Automorphic Forms in $GL(n)$ I:
Decomposition of the Space of Cusp Forms
and Some Finiteness Results

M.S. Raghunathan*

School of Mathematics, Tata Institute of Fundamental Research,
Mumbai, India

Lectures given at the
School on Automorphic Forms on $GL(n)$
Trieste, 31 July - 18 August 2000

LNS0821001

*msr@math.tifr.res.in
## Contents

0 Introduction ........................................... 5
1 Fundamental Domains .............................. 6
2 Automorphic Forms ................................ 13
3 Cusp Forms ........................................ 21
4 Proof of Theorem 2.13 ............................. 33
References ........................................... 37
0 Introduction

These are notes of lectures given by the author at an Instructional School on “Automorphic Forms in GL(n)” held at the Abdus Salam International Centre for Theoretical Physics, Italy, in August 2000. The author would like to thank the ICTP for their hospitality and their patience over the long delay in making these notes available.

In this first chapter we introduce some concepts in the theory of Automorphic Forms and prove some basic results. Although our central focus is on the general linear group $GL(n)$, in this chapter we formulate the results in the more general context of reductive algebraic groups over number fields. This is because the techniques for handling the general case are no different from those needed to handle $GL(n)$. We have also indicated what the notions used mean in the special case of $GL(n)$ (or $SL(n)$), for readers unfamiliar or uncomfortable with the general theory of algebraic groups. All the results about algebraic groups needed here are to be found in the paper [B-T] of Borel and Tits.

In §1 we describe a good fundamental domain for $G(k)$ in $G(\mathbb{A})$ where $k$ is a number field, $\mathbb{A}$ is its Adéle ring and $G$ is a reductive algebraic group over $k$. This description follows from a theorem of A. Borel [B] that describes a fundamental domain for an arithmetic subgroup $\Gamma$ in $\prod_{v \in \infty} G(k_v)$: here $\infty$ is the complete set of inequivalent archimeadean valuations of $k$ and for $v \in \infty, k_v$ is the completion of $k$ at $v$. Conversely Borel’s theorem can be deduced from this description (of a fundamental domain for $G(k)$ in $G(\mathbb{A})$). A quick and elegant proof of this (due to R. Godement and A. Weil) is to be found in [G].

In §2 the definition of automorphic forms is given and the central finite dimensionality results are formulated. These concepts and results are due to R. Langlands [L] (see also [H-C]).

In §3 we introduce the space of cusp-forms and prove that the representation of $G(\mathbb{A})$ in the space of $L^2$-cusp-forms is completely reducible, a result due to I. Gelfand and I. Piatetishi-Shapiro. This is needed in §4 where the finite dimensionality theorems formulated in §2 are proved. We need in §3 some well known results from Analysis; these can be found in R. Narasimhan’s book [N].
1 Fundamental Domains

1.1. Notations Throughout these notes we will adopt the following notations. We denote by \( k \) a number field and by \( V \) a complete set of mutually inequivalent valuations of \( k \). Let \( \infty \) (resp. \( V_f \)) denote the subset of archimedean (resp. non-archimedean) valuations in \( V \). For \( v \in V, k_v \) denotes the completion of \( k \) with respect to \( v \). We denote by \( O_v \) for \( v \in V_f \) the ring of integers in \( k_v \), and by \( p_v \) the maximal ideal in \( O_v \). The residue field \( O_v/p_v \) is denoted \( F_v \) and we set \( q_v = |F_v| \); also \( p_v \) is characteristic of \( F_v \). We denote by \(| \cdot |_v \) the absolute value on \( k_v \) determined by \( v \). We assume the absolute value \(| \cdot |_v \), \( v \in V \) chosen so that the following holds: Let \( \mathbb{Q}_v \) denote the closure of \( \mathbb{Q} \) in \( k_v \) and \( N_v : k_v \to \mathbb{Q}_v \) be the norm map. If \( v \) is archimedean we set \(|x|_v = N_v(x)| \) where \(| \cdot | \) is the usual absolute value on \( \mathbb{Q}_v \cong \mathbb{R} \). If \( v \in V_f \) and \( N_v(x) = p_v^r \cdot y \) with \( r \in \mathbb{Z} \) and \( y \) a unit in the ring of integers in \( \mathbb{Q}_v(\mathbb{Z}_{p_v}) \), \(|x|_v = p_v^{-r} \). With this definition we have the well known

1.2. Product formula If \( x \in k^x, |x|_v = 1 \) for all but finitely may \( v \in V \) and \( \prod_{v \in S} |x|_v = 1 \) where \( S = \{ v \in V \mid |x|_v \neq 1 \} \).

1.3. We denote by \( \mathbb{A} \) the ring of adeles: \( \mathbb{A} = \{ \underline{x} = (x_v)_{v \in V} \in \prod_{v \in V} k_v \mid x_v \in O_v \text{ for all } v \notin S, S \text{ a finite subset of } V_f \} \): \( \mathbb{A} \) is a ring under coordinate-wise addition and multiplication. Let \( \wedge \) denote the subset \( \prod_{v \in \infty} k_v \times \prod_{v \in V_f} O_v \); then \( \wedge \) is subring of \( \mathbb{A} \). Under the product topology \( \wedge \) is a locally compact ring topological ring. \( \mathbb{A} \) has a natural structure of a locally compact ring such that the inclusion of \( \wedge \) (with its product topology) in \( \mathbb{A} \) is an isomorphism of topological rings on to an open subring. We denote by \( A_f \) the subset \( \{ \underline{x} \in \mathbb{A} \mid x_v = 0 \text{ for } v \in \infty \} \). The group of Ideles \( I \) of \( k \) is the set of invertible elements in \( \mathbb{A} \). It is easy to see that \( I = \{ \underline{x} \in \mathbb{A} \mid x_v \neq 0 \text{ for } v \in V \text{ and } |x_v|_v = 1 \text{ for all but a finite number of } v \in V_f \} \). We also set \( I_f = \{ \underline{x} \in I \mid x_v = 1 \text{ for } v \in \infty \} \). The group \( I \) is given a topology as follows: Consider the group \( \prod_{v \in \infty} k_v^* \times \prod_{v \in V_f} O_v^* \) where for \( v \in V_f, O_v^* \) is the compact group of units in \( k_v^* \); this group has a natural structure of a locally compact group in the product topology. \( I \) is given the unique structure of a topological group for which the natural inclusion of \( \prod_{v \in \infty} k_v^* \prod_{v \in V_f} O_v^* \) in \( I \) is an isomorphism (of topological groups) onto an open subgroup. If \( \underline{x} \in I \) and \( S = \{ v \in V_f \mid x_v |_v \neq 1 \}, \) we set \(|\underline{x}| = \prod_{v \in V} |x_v|_v = \prod_{v \in S} |x_v|_v \); then \( \underline{x} \mapsto |\underline{x}| \) is a continuous homomorphism of \( I \) in \( \mathbb{R}^+ \) and the kernel of this
Automorphic Forms in $GL(n)$

Homomorphism is denoted $I^1$. Elements of $I^1$ will be called “Ideles of norm 1”. We state without proof the following well known

1.4. Theorem The inclusion of $k$ (resp. $k^*$) in $\mathbb{A}$ (resp. $I^1$) imbeds $k$ (resp. $k^*$) as a closed discrete subgroup of the (additive) group $\mathbb{A}$ (resp. (multiplicative) group $I^1$) such that $\mathbb{A}/k$ (resp. $I^1/k^*$) is compact.

1.5. Let $G$ denote a connected reductive linear algebraic group over $k$. (Our main interest is in the case when $G = GL(n)$ the general linear group and we will draw attention to what our terms and definitions mean in this special case). Let $S$ be a maximal $k$-split torus in $G$ (when $G = GL(n)$, $S$ can be taken to be the group of diagonal matrices in $G$). Let $X^*(S)$ denote the group of characters (all characters of $S$ are defined over $k$) of $S$ and $\Phi \subset X^*(S)$ the subset of $k$-roots of $G$ with respect to $S$: recall that a $k$-root is a non-trivial character for the adjoint action of $S$ on Lie algebra of $G$ (when $G = GL(n)$, $\Phi$ consists of the characters $d \mapsto d_i/d_j$ for some pair $(i,j), 1 \leq i,j \leq n, i \neq j$). We fix an ordering on $X^*(S)$ and denote by $\Phi^+$ (resp. $\Delta$) the set of positive (resp. simple) roots in $\Phi$ (when $G = GL(n)$, one usually chooses the ordering given as follows: any character $\chi$ on $S$ (= diagonal matrices) is of the form $\chi(d) = \prod_{1 \leq i \leq n} d_i^{n_i(\chi)}$ where $d = \text{diagonal} (d_1, \cdots, d_n)$ and $n_i(\chi)$ are integers; we set $\chi > 0$ if $n_i(\chi) > 0$ where $r(\chi) = \min \{i \mid 1 \leq i \leq n, n_i(\chi) \neq 0\}$. The set of simple roots for this order are the roots $d \mapsto d_i/d_{i+1}, 1 \leq i < n$. We fix once and for all a realisation of $G$ as an algebraic subgroup (over $k$) of $GL(N)$ for some $N$ (when $G = GL(n)$, we can take $N = n$). For $v \in V$ then $G(k_v)$ is a closed subset of $GL(N, k_v)$ hence a locally closed subset of $M(N, k_v) \simeq k_v^N$ and thus acquires a locally compact topology. For $v \in V_f$ we denote by $M_v$ the compact open subgroup $G(k_v) \cap GL(N, O_v)$ of $G(k_v)$ when $G = GL(n)$). $G(\mathbb{A})$ (resp. $G(\mathbb{A}_f)$) the $\mathbb{A}$ (resp. $k_f$) points of $G$ has a natural identification with \{ $g = \{g_v\}_{v \in V}(\text{resp.} V_f)$ $|$ $g_v \in G(k_v)$ and the set $\{v \in V_f \mid g_v \notin M_v\}$ is finite \}. We have a natural inclusion

$$\prod_{v \in V_f} M_v \hookrightarrow G(\mathbb{A}_f)$$
and \(G(\mathbb{A}_f)\) is made into a locally compact topological group by stipulating that the above inclusion is an isomorphism of topological groups of the group on the left onto an open subgroup (the topological group \(G(\mathbb{A}_f)\) obtained in this fashion is independent of the realisation of \(G\) as a \(k\)-algebraic subgroup of \(GL(N)\) for some \(N\)). We set \(G_\infty = \prod_{v \in \infty} G(k_v)\). We then have a natural identification of \(G(\mathbb{A})\) with \(G_\infty \times G(\mathbb{A}_f)\). We give to \(G_\infty\) the product topology on \(\prod_{v \in \infty} G(k_v)\) and to \(G(\mathbb{A})\) the product topology on \(G_\infty \times G(\mathbb{A}_f)\).

Let \(p_f\) (resp. \(p_\infty\)) be the projection of \(G(\mathbb{A})\) on \(G(\mathbb{A}_f)\) (resp. \(G_\infty\)). For \(g \in G(\mathbb{A})\), we set \(g_f = p_f(g)\) and \(g_\infty = p_\infty(g)\). Let \(d_f\) (resp. \(d_\infty\)) = \(p_f \circ d\) (resp. \(p_\infty \circ d\)). We have a natural inclusion \(d : G(k) \hookrightarrow G(\mathbb{A})\) of the \(k\)-points of \(G\) in \(G(\mathbb{A})\) as a closed discrete subgroup of \(G(\mathbb{A})\).

In the sequel we identify \(G(k)\) with \(d(G(k))\). We will now describe certain \(k\)-algebraic (resp. closed) subgroups of \(G\) and (resp. \(G(\mathbb{A})\)) that will be needed in the sequel. Let \(Ad\) denote the adjoint representation of \(G\) on its Lie algebra (over \(k\)) \(L(G)\). Under the adjoint action of \(S, L(G)\) decomposes into a direct sum of eigenspaces, the eigencharacters being the roots and the trivial character. For \(\alpha \in \Phi\), let \(L(G)(\alpha)\) be the eigenspace corresponding to \(\alpha\) and let \(L(U^+) = \prod_{\alpha \in \Phi^+} L(G)(\alpha)\). Then there is a unique \(k\)-subgroup \(U\) of \(G\) normalised by \(S\) and with \(L(U)\) as the Lie subalgebra of \(L(G)\) corresponding to \(U\). Also \(U\) is normalised by the \(Z(S)\) the centraliser of \(S\) and \(P = Z(S) \cdot U\) is a parabolic subgroup defined over \(k\); \(P\) is a minimal \(k\)-parabolic subgroup of \(G\). We denote by \(^o P\) the intersection of the kernels of all squares of characters on \(P\) defined over \(k\); then \(P = S \cdot ^o P\) and \(^o P = B \cdot U\) where \(B\) is a reductive algebraic \(k\)-subgroup of \(^o P\) which is anisotropic over \(k\) (i.e. \(B\) does not admit a non-trivial \(k\)-split toral subgroup). As \(S\) is \(k\)-split we may treat it as obtained from a \(Q\)-split torus by the base change \(Q \hookrightarrow k\)-th us in the sequel we will treat \(S\) as a \(Q\) split torus. We then have for each \(v \in \infty\) an inclusion \(S(Q_v) \hookrightarrow S(k_v)\) when \(Q_v\) is the closure of \(Q_v\) in \(k_v\); and since all the \(v \in \infty\) induce on \(Q\) the unique archimedean topology with \(R\) as the completion, we obtain a diagonal inclusion \(S(R) = \prod_{v \in \infty} S(k_v)\). Let \(\prod_{v \in \infty} S(k_v) \rightarrow S(\mathbb{A})\) be the inclusion \(\underline{s} = \{s_v\}_{v \in \infty} \rightarrow \{s_v\}_{v \in \infty}\) where for \(v \in \mathcal{V} \setminus \infty, s_v = 1\). We denote by \(A\) the image of the identity component of \(S(R)\) (a subgroup \(\prod_{v \in \infty} S(k_v)\)) in \(G(\mathbb{A})\). The group \(A\) is a closed subgroup of \(G(\mathbb{A})\) isomorphic to a product of \(\ell = \dim S\) copies of the multiplicative group \(\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}\). For a constant \(c > 0\) let \(A_c = \{x \in A \mid \alpha(x) \leq c\text{ for }\alpha \in \Delta\}: \alpha\) is a character (defined over \(k\)) on \(S\) and as is easily seen on \(A\) takes positive real values. With these notations we have
1.6. Theorem (R. Godement and A. Weil) There is a compact subset $\Omega$ of $^o\mathcal{P}(\mathbb{A})$ such that $\Omega$. $^o\mathcal{P}(k) = \mathcal{P}(\mathbb{A})$. There is a maximal compact (open) subgroup $K$ of $G(\mathbb{A})$ and a constant $c_0 > 0$ such that for all $c \geq c_0$ and compact subsets $\Omega'$ of $^o\mathcal{P}(\mathbb{A})$ with $\Omega' \supset \Omega$, one has $K.A_c.\Omega'.G(k) = G(\mathbb{A})$.

1.7. Corollary \( K_f.\Omega'_f.d_fG(k) = G(\mathbb{A}_f) \) where $\Omega'_f = p_f(\Omega_f)$.

1.8. Let $L_f$ be a compact open subgroup of $G(\mathbb{A}_f) = \{ g \in G(\mathbb{A}) \mid g_v = 1 \text{ for all } v \in \infty \}$. Let $\Gamma_{L_f} = \gamma \in G(k) \mid p_f(\gamma) \in L_f$. Now $G_\infty \times L_f$ has a natural identification with an open subgroup of $G(\mathbb{A})$ and $\Gamma_{L_f}$ is in a natural fashion a closed discrete subgroup of $G_\infty \times L_f$. Since $L_f$ is compact, the projection $d_\infty(\Gamma_{L_f})$ of $\Gamma_{L_f}$ on $G_\infty$ is a discrete subgroup of $G_\infty$. The maximal compact subgroup $K$ above decomposes as a direct product $K_\infty \times K_f$ following the product decomposition $G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$ with $K_\infty \text{ (resp. } K_f) \text{ a maximal compact subgroup of } G_\infty \text{ (resp. } G(\mathbb{A}_f)).$ We may also assume that $\Omega$ is of the form $\Omega_\infty \times \Omega_f$, where $\Omega_\infty \text{ (resp. } \Omega_f)$ is a compact subset of $^o\mathcal{P}_\infty \text{ (resp. } ^o\mathcal{P}_\mathbb{A}_f)$. It follows then that if $g \in G_\infty$, there exists $\gamma \in G(k)$ such that $g\gamma_\infty \in K_\infty A_c.\Omega_\infty$ and $\gamma_f \in K_f.\Omega_f$. Now $L_f$ being an open compact subgroup and $K_f.\Omega_f$ being a compact set of $G(\mathbb{A}_f)$, we can find finitely many elements $\theta_1, \ldots, \theta_r \in G(\mathbb{A}_f)$ such that $K_f\Omega_f \subset \cup_{1 \leq i \leq r} \theta_i L_f$. In particular we have $\gamma_f = \theta_i \ell_i$ for some $\ell_i \in L_f$ and $1 \leq i \leq r$. Now if $\gamma_f, \gamma_f'$ are such that $\gamma_f = \theta_i \ell_i$ and $\gamma_f' = \theta_i \ell_i'$ with $\ell_i, \ell_i' \in L_f$, then $\gamma_f' = \gamma_f.\ell_i^{-1}\ell_i' = \gamma_f \zeta$ with $\zeta \in L_f$. We thus see that we may assume that the $\theta_i = \xi_i$ with $\xi_i$ in $G(k)$. Let $\Xi = \{ \xi_1, \ldots, \xi_r \} \subset G(k)$. Then one finds that

$$K_\infty A_c.\Omega_\infty.\Xi_\infty \cdot d_\infty(\Gamma_{L_f}) = G_\infty.$$ 

We summarize the above discussion in

1.9. Theorem (A. Borel) Let $L$ be a compact open subgroup of $G(\mathbb{A}_f)$ and $\Gamma = d_\infty(\Gamma_{L})$, where $\Gamma_{L}$ is the subgroup $d_{L}^{-1}(L) \subset G(k)$. Then $\Gamma \text{ a discrete subgroup of } G_\infty$. There exists a finite subset $\Xi$ in $G(k)$ a constant $c_0 > 0$ and a compact set $\Omega_\infty \subset^o \mathcal{P}_\infty$ such that

$$K_\infty \cdot A_c \cdot \Omega'_\infty \cdot \Xi_\infty \cdot \Gamma = G_\infty$$

where $K_\infty$ is a maximal compact subgroup of $G_\infty$. Further for $c' > c$ if $S_{c'} = K_\infty \cdot A_{c'} \cdot \Omega'_\infty$, the set

$$\{ \gamma \in \Gamma \mid S_{c'} \xi \cap S_{c'} \xi' \neq \phi \}$$
is finite for \( \xi, \xi' \in \Xi_\infty \).

(The last assertion is an additional piece of information that cannot be deduced from Theorem 1.6.)

1.10. Suppose now that the centre of \( G \) does not contain a split torus (the centre of \( GL(n) \) is a 1-dimensional split torus so the considerations of this paragraph are not applicable to \( GL(n) \); they are however applicable to the group \( SL(n) \)). In this case by using the Iwasawa decomposition in \( G_\infty \) it can be shown that the Haar measure of the set \( S_{\varepsilon'} \) is finite. Since a left translation invariant Haar measure on \( G_\infty \) is also right translation invariant, one sees that a Haar measure on \( G_\infty \) defines a left translation (under \( G_\infty \)) invariant finite measure on \( G_\infty / T \). [We will elaborate on this in the case \( G = SL(n) \). In this case \( G_\infty = \prod_{v \in \Xi} SL(n, k_v) \). We can identify \( SL(n, k_v) \) with \( SL(n, \mathbb{R}) \) or \( SL(n, \mathbb{C}) \) according to \( k_v \simeq \mathbb{R} \) or \( \mathbb{C} \). The compact group \( K_\infty \) is a product \( \prod_{v \in \Xi} K_v \) where for \( v \in \Xi, K_v = SO(n) \) or \( SU(n) \) according to \( k_v \simeq \mathbb{R} \) or \( \mathbb{C} \). The group \( A \) can be identified with the group of diagonal matrices in \( SL(n, \mathbb{R}) \) with positive real diagonal entries - the inclusion of \( \mathbb{Q} \) in \( k \) induces for each \( v \in \Xi \), an inclusion \( SL(n, \mathbb{R}) \) in \( SL(n, k_v) \) and hence a diagonal inclusion of \( SL(n, \mathbb{R}) \) in \( G_\infty \). For \( c' > 0, A_{c'} = \{ \mathbf{d} = \text{diagonal} \ (d_1, \ldots, d_n) \mid d_i > 0 \text{ for } 1 \leq i \leq n, d_1 \cdot d_2 \cdots d_n = 1 \text{ and } d_i/d_{i+1} \leq c \text{ for } 1 \leq i < n \} \). The group \( P \) consists of upper triangular matrices in \( SL(n) \) and let \( U \) be the subgroup of upper triangular unipotent matrices in \( P \). The group \( ^v P_\infty = \prod_{v \in \Xi} ^v P(k_v) \) and \( ^v P(k_v) = B(k_v) \). \( U(k_v) \) where \( B(k_v) = \{ \mathbf{d} = \text{diagonal} \ (d_1 \cdots d_n) \mid d_i \in k_v^*, |d_i|_v = 1 \text{ for } 1 \leq i \leq n \} \). One sees then that \( B_\infty = \prod_{v \in \Xi} B(k_v) \) is compact and contained in \( K_\infty \) and as it centralises \( A \), we see that taking \( \Omega' \) to be left translation invariant under \( B_\infty, S_{\varepsilon'} = K_\infty \cdot A_{c'} \cdot \Omega_1' \) with \( \Omega_1' \) a suitable compact subset of \( U_\infty \). The natural map

\[
K_\infty \times A \times U_\infty \rightarrow G_\infty
\]

is an analytic isomorphism of manifolds. The Haar measure on \( G_\infty \) pulled back to \( K_\infty \times A \times U_\infty \) takes the form \( \rho^2(a) \cdot dk \cdot da \cdot du \) where \( dk \) (resp. \( da, du \)) is a Haar measure on \( K_\infty \) (resp. \( A, U \)) and \( \rho^2 \) is the homomorphism of \( A \) in \( \mathbb{R}^+ \) determined as follows: Inner conjugation by \( a \in A \) carries the Haar measure \( du \) on \( U_\infty^+ \) into \( \rho^2(a) \cdot du \) (\( U_\infty \) is normalised by \( A \)). It is also easy to see that \( \int_{A_c} \rho^2(a) da < \infty \). Since \( K_\infty \) and \( \Omega_1' \) are compact, it is immediate
that the Haar measure of \( S_c \) is finite.] We continue with the assumption
that \( G \) contains no nontrivial \( k \)-split central torus (so that \( G_\infty/\Gamma \) has finite
Haar measure). We assert now that in this case the measure induced on
\( G(\mathbb{A})/G(k) \) by the Haar measure on \( G(\mathbb{A}) \) is finite as well. To see this we
denote by \( G_1 \) the closure of \( G(k) \) (imbeddied diagonally in \( G(\mathbb{A}) \)). Now
\( G_1 = G(k) \) if \( G_\infty \) is compact; in this case one knows (Godement Criterion)
that \( G(\mathbb{A})/G(k) \) is compact and the finiteness of the Haar measure follows.
Thus we assume that \( G_\infty \) is not compact (this is the case when \( G = SL(n) \)
with \( n \geq 2 \)). Then one knows that \( G(\mathbb{A})/G_1 \) is compact and that \( G_1 \) is
normal in \( G(\mathbb{A}) \) (when \( G = SL(n) \) one has in fact \( G_1 = G(\mathbb{A}) \)-this is
seen easily using the fact that \( SL(n, k_v) \) is generated by upper and lower
triangular unipotent matrices and the fact (by the Chinese remainder the-
orem) that \( k \) is dense in \( \mathbb{A} \). \( \pi : G(\mathbb{A}) \to G_\infty \cdot G_1 \backslash G(\mathbb{A}) \simeq G_1 \backslash G(\mathbb{A}) \)
be the natural map. From the definition of \( G_1 \) it is clear that \( \pi \) factors
through \( G(\mathbb{A})/G(k) \). Since \( G_1 \backslash G(\mathbb{A}) \) is compact we conclude that in
order to show that \( G(\mathbb{A})/G(k) \) has finite Haar measure it suffices to show
that \( G_\infty \cdot G_1/G(k) \) has finite Haar measure. Now if \( L \) is any compact open
subgroup of \( G(\mathbb{A}), G_\infty \cdot G_1 \subset G_\infty \cdot L \cdot G(k) \) and the quotient \( L \cdot G_1/G_1 \)
is compact. Thus it suffices to show that \( G_\infty \cdot L \cdot G(k)/G(k) \) has finite
Haar measure for a compact open subgroup \( L \subset G(\mathbb{A}) \). Clearly one has a
natural identification of \( G_\infty \cdot L \cdot G(k)/G(k) \) with \( G_\infty \cdot L/(G_\infty \cdot L \cap G(k)) \);
since \( G_\infty/\Gamma \) (\( \Gamma = \) projection of \( \Gamma_L = G_\infty \cdot L \cap G(k) \) on \( G_\infty \)) has finite
Haar measure and \( L \) is compact \( G_\infty \cdot L \cdot G(k)/G(k) \) has finite Haar measure
proving our contention. We have thus

1.11. Theorem Assume that \( G \) has no non trivial \( k \) split torus in its
centre. Then \( G(\mathbb{A})/G(k) \) has finite Haar measure. If \( G \) contains no non
trivial \( k \)-split torus \( G(\mathbb{A})/G(k) \) is compact.

1.12. Suppose now that \( C \subset G \) is the maximal central \( k \) split torus. Then
\( \overline{G} = G/C \) contains no non trivial central split torus. In the case of main
interest to us viz when \( G = GL(n), C(\simeq GL(1)) \) is of dimension 1 and is the
entire centre and \( \overline{G} \) is the group \( PGL(n) \). By Theorem 1.10, \( \overline{G}(\mathbb{A})/\overline{G}(k) \)
has finite Haar measure. Now it is known that if \( \pi : G \to \overline{G} \) is the natural
map, \( \pi(G(\mathbb{A})) \) is a closed normal subgroup of \( \overline{G}(\mathbb{A}) \) and \( \overline{G}(\mathbb{A})/\pi(G(\mathbb{A})) \)
is compact. Also the kernel of \( \pi : G(\mathbb{A}) \to \overline{G}(\mathbb{A}) \) is evidently \( C(A) \). It follows
that \( \overline{G}(\mathbb{A})/\pi(G(k)) \) is compact so that \( G(\mathbb{A})/C(A) \cdot G(k) \) is compact as
well. Once again we elaborate on this in the special case \( G = GL(n) \).
Here for any field $k' \supset k$ the natural map $GL(n)(k') \to PGL(n)(k')$ is surjective - this is an immediate consequence of Hilbert Theorem 90. Using the realisation of $PGL(n)$ as an algebraic $k$ subgroup of $GL(n^2)$ got from the adjoint representation, one shows that $GL(n, O_v) \to PGL(n)(O_v)$ is surjective for all $v \in V_f$. It is then immediate that the map $GL(n, \mathbb{A}) \to PGL(n)(\mathbb{A})$ is surjective. Observe next that $C(\mathbb{A}) \simeq GL(1, \mathbb{A})$ is a maximal split torus in $\mathbb{G}$, and hence defines a function $f(\mathbb{G}(\mathbb{A}) \to C$ such that we have

\[
\begin{align*}
f(gc) &= \chi(c)^{-1} f(g) \text{ for } c \in C(\mathbb{A}) \text{ and } \\
f(gx) &= f(g) \text{ for } x \in \mathbb{G}(k)
\end{align*}
\]

Note that in order that $F^\chi(\mathbb{G}(\mathbb{A}), \mathbb{G}(k))$ be non-trivial we need $\chi$ to be trivial on $C(k)$. If $\chi$ is unitary i.e., maps $C(\mathbb{A})$ into $S^1 = \{ z \in \mathbb{C}^* \mid |z| = 1 \}$, then the function $g \mapsto f(g)$ on $\mathbb{G}(\mathbb{A})$ is invariant under $C(\mathbb{A}) \cdot \mathbb{G}(k)$ and hence defines a function $\int_{\mathbb{G}(\mathbb{A})/C(\mathbb{A})} f(\mathbb{G}(\mathbb{A}))$. We will say that $f \in F^\chi(\mathbb{G}(\mathbb{A}), \mathbb{G}(k))$ is square integrable if $f$ is square integrable on $\mathbb{G}(\mathbb{A})/C(\mathbb{A}) \cdot \mathbb{G}(k)$. Let $G' = [\mathbb{G}, \mathbb{G}]$ be the commutator subgroup. Then the connected component $S'$ of the identity in $S \cap G'$ is a maximal split torus in $G'$. The natural map $\mathbb{C} \times S' \to S$ is not in general an isomorphism (it is not in the case $\mathbb{G} = GL(n)$ where $G' = SL(n)$). However the group $A$ in $\mathbb{G}(\mathbb{A})$ breaks up into a direct product $(C(\mathbb{A}) \cap A) \times (S'(\mathbb{A}) \cap A)$ i.e., the natural map $(C(\mathbb{A}) \cap A) \times (S'(\mathbb{A}) \cap A) \to A$ is an isomorphism. Let $A_0 = C(\mathbb{A}) \cap A$ and $A' = S'(\mathbb{A}) \cap A$ so that $A \simeq A_0 \times A'$. We now define a subgroup $A_0^1$ in $A_0 : A_0^1 = \{ a \in A_0 \subset C(\mathbb{A}) \mid \prod_{v \in \infty} | \chi(a_v) | v = 1 \}$ for every character $\chi$ of $G$ defined over $k$. Since $\chi(a_v) \in \mathbb{R}^+$ for all $a \in A_0$ and any character $\chi$ on $C$, and the subgroup of characters on $C$ which are restrictions of characters on $G$ has finite index in the group of all characters on $C$, we see that $A_0^1 = \{ a \in A_0 \mid \prod_{v \in \infty} | \chi(a_v) | v = 1 \}$ for all characters $\chi$ on $C$. The sequence

\[
\{1\} \to A_0^1 \to A_0 \to A_0/A_0^1 \to \{1\}
\]

splits and has in fact a natural splitting. To see this fix an isomorphism of $C$ with a product $GL(1)^r$ of $r$ copies of $GL(1)$. Note that such an isomorphism yields compatible decompositions of $A_0$ and $A_0^1$ as products of $r$ copies of the same groups. This means that we need to give the 'natural splitting' in the case when $C = GL(1)$. Here $C(\mathbb{A}) = \mathbb{I}$ and $A_0$ is the product of
| $\infty$ | copies of $\mathbb{R}^+$ imbedded in $\prod_{v \in \infty} k_v^*$ through the inclusions $\mathbb{R} \subset k_v \simeq \mathbb{R}$ or $\mathbb{C}(v \in \infty)$. The group $A_0^1$ is then naturally isomorphic to the subgroup 
\{ $x \in \prod_{v \in \infty} k_v^* \mid x_v \in \mathbb{R}^+, \prod_{v \in \infty} x_v = 1$ \} and a natural supplement to $A_0^1$ in $A$ is the group \{ $x \in \prod_{v \in \infty} k_v^* \mid x_v = x$ for all $v \in \infty$ with $x \in \mathbb{R}^+$ \}. Thus we have obtained a splitting using an isomorphism of $\mathbb{C}$ with $(GL(1))^r$. It is easy to see that the splitting is independent of the choice of this isomorphism. Now let $G^* = \{ g \in G(\mathfrak{A}) \mid | \chi(g) | = 1$ for every character $\chi$ of $G$ defined over $k$. Then $G^* \supset A_0^1$ and if $'A_0$ is the natural supplement to $A_0^1$ in $A_0$, then $'A_0 \cap G^* = \{ 1 \}$ and the natural map

$'A_0 \times G^* \to G(\mathfrak{A})$

is an isomorphism (of locally compact groups). The product formula (1.2) shows that $G(k) \subset G^*$. Using the fact that $\mathbb{I}^1/k^*$ is compact, it is now easy to deduce the following from Theorem 1.10.

1.13. Theorem $G^*/G(k)$ has finite Haar measure.

1.14. It is easy to see that any compact subgroup of $G(\mathfrak{A})$ is contained in $G^*$. Consequently there is a $c > 0$ and a compact subset $\Omega \subset ^oP(\mathfrak{A}) \cap G^* =$ $P^*$ such that for all $\Omega \supset \Omega', \Omega \subset ^oP(\mathfrak{A}) \cap G^*$, and $c' \geq c$ we have

$G^* = K \cdot A_0^1 \cdot A_0' \cdot \Omega' \cdot G(k)$

where $A_0' = \{ a \in A^1 \mid \alpha(a) \leq c' \text{ for all } \alpha \in \triangle \}$. Let $C(\mathfrak{A})^1 = \{ g \in C(\mathfrak{A}) \mid | \chi(g) | = 1$ for all characters $\chi$ on $C \}$. Then $C(k) \subset C(\mathfrak{A})^1$ (product formula) and $C(\mathfrak{A})^1/C(k)$ is compact. Hence we can find a compact subset $C$ of $C(\mathfrak{A})^1$ such that $C \cdot C(k) = C(\mathfrak{A})^1$. Clearly $C \cdot \Omega' = \Omega_0$ is a compact subset of $P^*$ and one has

$G^* = K \cdot A_0' \cdot \Omega_0 \cdot G(k)$.

2 Automorphic Forms

2.1. We continue with the notations introduced in §1. We fix a unitary character $\chi : C(\mathfrak{A}) \to S^1$ on $C(\mathfrak{A})$ and assume that $\chi$ is trivial on $C(k)$. We introduce now some function spaces. Recall that we defined $\mathcal{F}_\chi(G(\mathfrak{A}), G(k))$
as the vector space of all functions \( f : G(A) \to \mathbb{C} \) satisfying the following condition
\[
f(xg\gamma) = \chi(x)^{-1}f(x)
\]
for all \( g \in G(A) \), \( x \in C(A) \) and \( \gamma \in G(k) \). In the sequel we often write \( \mathcal{F}_X \) for \( \mathcal{F}_X^X(G(A), G(k)) \) (where there is no ambiguity about the group \( G \) we are talking about). We set
\[
\mathcal{C}^X = \mathcal{C}_c^X(G(A), G(k)) = \{ f \in \mathcal{F}_X \mid f \text{ continuous} \}
\]
\[
\mathcal{C}_c^X = \mathcal{C}_c^X(G(A), G(k)) = \{ f \in \mathcal{C}^X \mid f \text{ has compact support modulo } C(A)G(k) \).
\]
A function \( f : G(A) \to \mathbb{C} \) in \( \mathcal{F}_X \) is square integrable if \( f \) is Borel measurable and
\[
\| f \|^2 = \int_{G(A)/C_\infty G(k)} |f(g)|^2 d\mu(g) < \infty \quad \text{for } g \in G(A), \quad x \in C(A) \quad \text{and } \quad \gamma \in G(k)
\]
so that \( |f(g)| \) can be treated as a function on \( G(A)/C(A)G(k) \); \( \mu \) is the Haar measure on this homogeneous space of the locally compact group \( G(A)/C(A) \). We denote by \( L^2_X = L^2_X(G(A), G(k)) \) the set of all square integrable functions in \( \mathcal{F}_X \) modulo the equivalence of equality almost everywhere with respect to \( \mu \). It is a Hilbert space under the inner product
\[
(f, f') \mapsto \int_{G(A)/C(A)G(k)} f(g)\overline{f'(g)}d\mu(g).
\]
We have (as usual) an inclusion \( \mathcal{C}_c^X \hookrightarrow L^2_X \). A function \( f : \Omega \to \mathbb{C} \), \( \Omega \) an open set in \( G(A) \) is said to be \( C^\infty \) if the following conditions hold:

(i) for any \( g_0 \in \Omega \), there is a neighbourhood \( \Omega' \) of \( g_0 \) in \( \Omega \) and a compact open subgroup \( L \subset G(A_f) \) such that \( L\Omega' \subset \Omega \) and \( f(xg) = f(g) \) for all \( g \in \Omega' \) and \( x \in L \).

(ii) For any \( g_0 \in \Omega \), there is a neighbourhood \( U \) of the identity in \( G_\infty \) such that \( Ug_0 \subset \Omega \) and the map \( x \mapsto f(xg_0) \) is \( C^\infty \) on \( U \).

Following the decomposition \( G_\infty \times G(A_f) \cong G(A) \) as a direct product we may regard any complex-valued function \( f \) on \( G(A) \) as a function of two variables (one in \( G_\infty \) and the other in \( G(A_f) \)). If \( f \) is \( C^\infty \), then it is \( C^\infty \) in the first variable and locally constant in the second variable; also the partial derivatives of \( f \) with respect to the first variable are themselves \( C^\infty \) on \( G(A) \). We denote by \( \mathcal{C}^\infty = \mathcal{C}^\infty(G(A), G(k)) \) (resp. \( \mathcal{C}_c^\infty = \mathcal{C}_c^\infty(G(A), G(k)) \)) the vector space of \( C^\infty \) functions (resp. \( C^\infty \) functions with support compact modulo \( C(A) \cdot G(k) \)). We have evidently inclusions
\[
\mathcal{C}_c^\infty \hookrightarrow \mathcal{C}_c^X \hookrightarrow L^2_X.
\]
2.2. In the sequel $E$ will denote any one of the function spaces $\mathcal{F}^\times, \mathcal{C}^\times, \mathcal{C}^\times_1, \mathcal{C}^\times_\infty, \mathcal{L}_2^\times$ introduced above and for an open compact subgroup $L$ of $G(\mathbb{A}_f), E^L$ will denote the subspace of functions in $E$ which are invariant under left translations by $L$. We observe that a function $f$ in $E$ is determined by its restriction to $G^\times$ (since $'A_0G^\times = G(\mathbb{A})$ and $'A_0 \subset C(\mathbb{A})$). On the other hand

$$G^* = K \cdot A_0^1 \cdot A_c' \cdot \Omega' \cdot C \cdot G(k)$$

with $C \subset C(\mathbb{A})$ (cf. 1.14). Further $K = K_\infty \cdot K_f$ with $K_\infty \subset G_\infty$ and $K_f \subset G(\mathbb{A}_f)$ and one may assume that $\Omega' = \Omega_\infty \cdot \Omega_f$ with $\Omega_f$ (resp. $\Omega_f$) a compact subset of $^*\mathbb{P}_\infty$ (resp. $^*\mathbb{P}(\mathbb{A}_f)$). Now $K_f \cdot \Omega_f$ is a compact subset of $G(\mathbb{A}_f)$. It follows that if $L_f \subset G(\mathbb{A}_f)$ is a compact open subgroup of $G(\mathbb{A}_f)$, there exists a finite set $\mathcal{H}^\prime \subset G(\mathbb{A}_f)$ such that $K_f \cdot \Omega_f \subset L_f \cdot \mathcal{H}^\prime$. It follows from Corollary 1.7 that we have $L_f \cdot \mathcal{H}^\prime \cdot d_f(G(k)) = G(\mathbb{A}_f)$. We introduce an equivalence relation on $\mathcal{H}^\prime$ as follows: for $\theta, \theta^\prime \in \mathcal{H}^\prime, \theta \sim \theta^\prime$ if there exists $\rho \in G(k)$ such that $\theta = \theta^\prime \cdot \rho \cdot \rho^{-1} \in L_f$ (it is easily checked that this is an equivalence relation). Let $\mathcal{H} \subset \mathcal{H}^\prime$ be a subset containing exactly one element in each equivalence class. We then assert that any $f \in E^{L_f \mathcal{H}}$ is determined by its restriction to $G_\infty L_f \mathcal{H}$ (and in view of $L_f$-invariance) by its restriction to $G_\infty \cdot \mathcal{H}$. Evidently to prove this it suffices to show that $G_\infty \cdot L_f \cdot \mathcal{H} \cdot C(\mathbb{A}) \cdot G(k) = G(\mathbb{A})$. We have seen that

$$G(\mathbb{A}) = 'A_0 \cdot G^* = 'A_0 \cdot K_\infty \cdot A^1_0 \cdot A'_c \cdot \Omega_\infty \cdot K_f \cdot \Omega_f \cdot C(\mathbb{A})G(k) \subset G_\infty \cdot L_f \cdot \mathcal{H} \cdot C(\mathbb{A}) \cdot G(k).$$

Thus it suffices to show that if $\theta \sim \theta^\prime$,

$$G_\infty \cdot L_f \cdot \theta \cdot G(k) \supset G_\infty \cdot L_f \cdot \theta^\prime.$$

Since $\theta \sim \theta^\prime$, there exists $\rho \in G(k)$ with $\theta \rho \theta^{-1} = \ell \in L_f$ so that $\theta^\prime = \ell^{-1} \theta \rho_f$. It follows that $G_\infty L_f \rho_f \rho^{-1} = G_\infty L_f \theta \rho_f \rho^{-1} = G_\infty \rho \rho^{-1} L_f \theta = G_\infty L_f \theta$. It follows that $L_f \cdot \mathcal{H} \cdot d_f(G(k)) = G(\mathbb{A}_f)$. For $\theta \in \mathcal{H}$, let $r_\theta(f) : G_\infty \to \mathbb{C}$ be the function $r_\theta(f)(g) = f(g \theta)$. Then $r_\theta(f)(g \gamma) = r_\theta(f)(g)$ for all $\gamma \in G(k)$ such that $\gamma \in \theta^{-1} L_f \gamma$. In fact one has

$$r_\theta(f)(g \gamma) = f(g \gamma \theta) = f(g \gamma \theta^{-1} \theta) = f(g \theta \gamma^{-1} \theta^{-1} \theta) = f(g \theta).$$

Let $\Gamma_\theta = d_\infty(G(k) \cap d_f^{-1}(\theta^{-1} L_f \theta))$ i.e.,

$$\Gamma_\theta = d_\infty \{ \gamma \mid \gamma \in \theta^{-1} L_f \theta \}.$$
Let $C^\chi(G_\infty/\Gamma_\theta)$ (resp. $C^\chi_c(G_\infty/\Gamma_\theta)$, resp. $C^{\chi\infty}(G_\infty/\Gamma_\theta)$, resp. $L^\chi_2(G_\infty/\Gamma_\theta)$) be the space of all continuous functions (resp. continuous functions with compact support modulo $C_\infty$ (resp. $C^\infty$, resp. $C^{\chi\infty}$, resp. $L^\chi_2$), and $f \in E^L, r_\theta(f) \in C^\chi(G_\infty/\Gamma_\theta)$ (resp. $C^\chi_c(G_\infty/\Gamma_\theta)$, resp. $C^{\chi\infty}(G_\infty/\Gamma_\theta)$, resp. $C^\chi_c^\infty(G_\infty/\Gamma_\theta)$, resp. $L^\chi_2(G_\infty/\Gamma_\theta)$). We thus obtain maps

$$
\prod_{\theta \in \mathcal{H}} f_\theta : C^\chi^L \rightarrow \prod_{\theta \in \mathcal{H}} C^\chi(G_\infty/\Gamma_\theta)
$$

$$
\prod_{\theta \in \mathcal{H}} f_\theta : C^\chi^L_c \rightarrow \prod_{\theta \in \mathcal{H}} C^\chi_c(G_\infty/\Gamma_\theta)
$$

$$
\prod_{\theta \in \mathcal{H}} f_\theta : C^{\chi\infty L} \rightarrow \prod_{\theta \in \mathcal{H}} C^{\chi\infty}(G_\infty/\Gamma_\theta)
$$

$$
\prod_{\theta \in \mathcal{H}} f_\theta : C^\chi^L_c \rightarrow \prod_{\theta \in \mathcal{H}} C^\chi_c^\infty(G_\infty/\Gamma_\theta)
$$

$$
\prod_{\theta \in \mathcal{H}} f_\theta : L^\chi_2 \rightarrow \prod_{\theta \in \mathcal{H}} L^\chi_2(G_\infty/\Gamma_\theta).
$$

### 2.3. Proposition

The maps above are isomorphisms of topological vector spaces.

We record here for future use the following fact proved above.

### 2.4. Lemma

$L_f \cdot \mathcal{H} \cdot d_f(G(k)) = G(\Lambda_f)$.

### 2.5. Proof of 2.3

The topologies on these spaces are the standard ones. On $C^\chi^L$ and $C^\chi(G_\infty/\Gamma_\theta)$ (resp. $C^{\chi\infty L}$ and $C^{\chi\infty}(G_\infty/\Gamma_\theta)$) it is the topology of uniform convergence (resp. together with all derivatives) on compact sets. On $C^\chi^L_c$ and $C^\chi_c^\infty$ (resp. $C^\chi^2(G_\infty/\Gamma_\theta)$ and $C^\chi_c^\infty(G_\infty/\Gamma_\theta)$) it is the inductive limit of the topology of uniform convergence (resp. together with all derivatives) on closed sets of $G(\Lambda)$ which are compact modulo $C(k)$ and closed subsets of $G_\infty$ which are compact modulo $C_\infty \cdot \Gamma_\theta$ respectively. The $L_2$ spaces are of course given the Hilbert space structure. That the maps are continuous injections is easy to see. We need to show that the maps are surjective - the open mapping theorem would then guarantee that the maps are isomorphisms. Let $\{f_\theta\}_{\theta \in \mathcal{H}}$ be a collection of functions on $G_\infty/\Gamma_\theta, \theta \in \mathcal{H}$ belonging to one of the above spaces. Define a function $f$ on $G_\infty L\mathcal{H}$ by setting $f(g \ell \theta) = f_\theta(g)$. We extend the function $f$ to all of
\[ \mathbf{G}(\mathbb{A}) \text{ by setting } f(g) = f(g\gamma) \] where \( \gamma \in \mathbf{G}(k) \) is an element such that \( g\gamma \in \mathbf{G}_{\infty}LH \). We need only check that \( f \) is well defined i.e., if \( g\gamma' \in \mathbf{G}_{\infty}LH \) also for some \( \gamma' \in \mathbf{G}(k), f(g\gamma) = f(g\gamma') \). Let \( g\gamma = h\ell\theta \) and \( g\gamma' = h'\ell'\theta' \). Then setting \( \gamma^{-1}\gamma' = \zeta \), we have \( h\ell\theta\zeta = h'\ell'\theta' \). It follows that \( \ell\theta\zeta f = \ell'\theta' f \) leading to \( \theta\zeta f\theta^{-1} = \ell^{-1}\ell' \in L \) i.e., \( \theta \sim \theta' \) and hence \( \theta = \theta' \); and when \( \theta = \theta', \zeta f \in \theta^{-1}L\theta \) so that \( \zeta \in \Gamma_\theta \), and \( f_\theta \) is \( \Gamma_\theta \)-invariant. This proves that \( f \) is well defined. That \( f \) has the required properties of continuity, smoothness etc. if the \( \{f_\theta\}_{\theta \in \mathcal{H}} \) have them is immediate from the definitions.

2.6. Let \( \mathcal{U} \) denote the universal enveloping algebra of the real Lie algebra \( L(G) \) of \( G \). Then \( \mathcal{U} \) operates on the space of \( C^\infty \) functions on \( \mathbf{G}(\mathbb{A}) \) leaving stable the subspaces \( \mathcal{C}^\infty(L(G)(\mathbb{A}), \mathbf{G}(k)) \) and \( \mathcal{C}^\infty_{\ell}(\mathbf{G}(\mathbb{A}), \mathbf{G}(k)) \) as well as the subspace of \( L \)-invariants in these spaces for any compact open subgroup \( L \) of \( \mathbf{G}(\mathbb{A}) \). A function \( f \in \mathcal{C}^\infty \) is \( K \)-finite if the \( C \)-linear span of \( \{L_k f \mid k \in K \} \) is finite dimensional. The group \( K \) acts continuously on this finite dimensional vector space \( V \) and hence the image of the profinite group \( K_f \) in \( GL(V) \) is finite. Thus if \( \varphi \) is \( K \)-finite, \( \varphi \) is invariant under an open compact subgroup of \( \mathbf{G}(\mathbb{A}) \); and the \( C \)-span of \( \{L_k \varphi \mid k \in K_\infty \} \) is finite dimensional. Conversely if \( \varphi \) satisfies these two conditions \( \varphi \) is \( K \)-finite. Let \( Z \) be the centre of \( \mathcal{U} \). A \( C^\infty \) function \( \varphi \) in \( \mathcal{C}^\infty_{\ell} \) is \( Z \)-finite iff the \( C \)-linear span of \( \{zf \mid z \in Z \} \) is finite dimensional. Functions relevant to harmonic analysis on \( \mathbf{G}(\mathbb{A})/\mathbf{G}(k) \) are those that do not grow too rapidly at infinity. To describe the kind of growth at infinity that we need to impose we need some preliminary definitions. We assume, as we may, that the imbedding of \( G \) in \( GL(N) \) maps \( G \) into \( SL(N) \) (when \( G = GL(n) \) we can take \( N = 2n \) and the imbedding to be

\[ g \mapsto \begin{pmatrix} g & 0 \\ 0 & t_g^{-1} \end{pmatrix} g \in GL(n). \]

This is done so that one takes care that the entries of \( g \) as well as those of \( g^{-1} \) are handled simultaneously. We define for \( g \in \mathbf{G}(\mathbb{A}) \) the height of \( g \) denoted \( \| g \| \) in the sequel as \( \prod_{v \in V} (\max_{1 \leq i,j \leq N} \| g_{vij} \|) \) where \( g_{vij}, 1 \leq i,j \leq N \) are the entries of the \( v \)-adic component \( g_v \in \mathbf{G}(k_v) \) of \( g \). Observe that since \( g_v \in GL(N,O_v) \) for all \( v \in V \backslash S \) for a finite subset \( S \) containing \( \{1 \leq i,j \leq N \mid \| g_{vij} \| = 1 \) for all but a finite number of \( v \) and thus the product above over all \( v \in V \) reduces to a finite product. It is easy to see that \( g \mapsto \| g \| \) is a continuous function of \( \mathbf{G}(\mathbb{A}) \) in \( \mathbb{R}^+ \). In the sequel we will say that a right \( \mathbf{G}(k) \)-invariant function \( f : \mathbf{G}(\mathbb{A}) \mapsto \mathbb{C} \) has moderate...
growth if there exist constants $c, r > 0$ such that for all $g \in G(\mathbb{A})$ one has

$$|f(g)| \leq c \|g\|^r$$

with this notion of moderate growth we have

2.7. Definition An automorphic form (with central unitary character $\chi$) on $G(\mathbb{A})$ is a $C^\infty$ function $f: G(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions

(i) $Uf$ is of moderate growth for all $U \in U$.

(ii) $f(gx) = \chi(x)^{-1} f(g)$ for all $g \in C(\mathbb{A})$ and all $x \in C(\mathbb{A})$.

(iii) $f(g\gamma) = f(g)$ for all $g \in G(\mathbb{A})$ and $\gamma \in G(k)$.

(iv) $f$ is $K$-finite

(v) $f$ is $\mathbb{Z}$ finite.

The automorphic forms with central character form a vector space which we denote $A^\chi$.

2.8. Remarks

(i) If a nonzero automorphic form is to exist, with $\chi$ as central character, evidently one must have $\chi(\rho) = 1$ for all $\rho \in C(k)$ in view of the condition (iii). Thus we will consider only characters $\chi$ on $C(\mathbb{A})$ that are trivial on $C(k)$. Also note that since $\chi$ is unitary, the growth condition does not lead to any contradiction.

(ii) We have assumed that the group $G$ is reductive. This means that the Lie algebra $L(G_\infty)$ of $G_\infty$ is a direct product of an abelian Lie algebra $h$ and a semisimple Lie algebra $s$. The Lie algebra $s$ has a Cartan - decomposition $s = k \oplus p$ with the Killing form $<,>$ of $s$ restricted to $k$ (resp. $p$) negative (resp. positive) definite. Let $X_i, 1 \leq i \leq r, Y_j, 1 \leq j \leq s$ and $Z_k, 1 \leq k \leq t$ be bases of $h, k$ and $p$ respectively so chosen that $<Y_j, Y_{j'}> = -\delta_{jj'}$ and $<Z_k, Z_{k'}> = \delta_{kk'}$. Then the element $C = \sum_{1 \leq i \leq r} X_i^2 + \sum_{1 \leq p \leq t} Z_k^2 - \sum_{1 \leq j \leq s} Y_j^2$ is a central element of the enveloping algebra $U$. Let $C' = \sum_{1 \leq j \leq s} Y_j^2$. Then $C + 2C'$ is an element of $U$ and the corresponding differential operator $\triangle$ on $G_\infty$ is an elliptic operator. Let $\mathcal{B}$ be the subalgebra of $\text{End} C^\chi(G(\mathbb{A}), G(k))$. 

generated by the centre $Z$ of $\mathcal{U}$ and $\{L_k \mid k \in K\}$ ($L_k$ is the left translation by $k$). Then the $\mathbb{C}$-span of $\{T(f) \mid T \in \mathcal{B}\}$ is finite dimensional for the automorphic form $f$. Now the span of the $L_k f, k \in K$ contains $U f$ for all $U$ in the subalgebra of $\mathcal{U}$ generated by $k$. Thus $C$ and $C'$ belong to $\mathcal{B}$ and hence so does $\Delta$. It follows that $\{\Delta^n f \mid n \in \mathbb{N}\}$ spans a finite dimensional vector space over $\mathbb{C}$. We conclude that there is a monic polynomial $P$ in one variable such that $P(\Delta)f = 0$. Now $P(\Delta)$ is an elliptic operator on $G_\infty$ with analytic coefficients. By the regularity theorem for elliptic operators with analytic coefficients we conclude that $f$ is analytic (in the variable in $G_\infty$).

(iii) The third comment is that if $U \in \mathcal{U}$ and $f$ is an automorphic form then so is $U f$. It is clear that it suffices to check this when $U \in L(G)$. Now the map $L(G_\infty) \otimes C^\infty$ given by $(U, f) \mapsto U f$ is compatible with the action of $G(\mathbb{A})$ on the two sides. $G(\mathbb{A})$ acts on $L(G)$ by the adjoint representations of the factor $G_\infty$ and on $C^\infty$ by left translation; we take the tensor product representation on the left-hand side. Suppose now $V$ is a finite dimensional subspace of $C^\infty$ which is $K$ as well as $Z$ stable. Then the image of $L(G) \otimes V$ in $C^\infty$ under the above map is finite dimensional $Z$-stable as well as $K$-stable. As this space contains $U f$, for $U \in L(G_\infty)$, we see that $U f$ is $K$-finite and $Z$-finite. That $U'(U f) = (U'U)(f)$ has moderate growth is clear from the definitions.

(iv) The vector space of all automorphic forms $A^\chi$ with central character $\chi$ is stable under left translations by elements of $G(\mathbb{A}_f)$. This is seen as follows: for $g \in G(\mathbb{A}_f)$, $gKg^{-1} \cap K$ has finite index in $K$ and $L_g f$ is $(g^{-1}Kg \cap K)$-finite and hence $K$-finite as well. Since the action of $G(\mathbb{A}_f)$ on the left and the action of $U$ on the space of $C^\infty$ functions commute, we see that $(L_g f)$ is $Z$-finite for $g \in G(\mathbb{A}_f)$. The growth condition is immediate from the fact that for $g_0 \in G(\mathbb{A}_f)$ one has a constant $C > 0$ such that for all $g \in G(\mathbb{A}), \| g_0 g \| \leq C \| g \|$. 

2.9. Lemma (Harish-Chandra: Acta Math. 116 p.118). If $f$ is a $K_\infty$-finite $Z$-finite $C^\infty$ function on $G_\infty$, there is an $\alpha \in C^\infty_c(G_\infty)$ ($= C^\infty$ functions with compact support in $G_\infty$) such that $\alpha * f = f$.

This lemma shows that the growth condition (i) in (2.6) can be replaced by the weaker condition: (i') $f$ is of moderate growth. In fact fix $\alpha \in C^\infty_c(G)$ such that $\alpha * f = f$. Then one has for $U \in \mathcal{U}, U f = U(\alpha * f) = U \alpha * f$; and
it is easy to see that for any $\beta \in C^\infty_c, \beta \ast f$ has moderate growth. In the sequel we will make use of Lemma 2.8 in other contexts too.

2.10. Let $\chi$ be a unitary character on $\mathbf{C}(\mathbb{A})/\mathbf{C}(k)$. Let $\sigma : K \to GL(W)$ be a finite dimensional representation of the compact subgroup $K$ of $\mathbf{G}(\mathbb{A})$ and $\lambda := \mathcal{Z} \to \text{End}_{\mathbb{C}}W'$ a finite dimensional representation of $\mathcal{Z}$. Let $A^\chi(\mathbf{G}(\mathbb{A}), \mathbf{G}(k), \sigma, \lambda)$ denote the space of complex values $C^\infty$ functions $f$ on $\mathbf{G}(\mathbb{A})$ satisfying the following conditions (cf. 2.8).

(i) $f$ has moderate growth.

(ii) $f(g\gamma) = f(g)$ for $g \in \mathbf{G}(\mathbb{A}), \gamma \in \mathbf{G}(k)$.

(iii) $f(gh) = \chi(h^{-1})f(g)$ for $g \in \mathbf{G}(\mathbb{A}), h \in \mathbf{C}(\mathbb{A})$.

(iv) The $K$-span of $f$ (in $F^\chi$) is a quotient of $W$ as a $K$-module.

(v) The $\mathcal{Z}$-span of $f$ (in $F^\chi$) is a quotient of $W'$ as a $\mathcal{Z}$-module.

With this notation our central result is

2.11. Theorem $\dim A^\chi(\mathbf{G}(\mathbb{A}), \mathbf{G}(k), \sigma, \lambda) < \infty$.

2.12. The representation $\sigma$ of $K$ when restricted to the totally disconnected subgroup $K_f$ has a kernel $L_f$ of finite index. Then it is easily seen that $A^\chi(\mathbf{G}(\mathbb{A}), \mathbf{G}(k), \sigma, \lambda)$ is contained in the space of $C^\infty$ functions $f$ on $\mathbf{G}(\mathbb{A})/\mathbf{G}(k)$ such that $f(gx) = \chi(x)^{-1}f(g)$ for $g \in \mathbf{G}(\mathbb{A}), x \in \mathbf{C}(\mathbb{A}), f(kg) = f(g)$ for $k \in L_f$, the $K_\infty$-span of $f$ is a quotient of $\sigma_\infty$ as a $K_\infty$-module and the $\mathcal{Z}$-span of $f$ is a quotient of $\lambda$. We can now appeal to Proposition 2.4 to conclude that Theorem 2.11 is equivalent to:

2.13. Theorem Let $\Gamma \subset \mathbf{G}(k)$ be an arithmetic subgroup. Let $A^\chi(\mathbf{G}_\infty, \Gamma, \sigma_\infty, \lambda)$ be the space of $C^\infty$ functions $f$ on $\mathbf{G}$ such that

(i) $f(g\gamma) = f(g)$ for all $g \in \mathbf{G}_\infty$ and $\gamma \in \Gamma$.

(ii) $f(gc) = \chi(c)f(g)$ for all $g \in \mathbf{G}_\infty$ and $c \in \mathbf{C}_\infty$.

(iii) The $K_\infty$-span of $f$ is a quotient of $\sigma_\infty$.

(iv) The $\mathcal{Z}$-span of $f$ is a quotient of $\lambda$. 
(v) there is a constant $c > 0$ and integer $r > 0$ such that $|f(g)| \leq c \|g\|^r$ for $g \in G_{\infty}$.

Then $A^v(G_{\infty}, \Gamma, \sigma_{\infty}, \lambda)$ is finite dimensional.

3 Cusp Forms

3.1. We need a number of facts from analysis which we collect together in 3.2 - 3.11 below for convenient later use. We refer to R. Narasimhan [N] for proofs of results not given here. We begin with the following

3.2. Proposition Let $X$ be a locally compact space and $\mu$ a Borel probability measure on $X$. Let $C_b(X)$ denote the space of bounded continuous functions on $X$ and for $\varphi \in C_b(X)$, let $\|\varphi\|_{\infty} = \lim \sup \{|\varphi(x)| : x \in X\}$. Suppose $T : L^2(X, \mu) \to L^2(X, \mu)$ is a linear map such that $T(\varphi) \in C_b(X)$ for all $\varphi \in L^2(X, \mu)$ and there is a constant $C > 0$ such that $\|T(\varphi)\|_{\infty} \leq C \|\varphi\|_2$. Then $T$ is an operator of the Hilbert Schmidt type (and hence compact).

Proof. For $x \in X$, the linear form $\varphi \mapsto T(\varphi)(x)$ on $L^2(= L^2(X, \mu))$ is bounded. It follows that there exists $k_x \in L^2$ such that $\|k_x\|_2 \leq C$ and $T(\varphi)(x) = <k_x, \varphi>$ for all $\varphi \in L^2$. Let $k(x, y) = k_x(y)$ for $x, y \in X : k(x, y)$ is defined for all most all $y$ for each $x$. Clearly for each fixed $x, k_x(y)$ is measurable so that $k(x, y)$ is measurable on $X \times X$. Now

$$\int_{X \times X} |k(x, y)|^2 \, d\mu(x) \cdot d\mu(y) = \int_X d\mu(x) \int_Y |k_x(y)|^2 \, d\mu(y) \leq \int_X d\mu(x)C^2 = C \quad \text{(since $\mu(X) = 1$)}$$

Thus $k(x, y) \in L^2(X \times X, \mu \times \mu)$ and

$$T(\varphi)(x) = \int_X k_x(y)\varphi(y)d\mu(y) = \int_X k(x, y)\varphi(y)d\mu(y).$$

Thus $T$ is a Hilbert-Schmidt operator.

3.3. The next result we want to state is the Sobolev inequality. We need to introduce some preliminary notation to state the result. Let $M$ be be a smooth manifold. Let $\Omega$, be a relatively compact subset of $M$; the compact closure of $\Omega$ is denoted $\overline{\Omega}$. Let $\{U_i, \ 1 \leq i \leq m\}$ be an open covering of
the closure $\overline{\Omega}$ of $\Omega$ by coordinate open sets; let $V_i, 1 \leq i \leq m$, be open subsets of $U_i$ such that the closure $\overline{V_i}$ of $V_i$ is compact and contained in $U_i$ and further $\bigcup_{1 \leq i \leq m} V_i \supseteq \overline{\Omega}$. We fix a $C^\infty$ volume form on $M$ and denote the corresponding measure by $\mu$. If $x_1, \cdots, x_n$ are the coordinates in $U_i$, we denote by $D^\beta$ the operator $\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ for a multi index $\beta = (\beta_1, \cdots, \beta_n)$ of non-negative integers. We introduce on the space of $C^\infty$ functions on $\overline{\Omega}$ (i.e. $C^\infty$ functions defined on a neighbourhood of $\overline{\Omega}$) the following norms $\| \cdot \|_{r,p}$ where $r$ is an integer $\geq 0$ and $p > 1$. For a $C^\infty$ function $f$ on $\overline{\Omega}$,

$$\| f \|_{r,p} = \text{Sup} \{ \| D^\beta f \|_{p,V_i} \mid \beta \mid \leq r, \quad 1 \leq i \leq m \}. $$

Here for $h$ a $C^\infty$ function on $\overline{\Omega}$,

$$\| h \|_{p,V_i} = \int_{V_i} | h |^p \, d\mu.$$  

The norm defined above depends on the covering and the shrinking chosen, the coordinates chosen in the covering open sets, and the volume form on $M$. However the equivalence class of the norm $\| \cdot \|_{r,p}$ depends only on $r$ and $p$ and not on the choices made above. With these definitions we can now state Sobolev’s inequality.

3.4. Theorem  There is a constant $C, C(p, \overline{\Omega}) > 0$ such that for all $f, C^\infty$ on $\overline{\Omega}, \| f \|_{\infty} \leq C \| f \|_{[n/p]^*,p}$ where $[n/p]^*$ is the minimal integer $\geq n/p$ ($n = \dim M$).

3.5. Remark  In the special case when $M = \mathbb{R}^n$ and we take the coordinates to be standard coordinates, the volume form as the standard one, and $\Omega$ is a disc of radius $\rho$ centred at a point $x_0$, we have for any $f, C^\infty$ on $\overline{\Omega}, \| f \|_{\infty} \leq C \| f \|_{[n/p]^*,p}$ with $C = C(\rho)$ independent of the point $x_0$. This is because the measure as well as the vector fields $\partial/\partial x_i$ are translation invariant.

3.6. Our next result is a very special case of a general theorem about elliptic operators on a compact manifold. We consider a connected real nilpotent Lie group $N$ with a discrete subgroup $\Phi$ such that $N/\Phi$ is compact. Let $L(N)$ denote the Lie algebra of $N$ and $X_1, \cdots, X_n$ be a basis of $L(N)$ over $\mathbb{R}$. We consider the $X_i$ to be right translation - invariant vector fields on $N$, hence they define vector fields on $N/\Phi$ which we continue to denote $X_i$. 
Then $\Delta = -\sum X_i^2$ is a non-negative self adjoint elliptic operator on the space of $C^\infty$ functions on $N/\Phi$ with respect to the inner product $\langle , \rangle: (f, g) \mapsto \int_{N/\Phi} f \overline{g} \, dn$ where $dn$ is the Haar measure on $N/\Phi$. One has in fact for $X \in L(N)$,

$$\langle Xf, g \rangle = -\langle f, -Xg \rangle$$

so that

$$-\langle X^2f, f \rangle = \langle Xf, Xf \rangle \geq 0$$

and hence $\langle \Delta f, f \rangle \geq 0$. In fact one has

$$\langle \Delta f, f \rangle = \sum <X_if, X_if>$$

so that $\Delta f = 0$ if and only if $X_if = 0$ for all $i$; and since the $X_i, 1 \leq i \leq r$ give a basis for the tangent space at every point this means that $f$ is a constant. Now the general theory of self adjoint elliptic operators applied here shows that $L^2(N/\Phi, dn)$ decomposes as an orthogonal direct sum

$$\mathbb{C} \oplus \bigoplus_{1 \leq n < \infty} H(\lambda_n)$$

where $\mathbb{C}$ is identified with the space of constant functions on $N/\Phi, \lambda_n, 1 \leq n < \infty$, a monotone sequence of positive real numbers tending to $\infty$ and $H(\lambda_n) = \{ f \text{ a } C^\infty \text{ function on } N/\Phi \mid \Delta f = \lambda_n f \}$ are finite dimensional subspaces of $L^2$. It is clear from this that if $f \in C^\infty(N/\Phi, \mathbb{C})$ and $\int_{N/\Phi} f \, dn = 0$ then $f \in \bigoplus_{1 \leq n < \infty} H(\lambda_n)$. It is also immediate from this that we have the following fact which has a crucial role in the sequel:

3.7. Lemma If $\varphi \in C^\infty(N/\Phi, \mathbb{C})$ is such that $\int_{N/\Phi} \varphi \, dn = 0$, then we have for every integer $r > 0$,

$$\| \Delta^r \varphi \|_2 \geq \lambda_1^r \| \varphi \|_{0,2}$$

where $\| \|_{2=\|} \|_{0,2}$ is the $L^2$ norm with respect to the Haar measure.

3.8. We now recall Friedrich’s inequality for elliptic operators as applied to our special case. For a multi-index $\beta = (\beta_1, \beta_2, \cdots \beta_n)$ of non-negative integers we set $X^\beta = X_1^{\beta_1} \cdots X_n^{\beta_n}$ - a differential operator on $N/\Phi$. As usual $| \beta |$ is the sum $\sum_{1 \leq i \leq n} \beta_i$. With this notation Friedrich’s inequality asserts the following
3.9. **Theorem**  There are constant $C, C' > 0$ such that for any $\alpha$ with $|\alpha| \leq 2r$ and any $\varphi \in C^\infty(N/\Phi)$,

$$
\| X^\alpha \varphi \|_2 \leq C \| \Delta^r \varphi \|_2 + C' \| \varphi \|_2.
$$

**Note**  The norm $\sup \{\| X^\beta \varphi \|_2 \mid |\beta| \leq r\}$ is equivalent to the norm $\| \| r, 2 \|$ introduced earlier; this is easily seen from the fact that the $X_i$ can be expressed as a linear combination of the $\partial/\partial x_j$ of the local coordinates with coefficients that are $C^\infty$.

3.10. **Corollary**  If $\varphi \in C^\infty(N/\Phi)$ is such that $\int_{N/\Phi} \varphi dn = 0$, then for $|\alpha| \leq 2r$, one has

$$
\| X^\beta \varphi \|_2 \leq C \| \Delta^r \varphi \|_2
$$

for a constant $C > 0$ (independent of $\varphi$).

3.11. **Corollary**  Let $(N, \Phi)$ be as in 3.6. Then there is a constant $C > 0$ such that for $\varphi \in C^\infty(N/\Phi)$ with $\int_{N/\Phi} \varphi(n) dn = 0$, we have $|\varphi(x)|^p \leq C \int_{N/\Phi} |\Delta^r \varphi(x)|^p dx$ for $r \geq [n/p]^*$. This is immediate from Theorem 3.4 and Corollary 3.10. From now on we go back to the notations of §1 and §2.

3.12.  A function $\varphi \in C^\infty$ (cf. 2.1) is **cuspidal** if the following holds: let $P'$ be a proper parabolic subgroup of $G$ defined over $k$ one $U'$ its unipotent radical. Then for almost all $g \in G(\mathbb{A})$ the function $u \mapsto \varphi(gu), u \in U'(\mathbb{A})$ is integrable for the Haar measure on $U'(\mathbb{A})/U'(k)$ and $\int_{U'(\mathbb{A})/U'(k)} \varphi(gu) du = 0$ for almost all $g \in G(\mathbb{A})$.

3.13. **Remarks**

(i)  ‘Almost all $g'$ above means for all $g \in G(\mathbb{A})$ outside a set of Haar measure zero.

(ii)  It is known that every parabolic subgroup defined over $k$ is conjugate to a parabolic subgroup of $G$ defined over $k$ containing a fixed minimal parabolic subgroup $P$ of $G$ defined over $k$. Using the $G(k)$ invariance of elements of $F^\chi$, it is easily seen that for $\varphi$ to be cuspidal, it suffices that $\int_{U'(\mathbb{A})/U'(k)} \varphi(gu) du$ vanish for almost all $g \in G(\mathbb{A})$ for $U'$ the unipotent radical of any of the (finitely many) $k$-parabolic subgroups $P'$ containing $P$. 
(iii) We observe next that for \( \varphi \) to be cuspidal, it sufficient that the integral vanish for maximal parabolic subgroups defined over \( k \) and containing \( P \). In fact if \( P' \) is a \( k \)-parabolic subgroup and \( P'' \) is a maximal \( k \)-parabolic subgroup containing \( P' \), then the unipotent radical \( U' \) of \( P' \) contains the unipotent radical \( U'' \) of \( P'' \) as a normal subgroup. We have thus a fibration \( U'((\A_k))/U'((k)) \rightarrow U''((\A_k))/U''(k) \) whose fibres are \( U''((\A_k))/U''(k) \); our contention now follows from Fubini's theorem which is valid for this fibration.

3.14. Notation If \( E \) denotes one of the spaces in \( \mathcal{E} = \{ \mathcal{F}^\chi, \mathcal{F}^\chi, \mathcal{C}^\chi, \mathcal{C}^\chi \} \) and \( L_2^\chi \) we denote by \( {}^oE \) the subspace

\[ \{ \varphi \in E \mid \varphi \text{ cuspidal} \}. \]

3.15. If \( E \) is one of the spaces in \( \mathcal{E} \), then \( {}^oE \) is invariant under the left action of \( G(\A) \) on \( G(\A)/G(k) \). This is clear from the definition of cuspidal functions. Also if \( E \in \mathcal{E} \) and \( E \) is not \( \mathcal{F}^\chi \), for \( \varphi \in E \) and \( \alpha \in \mathcal{C}^\chi(G(\A)), \alpha \ast \varphi \in E \) as is easily seen from the definition of the convolution operation \((\alpha \ast \varphi)(g) = \int_{G(\A)} \alpha(h) \varphi(h^{-1}g) dh \) (\( dh \) Haar measure on \( G(\A) \)). It is further easy to see that \( \varphi \mapsto \alpha \ast \varphi \) is a continuous linear operator on \( E \) which leaves the subspace \( {}^oE \) stable (note that \( {}^oE \) is closed in \( E \) for \( E \neq \mathcal{F}^\chi \)). By the definition of \( \mathcal{C}^\chi(G(\A)) \), there is a compact open subgroup \( K(\alpha) \) of \( G(\A_f) \) such that \( \alpha \) is invariant under left translations by \( K(\alpha) \). From the definition of \( \alpha \ast \varphi \), it is immediate that it is \( K(\alpha) \)-invariant on the left. Let \( A^\ast : E \rightarrow E(E \in \mathcal{E}, E \neq \mathcal{F}^\chi) \) be the averaging over \( K(\alpha) : 
( A^\ast \varphi)(g) = \int_{K(\alpha)} \varphi(kg) dk \) (\( dk \) Haar measure on \( K(\alpha), g \in G(\A) \)). Then \( A^\ast \) is a continuous projection of \( E \) on \( E^{K(\alpha)} \) and of \( {}^oE \) on \( {}^oE^{K(\alpha)} \). We note that for all \( \varphi \in E, \alpha \ast A^\ast(\varphi) = A^\ast(\alpha \ast \varphi) \alpha \ast \varphi \). In particular \( \alpha \ast \varphi = A(\alpha \ast A^\ast \varphi) \) on \( E \) as well as \( {}^oE \). We now state the central result of this chapter.

3.16. Theorem The operator \( \varphi \mapsto \alpha \ast \varphi \) of \( {}^oL_2^\chi \) into itself is a compact operator.

3.17. We will reformulate the theorem for a space of functions on \( G_\infty \). In the light of the remarks made at the end of 3.15, it is clear that it suffices to show that convolution with \( \alpha \) is a compact operator on \( {}^oL_2^{K(\alpha)} \). Consider now the isomorphism \( \mathcal{L} = \prod_{\theta \in \mathcal{H}} r_\theta \) (defined in 2.2: we take for \( L_f \) of 2.2 the group \( K(\alpha) \)) of \( L_2^{K(\alpha)}(G_\alpha/\Gamma_\theta) \) on \( \prod_{\theta \in \mathcal{H}} L_2^{\chi}(G_\alpha/\Gamma_\theta) \).
3.18. Claim  For $\theta \in \mathcal{H}$, let $\alpha L_2^X(G_{\infty}/\Gamma_{\theta})$ be the subspace of all those functions $\varphi \in L_2^X(G_{\infty}, \Gamma_{\theta})$ for which $\int_{U_{\infty}\cap \Gamma} \varphi(gu)du = 0$ for almost all $g \in G_{\infty}$ and for all $U'$, the unipotent radical of proper $k$-parabolic subgroup $P'$ of $G$. Then $\varphi$ maps $\alpha L_2^X$ isomorphically onto $\prod_{\theta \in \mathcal{H}} \alpha L_2^X(G_{\infty}/\Gamma_{\theta})$.

3.19. Proof  It is known that the group $d_f(U'(k))$ is dense in $U'((\mathbb{A}_f))$ (this is known as the strong approximation property). Let $M_\theta = U'(\mathbb{A}_f) \cap \theta^{-1}K(\alpha)\theta$; then $M_\theta$ is an open compact subgroup of $U'((\mathbb{A}_f))$. Consequently $U'(\mathbb{A}_f) = M_\theta \cdot d_f(U'(k))$. It follows that $U'(\mathbb{A}_f) = U'_\infty \cdot M_\theta \cdot U'(k)$ and hence the natural map

$$(U'_\infty \cdot M_\theta)/(U'_\infty \cdot M_\theta \cap U'(k)) \xrightarrow{\sim} U'(\mathbb{A}_f)/U'(k) \quad (\ast)$$

is a homeomorphism. For $\varphi \in \alpha L_2^X$, we have $\int_{U(\mathbb{A})/U'(k)} \varphi(\theta gu)du = 0$ for $\theta \in \mathcal{H}$ and $g \in G_{\infty}$. In view of the identification $(\ast)$ described above, we see that

$$\int_{U'_{\infty}/\Gamma_{\theta}} \varphi(\theta gu)du = 0.$$

Now if we set $u = u_{\infty} \cdot u_f$ with $u_{\infty} \in U'_{\infty}$ and $u_f \in U'(\mathbb{A}_f)$, we have $\varphi(\theta gu) = \varphi(\theta gu_{\infty}u_f) = \varphi(\theta u_f \theta^{-1} \cdot \theta gu_{\infty}) = \varphi(\theta gu_{\infty})$ (since by definition of $M_\theta$, $u_f \in M_\theta$ implies that $\theta u_f \theta^{-1} \in K(\alpha)$ and $\varphi$ is left $K(\alpha)$-invariant). Thus integrating over $M_\theta$ first we find that

$$0 = \int_{(U'_{\infty} \cdot M_\theta)/(U'_{\infty} \cdot M_\theta \cap U'(k))} \varphi(\theta gu)du = \int_{U'_{\infty}/U'_{\infty} \cap \Gamma_{\theta}} \varphi(\theta gu)du,$$

proving that $r$ maps $\alpha L_2^X$ into $\prod_{\theta \in \mathcal{H}} \alpha L_2^X(G_{\infty} | \Gamma_{\theta})$. We need to show that the map is onto this subspace. To see this let $\varphi \in L_2^X(K(\alpha))$ be such that $r_\varphi(\varphi)$ is in $\alpha L_2^X(G_{\infty} | \Gamma_{\theta})$ for all $\theta$-we know that such a $\varphi$ exists and is unique. We need to show that $\varphi \in \alpha L_2^X$ i.e. we have to show that

$$\int_{U'(\mathbb{A}_f)/U'(k)} \varphi(gu)du = 0 \quad (**)$$

for almost all $g \in G(\mathbb{A})$ for the unipotent radical $U'$ of any proper $k$-parabolic subgroup $P'$ of $G$. Now $g = g_{\infty} \cdot g_f$ where $g_{\infty} \in G_{\infty}$ and $g_f \in G(\mathbb{A}_f)$. By Lemma 2.4, $g_f = k_f \cdot \theta \cdot d_f(\gamma)$ with $k_f \in K(\alpha)$, $\theta \in \mathcal{H}$ and $\gamma \in G(k)$. Since $\varphi$ is left $K(\alpha)$-invariant, $\varphi(gu) = \varphi(g_{\infty} g_f u) = \varphi(g_{\infty} k_f \theta d_f(\gamma) u) = \varphi(g_{\infty} \theta d_f(\gamma) u)$ (since $\varphi$ is left $K(\alpha)$-invariant) $\varphi(g_{\infty} \gamma_{\infty}^{-1} \gamma \gamma_f u) = \varphi(g_{\infty} \theta \gamma u)$. 

Theorem 2.5. If $\varphi \in \alpha L_2^X$ is such that $r_\varphi(\varphi)$ is in $\alpha L_2^X(G_{\infty} | \Gamma_{\theta})$ for all $\theta$, then $\varphi \in \alpha L_2^X$ for some $\theta$. 

Theorem 2.6. If $\mathcal{U}(\mathbb{A}_f)$ is the unipotent radical of $P'$ of $G$, then $\mathcal{U}(\mathbb{A}_f) = U'_{\infty} \cdot M_{\theta} \cdot U'(k)$ for $\theta \in \mathcal{H}$.
Set $U'' = \gamma u \gamma^{-1}$. Then one has
\[
\int_{U''(\mathbb{A})/U'(k)} \varphi(g'_{\infty} \theta \gamma u) du = \int_{U''(\mathbb{A})/U'(k)} \varphi(g'_{\infty} \theta \cdot \gamma u \gamma^{-1}) du = \int_{(U'' \cdot M'_{\theta})(U'' \cap U''(k))} \varphi(\theta g'_{\infty} u') du'
\]
where $M'_{\theta} = (U''(\mathbb{A}) \cap \theta^{-1} K(\alpha) \theta)$; and the integral in the last integral
is an $M'_{\theta}$-invariant function on $(U'' \cdot M'_{\theta})(U'' \cap U''(k))$. It follows
that integral equals $\int_{U''/U'' \Gamma_{\theta}} \varphi(\theta g'_{\infty} u) du$ and is thus zero since $r_{\theta} \varphi \in {}^o L^2_2(G_{\infty}/\Gamma_{\theta})$. Thus Claim 3.15 is established.

3.20. Lemma For $\alpha \in C_c(G(\mathbb{A}))$, let $\alpha_{\infty} \in C_c(G_{\infty})$ be the function
defined by $\alpha_{\infty}(g) = \int_{G(\mathbb{A})} \alpha(gh) dh$. Then $r_{\theta}(\alpha \ast \varphi) = \alpha_{\infty} \ast \varphi$ for all $\varphi \in L^2_\lambda K(\alpha)$. This is immediate from the definition of $r_{\theta}$ and the $K(\alpha)$-invariance
of $\alpha$. The lemma shows that Theorem 3.13 is equivalent to the following:

3.21. Theorem Let $L_f$ be a compact open subgroup of $G(\mathbb{A})$ and set
$\Gamma = d_\infty(\Gamma_{L_f})$. Let $\chi$ be a unitary character on $C(\mathbb{A})(C = \text{centre of } G)$
which is trivial on $C(k)$. Let $\alpha \in C_c(G_{\infty})$. Then $\varphi \mapsto \alpha \ast \varphi$ is a compact
operator on $^o L^2_2(G_{\infty}/\Gamma)$.

3.22. We begin with the observation that there is no loss of generality in
assuming that the centre of $G$ is finite. To see this observe first that if $G'$ is
the commutator subgroup of the identity connected component of $G$, then
$C_{\infty} \cdot G'_{\infty} = H$ has finite index in $G_{\infty}$. Let $\xi_1 \ldots \xi_r$ be a complete set of
inequivalent representatives for $G_{\infty}/H$ in $G_{\infty}$. For $1 \leq i \leq r$ and $\varphi \in L^2_{\chi}$,
let $\varphi_i : C_{\infty} \cdot G_{\infty} \to \mathbb{C}$ be the function defined by $\varphi_i(g) = \varphi(\xi_i \cdot g)$. Then
$\varphi_i(g \gamma) = \varphi_i(g)$ for $g \in C_{\infty} \cdot G_{\infty} \overset{'}{=} = H$ and $\gamma \in H \cap \Gamma$. Also $\varphi_i(gc) = \chi(c) \varphi_i(g)$
for $c \in C_{\infty}$. Now every unipotent element of $G_{\infty}$ is contained in $G_{\infty}'$. From
this we see immediately that if $\varphi$ is a cuspidal function so is $\varphi_i$ restricted
to $G_{\infty}'$. Further $\varphi_i$ is completely determined by its restriction $\varphi_i'$ to $G_{\infty}'$.
Evidently the map
\[
^o L^2_2(G_{\infty}/\Gamma) \longrightarrow \prod_{r \text{ copies}} ^o L^2_2(G_{\infty}'/\Gamma')
\]
where $\chi' = \chi \mid_{G' \cap C}$ and $\Gamma' = G'_\infty \cap \Gamma$ is an isomorphism onto a closed subset. Moreover for $1 \leq i \leq r$

$$
(a \ast \varphi)_i(g) = \int_{G'_\infty} \alpha(h) \varphi(h^{-1} \xi_i g) dh = \int_{G'_\infty} \alpha(\xi_i^{-1} h) \varphi(h^{-1} g) dh = \int_H \sum_{1 \leq j \leq r} \alpha_{ij}(h) \varphi_j(h^{-1} g) dh = \sum_{1 \leq j \leq r} \alpha_{ij} \ast \varphi_j
$$

where $\alpha_{ij} \in C^\infty_c(H)$ is the function $\alpha_{ij}(h) = \alpha(\xi_i^{-1} \xi_j h)$. The natural map $\pi : C^\infty_t \times G'_\infty \to H$ has for kernel the finite group $C^\infty_t \cap G'_\infty$ whose order we denote by $q$. We regard functions on $H$ as functions on $C^\infty_t \times G'_\infty$ by composing with $\pi$. With this convention we have for $\alpha \in C^\infty_c(H)$ and $\varphi \in L^2_k(H/H \cap \Gamma)$, $(a \ast \varphi)(g) = \int_H \alpha(h) \varphi(h^{-1} g) dh = \frac{1}{q} \int_{C^\infty_t \times G'_\infty} \alpha(h) \varphi(h^{-1} g) dh$

and setting $h = c x$ with $c \in C^\infty_t$ and $x \in G'_\infty$ we have for $g \in G'_\infty$

$$
(a \ast \varphi)'(g) = \frac{1}{q} \int_{G'_\infty} dx \int_{C^\infty_t} \alpha(xc) \varphi(c^{-1} x^{-1} g) dc = \frac{1}{q} \int_{G'_\infty} dx \int_{C^\infty_t} \alpha(xc) \chi(c^{-1} g) dc = \alpha' \ast \varphi'(g)
$$

where $\alpha'$ in $C^\infty_c(G'_\infty)$ is defined by $\alpha'(x) = \frac{1}{q} \int_{C^\infty_t} \alpha(xc) \chi(c) dc$. This discussion shows that we need to prove the theorem only in the case when $G'_\infty = G'_\infty$ has a finite centre. Now the group $\Gamma$ admits a torsion free subgroup $\Gamma'$ of finite index and $\Gamma'$ evidently intersects the centre trivially. We have a natural inclusion of $^oL^2_k(G'_\infty/\Gamma)$ in $^oL^2_k(G'_\infty/\Gamma')$ (where $^oL^2_k(G'_\infty/\Gamma') = \{ \varphi \in L^2_k(G'_\infty/\Gamma') \mid \int_{U \cap \Gamma'} \varphi(gu) du = 0 \text{ for all unipotent radicals of proper } k\text{-parabolic subgroups of } G'_\infty \}$) compatible with convolution by elements of $C^\infty_c(G'_\infty)$ on both the spaces. We have thus to prove the following:

3.23. Theorem Let $G$ be a connected semisimple algebraic $k$-group and $\Gamma$ an arithmetic subgroup of $G'_\infty$. Let $^oL^2_k(G'_\infty/\Gamma)$ be the space of all $\Gamma$-unvariant functions $\varphi$ on $G'_\infty$ which are square summable on $G'_\infty/\Gamma$ and in addition are cuspidal i.e. satisfy the following condition: If $U$ is the unipotent radical of a proper $k$-parabolic subgroup $P$ of $G$, $\int_{U \cap \Gamma'} \varphi(gu) du = 0$ for almost all $g \in G'_\infty$. Then the operator $\varphi \mapsto \alpha \ast \varphi$ on $^oL^2_k(G'/\Gamma)$ is a compact operator for all $\alpha \in C^\infty_c(G'_\infty)$.

Since $G'_\infty$ is semisimple, $G'_\infty/\Gamma$ has finite Haar measure. We now see from Proposition 3.2 that Theorem 3.20 is a consequence of the following:
3.24. Proposition  Let $G_{\infty}, \Gamma, \alpha$ be as above. Then there is a constant $C = C(\alpha) > 0$ such that for $\varphi \in L_2(G_{\infty}/\Gamma)$, $|\alpha \ast \varphi(x)| \leq C \parallel \varphi \parallel_2$ for all $x \in G_{\infty}$ (note that $\alpha \ast \varphi$ is $C^\infty$ for $\varphi \in L_2(G_{\infty}/\Gamma)$).

3.25. For $c > 0$ and a compact subset $\Omega \subset o\mathcal{P}_{\infty}$ (notation introduced in §1.5 - 1.9) we define $S(c, \Omega)$ to be the subset $K_{\infty} \cdot A_c \cdot \Omega$ of $G_{\infty}$. According to Theorem 1.9, there is a finite set $\Xi \subset G(k)$, a constant $c_0 > 0$ and a compact set $\Omega_0 \subset o\mathcal{P}_{\infty}$ with the following properties:

(i) $(\xi \Omega_0 \xi^{-1})(\xi o\mathcal{P}_{\infty} \xi^{-1} \cap \Gamma) = \xi o\mathcal{P}_{\infty} \xi^{-1}$.

(ii) If $c > c_0$ and $\Omega \supset \Omega_0$ is a compact subset of $o\mathcal{P}$, then $S(c, \Omega) \Xi \Gamma = G_{\infty}$.

(iii) If $c, \Omega$ are as above the set $\{ \gamma \in \Gamma \mid S(c, \Omega) \Xi \gamma \cap S(c, \Omega) \Xi \neq \phi \}$ is finite.

Condition (iii) implies in particular the following. There is a constant $M = M(c, \Omega) > 0$ such that for $\xi \in \Xi$ and all $\varphi \in L_2(G_{\infty}/\Gamma)$, we have

$$\int_{S(c, \Omega)\xi} |\varphi(g)|^2 \, dg \leq M \parallel \varphi \parallel_2^2 \quad (\ast)$$

3.26. We have seen that $o\mathcal{P}$ is a semi-direct product $B \cdot U$ and we have correspondingly a semi-direct product decomposition $o\mathcal{P}_{\infty} = B_{\infty} \cdot U_{\infty}$. The group $B_{\infty}$ is reductive and $K_{\infty} \cap B_{\infty}$ is a maximal compact subgroup of $B_{\infty}$. We then have an Iwasawa decomposition of $B_{\infty}$ as $(K_{\infty} \cap B_{\infty}) \cdot F$ where $F$ is a connected solvable closed Lie subgroup of $B_{\infty}$. Moreover as $B_{\infty}$ commutes with $A$, we see that for $\xi \in \Xi$ the map

$$K_{\infty} \times F \times A \times U_{\infty} \to G_{\infty} \quad (\ast)$$

given by $(k, f, a, u) \mapsto k.f.a.u.\xi$ is an analytic diffeomorphism. Suppose now that $P'$ is a proper maximal $k$-parabolic subgroup of $G$ containing $P$. Then $P'$ is the semidirect product $Z(S') \cdot U'$ where $Z(S')$ is the centraliser of a suitable 1-dimensional sub-torus of $S$ and $U'$ is the unipotent radical of $P'$. We have correspondingly a semidirect product decomposition $Z(S'_{\infty}) U'_{\infty}$ of $P'_{\infty}$. Now $U'_{\infty}$ is a normal subgroup of $U_{\infty}$ and (hence) $U_{\infty}$ is the semidirect product $(Z(S'_{\infty}) \cap U_{\infty}) \cdot U'_{\infty}$. Let $Z(S'_{\infty}) \cap U_{\infty} = U''_{\infty}$. Then we have a further refinement of the product decomposition (\ast) above:

$$\Phi_{\xi} : K_{\infty} \times F \times A \times U''_{\infty} \times U'_{\infty} \to G$$
given by \((k, f, a, u'', u') \mapsto k, f, a, u'', u', \xi\). It is convenient to denote by \(H_1, H_2, H_3, H_4\) and \(H_5\) the groups \(K_\infty, F, A, U''_\infty\) and \(U'''_\infty\) respectively. We assume as we may say that the compact set \(\Omega_0\) (resp. \(\Omega\)) in 3.22 is chosen to be of the form \((K_\infty \cap oP_\infty)\Omega_{i02}\Omega_{i04}\Omega_{i05}\) (resp. \((K_\infty \cap oP_\infty)\Omega_{i02}\Omega_{i04}\Omega_{i05}\) with \(\Omega_{i0i}\) (resp. \(\Omega_i\)) a compact subset of \(H_i, 1 = 2, 4, 5\) with \(\Omega_{i0i}\) contained in the interior of \(\Omega_i\). We also assume \(\Omega_{i05}\) so chosen that \(\xi^{-1}\Omega_{i05}, \xi, (\xi^{-1}H_5\xi \cap \Gamma) = \xi^{-1}H_5\xi\). The Lie algebra \(L(G_\infty)\) of \(G_\infty\) is identified with the Lie algebra of right translation invariant vector fields on \(G_\infty\). The Lie algebras \(L(H_i)\) of the \(H_i, 1 \leq i \leq 5\) are identified with subalgebras of \(L(G_\infty)\) and hence their elements will also be regarded as right translation invariant vector fields on \(G_\infty\). On the other hand an element \(X\) of \(L(H_i)\) determines a right translation invariant vector field on the group \(H_i\). We denote this vector field by \(X'\) (to distinguish it from the vector field \(X\) on \(G_\infty\)). For \(X \in L(H_i), 1 \leq i \leq 5\) we define a vector field \(\tilde{X}\) on \(G_\infty\) as follows: \(\tilde{X}\) is the image under the analytic diffeomorphism \(\Phi_\xi\) of the vector field on the product \(\prod_{1 \leq i \leq 5} H_i\) whose component in \(H_i\) is \(X\) while all the other components are zero. If \(g \in G_\infty\) is such that \(g = h_1h_2h_3h_4h_5\xi\) with \(h_i \in H\) then we have - as is easily checked - for \(X_i \in L(H_j)\),

\[
\tilde{X}(g) = (Ad(\prod_{i<j} h_i)(X))(g)
\]  

\((*)\)

Here when \(j = 1, \prod_{i<j} h_i\) is the identity element while for \(j > 1\), the product is taken in the order of increasing \(i : h_1h_2\cdots h_{j-1}\).

3.27. We now fix a basis \(B = \{X_i\}_{1 \leq i \leq p} \cup \{Y_i\}_{1 \leq i \leq q} \cup \{Z_i\}_{1 \leq i \leq r}\) of \(L(G_\infty)\) with the following properties:

(i) \(X_i\) and \(Y_j\) are eigenvectors for \(Ad(S)\), the corresponding character on \(S\) being denoted \(\eta_i\) and \(\zeta_j\) respectively: \(Ad(t)X_i = \eta_i(t)X_i\) and \(Ad(t)Y_i = \zeta_j(t)Y_i\).

(ii) \(\{Z_i \mid 1 \leq i \leq r\} \cap L(H_i)\) is a basis for \(L(H_i)\) for \(1 \leq i \leq 3\).

(iii) \(\{X_i \mid 1 \leq i \leq p\}\) (resp. \(\{Y_i \mid 1 \leq i \leq q\}\)) is a basis for \(L(H_5)\) (resp. \(L(H_4)\)).

From our choice of ordering on the character group of \(S, \eta_i, 1 \leq i \leq p\) and \(\zeta_i, 1 \leq i \leq q\), are positive roots so that \(\eta_i = \prod_{\alpha \in \Delta} g^{m_{i\alpha}}, \zeta_i = \prod_{\alpha \in \Delta} g^{n_{i\alpha}}\) with \(m_{i\alpha}, n_{i\alpha} \geq 0\). For a multi-index \(\beta = (\beta_1, \cdots, \beta_p)\) (resp. \(\gamma = (\gamma_1, \cdots, \gamma_1)\), resp. \(\delta = (\delta_1, \cdots, \delta_r), (\beta_1, \gamma_j, \delta_k\) non-negative integers) we set
\[
X^\beta = X_1^{\beta_1}X_2^{\beta_2} \cdots X_p^{\beta_p}, \quad \tilde{X}^{1\beta} = \tilde{X}_1^{1\beta_1}\tilde{X}_2^{1\beta_2} \cdots \tilde{X}_p^{1\beta_p}, \\
Y^\gamma = Y_1^{\gamma_1}Y_2^{\gamma_2} \cdots Y_q^{\gamma_q}, \quad \tilde{Y}^{1\gamma} = \tilde{Y}_1^{1\gamma_1}\tilde{Y}_2^{1\gamma_2} \cdots \tilde{Y}_q^{1\gamma_q}, \\
Z^\delta = Z_1^{\delta_1}Z_2^{\delta_2} \cdots Z_r^{\delta_r}, \quad \tilde{Z}^{1\delta} = \tilde{Z}_1^{1\delta_1}\tilde{Z}_2^{1\delta_2} \cdots \tilde{Z}_r^{1\delta_r}.
\]

With this notation we have

3.28. Lemma Let \( g = h_1, h_2, h_3, h_4, h_5, \xi \in G_{\infty} \) with \( h_i \in H_i \). Then for multi-indices \( \beta, \gamma, \delta \) as above we have, setting \( m = |\beta| + |\gamma| + |\delta| \),

\[
\tilde{Z}^{\delta}\tilde{Y}^{\gamma}\tilde{X}^{\beta} = \eta^\beta(h_3)\zeta^\gamma(h_3) \sum_{|\beta'| + |\gamma'| + |\delta'| \leq m} C_{\beta', \gamma', \delta'}(h_1, h_2, h_3, h_4, h_3) \quad \text{with } C_{\beta', \gamma', \delta'}, C_{\infty}-functions on } H_1 \times H_2 \times H_4 \text{ (note that } H_3 \text{ normalises } H_4). \]

Here \( \eta^\beta = \prod_{1 \leq i \leq p} \eta_i^\beta \) and \( \zeta^\gamma = \prod_{1 \leq i \leq q} \zeta_i^\gamma \).

Proof. One argues by induction on \( |\beta| + |\gamma| + |\delta| \). When \( |\beta| + |\gamma| + |\delta| = 1 \), this follows from (*) of 3.23 and the fact that the \( X_i \) and \( Y_j \) are eigenvectors with eigencharacter \( \eta_i \) and \( \zeta_j \) respectively. Suppose that \( |\beta| + |\gamma| + |\delta| > 1 \); then we can find \( \beta', \gamma', \delta' \) and with \( |\beta'| + |\gamma'| + |\delta'| = |\beta| + |\gamma| + |\delta| - 1 \) and \( \tilde{Z}^{\delta'}\tilde{Y}^{\gamma'}\tilde{X}^{\beta'} \) with \( T \in B \). Using the induction hypothesis we have an expansion for \( \tilde{Z}^{\delta'}\tilde{Y}^{\gamma'}\tilde{X}^{\beta'} \) in terms of the \( X, Y, Z \) of the desired kind. The result for \( \tilde{Z}^{\delta}\tilde{Y}^{\gamma}\tilde{X}^{\beta} \) now follows from the following observations: \( \tilde{T}' \) has the desired kind of expansion; secondly for a character \( \chi \) on \( H_3 \) treated as a function on \( G_{\infty}, \tilde{T}'(\chi)(h_3) = 0 \) unless \( T \in LH_3 \) and if \( T \in LH_3, \tilde{T}'(\chi)(h_3) = \chi(T) \chi(h_3) \) where \( \chi \) is the tangent map \( L(H_3) \to \mathbb{R} \) induced by \( \chi \). This proves the lemma.

3.29. We fix an element \( g^0 = h_1^0, h_2^0, h_3^0, h_5^0, \xi \in G_{\infty} \) with \( h_i \in H_i, 1 \leq i \leq 5 \). We take \( P' \) to be the parabolic subgroup defined as follows: let \( \alpha_0 \in \Delta \) be such that \( \alpha_0(h_0) \leq \alpha(h'_3) \) for all \( \alpha \in \Delta \); then \( P' \) is the proper maximal \( p \)-parabolic subgroup determined by \( \alpha_0 \). The Lie algebra \( L(H_3) \) is the sum of the eigenspace in \( L(G_{\infty}) \) for \( AdH_3 \) corresponding to the eigencharacters (i.e. \( k \)-roots) \( \psi \) of the form \( \prod_{\alpha \in \Delta} \alpha^{m_\alpha} \) with \( m_\alpha \) integers \( \geq 0 \) and \( m_{\alpha_0} > 0 \) in other words in the notation of Lemma 3.25, \( \eta_3 = \prod_{\alpha \in \Delta} \alpha^{m_\alpha} \) with \( m_{\alpha_0} > 0 \). Let \( \langle c_0, \Omega_0 \rangle \) and \( (c, \Omega) \) be as in 3.22. Then there is neighbourhood \( V \) of the identity in \( A \) such that for \( g^0 x^{-1} = h_1^0, h_2^0, h_3^0, h_4^0, h_5^0 \in S(c_0, \Omega_0) \) and \( x \in V, h_1^0, h_2^0, h_3^0 x, h_4^0, h_5^0 \in S(c, \Omega) \). Fix such a neighbourhood \( V \) of 1 in
A once and for all (for a fixed \((c, \Omega)\)). Let \(\Delta\) denote the elliptic operator 
\[
\sum_{1 \leq i \leq p} X_i^2
\]
on \(H_5\) and let \(\Delta' = \sum_{1 \leq i \leq p} \tilde{X}_i^2\). Then we have the \(L^2\)-Sobolev inequality (Theorem 3.4) if \(2p > r/2\),
\[
| \varphi(g^0) |^2 \leq C \int_{\Omega_5} | \tilde{\Delta}'^p \varphi(h^0_1, h^0_2, h^0_3, h^0_4, \xi) |^2 \, dh_5 \ldots
\]
where \(\varphi \in C_c^\infty(\mathbf{G}_\infty)\). Treating the right-hand side as a function on \(H_1 \times H_2 \times H_3 \times H_4\) and applying the \(L^1\) Sobolev inequality combined with Schwarz’s inequality we have
\[
| \varphi(g^0) |^2 \leq C' \int_{K_\infty \times \Omega_2 \times h^0_3 V \times \Omega_4 \times \Omega_5} \sum_{|\delta| + |\gamma| \leq q+4} | \tilde{X}^\delta \tilde{Y}^\gamma \tilde{\Delta}'^p \varphi |^2 
\]
\[
dh_1, dh_2, dh_3, dh_4, dh_5
\]
Now \(\tilde{\Delta}'^p = \sum_{1 \leq i_1, i_2, \ldots, i_p \leq p} a_{i_1} \cdots a_{i_p} \tilde{X}_{i_1}' \tilde{X}_{i_2}' \cdots \tilde{X}_{i_p}'.\) It is now immediate from Lemma 3.25 that we have
\[
| \varphi(g^0) |^2 \leq C'' \int_E \sum_{1 \leq i_1, i_2, \ldots, i_p \leq p} \sum_{|\beta|+|\gamma|+|\delta| \leq 2p+q+r} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_p} | \tilde{X}^\delta \tilde{Y}^\gamma X^\beta \phi |^2 
\]
\[
dh_1, dh_2, dh_3, dh_4, dh_5 \quad \text{(III)}
\]
where \(E = K_\infty \times \Omega_2 \times h^0_3 V \times \Omega_4 \times \Omega_5\). Now for \(h_3'\), \(h_3 V\) one has \((\eta_{i_1} \eta_{i_2} \cdots \eta_{i_p})^2 (h_3')^2 \leq \lambda_p \cdot a_0^2(h_3')^2\) for a suitable constant \(\lambda_p > 0\). This is because we have for any \(i\), with \(1 \leq i \leq r\), \(\eta_i = \prod_{\alpha \in \Delta} \alpha_i^{m_{i\alpha}}\) with \(m_{i\alpha} \geq 1\) integers; also \(\alpha(h_3')^2 \leq c\) for \(\alpha \in \Delta\) so that \((\eta_{i_1} \eta_{i_2} \cdots \eta_{i_p} \cdots \eta_{i_p})^2 (h_3')^2 \leq c^N a_0(h_3')^p\) where \(N = (\sum_{1 \leq i \leq p} \eta_i \prod_{1 \leq j \leq q} \xi_j).\) This is seen as follows: since \(a_0(h_3')^2 \leq \alpha(h_3')^2\) for all \(\alpha \in \Delta\), we see that there is a constant \(b > 0\) such that \(a_0(h_3) \leq b a_0(h_3)\) for all \(\alpha \in \Delta\) and \(h_3 \in h^0_3 V\). Now \(\delta^2(h_3) = \prod_{\alpha \in \Delta} \alpha(h_3)^{\nu_{i\alpha}}\) with \(\nu_{i\alpha} \) integers \(\geq 1\) so that \(\delta^2(h_3) \geq \prod_{\alpha \in \Delta} (b^{-1} a_0(h_3))^{\nu_{i\alpha}} \geq C_1^{-1} a_0(h_3)^\nu\) where \(\nu = \sum_{\alpha \in \Delta} \nu_{i\alpha}.\) Now if \(p \geq \nu\), we have \(a_0(h_3)^\nu \leq a_0(h_3)^p \cdot \varepsilon^p \leq \varepsilon^p C_1 \delta^2(h_3).\) We see from (III) above now that we have
\[
| \varphi(g_0) |^2 \leq C' a_1 \int_{S(c, \Omega)} \sum_{|\beta|+|\gamma|+|\delta| \leq 2p+q+r} \left| (Z^\delta Y^\gamma X^\beta) \varphi \right|^2 dg \quad \text{(IV)}
\]
Note that $E \subset S(c, \Omega)$ and $\text{dh}_1, \text{dh}_2, \text{dh}_3, \text{dh}_4, \text{dh}_5, \delta^2(h_3) = dg$. Proposition 3.21 now follows for the inequality IV above: We have for $\alpha \in \mathcal{A}^\infty(G_\infty)$ and $D$ in the enveloping algebra of $L(G_\infty)$, $D(\alpha * \varphi) = D\alpha * \varphi$ so that by (IV)

$$(\alpha * \varphi)(g_0)^2 \leq C'' C_1 \int_{S(c,\Omega)} \sum_{|\beta| + |\gamma| + |\delta| \leq 2p + q + r} |(Z^\delta Y^\gamma X^\beta \alpha) * \varphi|^2 \, dg$$

where $C(\alpha) = C'' C_1 M(\alpha)$ with $M(\alpha) = \sum_{|\beta| + |\gamma| + |\delta| \leq 2p + q + r} \| Z^\delta Y^\gamma X^\beta \alpha \|_01$. Note that we have for $\alpha \in \mathcal{A}^\infty(G_\infty)$ and $\varphi \in L^2(G_\infty/\Gamma)$, $\| \alpha \|_2 \leq \| \alpha \|_2$. This proves the proposition and hence the Theorem.

4 Proof of Theorem 2.13

We begin by proving the following:

4.1. Theorem Let $^0\mathcal{A}^\chi(G_\infty, \Gamma, \sigma_\infty, \lambda) = \{f \in \mathcal{A}^\chi(G_\infty, \Gamma, \sigma_\infty, \lambda) \mid f \text{ cuspidal} \}$. Then $^0\mathcal{A}^\chi(G_\infty, \Gamma, \sigma_\infty, \lambda)$ is finite dimensional.

4.2. We will first show that any $\varphi \in ^0\mathcal{A}^\chi(G_\infty, \Gamma, \sigma_\infty, \lambda)$ decreases to zero rapidly at $\infty$ so that in particular $\varphi \in L_2(G_\infty/\Gamma)$ (and hence in $^0L_2^0(G_\infty/\Gamma)$). By Lemma 3.25 combined with (I) of 3.26, we have for $p' \geq p$ and $g^0 = h_1^0, h_2^0, h_3^0, h_4^0, h_5^0, \xi$,

$$\| \varphi(g^0) \|_2^2 \leq \int_{\Omega_5} \sum_{1 \leq i_1 \leq \cdots i_{p'} \leq p'} |(\eta_1 \eta_2 \cdots \eta_{p'} (h_3^0))|^2$$

$$\sum_{|\beta| + |\gamma| + |\delta| \leq 2p'} c_{\beta, \gamma, \delta} Z^\delta Y^\gamma X^\beta \varphi(g^0 \xi - 1) (h_5 \xi) \| dh_5$$

Let $I$ be the set of $p'$-tuples $i_1, \ldots, i_{p'}$ with $1 \leq i_1 \leq i_2 \cdots \leq i_{p'} \leq p'$ and for $I \in I$, let $\eta_1 \eta_2 \cdots \eta_{p'}$. Since $\varphi$ and its derivatives have moderate growth there is constant $B > 0$ and an integer $r > 0$ such that $|(Z^\delta Y^\gamma X^\beta \varphi(g^0)|^2 \leq B \| g^0 \|^r$ for $|\alpha| + |\beta| + |\gamma| \leq 2p'$. Thus, we have any $p' \geq p$

$$\| \varphi(g^0) \|^2 \leq C'' \int_{\Omega_5} \sum_{I \in I} |(\eta_1 (h_3^0))|^2 \sum_{|\beta| + |\gamma| + |\delta| \leq 2p'} |(Z^\delta Y^\gamma X^\beta \varphi)(g^0 \xi - 1) h_5 \xi)|^2 \, dh_5$$

$$\leq C'' \alpha_0 (h_3^0)^{p'} \| g^0 \|^r$$
Now \( \| g^0 \| \leq b' \sup_{\alpha \in \Delta} \alpha(h_3^0)^{-s} \) for some integer \( s \) for \( g_0 \in \mathcal{S}(c_0, \Omega_0) \). It follows that we have
\[
| \varphi(g_0) |^2 \leq b'' \alpha_0(h_0^0)^{p'} \cdot \alpha_0(h_3^0)^{-s} = b'' \alpha(h_3^0)^{p'-s}.
\]
Since \( p' \) is at our choice we see that \( | \varphi(g_0) |^2 \) is bounded in \( \mathcal{S}(c_0, \Omega_0) \). Hence \( \varphi \in \mathcal{A} L^2(G_{\infty}/\Gamma) \).

#### 4.3. Lemma (Godement)
Let \( X \) be a locally compact space and \( \mu \) a probability measure. Let \( V \subset L^2(X, \mu) \) be a closed subspace. Suppose that every \( \varphi \in V \) is essentially bounded. Then \( \dim H < \infty \).

**Proof** \( V \) is a closed subspace of \( L^\infty \) as well since \( L^\infty \)-convergence implies \( L^2 \) convergence. It follows also that the identity map of \( V \) is a continuous homomorphism of \( V \) with \( L^\infty \)-topology on \( V \) with the \( L^2 \)-topology. By the open mapping theorem there is a constant \( c > 0 \) such that \( \| \varphi \|_\infty < c \| \varphi \|_2 \) for all \( \varphi \in V \). Suppose \( \varphi_1, \ldots, \varphi_n \) is an orthonormal set in \( V \), and \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n \); then we have for almost all \( x \in X \),
\[
| \sum_{i=1}^n a_i \varphi_i(x) | \leq C \| \sum_{i=1}^n | a_i \varphi_i |^2 \|^\frac{1}{2}
\]
It follows that if \( D \) is a dense countable subset of \( \mathbb{C}^n \), there is a set \( Y \) of measure zero in \( X \) such that we have for all \( x \in X \setminus Y \) and all \( a \in D \),
\[
| \sum_{i=1}^n a_i \varphi_i(x) | \leq c \sum_{i=1}^n | a_i |^2 \|^\frac{1}{2}
\]
Since \( D \) is dense in \( \mathbb{C}^n \), the inequality holds for all \( a \in \mathbb{C}^n \). Taking \( a_i = \overline{f_i}(x) \), we conclude that
\[
\sum_{i=1}^n | f_i(x) |^2 \leq c^2
\]
Integrating over \( X \), we have \( n \leq c^2 \). Thus \( \dim V < \infty \).

#### 4.4. Proof of 4.1
We have seen that we may assume that \( G \) has no central split torus so that \( G_{\infty}/\Gamma \) has finite measure and further that \( \chi_i \) trivial. We claim that \( \mathcal{A}(G_{\infty}, \Gamma, \sigma_{\infty}, \lambda) \) is a closed subspace of \( L_2(G_{\infty}/\Gamma) \). In fact if \( \varphi_n \in \mathcal{A}(G_{\infty}, \Gamma, \sigma_{\infty}, \lambda) \) converges to \( \varphi \) in \( L_2 \), then \( \varphi_n \rightarrow \varphi \) in
the sense of distributions. All the \( \varphi_n \) are in the kernel of a fixed elliptic operator with \( C^\infty \) coefficients on \( G_\infty/\Gamma \) hence the distribution \( \varphi \) is also in this kernel. But any distribution in the kernel of an elliptic operator with \( C^\infty \) coefficients is a \( C^\infty \) function. Thus \( \varphi \) is a \( C^\infty \) function in \( oL_2 \).

Since it is the limit of the \( \varphi_n \) as a distribution, the \( K_\infty \)-span of \( \varphi \) is a quotient of \( \sigma_\infty \) and the \( \mathbb{Z} \)-span a quotient of \( \lambda \). Further as \( \varphi \) is \( k \)-finite and \( \mathbb{Z} \) finite there is an \( \alpha \in \mathbb{Z}^\infty(G) \) such that \( \alpha \ast \varphi = \varphi \) and one has thus \( \| \varphi \|_{\infty} \leq \| \alpha \ast \varphi \|_{\infty} \leq C \| \alpha \|_{2G_\infty} \| \varphi \|_{G_\infty/\Gamma} \). It follows that \( \varphi \) has moderate growth so that \( \varphi \in \alpha, \mathbb{A}((G_\infty, \Gamma, \sigma_\infty, \lambda)) \). Lemma 4.3 now clearly implies Theorem 4.1.

4.5. Clearly Theorem 4.1 is equivalent to the assertion that \( \alpha, \mathbb{A}((G(\mathbb{A}), G(k), \sigma, \lambda)) \) is finite dimensional. To prove Theorem 2, we will argue by induction on the \( k \)-rank of \( [G, G] \). Let \( \{P_i\}_{i \in I} \) be the collection of all the maximal parabolic subgroups of \( G \) containing \( P \). Let \( U_i \) be the cuspidal radical of \( P_i \) and \( G_i \supset S \). Let supplement to \( U_i, P_i \). Let \( r_i : C^\infty(G(\mathbb{A})/G(k)) \rightarrow C^\infty(G_i(\mathbb{A})/G_i(k)) \) be the map defined as follows:

\[
\gamma_i(\varphi_i)(x) = \int_{U_i(\mathbb{A})/U_i(k)} \varphi(xu) du.
\]

We assert that for a suitable \( \chi_i \) and \( \lambda_i \) and \( \sigma_{i,\infty}, \lambda_i(\varphi) \in \mathbb{A}^\chi(G_{i,\infty}(\mathbb{A}), G_{i,\infty}(k), \lambda_i, \sigma_{i,\infty}) \) if \( \varphi \in \mathbb{A}^\chi(G(\mathbb{A}), G(k), \lambda, \sigma_{\infty}) \). Assume that this is true. Observe that \( k \)-rank \( ([G_i, G_i]) = k \) rank \( G - 1 \). Thus by induction hypothesis \( \mathbb{A}^\chi(G_i(\mathbb{A}), G_i(k), \lambda_i, \lambda_{i,\infty}) \) is finite dimensional for all \( i \in I \). On the other hand if \( \varphi \) is in \( \cap_{i \in I} \ker \lambda_i \), \( \varphi \) is a cuspidal automorphic form in \( \alpha, \mathbb{A}^\chi(G(\mathbb{A}), G(k), \sigma, \lambda_{\infty}) \); and this last space is finite dimensional by Theorem 4.1. This completes the proof of Theorem 2 assuming the following:

4.6. Lemma Given \( \chi, \sigma, \lambda \) there exists for every \( i \in I, \chi_i, \sigma_i, \lambda_i \), \( \chi_i = \chi \) (note that \( C \subset G_i(\sigma_i) \)), representation of \( K_\infty \cap G_i \) and \( \lambda_i : Z_i \rightarrow \text{End}W_i' \), \( (Z_i \) the centre of the enveloping algebra \( U_i \) of \( L(G_i) \) ) such that

\[
r_i(\mathbb{A}^\chi(G(\mathbb{A}_f), G(k), \sigma, \lambda)) \subset A^{\chi_i}(G_i(\mathbb{A}_f), G_i(k), \sigma_i, \lambda_i)
\]

Proof Let \( U_-^\mathbb{C} \) be the unipotent algebraic \( k \)-subgroup of \( G \) normalised by \( S \) and whose Lie algebra is spanned by \( \{L(G)(\beta) \mid \beta \text{ a root}, \beta = \sum_{\alpha \in \Delta} m_\alpha \alpha, m_\alpha \in \mathbb{C} \} \). Let \( P_i^- \) be the normaliser of \( U_i^- \). One then has
\( \mathbf{P}_i \cap \mathbf{P}_i^- = \mathbf{G}_i \) and \( L(\mathbf{G}) = L(\mathbf{U}_i^-) \otimes L(\mathbf{G}_i) \otimes L(\mathbf{U}_i) \). Correspondingly the enveloping algebra \( \mathcal{U} \) of \( L(\mathbf{G}) \) can be written as

\[
\mathcal{U}(L(\mathbf{U}_i^-)) \cdot \mathcal{U}_i \cdot \mathcal{U}(L(\mathbf{U}_i))
\]

where \( \mathcal{U}_i \) is the enveloping algebra of \( L(\mathbf{G}_i) \). Now \( \mathcal{U}(L(\mathbf{U}_i^-)) = L(\mathbf{U}_i^-)(\mathcal{U}(L(\mathbf{U}_i^-)) \oplus \mathbb{C} \) and similarly \( \mathcal{U}(L(\mathbf{U}_i)) = \mathcal{U}(L(\mathbf{U}_i))\mathcal{U}(\mathbf{U}_i) \oplus \mathbb{C} \). Consequently we have

\[
\mathcal{U} = L(\mathbf{U}_i^-)\mathcal{U}(L(\mathbf{U}_i^-)) : \mathcal{U}_i \cdot \mathcal{U}(L(\mathbf{U}_i))L(\mathbf{U}_i)
\]

\[
\oplus L(\mathbf{U}_i^-)\mathcal{U}(L(\mathbf{U}_i^-)) : \mathcal{U}_i
\]

\[
\oplus \mathcal{U}_i \cdot \mathcal{U}(L(\mathbf{U}_i))L(\mathbf{U}_i) \oplus \mathcal{U}_i
\]

Let \( h_i \) denote the projection of \( \mathcal{U} \) on \( \mathcal{U}_i \) following this direct sum decomposition. Then according to a theorem of Harish-Chandra \( h_i \) restrict to \( \mathcal{U} \) maps \( \mathcal{Z} \) injectively into \( \mathcal{Z}_i \) and \( \mathcal{Z}_i \) is finitely generated as a \( \mathcal{Z} \)-module. Also the component of \( z \in \mathcal{C}U \) in the direct factors \( L(\mathbf{U}_i^-)\mathcal{U}(L(\mathbf{U}_i^-)) : \mathcal{U}_i \) and \( \mathcal{U}(L(\mathbf{U}_i))L(\mathbf{U}_i) \) are trivial. From this it is easy to see that \( r_i(z\varphi) = h_i(z)r_i(\varphi) \) for \( \varphi \in (\mathcal{A}^\chi(\mathbf{G}(\mathbf{A}), \mathbf{G}(k), \sigma, \lambda)) \). It follows that if we set \( W'_i = W \otimes \mathcal{Z}_i \), \( W'_i \) is a finite dimensional vector space and we have a homomorphism \( \lambda_i : \mathcal{Z}_i \rightarrow \text{End } W'_i \). Evidently \( \mathcal{U}_i(r_i(\varphi)) \) is a quotient of \( W'_i \) as a \( \mathcal{Z}_i \)-module. Also, if \( K_i = K \cap \mathbf{G}_i(\mathbf{A}_f) \), it is clear that the \( K_i \) (\( K \)-span of \( \varphi \)) and thus is a quotient of the \( K_i \)-module \( W \) (obtained by restriction of the \( K \)-module structure to \( K_0 \)). Lemma 4.5 is immediate from this.
References


