y)\; u_{2\alpha+\beta}(-x^2\, y)\; u_{3\alpha+\beta}(-x^3\, y)\; u_{3\alpha+2\beta}(x^3\, y^2)$$

  and this is the longest product which might occur (cf. [27], p. 151 – please note that long and short roots are differently named there).

  For groups of type different from G_2, at most two factors do occur on the right of (B).

- Each relation involves only generators of some almost simple rank 2 subgroups (generated by $u_\alpha(x), u_\beta(y)$ involved, hence the theorem implies that $\hat{G}(k)$ is an amalgamated product of its almost simple rank 2 subgroups.

Theorem 4.2 (Matsumoto, cf. [12], Prop. 5.5 ff., Thm. 5.10, Cor. 5.11, see also Moore[16])*:*

For $\alpha \in \Sigma$, define $c_\alpha(x,y) := h_\alpha(x)h_\alpha(y)h_\alpha(xy)^{-1} \in \tilde{G}$.

Then, for each long root $\alpha \in \Sigma$, the values $c_\alpha(x,y), x, y \in k^*$ generate the kernel of $\pi : \tilde{G} \to \hat{G}(k)$.

They fulfill the following set of relations (setting $c(x,y) := c_\alpha(x,y)$), which gives a presentation of kernel π*:*

rank $G = 1$ *or G symplectic: (cf.* [12], Prop. 5.5)

(S1)	$c(x,y)\, c(xy,z) = c(x,yz)\, c(y.z)$	
(S2)	$c(1,1) = 1,\ c(x,y) = c(x^{-1}, y^{-1})$	
(S3)	$c(x,y) = c(x,(1-x)y)$	*for* $x \neq 1$

rank ≥ 2 *and G not symplectic: (cf.* [12], Lemme 5.6)

(S°1)	$c(x,yz) = c(x,y)\, c(x,z)$	
(S°2)	$c(xy,z) = c(x,z)\, c(y,z)$	
(S°3)	$c(x,1-x) = 1$	*for* $x \neq 1$

Define $c^\natural(x,y) := c(x,y^2)$. Then $c^\natural(x,y)$ fulfills the relations (S°1)*,* (S°2)*,* (S°3)*,*

Remarks:

Hence we have two groups, the "usual" $K_2(k)$, occurring as the kernel of π for non-symplectic groups (excluding SL_2) and the "symplectic: K_2^{sym}, occurring as the kernel of π for the symplectic groups as well as for SL_2:

$K_2^{\mathrm{sym}}(k)$	defined by generators and relations as in (S1), (S2), (S3)
$K_2(k)$	defined by generators and relations (S°1), (S°2), (S°3)

From the theorem, we also obtain two homomorphisms

$$\begin{array}{ll} K_2^{\mathrm{sym}}(k) \to K_2(k) & c(x,y) \mapsto c(x,y), \\ K_2(k) \to K_2^{\mathrm{sym}}(k) & c(x,y) \mapsto c^\natural(x,y) = c(x,y^2). \end{array}$$

These maps are interrelated in a surprising way with the Witt ring of the underlying field K as described by Suslin ([28], §6 (p. 26)).

The Witt ring $W(k)$ of nondegenerate symmetric bilinear forms over k contains the maximal ideal $I(k)$ of even dimensional forms as well as its powers $I^r(k), r = 1, 2, \dots$

It is well known that $I^r(k)$ is generated by so-called Pfister forms $\langle\langle a_1, \dots, a_r\rangle\rangle = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_r\rangle$.

Suslin observed two things:

i) There is a natural homomorphism $\varphi : K_2^{\mathrm{sym}} \to I^2(k), \quad c(x, y) \mapsto \langle\langle x, y\rangle\rangle$, and kernel φ is generated by $(x, y) = c(x, y^2)$. This gives rise to an exact sequence

$$1 \to K_2(k)/\{c(x, -1) \mid x \in k^*\} \to K_2^{\mathrm{sym}}(k) \xrightarrow{\varphi} I^2(k) \to 1.$$

ii) There is an isomorphism ψ from the kernel of $K_2^{\mathrm{sym}}(k) \to K_2(k)$ onto $I^3(k)$, sending each element $c(x, y)\ c(x, z)\ c(x, yz)^{-1}$ to the Pfister form $\langle\langle x, y, z\rangle\rangle$. This yields an exact sequence

$$1 \to I^3(k) \xrightarrow{\psi} K_2^{\mathrm{sym}}(k) \to K_2(k) \to 1.$$

Combining this for char $k \neq 2$, one gets the following diagram with exact rows and columns, where $\mathrm{Br}_2(k)$ denotes the 2-torsion part of the Brauer group of k:

The exactness of the third vertical sequence is just the norm residue theorem by Merkurjev and Suslin (cf. [14]).

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & \dfrac{K_2(k)}{\langle c(x,-1) \mid x \in k^*\rangle} & \xrightarrow{2} & 2K_2(k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle 2} & & \downarrow & & \\
1 & \longrightarrow & I^3(k) & \xrightarrow{\psi} & K_2^{\mathrm{sym}}(k) & \longrightarrow & K_2(k) & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
1 & \longrightarrow & I^3(k) & \longrightarrow & I^2(k) & \longrightarrow & \mathrm{Br}_2(\mathrm{k}) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

This diagram can be found as well in [10], 6.5.13.

Since $K_2(2,k) = K_2^{\mathrm{sym}}(k)$, we find, as a corollary, the following exact sequence:

$$1 \longrightarrow I^3(k) \longrightarrow K_2(2,k) \longrightarrow K_2(k) \longrightarrow 1$$

This has been generalized: In [17], it is shown that for the Laurent polynomial ring $k[\xi, \xi^{-1}]$, there are the following exact sequences:

$$1 \longrightarrow I^3(k) \oplus I^2(k) \longrightarrow K_2^{\mathrm{sym}}(k[\xi,\xi^{-1}]) \longrightarrow K_2(k[\xi,\xi^{-1}]) \longrightarrow 1,$$
$$1 \longrightarrow \mathrm{N}(k) \oplus \{\pm 1 \in k\} \longrightarrow K_2(k[\xi,\xi^{-1}]) \longrightarrow K_2^{\mathrm{sym}}(k[\xi,\xi^{-1}]) \longrightarrow I^2(k) \oplus I(k) \longrightarrow 1.$$

Here $\mathrm{N}(k)$ denotes the subgroup of $K_2(k)$ generated by $c(x,-1)$ with $x \in k^*$.

These sequences give rise to a similar commutative diagram as above, cf. [17].

5 Structure and classification of almost simple algebraic groups

Let k be a field, k_s a separable closure of k and $\Gamma = \mathrm{Gal}(k_s/k)$ and G a semisimple k-group.

The anisotropic kernel of G:

Denote by S a maximal k-split torus of G, and assume that T is a maximal torus of G defined over k and containing S. This can be achieved by taking T as a maximal k-torus in the centralizer of S in G, which is a connected reductive k-subgroup of G by [24], 15.3.2.

A proof of the existence of T can be found in [24], 13.3.6ff. This important theorem had first been proved by Chevalley for characteristic 0, and later in general by Grothendieck [7] Exp XIV]. Since every k-torus is an almost direct product of a unique anisotropic and a unique split k-subtorus (cf, [24] 13.2.4), we obtain S as the split part of T.

Since all maximal split k-tori of G are conjugate over k (cf. [24], 15.2.6), they are isomorphic over k, and hence the dimension of S is an invariant of G, called the *k-rank* of G and denoted by $\mathrm{rank}_k\, G$.

T can be split over a finite separable extension, and since any semisimple group with a split maximal torus is a Chevalley group, this is true for $G \times_k k_s$

Remark: T and S are usually different, as seen by the following example:

> Take $G = \mathrm{SL}_{r+1}(D)$[3], where D/k is a central k-division algebra of degree $d > 1$.
>
> By a theorem of Wedderburn, $G \times_k k_s(k_s) \cong \mathrm{SL}_{r+1}(M_d(k_s)) \cong \mathrm{SL}_{d(r+1)}(k_s)$

[3] It should be mentioned here that $\mathrm{SL}_{r+1}(D)$, as an algebraic group, is defined to be the kernel of the "reduced norm" $\mathrm{RN} : \mathrm{M}_{r+1}(D) \longrightarrow k$, defined as follows. Let A be any finite dimensional central simple k-algebra. Then, by Wedderburn's theorem, $A \otimes_k \bar{k} \cong \mathrm{M}_m(\bar{k})$ for some m, hence $\dim_k A = m^2$. The characteristic polynomial $\chi_{a\otimes 1}$ of the matrix $a \otimes 1$ for any $a \in A$ has coefficients in k and is independent of the embedding of A in $\mathrm{M}_m(\bar{k})$ (cf. [5], Algèbre...):

$$\chi_{a\otimes 1}(X) = X^m - s_1(a)X^{m-1} + s_2(a)X^{m-2} \mp \cdots + (-1)^m s_m(a).$$

Clearly, $s_1(a), s_m(a)$ are trace and determinant of $a \otimes 1$, they are called the *reduced trace* and the *reduced norm* of a, and are obtained as polynomials with coefficients in k: $\mathrm{RS}(a) = s_1(a), \mathrm{RN}(a) = s_m(a)$, hence $\mathrm{RS} : A \longrightarrow k$ is k-linear and $\mathrm{RN} : A \longrightarrow k$ is multiplicative.

In our example above, $A = M_{r+1}(D)$.

A maximal split torus $S \subset \mathrm{SL}_{r+1}(D)$ is given by the diagonal matrices with entries in k and of determinant 1, hence $S \cong \mathbf{G}_{\mathrm{m}}^r$. A maximal torus in $\mathrm{SL}_{d(r+1)}(k_s)$ also consists of the diagonal matrices of determinant 1 and hence has dimension $(r+1)d-1$.

Hence we have $\operatorname{rank}_k \mathrm{SL}_{r+1}(D) = r$, and $\operatorname{rank} \mathrm{SL}_{r+1}(D) = (r+1)d-1$.

We see in this case that $\operatorname{rank}_k \mathrm{SL}_{r+1}(D) = \operatorname{rank} \mathrm{SL}_{r+1}(D)$ if and only if $d = 1$, i.e., if and only if $D = k$.

The other extreme case is $\operatorname{rank}_k \mathrm{SL}_{r+1}(D) = 0$, which means that $r = 0$, hence our group is $\mathrm{SL}_1(D)$, the group of elements in D which are of reduced norm 1.

Definition

The group G is called *isotropic* if it contains a non trivial k-split torus (i.e., if $\operatorname{rank}_k G > 0$), and *anisotropic* otherwise.

Examples:

i) Let D/k be as above and $G = \mathrm{SL}_{r+1}(D)$. As we have seen, this is isotropic if and only if $r > 0$.

ii) Let q be a regular quadratic form over k. Then we have a Witt decomposition

$$q = q_{\mathrm{an}} \perp \mathbb{H}^r$$

into the “anisotropic kernel” q_{an} of q and a direct sum of $r \geq 0$ hyperbolic planes. The anisotropic kernel q_{an} is uniquely determined by q up to isometry. r is called the *Witt index* of q and is of course also uniquely determined by q.

We have $\mathrm{SO}(\mathbb{H}) \cong \mathbf{G}_{\mathrm{m}}$, hence SO_q admits the embedding of a split k-torus $S \cong \mathbf{G}_{\mathrm{m}}^i$ (one for each summand $\mathbb{H}$), and in fact this is maximal: $\operatorname{rank}_k \mathrm{SO}_q = r$.

Over k_s the number of $\mathbb{H}$-summands in the Witt decomposition attains the maximal possible value $[\dim q/2]$.

Hence we have $\operatorname{rank} \mathrm{SO}_q = [\dim q/2]$, and SO_q is anisotropic if and only if q is anisotropic as a quadratic form.

Arbitrary semisimple k-groups behave similarly as quadratic forms do under the Witt decomposition as seen in the preceding example.

In order to understand the following construction it is helpful to think of the example SO_q as being represented using the Witt decomposition.

We introduce the following notation for certain k-subgroups of G:

$\mathrm{Z}(S)$	the centralizer of S in G	$(=$ reductive$)$
$\mathrm{DZ}(S) = [\mathrm{Z}(S), \mathrm{Z}(S)]$	its derived group	$(=$ semisimple$)$
Z_{a}	max. anisotropic subtorus of center$(\mathrm{Z}(S))$	

If we "visualize" these groups for SO_q by matrices with respect to a basis aligned along a Witt decomposition for q as above, $q = q_{\mathrm{an}} \perp \mathbb{H}^r$, the matrices for $\mathrm{Z}(S)$ look as follows:

$$\begin{pmatrix} \mathrm{DZ}(S) & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & s_1 & 0 & & & \vdots \\ \vdots & 0 & s_1^{-1} & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & s_r & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & s_r^{-1} \end{pmatrix}$$

$\mathrm{DZ}(S)$ is a $\dim q_\alpha \times \dim q_\alpha$-matrix, and each s_i, s_i^{-1} pair is a torus component and belongs to a copy of $\mathbb{H}$. The torus Z_{a} is a maximal k-torus in $\mathrm{DZ}(S)$, hence $T = Z_{\mathrm{a}}\, S$ as an almost direct product.

Definition

i) The group $\mathrm{DZ}(S)$ is called a *semisimple anisotropic kernel of* G,

ii) the group $\mathrm{DZ}(S) \cdot Z_a$ is called a *reductive anisotropic kernel of* G.

Proposition 5.1 i) *The semisimple anisotropic kernels of G are precisely the subgroups occurring as derived group of the Levi-k-subgroups (=semi simple parts) of the minimal parabolic k-subgroups. Any two such are conjugate under $G(k)$.*

ii) *The anisotropic kernels of G are anisotropic k-groups.*

iii) *The group G is quasi-split (i.e., has a k-Borel subgroup) if and only if its semisimple anisotropic kernel is trivial.*

The Tits index of a semisimple group G:

Let Δ be a system of simple roots of $G \times_k k_s$ with respect to T (and some ordering) and define $\Delta_0 = \{ \alpha \in \Delta \text{ such that } \alpha|_S = 0 \}$

Then Δ_0 is the set of simple roots of $\mathrm{DZ}(S)$ with respect to $T \cap \mathrm{DZ}(S)$.

The group $\Gamma = \mathrm{Gal}(k_s/k)$ acts on Δ as follows: To each $\alpha \in \Delta$, we associate the maximal proper parabolic subgroup $P_{\Delta\setminus\alpha}$ (each represents one of these conjugacy classes).

The group Γ operates on the set of conjugacy classes of parabolic subgroups, and thereby on Δ:

$$(\gamma, \alpha) \longrightarrow \gamma * \alpha \qquad (\gamma \in \Delta)$$

This operation is called the $*$-*operation* or *star-operation*.

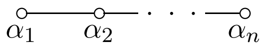
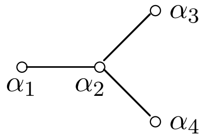
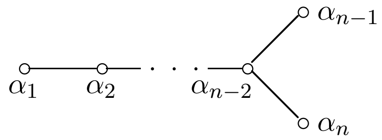
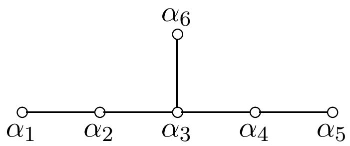
Beware! This is not(!) the same as ${}^{\gamma}\alpha$, as this root usually is not in Δ. Namely, γ induces a switch of the ordering of Δ (which is defined by its underlying Weyl chamber), but then there is a unique $w \in W$ with $w({}^{\gamma}\Delta) = \Delta$, as W operates simply transitively on the Weyl chambers. Hence $\gamma * \alpha = w\,{}^{\gamma}\alpha$.

Definition (Tits index):

- The group G is said to be of *inner type* if the $*$-operation is trivial.
- The group G is said to be of *outer type* if not.
- The *Tits index of G* is given by (Δ, Δ_0) together with $*$-operation (leaving Δ_0 invariant).

  The subset $\Delta_0 = \{ \alpha \in \Delta \text{ such that } \alpha|_S = 0 \}$ is called the system of *distinguished roots* of G.

Remark: Only the following Dynkin diagrams allow non-trivial automorphisms and therefore are candidates for the $*$-operation:

$\mathrm{A}_{n(n\geq 2)}$		$\alpha_i \leftrightarrow \alpha_{n+1-i}$ $(i = 1, \dots, n)$	$\mathrm{Aut}(\mathrm{A}_n) \cong \mathbb{Z}_2$
D_4			$\mathrm{Aut}(\mathrm{D}_4) \cong \mathfrak{S}_3$
$\mathrm{D}_{n(n\geq 5)}$		$\alpha_{n-1} \leftrightarrow \alpha_n$	$\mathrm{Aut}(\mathrm{D}_n) \cong \mathbb{Z}_2$
E_6		$\alpha_1 \leftrightarrow \alpha_5$ $\alpha_2 \leftrightarrow \alpha_4$	$\mathrm{Aut}(\mathrm{D}_6) \cong \mathbb{Z}_2$

Therefore, only groups with these Dynkin diagrams may allow outer types, all the others are a priori of inner type.

Theorem 5.2 i) (Pre-classification Witt-type theorem, cf. [24], 16.4.2) *The group G is uniquely determined by its Tits index and by its anisotropic kernel, if the semisimple anisotropic kernel is nontrivial (i.e., if G is not quasisplit).*

Otherwise (in the split or quasi-split case) G is determined by its Tits index and by its anisotropic kernel up to strict isogeny.

ii) (Pre-structural theorem, this is a special case of [24], 16.1.3) *Let P be a minimal parabolic subgroups of G, then there is a Bruhat decomposition*

$$G(k) = \bigcup_{w \in \Delta\backslash\Delta_0} P(k)WP(k)$$

into pairwise disjoint cosets mod P.

Remarks:

ii) holds in fact for arbitrary k-parabolic subgroups, as [24] 16.1.3 says. However, the case of a minimal parabolic gives most of the structural information for $G(k)$ and "burns everything down" to anisotropic groups, which are structurally widely unknown.

All the results above don't say anything about anisotropic groups or about anisotropic kernels of G, since that kernel is hidden in the Levi-group of P.

Notation for the Tits index of G:

The notation for the Dynkin type is enriched as follows: The symbol

$$ {}^{g}X^{t}_{n.r} $$

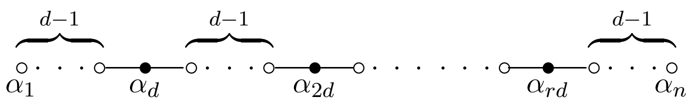
is used in order to describe a group G over k of Dynkin type X, where $n = \text{rank } G$, $r = \text{rank}_k\ G$, g is the order of the outer automorphism group employed by the $*$-operation (left out in case this is 1, i.e., if G is of inner type), and t is either(for groups of type A, C, D) the index of a central k-division algebra involved in the definition of G, or the dimension of its anisotropic kernel (for the "exceptional" groups defined by the exceptional root systems as explained in section 3). To distinguish both cases, if t denotes an index of a division algebra, it is put between parentheses.

In the Dynkin diagram, roots which are rational over k are marked as bullets, the others, which occur over k_{s}, are marked as circles.

Examples:

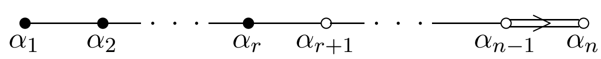
- Type: ${}^{1}\mathrm{A}^{(d)}_{n,r}$: simply connected group $\mathrm{SL}_{r+1}(D)$, D/k a central division algebra of degree d. Conditions: $d(r+1) = n+1$.

  Tits-Dynkin diagram:



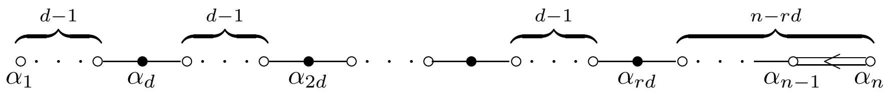
- Type: $\mathrm{B}_{n,r}$: Special orthogonal groups of regular quadratic forms with Witt index r and dimension $2n+1$.

  Tits-Dynkin diagram:



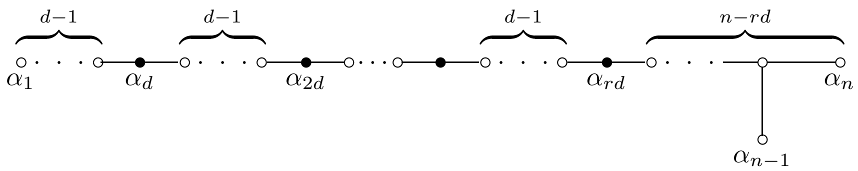
- Type: $C^d_{n,r}$: Special unitary group $\mathrm{SU}_{2n/d}(D,h)$, where D is a division algebra of degree d over k, and H is a non degenerate antihermitian sesquilinear form of index r relative to an involution of the first kind of D. For $d=1$ this group is just $\mathrm{Sp}_{2n}(k)$.

  Tits-Dynkin diagram:



- Type: ${}^1D^d_{n,r}$: char$k \neq 2$: Special unitary group $\mathrm{SU}_{2n/d}(D,h)$, where D is a division algebra of degree 2 over k, and H is a non degenerate hermitian form of discriminant 1 and index r relative to an involution of the first kind of D. In case $d=1$ this becomes $SO_q(k)$ for a regular quadratic form of Witt index r, dimension $2n$ and trivial discriminant.

  Tits-Dynkin diagram:



6 Further K-theoretic results for simple algebraic groups

Most K-theoretic results deal with so-called "classical groups", these are the groups SL_n over a skew field, and unitary groups for various (skew)-hermitian forms. These groups are not necessarily algebraic – for example, the group $\mathrm{SL}_n(D)$ as defined by the Dieudonné determinant, is in general not algebraic, when D is not of finite dimension over its center. A very good accounting of these results is given in the book by Hahn and O'Meara ("The classical groups and K-theory", [10]).

The results are also mostly under the assumption that the groups under consideration have "many" transvections. Insomuch the groups are algebraic, this amounts to assuming that they are split (Chevalley groups) or at least quasi-split (i.e., having a Borel group over the field of definition).

The methods are variations of classical K-theory methods as well as those used by Steinberg-Matsumoto, but also, the study of generalized Witt groups plays a big role.

However, a few results concern groups with a non-trivial anisotropic kernel, which we will discuss here.

On K_2 of skew fields:

For a skew field D, there is the "Dieudonné Determinant"

$$\det : \mathrm{GL}_n(D) \longrightarrow D^*/[D^*, D^*]$$

which has essentially the same properties as the determinants for fields (Np. [8], or for an alternative approach, [9], Teil 1, 2. Vortrag). Its definition specifies to the ordinary determinant if D is commutative. Its kernel $\mathrm{E}_n(D)$ is generated by the elementary matrices $u_{ij}(x) = 1 + xe_{ij}$, $i \neq j$ and $x \in D$.

Again we have for $n \geq 3$ (we omit the technically more involved case $n = 2$):

(A) $\quad u_{ij}(x + y) = u_{ij}(x)u_{ij}(y)$

(B) $\quad [u_{ij}(x), u_{kl}(y)] = \begin{cases} u_{il}(xy) & \text{if } i \neq l, j = k, \\ u_{jk}(-yx) & \text{if } i = l, j \neq k, \\ \mathbf{1} & \text{otherwise, provided } (i,j) \neq (j,i) \end{cases}$

(C) $\quad h_{ij}(xy) = h_{ij}(x)h_{ij}(y)$

Please observe here the order of the factors x and y in the right-hand side of (B).

We have the standard definitions:

$$\begin{aligned} w_{ij}(x) &= u_{ij}(x)u_{ji}(-x^{-1})u_{ij}(x) \quad (x \in k, x \neq 0) \\ h_{ij}(x) &= w_{ij}(x)w_{ij}(-1) \quad (x \in k, x \neq 0) \end{aligned}$$

By Milnor ([15], 5.10), the group $\mathrm{St}_n(D)$ presented by (A), (B) is a universal central extension of the perfect group $\mathrm{E}_n(D)$, again its kernel is denoted by $K_2(n, D)$.

(In [12], the universality of the central extension is only proved for $n \geq 5$, for $n \geq 3$, compare [19], Kor. 1 and "Bemerkung", p. 101.)

One also has a Bruhat double coset decomposition $\mathrm{E}_n(D) = UMU$, here M denotes the monomial matrices and U the upper triangular matrices.

Let $\pi : St_n(D) \longrightarrow E_n(D)$ be the canonical map, let us denote the generators of $St_n(D)$ by $\tilde{u}_{ij}(x)$,$\tilde{w}_{ij}(x)$,and $\tilde{h}_{ij}(x)$. Now the elements

$$c_{ij}(xy) = \tilde{h}_{ij}(xy)\tilde{h}_{ij}(x)^{-1}\tilde{h}_{ij}(y)^{-1} \in St_n(D)$$

are not any more in kernel π, as they map to the diagonal matrix

$$\begin{array}{cc} & \begin{matrix} & & & i & & & \end{matrix} \\ i & \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& xyx^{-1}y^{-1} &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix} \end{array}$$

However, for $x_\nu, y_\nu \in D^*$ such that $\prod\limits_\nu [x_\nu, y_\nu] = 1$, one has obviously $\prod\limits_\nu c_{ij}(x_\nu, y_\nu) \in Ker\ \pi$, and all elements of $\ker \pi$ are of this type. It can be shown that these elements are independent of the choice of (i,j), and one has the following replacement for Matsumoto's theorem ([19]):

Theorem 6.1 *Let U_D denote the group generated by $c(x,y)$, $x, y \in D^*$, subject to the following relations:*

(U0)	$c(x, 1-x) = 1$	$x \neq 0, 1$
(U1)	$c(xy, z) = c({}^x y, {}^x z)\ c(x,z)$	
(U2)	$c(x, yz) = c(x,y)\ c({}^y x, {}^y x)$	

(Here the abbreviation ${}^x y := xyx^{-1}$ is used.) Then the map

$$U_D \longrightarrow [D^*, D^*] \qquad c(x,y) \mapsto [x,y]$$

defines a central extension of $[D^\star, D^\star]$.

Moreover, $U_D \longrightarrow \mathrm{St}_n(D)$ injects via

$$c(x,y) \mapsto c_{12}(x,y) = \tilde{h}_{12}(xy)\tilde{h}_{12}(x)^{-1}\tilde{h}_{12}(y)^{-1}$$

This implies that $K_2(n,D) = K_2(D)$ for $n \geq 3$.) Hence one has an exact sequence

$$0 \longrightarrow K_2(D) \longrightarrow U_D \longrightarrow [D^*, D^*] \longrightarrow 0$$

Remarks:

- Obviously, this gives Matsumoto's theorem for D commutative and in general relates $K_2(D)$ to a central extension of $[D^*, D^*]$.
- The relations U1, U2 together with the relations $c(x,x) = 1$ give a generating set for all formal commutator relations of an arbitrary group H and $x, y \in H$.

  Applying the Hochschild-Serre spectral sequence to the central extension above we obtain the exact sequence:

$$H_2(U_D) \longrightarrow H_2[D^*, D^*] \longrightarrow K_2(D) \longrightarrow H_1(U_D) \longrightarrow H_1[D^*, D^*] \longrightarrow 1$$

In general, $\mathrm{E}_n(D)$ is not an algebraic group; the Dieudonné determinant is not a polynomial function.

However, in certain cases this becomes true if D is a finite central k-division algebra: then both, D and $\mathrm{M}_n(D)$ are central simple k-algebras of finite dimension over k. The Dieudonné determinant factors through to the reduced norm (see the previous section for the definition of the reduced norm). As this is a polynomial, $\mathrm{E}_n(D) \subset \mathrm{SL}_n(D)$ in general, and we will below discuss some conditions which guarantee equality.

The exact sequence above about $K_2(D)$ in this case should be understood as a statement which relates the central extensions of $G = \mathrm{SL}_{r+1}(D)$ to those of the anisotropic kernel of G.

Therefore we will investigate this group in more detail here:

Let D/k be a finite central algebra of index d, then $\dim_k D = d^2$. To understand the structure of $\mathrm{SL}_{r+1}(D)$ (up to its anisotropic kernel), we consider the subgroup of upper triangular matrices which in this case is a minimal parabolic subgroup:

$$P := \left\{ \begin{pmatrix} * & * & * \\ & \ddots & * \\ 0 & & * \end{pmatrix} \in \mathrm{SL}_{r+1}(D) \right\}$$

Its Levi decomposition looks as follows:

$$P = L \ltimes R_n(P) = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{pmatrix} \right\}$$

Each $(*)$ in the Levi-group L is a copy of $D^* = \mathrm{GL}_1(D)$ with a central torus $\mathbf{G}_{\mathrm{m}}$ (the center of $D^* = k^*$, there are $r+1$ copies, but with reduced norm 1, hence the central torus in L has dimension r:

$$S = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_{r+1} \end{pmatrix} \mid a_i \in k^*, \prod_{i=1}^{r+1} \alpha_i = 1 \right\}.$$

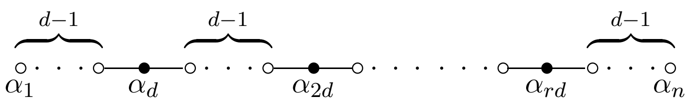
This is a maximal k-split torus of $G = \mathrm{SL}_{r+1}(D)$, and obviously L is its centralizer.

The semi-simple anisotropic kernel consists of the diagonal matrices with $r+1$ copies of $\mathrm{SL}_1(D)$.

The group $\mathrm{SL}_{r+1}(D)$ is of inner type, because all roots are kept invariant under $\mathrm{Gal}(k_s/k)$, hence its type is ${}^1\mathrm{A}^d_{n,r}$.

Since $\mathrm{SL}_{r+1}(D) \times_k k_{\mathrm{sep}} \cong \mathrm{SL}_{r+1}(\mathrm{M}_d(k_{sep})) \cong \mathrm{SL}_{(r+1)d}(k_{sep})$, we have $n = \mathrm{rank}\, \mathrm{SL}_n(D) = (r+1)d - 1$.

The Tits-Dynkin diagram for this group looks like this:



The distinguished roots are: $\Delta_0 = \{\alpha_d, \alpha_{2d}, \ldots, \alpha_{rd}\}$, and one has $n+1 = (r+1)d$,

Consequences for the determination of $K_2(D)$:

There are results only for special classes of fields k:

Theorem 6.2 (Alperin/Dennis, cf. [1]) *Let* $\mathbf{H}$ *the skew field of Hamilton's quaternions over* $\mathbb{R}$*. Then the natural embedding* $\mathbb{R} \longrightarrow \mathbf{H}$ *induces an isomorphism* $K_2(\mathbb{R})/c(-1,-1) \longrightarrow K_2(\mathbf{H})$.

For division algebras D over local or global fields k, several results have been achieved by Rehmann and Stuhler in [20]. The methods and results there are as follows:

Define $N_{D/k} := \mathrm{Image}(\mathrm{RN} : D^* \longrightarrow k^*)$, then there exists (under the assumption that $\mathrm{kernel}(\mathrm{RN}) = [D^*, D^*]$, which is true for local and global fields – see the next section about this assumption) a bimultiplicative map $\psi_0 : N_{D/k} \times k^* \longrightarrow K_2(D)$, defined by $\psi_0(\mathrm{RN}(x), z) = c(x, z)$ for $x \in D^*, z \in k^*$. (The right-hand side indeed only depends on z and on $\mathrm{RN}(x)$, not on the choice of x in the pre-image of $\mathrm{RN}(x)$.) Under certain conditions on D, which always hold for local and global fields, ψ_0 turns out to be a symbol, i.e., $\psi_0(\alpha, 1 - \alpha) = 0$ for any $\alpha \in N_{D/k}$.

We define $Y(D/k) := N_{D/k} \otimes k^*/\langle\ c(a, 1 - \alpha) \mid \alpha \in N_{D/k}\ \rangle$. Then we obtain a homomorphism $\psi : Y(D/k) \longrightarrow K_2(D)$

Of course there is also a natural map $\iota : Y(D/k) \longrightarrow K_2(k) \longrightarrow K_2(D)$, given by $x \otimes y \mapsto c(x, y) \in K_2(D)$. It turns out that

$$\psi = \iota^d \text{ for } d = \text{degree } D. \tag{$*$}$$

Now if RN is an epimorphism (which is true for non-Archimedean local fields or for global fields with no real places), then $Y(D/k) = K_2(k)$.

By another result of Alperin-Dennis ([1]) it can be shown that for quaternion algebras, ψ is always surjective.

If k is a global function field, then it is shown in [20] that:

$$K_2(D) = K_2(k) \times \text{finite group},$$

and if D in addition is a quaternion division algebra (i.e., $\dim_k D = 4$), then:

$$K_2(k) \cong K_2(D)$$

But it is important to realize that this isomorphism is <u>not</u> induced by the natural embedding $k \longrightarrow D$ – it is something like $1/d$ times this map, by the fact $(*)$ above.

There are similar results of this type for number fields in [20].

The map ψ seems to something like the inverse of the "reduced K_2-norm" for division algebras of square-free degree as constructed by Merkurjev and Suslin:

Theorem 6.3 *(*Merkurjev, Suslin, cf. [14]): *If D is of square-free degree, then there exists a unique homomorphism* $\mathrm{RN}_{K_2} : K_2(D) \longrightarrow K_2(k)$*, such that for every splitting field L/k of D the transfer map $N_{L/k} : K_2(L) \longrightarrow K_2(k)$ factorizes through* RN_{K_2}*, such that the following diagram commutes:*

$$\begin{tikzcd} K_2(L) \arrow[r, "\iota"] \arrow[dr, "N_{L|F}"'] & K_2(D) \arrow[d, "\mathrm{RN}_{\mathrm{K}_2}"] \\ & K_2(k) \end{tikzcd}$$

Here ι denotes the map induced by some embedding of L into some $\mathrm{M}_l(D)$*.*

From this, the following is deduced:

Theorem 6.4 (Merkurjev, Suslin, cf. [14]) *If D is a division algebra of square-free degree over a local or global field, then the sequence*

$$0 \longrightarrow K_2(D) \stackrel{\mathrm{RN}_{K_2}}{\longrightarrow} K_2(F) \longrightarrow \bigoplus_v \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact, where the sum is taken over all real places of k for which D_v is non-trivial.

This generalizes some of the results from [20] to a certain extent, but only under the assumption of a square-free index. It is expected that, at least for results on local or global fields, this restriction should not be necessary.

Results and open questions on $\mathrm{SK}_1(D)$:

We have $\mathrm{SL}_1(D) = \mathrm{kernel}(\mathrm{RN} : D^* \longrightarrow k^*)$, hence $[D^*, D^*]$ is contained in $\mathrm{SL}_1(D)$.

The abelian quotient group $\mathrm{SL}_1(D)/[D^*, D^*]$ is denoted by $\mathrm{SK}_1(D)$[4]

The question in general is still open: What is $\mathrm{SK}_1(D)$?

Wang(1950) proved ([29]):

Theorem 6.5 *For any finite dimensional central k-division algebra D of degree d one has* $\mathrm{SK}_1(D) = 1$*, if*

i) *either k is arbitrary and d is square free,*

ii) *or k is local or global and d is arbitrary.*

It was a long standing open question whether $\mathrm{SK}_1(D)$ is always trivial, in fact this was stated as the “Artin-Tannaka”-conjecture.

But around 1975, V. P. Platonov (in [18] and several subsequent articles) gave first examples of a finite central algebra D/k and $\mathrm{SK}_1(D) \neq 1$, which were constructed over a twofold valuated field k.

The theory was further developed by P. K. J. Draxl, who proved that, for any finite abelian group A, there exists D such that $\mathrm{SK}_1(D) = A$ (cf. [9]).

A. Suslin conjectured in 1990 that only in the case of square free degree, $\mathrm{SK}_1(D)$ should “generically vanish”:

Conjecture: $\mathrm{SK}_1(D_{k(SL_1(D)})) = 1$ if and only if the degree of D is square free.

So far, there is just this result:

---

[4]The reason for this is as follows: For an arbitrary ring R, one defines the groups $\mathrm{GL}(R) = \varinjlim_{n} \mathrm{GL}_n(R), \mathrm{E}(R) = \varinjlim_{n} \mathrm{E}_n(R)$ as the inductive limits via the embeddings given by $\mathrm{GL}_n(R) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R)$, (here $\mathrm{E}_n(R)$ being the group generated by the elementary matrices $\mathbf{1}_n + x\, e_{ij}$). Then $K_1(A) := \mathrm{GL}(R)/\mathrm{E}(R)$ is an abelian group. If R is commutative, then one has as well the inductive limit $\mathrm{SL}(R) = \varinjlim_{n} \mathrm{SL}_n(R)$, and the determinant gives an epimorphism $\det : K_1(R) \twoheadrightarrow R^*$, since $\mathrm{E}_n(R) \subset \mathrm{SL}_n(R)$ with the kernel $\mathrm{SK}_1(R) := \mathrm{SL}(R)/\mathrm{E}(A)$. This map splits because of $R^* \cong \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}(R) \twoheadrightarrow R^*$, hence we get $K_2(R) = \mathrm{SK}_1(R) \oplus R^*$.

If $R = D$ is a division algebra over a field k, and if det is replaced by the reduced norm RN, then $\mathrm{SL}_n(D) \supset \mathrm{E}_n(D)$ and the above definitions amount to $K_1(D) = \varinjlim_{n} \mathrm{GL}_n(D)/E_n(D)$, $\mathrm{SK}_1(D) = \varinjlim_{n} \mathrm{SL}_n(D)/E_n(D)$. From the properties of the Dieudonné determinant we obtain $K_1(D) \cong D^*/[D^*, D^*]$ and $\mathrm{SK}_1(D) = \mathrm{SL}_1(D)/[D^*, D^*]$.

Theorem 6.6 (Merkurjev, [13]) *Let D be a division algebra over a field k. If the degree of D is divisible by 4, then* $\mathrm{SK}_1(D_{k(SL_1(D))}) \neq 1$.

In fact, in [13] the assumption was that char $k \neq 2$, but this has meanwhile been removed.

The main tool in the proof is a theorem by Rost, who had proved an exact sequence $0 \longrightarrow \mathrm{SK}_1(D) \longrightarrow H^4(F, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^4(F(q), \mathbf{Z}/2\mathbf{Z})$ which compares $\mathrm{SK}_1(D)$ for a tensor product D of two quaternion algebras with the Galois cohomology of a quadratic Albert q form which is related to the two quaternion forms involved.

Further details can be obtained from the Book of Involutions [11], §17.

References

[1] R.C. Alperin and R. Keith Dennis, K_2 of quaternion algebras, J. Algebra, no.1 **56** (1979) 262-273.

[2] Armand Borel, *Linear Algebraic Groups*, Second edition. Graduate Texts in Mathematics, 126 (Springer-Verlag, New York, 1991). xii+288 pp. ISBN: 0-387-97370-2.

[3] Armand Borel and Jacques Tits, Groupes réductifs. Publications Mathématiques de l'IHÉS, **27** (1965) pp.55-151. http://www.numdam.org/item?id=PMIHES_1965__7__55_0

[4] Armand Borel and Jacques Tits, Compléments à l'article "Groupes réductifs", Publications Mathématiques de l'IHÉS, **41** (1972) pp.253-276. http://www.numdam.org/item?id=PMIHES_1972__41__253_0

[5] N. Bourbaki and N. Bourbaki, *Éléments de Mathématique. Algèbre* Chapitre 9, Reprint of the 1959 original (Springer-Verlag, Berlin, 2007).

[6] N. Bourbaki, *Éléments de mathématique* Fasc. XXXIV. Groupes et algèbres de Lie, Chapitre IV: Groupes de Coxeter et systèmes de Tits, Chapitre V: Groupes engendrés par des réflexions, Chapitre VI: Systèmes de racines, Actualités Scientifiques et Industrielles, No.1337 (Hermann, Paris 1968).

[7] Michel Demazure and Alexandre Grothendieck, eds., Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA 3) - vols.1,2,3 (Lecture Notes in Mathematics 151, 152, 153) (1970).

[8] Jean Dieudonné, Les déterminants sur un corps non commutatif, Bulletin de la Société Mathématique de France **71** (1943), pp.27-45.

[9] P. Draxl and M. Kneser, SK_1 von Schiefkörpern, Seminar Bielefeld-Göttingen (1976). Lecture Notes in Math. 778 (1980).

[10] Alexander J. Hahn and O. Timothy O'Meara, *The Classical Groups and K-theory*, with a foreword by J. Dieudonné. Grundlehren der Mathematischen Wissenschaften, 291 (Springer-Verlag, Berlin, 1989). ISBN: 3-540-17758-2.

[11] Max-Albert Knus, Alexander Merkurjev, Markus Rost and Jean-Pierre Tignol, "The Book of Involutions", with a preface in French by J. Tits. American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI (1998). xxii+593 pp. ISBN: 0-8218-0904-0.

[12] Hideya Matsumoto, Sur les groupes semi-simples déployés sur un anneau principal, C.R. Acad. Sci. Paris Sér. A-B 262 (1966) A1040–A1042. http://www.numdam.org/item?id=ASENS_1969_4_2_1_1_0

[13] A.S. Merkurjev, Generic element in SK_1 for simple algebras, K-Theory 7 (1993) no.1, 1-3.

[14] A.S. Merkurjev, A.A. Suslin, K-cohomology of Severi-Brauer varieties and the norm residue homomorphism, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982) no.5, 1011-1046, 1135-1136.

[15] John Milnor, *Introduction to Algebraic K-theory*, Annals of Mathematics Studies, No.72 (Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971).

[16] Calvin C. Moore, Group extensions of p-adic and adelic linear groups, Inst. Hautes etudes Sci. Publ. Math. No. 35 1968 157-222. http://www.numdam.org/item?id=PMIHES_1968__35__5_0

[17] Jun Morita and Ulf Rehmann, Symplectic K_2 of Laurent polynomials, associated Kac-Moody groups and Witt rings, Math. Z. **206** (1991) no.1, 57-66.

[18] V.P. Platonov, The Tannaka-Artin problem, and groups of projective conorms, Dokl. Akad. Nauk SSSR **222** (1975) no.6, 1299-1302.

[19] Ulf Rehmann, Zentrale Erweiterungen der speziellen linearen Gruppe eines Schiefkörpers, J. Reine Angew. Math. **301** (1978) 77-104.

[20] Ulf Rehmann and Ulrich Stuhler, On K_2 of finite-dimensional division algebras over arithmetical fields, Invent. Math. **50** (1978/79) no.1, 75-90.

[21] Ulf Rehmann, Central extensions of SL_2 over division rings and some metaplectic theorems, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 561-607, Contemp. Math. **55**, Amer. Math. Soc., Providence, RI (1986).

[22] J. Tits, Classification of algebraic semisimple groups, 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 33-62 Amer. Math. Soc., Providence, R.I. (1966). http://www.ams.org/online_bks/pspum9

[23] Jean-Pierre Serre, *Algèbres de Lie Semi-simples Complexes* (W.A. Benjamin, Inc. New York, Amsterdam, 1966).

[24] T. A. Springer, *Linear Algebraic Groups*, 2nd edition, Progress in Mathematics, 9 (Birkhäuser Boston, Inc., Boston, MA, 1998). xiv+334 pp. ISBN: 0-8176-4021-5.

[25] Robert Steinberg, Générateurs, relations et revêtements de groupes algébriques, 1962 Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962) pp. 113-127 Librairie Universitaire, Louvain; Gauthier-Villars, Paris.

[26] Robert Steinberg, Generators, relations and coverings of algebraic groups. II, J. Algebra **71** (1981) no.2, 527-543.

[27] Robert Steinberg, Lectures on Chevalley Groups, Yale University 1967, Notes prepared by John Faulkner and Robert Wilson, Mimeographic Notes.

[28] A.A. Suslin, Torsion in K_2 of fields, K-Theory 1 (1987) no.1, 5-29.

[29] Shianghaw Wang, On the commutator group of a simple algebra, Amer. J. Math. **72** (1950) 323-334. http://www.jstor.org/journals/00029327.html