Notes on L-functions for GL_n

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Lectures given at the
School on Automorphic Forms on $GL(n)$
Trieste, 31 July - 18 August 2000

LNS0821003

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Abstract

The theory of L-functions of automorphic forms (or modular forms) via integral representations has its origin in the paper of Riemann on the ζ-function. However the theory was really developed in the classical context of L-functions of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Hecke and his school. Much of our current theory is a direct outgrowth of Hecke's. L-functions of automorphic representations were first developed by Jacquet and Langlands for GL_2. Their approach followed Hecke combined with the local-global techniques of Tate's thesis. The theory for GL_n was then developed along the same lines in a long series of papers by various combinations of Jacquet, Piatetski-Shapiro, and Shalika. In addition to associating an L-function to an automorphic form, Hecke also gave a criterion for a Dirichlet series to come from a modular form, the so-called Converse Theorem of Hecke. In the context of automorphic representations, the Converse Theorem for GL_2 was developed by Jacquet and Langlands, extended and significantly strengthened to GL_3 by Jacquet, Piatetski-Shapiro, and Shalika, and then extended to GL_n.

What we have attempted to present here is a synopsis of this work and in doing so present the paradigm for the analysis of automorphic L-functions via integral representations. We begin with the Fourier expansion of a cusp form and results on Whittaker models since these are essential for defining Eulerian integrals. We then develop integral representations for L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$ which have nice analytic properties (meromorphic continuation, finite order of growth, functional equations) and have Eulerian factorization into products of local integrals. We next turn to the local theory of L-functions for GL_n, in both the archimedean and non-archimedean local contexts, which comes out of the Euler factors of the global integrals. We finally combine the global Eulerian integrals with the definition and analysis of the local L-functions to define the global L-function of an automorphic representation and derive their major analytic properties. We next turn to the various Converse Theorems for GL_n and their applications to Langlands liftings.

Contents

Introduction

The purpose of these notes is to develop the analytic theory of L-functions for cuspidal automorphic representations of GL_n over a global field. There are two approaches to L-functions of GL_n: via integral representations or through analysis of Fourier coefficients of Eisenstein series. In these notes we develop the theory via integral representations.

The theory of L-functions of automorphic forms (or modular forms) via integral representations has its origin in the paper of Riemann on the ζ-function [53]. However the theory was really developed in the classical context of L-functions of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Hecke and his school [25]. Much of our current theory is a direct outgrowth of Hecke's. L-functions of automorphic representations were first developed by Jacquet and Langlands for GL_2 [21, 28, 30]. Their approach followed Hecke combined with the local-global techniques of Tate's thesis [64]. The theory for GL_n was then developed along the same lines in a long series of papers by various combinations of Jacquet, Piatetski-Shapiro, and Shalika [31–38, 47, 48, 62]. In addition to associating an L-function to an automorphic form, Hecke also gave a criterion for a Dirichlet series to come from a modular form, the so-called Converse Theorem of Hecke [26]. In the context of automorphic representations, the Converse Theorem for GL_2 was developed by Jacquet and Langlands [30], extended and significantly strengthened to GL_3 by Jacquet, Piatetski-Shapiro, and Shalika [31], and then extended to GL_n [7, 9].

What we have attempted to present here is a synopsis of this work and in doing so present the paradigm for the analysis of automorphic L-functions via integral representations. Section 1 deals with the Fourier expansion of automorphic forms on GL_n and the related Multiplicity One and Strong Multiplicity One theorems. Section 2 then develops the theory of Eulerian integrals for GL_n. In Section 3 we turn to the local theory of L-functions for GL_n, in both the archimedean and non-archimedean local contexts, which comes out of the Euler factors of the global integrals. In Section 4 we finally combine the global Eulerian integrals with the definition and analysis of the local L-functions to define the global L-function of an automorphic representation and derive their major analytic properties. In Section 5 we turn to the various Converse Theorems for GL_n.

We have tried to keep the tone of the notes informal for the most part. We have tried to provide complete proofs where feasible, at least sketches of most major results, and references for technical facts.

There is another body of work on integral representations of L-functions for GL_n which developed out of the classical work on zeta functions of algebras. This is the theory of principal L-functions for GL_n as developed by Godement and Jacquet [22, 28]. This approach is related to the one pursued here, but we have not attempted to present it here.

The other approach to these L-functions is via the Fourier coefficients of Eisenstein series. This approach also has a classical history. In the context of automorphic representations, and in a broader context than GL_n, this approach was originally laid out by Langlands [43] but then most fruitfully pursued by Shahidi. Some of the major papers of Shahidi on this subject are [55–61]. In particular, in [58] he shows that the two approaches give the same L-functions for GL_n. We will not pursue this approach in these notes.

For a balanced presentation of all three methods we recommend the book of Gelbart and Shahidi [16]. They treat not only L-functions for GL_n but L-functions of automorphic representations of other groups as well.

We have not discussed the arithmetic theory of automorphic representations and L-functions. For the connections with motives, we recommend the surveys of Clozel [5] and Ramakrishnan [50].

1 Fourier Expansions and Multiplicity One Theorems

In this section we let k denote a global field, $\mathbb{A}$, its ring of adeles, and ψ will denote a continuous additive character of $\mathbb{A}$ which is trivial on k. For the basics on adeles, characters, etc. we refer the reader to Weil [68] or the book of Gelfand, Graev, and Piatetski-Shapiro [18].

We begin with a cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. For us, automorphic forms are assumed to be smooth (of uniform moderate growth) but not necessarily K_∞–finite at the archimedean places. This is most suitable for the analytic theory. For simplicity, we assume the central character ω_π of π is unitary. Then V_π is the space of smooth vectors in an irreducible unitary representation of $\mathrm{GL}_n(\mathbb{A})$. We will always use cuspidal in this sense: the smooth vectors in an irreducible unitary cuspidal automorphic representation. (Any other smooth cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ is necessarily of the form $\pi = \pi^\circ \otimes |\det|^t$ with π° unitary and t real, so there is really no loss of generality in the unitarity assumption. It merely provides us with a convenient normalization.) By a cusp form on $\mathrm{GL}_n(\mathbb{A})$ we will mean a function lying in a cuspidal representation. By a cuspidal function we will simply mean a smooth function φ on $\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})$ satisfying $\int_{\mathrm{U}(k)\backslash\,\mathrm{U}(\mathbb{A})} \varphi(ug)\, du \equiv 0$ for every unipotent radical U of standard parabolic subgroups of GL_n.

The basic references for this section are the papers of Piatetski-Shapiro [47, 48] and Shalika [62].

1.1 Fourier Expansions

Let $\varphi(g) \in V_\pi$ be a cusp form in the space of π. For arithmetic applications, and particularly for the theory of L-functions, we will need the *Fourier expansion* of $\varphi(g)$.

If $f(\tau)$ is a holomorphic cusp form on the upper half plane $\mathfrak{H}$, say with respect to $\mathrm{SL}_2(\mathbb{Z})$, then f is invariant under integral translations, $f(\tau+1) = f(\tau)$ and thus has a Fourier expansion of the form

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}.$$

If $\varphi(g)$ is a smooth cusp form on $\mathrm{GL}_2(\mathbb{A})$ then the translations correspond to the maximal unipotent subgroup $\mathrm{N}_2 = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ and $\varphi(ng) = \varphi(g)$ for $n \in \mathrm{N}_2(k)$. So, if ψ is any continuous character of $k\backslash\mathbb{A}$ we can define the ψ-Fourier coefficient or ψ-Whittaker function by

$$W_{\varphi,\psi}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi^{-1}(x)\, dx.$$

We have the corresponding Fourier expansion

$$\varphi(g) = \sum_{\psi} W_{\varphi,\psi}(g).$$

(Actually from abelian Fourier theory, one has

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \sum_{\psi} W_{\varphi,\psi}(g)\psi(x)$$

as a periodic function of $x \in \mathbb{A}$. Now set $x = 0$.)

If we fix a single non-trivial character ψ of $k\backslash\mathbb{A}$, then by standard duality theory [18, 68] the additive characters of the compact group $k\backslash\mathbb{A}$ are isomorphic to k via the map $\gamma \in k \mapsto \psi_\gamma$ where ψ_γ is the character $\psi_\gamma(x) = \psi(\gamma x)$. Now, an elementary calculation shows that $W_{\varphi,\psi_\gamma}(g) = W_{\varphi.\psi}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)$ if $\gamma \neq 0$. If we set $W_\varphi = W_{\varphi,\psi}$ for our fixed ψ, then the Fourier expansion of φ becomes

$$\varphi(g) = W_{\varphi,\psi_0}(g) + \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

Since φ is cuspidal

$$W_{\varphi,\psi_0}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx \equiv 0$$

and the Fourier expansion for a cusp form φ becomes simply

$$\varphi(g) = \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

We will need a similar expansion for cusp forms φ on $\mathrm{GL}_n(\mathbb{A})$. The translations still correspond to the maximal unipotent subgroup

$$\mathrm{N}_n = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & & & * \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & x_{n-1,n} \\ 0 & & & & 1 \end{pmatrix} \right\},$$

but now this is non-abelian. This difficulty was solved independently by Piatetski-Shapiro [47] and Shalika [62]. We fix our non-trivial continuous character ψ of $k\backslash\mathbb{A}$ as above. Extend it to a character of N_n by setting $\psi(n) = \psi(x_{1,2} + \cdots + x_{n-1,n})$ and define the associated Fourier coefficient or Whittaker function by

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(ng)\psi^{-1}(n)\, dn.$$

Since φ is continuous and the integration is over a compact set this integral is absolutely convergent, uniformly on compact sets. The Fourier expansion takes the following form.

Theorem 1.1 *Let* $\varphi \in V_\pi$ *be a cusp form on* $\mathrm{GL}_n(\mathbb{A})$ *and* W_φ *its associated* ψ*-Whittaker function. Then*

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)$$

with convergence absolute and uniform on compact subsets.

The proof of this fact is an induction. It utilizes the *mirabolic subgroup* P_n of GL_n which seems to be ubiquitous in the study of automorphic forms on GL_n. Abstractly, a mirabolic subgroup of GL_n is simply the stabilizer of a non-zero vector in (either) standard representation of GL_n on k^n. We denote by P_n the stabilizer of the row vector $e_n = (0, \ldots, 0, 1) \in k^n$. So

$$\mathrm{P}_n = \left\{ p = \begin{pmatrix} h & y \\ & 1 \end{pmatrix} \middle| h \in \mathrm{GL}_{n-1}, y \in k^{n-1} \right\} \simeq \mathrm{GL}_{n-1} \ltimes \mathrm{Y}_n$$

where

$$\mathrm{Y}_n = \left\{ y = \begin{pmatrix} I_{n-1} & y \\ & 1 \end{pmatrix} \middle| y \in k^{n-1} \right\} \simeq k^{n-1}.$$

Simply by restriction of functions, a cusp form on $\mathrm{GL}_n(\mathbb{A})$ restricts to a smooth cuspidal function on $\mathrm{P}_n(\mathbb{A})$ which remains left invariant under $\mathrm{P}_n(k)$. (A smooth function φ on $\mathrm{P}_n(\mathbb{A})$ which is left invariant under $\mathrm{P}_n(k)$ is called cuspidal if $\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi(up)\, du \equiv 0$ for every standard unipotent subgroup $\mathrm{U} \subset \mathrm{P}_n$.) Since $\mathrm{P}_n \supset \mathrm{N}_n$ we may define a Whittaker function attached to a cuspidal function φ on $\mathrm{P}_n(\mathbb{A})$ by the same integral as on $\mathrm{GL}_n(\mathbb{A})$, namely

$$W_\varphi(p) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(np)\psi^{-1}(n)\, dn.$$

We will prove by induction that for a cuspidal function φ on $\mathrm{P}_n(\mathbb{A})$ we have

$$\varphi(p) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right)$$

with convergence absolute and uniform on compact subsets.

The function on $\mathrm{Y}_n(\mathbb{A})$ given by $y \mapsto \varphi(yp)$ is invariant under $\mathrm{Y}_n(k)$ since $\mathrm{Y}_n(k) \subset \mathrm{P}_n(k)$ and φ is automorphic on $\mathrm{P}_n(\mathbb{A})$. Hence by standard abelian Fourier analysis for $\mathrm{Y}_n \simeq k^{n-1}$ we have as before

$$\varphi(p) = \sum_{\lambda \in (\widehat{k^{n-1}\backslash \mathbb{A}^{n-1}})} \varphi_\lambda(p)$$

where

$$\varphi_\lambda(p) = \int_{\mathrm{Y}_n(k)\backslash \mathrm{Y}_n(\mathbb{A})} \varphi(yp)\lambda^{-1}(y)\ dy.$$

Now, by duality theory [68], $(\widehat{k^{n-1}\backslash\mathbb{A}^{n-1}}) \simeq k^{n-1}$. In fact, if we let $\langle\ ,\ \rangle$ denote the pairing $k^{n-1}\times k^{n-1}\to k$ by $\langle x,y\rangle = \sum x_i y_i$ we have

$$\varphi(p) = \sum_{x\in k^{n-1}} \varphi_x(p)$$

where now we write

$$\varphi_x(p) = \int_{k^{n-1}\backslash\mathbb{A}^{n-1}} \varphi(yp)\psi^{-1}(\langle x,y\rangle)\ dy.$$

$\mathrm{GL}_{n-1}(k)$ acts on k^{n-1} with two orbits: $\{0\}$ and $k^{n-1}-\{0\} = \mathrm{GL}_{n-1}(k)\cdot{}^t e_{n-1}$ where $e_{n-1}=(0,\ldots,0,1)$. The stabilizer of ${}^te_{n-1}$ in $\mathrm{GL}_{n-1}(k)$ is ${}^t\mathrm{P}_{n-1}$. Therefore, we may write

$$\varphi(p) = \varphi_0(p) + \sum_{\gamma\in \mathrm{GL}_{n-1}(k)/{}^t\mathrm{P}_{n-1}(k)} \varphi_{\gamma\cdot{}^te_{n-1}}(p).$$

Since $\varphi(p)$ is cuspidal and Y_n is a standard unipotent subgroup of GL_n, we see that

$$\varphi_0(p) = \int_{\mathrm{Y}_n(k)\backslash \mathrm{Y}_n(\mathbb{A})} \varphi(yp)\ dy \equiv 0.$$

On the other hand an elementary calculation as before gives

$$\varphi_{\gamma\cdot{}^te_{n-1}}(p) = \varphi_{{}^te_{n-1}}\left(\begin{pmatrix} {}^t\gamma & 0\\ 0 & 1\end{pmatrix} p\right).$$

Hence we have

$$\varphi(p) = \sum_{\gamma\in \mathrm{P}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \varphi_{{}^te_{n-1}}\left(\begin{pmatrix} \gamma & 0\\ 0 & 1\end{pmatrix} p\right)$$

and the convergence is still absolute and uniform on compact subsets.

Note that if $n=2$ this is exactly the fact we used previously for GL_2. This then begins our induction.

Next, let us write the above in a form more suitable for induction. Structurally, we have $\mathrm{P}_n = \mathrm{GL}_{n-1}\ltimes \mathrm{Y}_n$ and $\mathrm{N}_n = \mathrm{N}_{n-1}\ltimes \mathrm{Y}_n$. Therefore, $\mathrm{N}_{n-1}\backslash \mathrm{GL}_{n-1} \simeq \mathrm{N}_n\backslash \mathrm{P}_n$. Furthermore, if we let $\widetilde{\mathrm{P}}_{n-1} = \mathrm{P}_{n-1}\ltimes \mathrm{Y}_n \subset \mathrm{P}_n$, then $\mathrm{P}_{n-1} \backslash \mathrm{GL}_{n-1} \simeq \widetilde{\mathrm{P}}_{n-1} \backslash \mathrm{P}_n$. Next, note that the function $\varphi_{{}^te_{n-1}}(p)$ satisfies, for $y \in \mathrm{Y}_n(\mathbb{A}) \simeq \mathbb{A}^{n-1}$,

$$\varphi_{{}^te_{n-1}}(yp) = \psi(y_{n-1})\varphi_{{}^te_{n-1}}(p).$$

Since ψ is trivial on k we see that $\varphi_{{}^te_{n-1}}(p)$ is left invariant under $\mathrm{Y}_n(k)$. Hence

$$\varphi(p) = \sum_{\gamma \in \mathrm{P}_{n-1}(k) \backslash \mathrm{GL}_{n-1}(k)} \varphi_{{}^te_{n-1}}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) = \sum_{\delta \in \widetilde{\mathrm{P}}_{n-1}(k) \backslash \mathrm{P}_n(k)} \varphi_{{}^te_{n-1}}(\delta p).$$

To proceed with the induction, fix $p \in \mathrm{P}_n(\mathbb{A})$ and consider the function $\varphi'(p') = \varphi'_p(p')$ on $\mathrm{P}_{n-1}(\mathbb{A})$ given by

$$\varphi'(p') = \varphi_{{}^te_{n-1}}\left(\begin{pmatrix} p' & 0 \\ 0 & 1 \end{pmatrix} p\right).$$

φ' is a smooth function on $\mathrm{P}_{n-1}(\mathbb{A})$ since φ was smooth. One checks that φ' is left invariant by $\mathrm{P}_{n-1}(k)$ and cuspidal on $\mathrm{P}_{n-1}(\mathbb{A})$. Then we may apply our inductive assumption to conclude that

$$\begin{aligned} \varphi'(p') &= \sum_{\gamma' \in \mathrm{N}_{n-2} \backslash \mathrm{GL}_{n-2}} W_{\varphi'}\left(\begin{pmatrix} \gamma' & 0 \\ 0 & 1 \end{pmatrix} p'\right) \\ &= \sum_{\delta' \in \mathrm{N}_{n-1} \backslash \mathrm{P}_{n-1}} W_{\varphi'}(\delta' p'). \end{aligned}$$

If we substitute this into the expansion for $\varphi(p)$ we see

$$\begin{aligned} \varphi(p) &= \sum_{\delta \in \widetilde{\mathrm{P}}_{n-1}(k) \backslash \mathrm{P}_n(k)} \varphi_{{}^te_{n-1}}(\delta p) \\ &= \sum_{\delta \in \widetilde{\mathrm{P}}_{n-1}(k) \backslash \mathrm{P}_n(k)} \varphi'_{\delta p}(1) \\ &= \sum_{\delta \in \widetilde{\mathrm{P}}_{n-1}(k) \backslash \mathrm{P}_n(k)} \sum_{\delta' \in \mathrm{N}_{n-1} \backslash \mathrm{P}_{n-1}} W_{\varphi'_{\delta p}}(\delta'). \end{aligned}$$

Now, as before, $\mathrm{N}_{n-1} \backslash \mathrm{P}_{n-1} \simeq \mathrm{N}_n \backslash \widetilde{\mathrm{P}}_{n-1}$ and $\mathrm{N}_n \simeq \mathrm{N}_{n-1} \ltimes \mathrm{Y}_{n-1}$. Thus

$$\begin{aligned}
W_{\varphi'_{\delta p}}(\delta') &= \int_{\mathrm{N}_{n-1}(k)\backslash \mathrm{N}_{n-1}(\mathbb{A})} \varphi'_{\delta p}(n'\delta')\psi^{-1}(n')\, dn' \\
&= \int_{\mathrm{N}_{n-1}(k)\backslash \mathrm{N}_{n-1}(\mathbb{A})} \int_{\mathrm{Y}_n(k)\backslash \mathrm{Y}_n(\mathbb{A})} \varphi(yn'\delta'\delta p)\psi^{-1}(y_{n-1})\psi^{-1}(n')\, dy\, dn' \\
&= \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(n\delta'\delta p)\psi^{-1}(n)\, dn \\
&= W_\varphi(\delta'\delta p)
\end{aligned}$$

and so

$$\begin{aligned}
\varphi(p) &= \sum_{\delta \in \widetilde{\mathrm{P}}_{n-1}(k)\backslash \mathrm{P}_n(k)} \sum_{\delta' \in \mathrm{N}_n \backslash \widetilde{\mathrm{P}}_{n-1}} W_\varphi(\delta'\delta p) \\
&= \sum_{\delta \in \mathrm{N}_n(k)\backslash \mathrm{P}_n(k)} W_\varphi(\delta p) \\
&= \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right)
\end{aligned}$$

which was what we wanted.

To obtain the Fourier expansion on GL_n from this, if φ is a cusp form on $\mathrm{GL}_n(\mathbb{A})$, then for $g \in \Omega$ a compact subset the functions $\varphi_g(p) = \varphi(pg)$ form a compact family of cuspidal functions on $\mathrm{P}_n(\mathbb{A})$. So we have

$$\varphi_g(1) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_{\varphi_g}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\right)$$

with convergence absolute and uniform. Hence

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

again with absolute convergence, uniform for $g \in \Omega$.

1.2 Whittaker Models and the Multiplicity One Theorem

Consider now the functions W_φ appearing in the Fourier expansion of a cusp form φ. These are all smooth functions $W(g)$ on $\mathrm{GL}_n(\mathbb{A})$ which satisfy

$W(ng) = \psi(n)W(g)$ for $n \in N_n(\mathbb{A})$. If we let $\mathcal{W}(\pi,\psi) = \{W_\varphi \mid \varphi \in V_\pi\}$ then $\mathrm{GL}_n(\mathbb{A})$ acts on this space by right translation and the map $\varphi \mapsto W_\varphi$ intertwines V_π with $\mathcal{W}(\pi,\psi)$. $\mathcal{W}(\pi,\psi)$ is called the *Whittaker model* of π.

The notion of a Whittaker model of a representation makes perfect sense over a local field or even a finite field. Much insight can be gained by pursuing these ideas over a finite field [20,49], but that would take us too far afield. So let k_v be a local field (a completion of k for example [18,68]) and let (π_v, V_{π_v}) be an irreducible admissible smooth representation of $\mathrm{GL}_n(k_v)$. Fix a non-trivial continuous additive character ψ_v of k_v. Let $\mathcal{W}(\psi_v)$ be the space of all smooth functions $W(g)$ on $\mathrm{GL}_n(k_v)$ satisfying $W(ng) = \psi(n)W(g)$ for all $n \in \mathrm{N}_k(k_v)$, that is, the space of all smooth Whittaker functions on $\mathrm{GL}_n(k_v)$ with respect to ψ_v. This is also the space of the smooth induced representation $\mathrm{Ind}_{\mathrm{N}_v}^{\mathrm{GL}_n}(\psi_v)$. $\mathrm{GL}_n(k_v)$ acts on this by right translation. If we have a non-trivial continuous intertwining $V_{\pi_v} \to \mathcal{W}(\psi_v)$ we will denote its image by $\mathcal{W}(\pi_v,\psi_v)$ and call it a Whittaker model of π_v.

Whittaker models for a representation (π_v, V_{π_v}) are equivalent to continuous Whittaker functionals on V_{π_v}, that is, continuous functionals Λ_v satisfying $\Lambda_v(\pi_v(n)\xi_v) = \psi_v(n)\Lambda_v(\xi_v)$ for all $n \in \mathrm{N}_n(k_v)$. To obtain a Whittaker functional from a model, set $\Lambda_v(\xi_v) = W_{\xi_v}(e)$, and to obtain a model from a functional, set $W_{\xi_v}(g) = \Lambda_v(\pi_v(g)\xi_v)$. This is a form of Frobenius reciprocity, which in this context is the isomorphism between $\mathrm{Hom}_{\mathrm{N}_n}(V_{\pi_v}, \mathbb{C}_{\psi_v})$ and $\mathrm{Hom}_{\mathrm{GL}_n}(V_{\pi_v}, \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{GL}_n}(\psi_v))$ constructed above.

The fundamental theorem on the existence and uniqueness of Whittaker functionals and models is the following.

Theorem 1.2 *Let (π_v, V_{π_v}) be a smooth irreducible admissible representation of $\mathrm{GL}_n(k_v)$. Let ψ_v be a non-trivial continuous additive character of k_v. Then the space of continuous ψ_v–Whittaker functionals on V_{π_v} is at most one dimensional. That is, Whittaker models, if they exist, are unique.*

This was first proven for non-archimedean fields by Gelfand and Kazhdan [19] and their results were later extended to archimedean local fields by Shalika [62]. The method of proof is roughly the following. It is enough to show that $\mathcal{W}(\pi_v) = \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{GL}_n}(\psi_v)$ is multiplicity free, i.e., any irreducible representation of $\mathrm{GL}_n(k_v)$ occurs in $\mathcal{W}(\psi_v)$ with multiplicity at most one. This in turn is a consequence of the commutativity of the endomorphism algebra $\mathrm{End}(\mathrm{Ind}(\psi_v))$. Any intertwining map from $\mathrm{Ind}(\psi_v)$ to itself is given by convolution with a so-called Bessel distribution, that is, a distribution B on $\mathrm{GL}_n(k_v)$ satisfying $B(n_1gn_2) = \psi_v(n_1)B(g)\psi_v(n_2)$ for $n_1, n_2 \in \mathrm{N}_n(k_v)$.

Such quasi-invariant distributions are analyzed via Bruhat theory. By the Bruhat decomposition for GL_n, the double cosets $\mathrm{N}_n \setminus \mathrm{GL}_n / \mathrm{N}_n$ are parameterized by the monomial matrices. The only double cosets that can support Bessel distributions are associated to permutation matrices of the form $\begin{pmatrix} & & I_{r_k} \\ & \cdot^{\cdot^{\cdot}} & \\ I_{r_1} & & \end{pmatrix}$ and the resulting distributions are then stable under the involution $g \mapsto g^\sigma = w_n\ {}^t g\ w_n$ with $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ the long Weyl element of GL_n. Thus for the convolution of Bessel distributions we have $B_1 * B_2 = (B_1 * B_2)^\sigma = B_2^\sigma * B_1^\sigma = B_2 * B_1$. Hence the algebra of intertwining Bessel distributions is commutative and hence $\mathcal{W}(\psi_v)$ is multiplicity free.

A smooth irreducible admissible representation (π_v, V_{π_v}) of $\mathrm{GL}_n(k_v)$ which possesses a Whittaker model is called *generic* or *non-degenerate*. Gelfand and Kazhdan in addition show that π_v is generic iff its contragredient $\widetilde{\pi}_v$ is generic, in fact that $\widetilde{\pi} \simeq \pi^\iota$ where ι is the outer automorphism $g^\iota = {}^t g^{-1}$, and in this case the Whittaker model for $\widetilde{\pi}_v$ can be obtained as $\mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1}) = \{\widetilde{W}(g) = W(w_n\ {}^t g^{-1}) \mid W \in \mathcal{W}(\pi, \psi_v)\}$.

As a consequence of the local uniqueness of the Whittaker model we can conclude a global uniqueness. If (π, V_π) is an irreducible smooth admissible representation of $\mathrm{GL}_n(\mathbb{A})$ then π factors as a restricted tensor product of local representations $\pi \simeq \otimes' \pi_v$ taken over all places v of k [14, 18]. Consequently we have a continuous embedding $V_{\pi_v} \hookrightarrow V_\pi$ for each local component. Hence any Whittaker functional Λ on V_π determines a family of local Whittaker functionals Λ_v on each V_{π_v} and conversely such that $\Lambda = \otimes' \Lambda_v$. Hence global uniqueness follows from the local uniqueness. Moreover, once we fix the isomorphism of V_π with $\otimes' V_{\pi_v}$ and define global and local Whittaker functions via Λ and the corresponding family Λ_v we have a factorization of global Whittaker functions

$$W_\xi(g) = \prod_v W_{\xi_v}(g_v)$$

for $\xi \in V_\pi$ which are factorizable in the sense that $\xi = \otimes' \xi_v$ corresponds to a pure tensor. As we will see, this factorization, which is a direct consequence of the uniqueness of the Whittaker model, plays a most important role in the development of Eulerian integrals for GL_n.

Now let us see what this means for our cuspidal representations (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. We have seen that for any smooth cusp form $\varphi \in V_\pi$ we have the Fourier expansion

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

We can thus conclude that $\mathcal{W}(\pi, \psi) \neq 0$ and that π is (globally) generic with Whittaker functional

$$\Lambda(\varphi) = W_\varphi(e) = \int \varphi(ng)\psi^{-1}(n)\ dn.$$

Thus φ is completely determined by its associated Whittaker function W_φ. From the uniqueness of the global Whittaker model we can derive the Multiplicity One Theorem of Piatetski-Shapiro [48] and Shalika [62].

Theorem (Multiplicity One) *Let* (π, V_π) *be an irreducible smooth admissible representation of* $\mathrm{GL}_n(\mathbb{A})$*. Then the multiplicity of* π *in the space of cusp forms on* $\mathrm{GL}_n(\mathbb{A})$ *is at most one.*

Proof: Suppose that π has two realizations (π_1, V_{π_1}) and (π_2, V_{π_2}) in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$. Let $\varphi_i \in V_{\pi_i}$ be the two cusp forms associated to the vector $\xi \in V_\pi$. Then we have two nonzero Whittaker functionals on V_π, namely $\Lambda_i(\xi) = W_{\varphi_i}(e)$. By the uniqueness of Whittaker models, there is a nonzero constant c such that $\Lambda_1 = c\Lambda_2$. But then we have $W_{\varphi_1}(g) = \Lambda_1(\pi(g)\xi) = c\Lambda_2(\pi(g)\xi) = cW_{\varphi_2}(g)$ for all $g \in \mathrm{GL}_n(\mathbb{A})$. Thus

$$\begin{aligned} \varphi_1(g) &= \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_{\varphi_1}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \\ &= c \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_{\varphi_2}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) = c\varphi_2(g). \end{aligned}$$

But then V_{π_1} and V_{π_2} have a non-trivial intersection. Since they are irreducible representations, they must then coincide. □

1.3 Kirillov Models and the Strong Multiplicity One Theorem

The Multiplicity One Theorem can be significantly strengthened. The Strong Multiplicity One Theorem is the following.

Theorem (Strong Multiplicity One) *Let* (π_1, V_{π_1}) *and* (π_2, V_{π_2}) *be two cuspidal representations of* $\mathrm{GL}_n(\mathbb{A})$*. Suppose there is a finite set of places* S *such that for all* $v \notin S$ *we have* $\pi_{1,v} \simeq \pi_{2,v}$*. Then* $\pi_1 = \pi_2$*.*

There are two proofs of this theorem. One is based on the theory of L-functions and we will come to it in Section 4. The original proof of Piatetski-Shapiro [48] is based on the *Kirillov model* of a local generic representation.

Let k_v be a local field and let (π_v, V_{π_v}) be an irreducible admissible smooth generic representation of $\mathrm{GL}_n(k_v)$, such as a local component of a cuspidal representation π. Since π_v is generic it has its Whittaker model $\mathcal{W}(\pi_v, \psi_v)$. Each Whittaker function $W \in \mathcal{W}(\pi_v, \psi_v)$, since it is a function on $\mathrm{GL}_n(k_v)$, can be restricted to the mirabolic subgroup $\mathrm{P}_n(k_v)$. A fundamental result of Bernstein and Zelevinsky in the non-archimedean case [1] and Jacquet and Shalika in the archimedean case [36] says that the map $\xi_v \mapsto W_{\xi_v}|_{\mathrm{P}_n(k_v)}$ is injective. Hence the representation has a realization on a space of functions on $\mathrm{P}_n(k_v)$. This is the Kirillov model

$$\mathcal{K}(\pi_v, \psi_v) = \{W(p) | W \in \mathcal{W}(\pi_v, \psi_v)\}.$$

$\mathrm{P}_n(k_v)$ acts naturally by right translation on $\mathcal{K}(\pi_v, \psi_v)$ and the action of all of $\mathrm{GL}_n(k_v)$ can be obtained by transport of structure. But for now, it is the structure of $\mathcal{K}(\pi_v, \psi_v)$ as a representation of $\mathrm{P}_n(k_v)$ which is of interest.

For k_v a non-archimedean field, let (τ_v, V_{τ_v}) be the compactly induced representation $\tau_v = \mathrm{ind}_{\mathrm{N}_n(k_v)}^{\mathrm{P}_n(k_v)}(\psi_v)$. Then Bernstein and Zelevinsky have analyzed the representations of $\mathrm{P}_n(k_v)$ and shown that whenever π_v is an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$ then $\mathcal{K}(\pi_v, \psi_v)$ contains V_{τ_v} as a $\mathrm{P}_n(k_v)$ sub-representation and if π_v is supercuspidal then $\mathcal{K}(\pi_v, \psi_v) = V_{\tau_v}$ [1].

For k_v archimedean, we then let (τ_v, V_{τ_v}) be the smooth vectors in the irreducible smooth unitarily induced representation $\mathrm{Ind}_{\mathrm{N}_n(k_v)}^{\mathrm{P}_n(k_v)}(\psi_v)$. Then Jacquet and Shalika have shown that as long as π_v is an irreducible admissible smooth unitary representation of $\mathrm{GL}_n(k_v)$ then in fact $\mathcal{K}(\pi_v, \psi_v) = V_{\tau_v}$ as representations of $\mathrm{P}_n(k_v)$ [36, Remark 3.15].

Therefore, for a given place v the local Kirillov models of any two irreducible admissible generic smooth unitary representations have a certain $\mathrm{P}_n(k_v)$-submodule in common, namely V_{τ_v}.

Let us now return to Piatetski-Shapiro's proof of the Strong Multiplicity One Theorem [48].

Proof: We begin with our cuspidal representations π_1 and π_2. Since π_1 and π_2 are irreducible, it suffices to find a cusp form $\varphi \in V_{\pi_1} \cap V_{\pi_2}$. If we let B_n denote the Borel subgroup of upper triangular matrices in GL_n, then $\mathrm{B}_n(k)\backslash \mathrm{B}_n(\mathbb{A})$ is dense in $\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})$ and so it suffices to find two cusp forms $\varphi_i \in V_{\pi_i}$ which agree on $\mathrm{B}_n(\mathbb{A})$. But $\mathrm{B}_n \subset \mathrm{P}_n Z_n$ with Z_n the center. If we let ω_i be the central character of π_i then by assumption $\omega_{1,v} = \omega_{2,v}$ for $v \notin S$ and the weak approximation theorem then implies $\omega_1 = \omega_2$. So it suffices to find two φ_i which agree on $\mathrm{P}_n(\mathbb{A})$. But as in the proof of the Multiplicity One Theorem, via the Fourier expansion, to show that $\varphi_1(p) = \varphi_2(p)$ for $p \in \mathrm{P}_n(\mathbb{A})$ it suffices to show that $W_{\varphi_1}(p) = W_{\varphi_2}(p)$. Since we can take each W_{φ_i} to be of the form $\prod_v W_{\varphi_{i,v}}$ this then reduces to a question about the local Kirillov models. For $v \notin S$ we have by assumption that $\mathcal{K}(\pi_{1,v}, \psi_v) = \mathcal{K}(\pi_{2,v}, \psi_v)$ and for $v \in S$ we have seen that $V_{\tau_v} \subset \mathcal{K}(\pi_{1,v}, \psi_v) \cap \mathcal{K}(\pi_{2,v}, \psi_v)$. So we can construct a common Whittaker function in the restriction of $\mathcal{W}(\pi_i, \psi)$ to $\mathrm{P}_n(\mathbb{A})$. This completes the proof. □

2 Eulerian Integrals for GL_n

Let $f(\tau)$ again be a holomorphic cusp form of weight k on $\mathfrak{H}$ for the full modular group with Fourier expansion

$$f(\tau) = \sum a_n e^{2\pi i n \tau}.$$

Then Hecke [25] associated to f an L-function

$$L(s, f) = \sum a_n n^{-s}$$

and analyzed its analytic properties, namely continuation, order of growth, and functional equation, by writing it as the Mellin transform of f

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f) = \int_0^\infty f(iy) y^s d^\times y.$$

An application of the modular transformation law for $f(\tau)$ under the transformation $\tau \mapsto -1/\tau$ gives the functional equation

$$\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f).$$

Moreover, if f was an eigenfunction of all Hecke operators then $L(s,f)$ had an Euler product expansion

$$L(s,f)=\prod_p (1-a_p p^{-s}+p^{k-1-2s})^{-1}.$$

We will present a similar theory for cuspidal automorphic representations (π,V_π) of $\mathrm{GL}_n(\mathbb{A})$. For applications to functoriality via the Converse Theorem (see Lecture 5) we will need not only the standard L-functions $L(s,\pi)$ but the twisted L-functions $L(s,\pi\times\pi')$ for $(\pi',V_{\pi'})$ a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ for $m<n$ as well. One point to notice from the outset is that we want to associate a single L-function to an infinite dimensional representation (or pair of representations). The approach we will take will be that of integral representations, but it will be broadened in the sense of Tate's thesis [64].

The basic references for this section are Jacquet-Langlands [30], Jacquet, Piatetski-Shapiro, and Shalika [31], and Jacquet and Shalika [36].

2.1 Eulerian Integrals for GL_2

Let us first consider the L-functions for cuspidal automorphic representations (π,V_π) of $\mathrm{GL}_2(\mathbb{A})$ with twists by an idele class character χ, or what is the same, a (cuspidal) automorphic representation of $\mathrm{GL}_1(\mathbb{A})$, as in Jacquet-Langlands [30].

Following Jacquet and Langlands, who were following Hecke, for each $\varphi\in V_\pi$ we consider the integral

$$I(s;\varphi,\chi)=\int_{k^\times\backslash\mathbb{A}^\times}\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\,d^\times a.$$

Since a cusp form on $\mathrm{GL}_2(\mathbb{A})$ is rapidly decreasing upon restriction to $\mathbb{A}^\times$ as in the integral, it follows that the integral is absolutely convergent for all s, uniformly for $\mathrm{Re}(s)$ in an interval. Thus $I(s;\varphi,\chi)$ is an entire function of s, bounded in any vertical strip $a\le\mathrm{Re}(s)\le b$. Moreover, if we let $\widetilde{\varphi}(g)=\varphi({}^tg^{-1})=\varphi(w_n\,{}^tg^{-1})$ then $\widetilde{\varphi}\in V_{\widetilde{\pi}}$ and the simple change of variables $a\mapsto a^{-1}$ in the integral shows that each integral satisfies a functional equation of the form

$$I(s;\varphi,\chi)=I(1-s;\widetilde{\varphi},\chi^{-1}).$$

So these integrals individually enjoy rather nice analytic properties.

If we replace φ by its Fourier expansion from Lecture 1 and unfold, we find

$$\begin{aligned} I(s;\varphi,\chi) &= \int_{k^\times\backslash\mathbb{A}^\times} \sum_{\gamma\in k^\times} W_\varphi\begin{pmatrix}\gamma a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\,d^\times a \\ &= \int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\,d^\times a \end{aligned}$$

where we have used the fact that the function $\chi(a)|a|^{s-1/2}$ is invariant under $k^\times$. By standard gauge estimates on Whittaker functions [31] this converges for $Re(s) >> 0$ after the unfolding. As we have seen in Lecture 1, if $W_\varphi \in \mathcal{W}(\pi,\psi)$ corresponds to a decomposable vector $\varphi \in V_\pi \simeq \otimes' V_{\pi_v}$ then the Whittaker function factors into a product of local Whittaker functions

$$W_\varphi(g) = \prod_v W_{\varphi_v}(g_v).$$

Since the character χ and the adelic absolute value factor into local components and the domain of integration $\mathbb{A}^\times$ also factors we find that our global integral naturally factors into a product of local integrals

$$\int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\,d^\times a = \prod_v \int_{k_v^\times} W_{\varphi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix}\chi_v(a_v)|a_v|^{s-1/2}\,d^\times a_v,$$

with the infinite product still convergent for $\mathrm{Re}(s) >> 0$, or

$$I(s;\varphi,\chi) = \prod_v \Psi_v(s;W_{\varphi_v},\chi_v)$$

with the obvious definition of the local integrals

$$\Psi_v(s;W_{\varphi_v},\chi_v) = \int_{k_v^\times} W_{\varphi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix}\chi_v(a_v)|a_v|^{s-1/2}\,d^\times a_v.$$

Thus each of our global integrals is Eulerian.

In this way, to π and χ we have associated a family of global Eulerian integrals with nice analytic properties as well as for each place v a family of local integrals convergent for $\mathrm{Re}(s) >> 0$.

2.2 Eulerian Integrals for $\mathrm{GL}_n \times \mathrm{GL}_m$ with $m < n$

Now let (π, V_π) be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Take $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$. At first blush, a natural analogue of the integrals we considered for GL_2 with GL_1 twists would be

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \varphi \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} \varphi'(h) |\det(h)|^{s-(n-m)/2} \, dh.$$

This family of integrals would have all the nice analytic properties as before (entire functions of finite order satisfying a functional equation), but they would not be Eulerian except in the case $m = n-1$, which proceeds exactly as in the GL_2 case.

The problem is that the restriction of the form φ to GL_m is too brutal to allow a nice unfolding when the Fourier expansion of φ is inserted. Instead we will introduce projection operators from cusp forms on $\mathrm{GL}_n(\mathbb{A})$ to cuspidal functions on on $P_{m+1}(\mathbb{A})$ which are given by part of the unipotent integration through which the Whittaker function is defined.

2.2.1 The projection operator

In GL_n, let $\mathrm{Y}_{n,m}$ be the unipotent radical of the standard parabolic subgroup attached to the partition $(m+1, 1, \dots, 1)$. If ψ is our standard additive character of $k\backslash\mathbb{A}$, then ψ defines a character of $\mathrm{Y}_{n,m}(\mathbb{A})$ trivial on $\mathrm{Y}_{n,m}(k)$ since $\mathrm{Y}_{n,m} \subset \mathrm{N}_n$. The group $\mathrm{Y}_{n,m}$ is normalized by $\mathrm{GL}_{m+1} \subset \mathrm{GL}_n$ and the mirabolic subgroup $\mathrm{P}_{m+1} \subset \mathrm{GL}_{m+1}$ is the stabilizer in GL_{m+1} of the character ψ.

Definition *If $\varphi(g)$ is a cusp form on $\mathrm{GL}_n(\mathbb{A})$ define the projection operator $\mathbb{P}^n_m$ from cusp forms on $\mathrm{GL}_n(\mathbb{A})$ to cuspidal functions on $\mathrm{P}_{m+1}(\mathbb{A})$ by*

$$\mathbb{P}^n_m \varphi(p) = |\det(p)|^{-\left(\frac{n-m-1}{2}\right)} \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} p & \\ & I_{n-m-1} \end{pmatrix} \right) \psi^{-1}(y) \, dy$$

for $p \in \mathrm{P}_{m+1}(\mathbb{A})$.

As the integration is over a compact domain, the integral is absolutely convergent. We first analyze the behavior on $\mathrm{P}_{m+1}(\mathbb{A})$.

Lemma *The function $\mathbb{P}^n_m \varphi(p)$ is a cuspidal function on $\mathrm{P}_{m+1}(\mathbb{A})$.*

Proof: Let us let $\varphi'(p)$ denote the non-normalized projection, i.e., for $p \in \mathrm{P}_{m+1}(\mathbb{A})$ set

$$\varphi'(p) = |\det(p)|^{\left(\frac{n-m-1}{2}\right)} \mathbb{P}^n_m \varphi(p).$$

It suffices to show this function is cuspidal. Since $\varphi(g)$ was a smooth function on $\mathrm{GL}_n(\mathbb{A})$, $\varphi'(p)$ will remain smooth on $\mathrm{P}_{m+1}(\mathbb{A})$. To see that $\varphi'(p)$ is automorphic, let $\gamma \in \mathrm{P}_{m+1}(k)$. Then

$$\varphi'(\gamma p) = \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(y)\ dy.$$

Since $\gamma \in \mathrm{P}_{m+1}(k)$ and P_{m+1} normalizes $\mathrm{Y}_{n,m}$ and stabilizes ψ we may make the change of variable $y \mapsto \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} y \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ in this integral to obtain

$$\varphi'(\gamma p) = \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} y \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(y)\ dy.$$

Since $\varphi(g)$ is automorphic on $\mathrm{GL}_n(\mathbb{A})$ it is left invariant under $\mathrm{GL}_n(k)$ and we find that $\varphi'(\gamma p) = \varphi'(p)$ so that φ' is indeed automorphic on $\mathrm{P}_{m+1}(\mathbb{A})$.

We next need to see that φ' is cuspidal on $\mathrm{P}_{m+1}(\mathbb{A})$. To this end, let $\mathrm{U} \subset \mathrm{P}_{m+1}$ be the standard unipotent subgroup associated to the partition $(n_1, \dots, n_r)$ of $m+1$. Then we must compute the integral

$$\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du.$$

Inserting the definition of φ' we find

$$\begin{aligned} &\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du \\ &\quad = \int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \psi^{-1}(y)\ dy\ du. \end{aligned}$$

The group $\mathrm{U}' = \mathrm{U} \ltimes \mathrm{Y}_{n,m}$ is the standard unipotent subgroup of GL_n associated to the partition $(n_1, \dots, n_r, 1, \dots, 1)$ of n. We may decompose this group in a second manner. If we let U'' be the standard unipotent subgroup of GL_n associated to the partition $(n_1, \dots, n_r, n-m-1)$ of n and let $\widetilde{\mathrm{N}}_{n-m-1}$ be the subgroup of GL_n obtained by embedding N_{n-m-1} into GL_n by

$$n \mapsto \begin{pmatrix} I_{m+1} & 0 \\ 0 & n \end{pmatrix}$$

then $\mathrm{U}' = \widetilde{\mathrm{N}}_{n-m-1} \ltimes \mathrm{U}''$. If we extend the character ψ of $\mathrm{Y}_{m,n}$ to U' by making it trivial on U, then in the decomposition $\mathrm{U}' = \widetilde{\mathrm{N}}_{n-m-1} \ltimes \mathrm{U}''$, ψ is dependent only on the $\widetilde{\mathrm{N}}_{n-m-1}$ component and there it is the standard character ψ on N_{n-m-1}. Hence we may rearrange the integration to give

$$\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du$$
$$= \int_{\mathrm{N}_{n-m-1}(k)\backslash \mathrm{N}_{n-m-1}(\mathbb{A})} \int_{\mathrm{U}''(k)\backslash \mathrm{U}''(\mathbb{A})} \varphi\left(u''\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right)\ du''\psi^{-1}(n)\ dy.$$

But since φ is cuspidal on GL_n and U'' is a standard unipotent subgroup of GL_n then

$$\int_{\mathrm{U}''(k)\backslash \mathrm{U}''(\mathbb{A})} \varphi\left(u''\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right)\ du'' \equiv 0$$

from which it follows that

$$\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du \equiv 0$$

so that φ' is a cuspidal function on $\mathrm{P}_{m+1}(\mathbb{A})$. □

From Lecture 1, we know that cuspidal functions on $\mathrm{P}_{m+1}(\mathbb{A})$ have a Fourier expansion summed over $\mathrm{N}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})$. Applying this expansion to our projected cusp form on $\mathrm{GL}_n(\mathbb{A})$ we are led to the following result.

Lemma *Let φ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$. Then for $h \in \mathrm{GL}_m(\mathbb{A})$, $\mathbb{P}^n_m\varphi\begin{pmatrix}h & \\ & 1\end{pmatrix}$ has the Fourier expansion*

$$\mathbb{P}^n_m\varphi\begin{pmatrix}h & \\ & 1\end{pmatrix} = |\det(h)|^{-\left(\frac{n-m-1}{2}\right)} \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix}h & \\ & I_{n-m}\end{pmatrix}\right)$$

with convergence absolute and uniform on compact subsets.

Proof: Once again let

$$\varphi'(p) = |\det(p)|^{\left(\frac{n-m-1}{2}\right)}\mathbb{P}^n_m\varphi(p)$$

with $p \in \mathrm{P}_{m+1}(\mathbb{A})$. Since we have verified that $\varphi'(p)$ is a cuspidal function on $\mathrm{P}_{m+1}(\mathbb{A})$ we know that it has a Fourier expansion of the form

$$\varphi'(p) = \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_{\varphi'}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}p\right)$$

where

$$W_{\varphi'}(p) = \int_{N_{m+1}(k)\backslash \mathrm{N}_{m+1}(\mathbb{A})} \varphi'(np)\psi^{-1}(n)\ dn.$$

To obtain our expansion for $\mathbb{P}^n_m\varphi$ we need to express the right-hand side in terms of φ rather than φ'.

We have

$$\begin{aligned} W_{\varphi'}(p) &= \int_{N_{m+1}(k)\backslash \mathrm{N}_{m+1}(\mathbb{A})} \varphi'(n'p)\psi^{-1}(n')\ dn' \\ &= \int_{N_{m+1}(k)\backslash \mathrm{N}_{m+1}(\mathbb{A})} \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y\begin{pmatrix} n'p & 0 \\ 0 & 1\end{pmatrix} g\right) \cdot \\ &\qquad\qquad \cdot \psi^{-1}(y)\ dy\ \psi^{-1}(n')\ dn'. \end{aligned}$$

It is elementary to see that the maximal unipotent subgroup N_n of GL_n can be factored as $\mathrm{N}_n = \mathrm{N}_{m+1} \ltimes \mathrm{Y}_{n,m}$ and if we write $n = n'y$ with $n' \in \mathrm{N}_{m+1}$ and $y \in \mathrm{Y}_{n,m}$ then $\psi(n) = \psi(n')\psi(y)$. Hence the above integral may be written as

$$W_{\varphi'}(p) = \int_{N_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi\left(n\begin{pmatrix} p & 0 \\ 0 & I_{n-m-1}\end{pmatrix}\right)\psi^{-1}(n)\, dn = W_\varphi\begin{pmatrix} p & 0 \\ 0 & I_{n-m-1}\end{pmatrix}.$$

Substituting this expression into the above we find that

$$\begin{aligned} &\mathbb{P}^n_m\varphi\begin{pmatrix} h & \\ & 1\end{pmatrix} \\ &\quad = |\det(h)|^{-\left(\frac{n-m-1}{2}\right)} \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}\right) \end{aligned}$$

and the convergence is absolute and uniform for h in compact subsets of $\mathrm{GL}_m(\mathbb{A})$. □

2.2.2 The global integrals

We now have the prerequisites for writing down a family of Eulerian integrals for cusp forms φ on GL_n twisted by automorphic forms on GL_m for $m < n$. Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $\mathrm{GL}_m(\mathbb{A})$. (Actually, we could take φ' to be an arbitrary automorphic form on $\mathrm{GL}_m(\mathbb{A})$.) Consider the integrals

$$I(s;\varphi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m\varphi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\varphi'(h)|\det(h)|^{s-1/2}\ dh.$$

The integral $I(s;\varphi,\varphi')$ is absolutely convergent for all values of the complex parameter s, uniformly in compact subsets, since the cusp forms are rapidly decreasing. Hence it is entire and bounded in any vertical strip as before.

Let us now investigate the Eulerian properties of these integrals. We first replace $\mathbb{P}^n_m\varphi$ by its Fourier expansion.

$$\begin{aligned} I(s;\varphi,\varphi') &= \int_{\mathrm{GL}_m(k)\backslash\,\mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m\varphi\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\varphi'(h)|\det(h)|^{s-1/2}\,dh\\ &= \int_{\mathrm{GL}_m(k)\backslash\,\mathrm{GL}_m(\mathbb{A})}\ \sum_{\gamma\in \mathrm{N}_m(k)\backslash\,\mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\right)\cdot\\ &\qquad\qquad \cdot\varphi'(h)|\det(h)|^{s-(n-m)/2}\,dh. \end{aligned}$$

Since $\varphi'(h)$ is automorphic on $\mathrm{GL}_m(\mathbb{A})$ and $|\det(\gamma)|=1$ for $\gamma\in\mathrm{GL}_m(k)$ we may interchange the order of summation and integration for $\mathrm{Re}(s)>>0$ and then recombine to obtain

$$I(s;\varphi,\varphi') = \int_{\mathrm{N}_m(k)\backslash\,\mathrm{GL}_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\varphi'(h)|\det(h)|^{s-(n-m)/2}\,dh.$$

This integral is absolutely convergent for $\mathrm{Re}(s)>>0$ by the gauge estimates of [31, Section 13] and this justifies the interchange.

Let us now integrate first over $\mathrm{N}_m(k)\backslash\,\mathrm{N}_m(\mathbb{A})$. Recall that for $n\in \mathrm{N}_m(\mathbb{A})\subset\mathrm{N}_n(\mathbb{A})$ we have $W_\varphi(ng)=\psi(n)W_\varphi(g)$. Hence we have

$$\begin{aligned} &I(s;\varphi,\varphi') =\\ &\int_{\mathrm{N}_m(\mathbb{A})\backslash\,\mathrm{GL}_m(\mathbb{A})}\int_{N_m(k)\backslash\,\mathrm{N}_m(\mathbb{A})} W_\varphi\left(\begin{pmatrix} n & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\right)\cdot\\ &\qquad\qquad \cdot\varphi'(nh)\,dn\,|\det(h)|^{s-(n-m)/2}\,dh\\ &= \int_{\mathrm{N}_m(\mathbb{A})\backslash\,\mathrm{GL}_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\int_{\mathrm{N}_m(k)\backslash\,\mathrm{N}_m(\mathbb{A})}\psi(n)\varphi'(nh)\,dn\\ &\qquad\qquad \cdot|\det(h)|^{s-(n-m)/2}\,dh\\ &= \int_{\mathrm{N}_m(\mathbb{A})\backslash\,\mathrm{GL}_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}W'_{\varphi'}(h)|\det(h)|^{s-(n-m)/2}\,dh\\ &= \Psi(s;W_\varphi,W'_{\varphi'}) \end{aligned}$$

where $W'_{\varphi'}(h)$ is the ψ^{-1}-Whittaker function on $\mathrm{GL}_m(\mathbb{A})$ associated to φ', i.e.,

$$W'_{\varphi'}(h) = \int_{\mathrm{N}_m(k)\backslash\,\mathrm{N}_m(\mathbb{A})}\varphi'(nh)\psi(n)\,dn,$$

and we retain absolute convergence for $\mathrm{Re}(s) >> 0$.

From this point, the fact that the integrals are Eulerian is a consequence of the uniqueness of the Whittaker model for GL_n. Take φ a smooth cusp form in a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$. Assume in addition that φ is factorizable, i.e., in the decomposition $\pi = \otimes' \pi_v$ of π into a restricted tensor product of local representations, $\varphi = \otimes \varphi_v$ is a pure tensor. Then as we have seen there is a choice of local Whittaker models so that $W_\varphi(g) = \prod W_{\varphi_v}(g_v)$. Similarly for decomposable φ' we have the factorization $W'_{\varphi'}(h) = \prod W'_{\varphi'_v}(h_v)$.

If we substitute these factorizations into our integral expression, then since the domain of integration factors $\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A}) = \prod \mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)$ we see that our integral factors into a product of local integrals

$$\Psi(s; W_\varphi, W'_{\varphi'}) = \prod_v \int_{\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)} W_{\varphi_v}\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'_v}(h_v) \cdot \\ \cdot |\det(h_v)|_v^{s-(n-m)/2}\, dh_v.$$

If we denote the local integrals by

$$\Psi_v(s; W_{\varphi_v}, W'_{\varphi'_v}) = \int_{\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)} W_{\varphi_v}\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'_v}(h_v) \cdot \\ \cdot |\det(h_v)|_v^{s-(n-m)/2}\, dh_v,$$

which converges for $\mathrm{Re}(s) >> 0$ by the gauge estimate of [31, Prop. 2.3.6], we see that we now have a family of Eulerian integrals.

Now let us return to the question of a functional equation. As in the case of GL_2, the functional equation is essentially a consequence of the existence of the outer automorphism $g \mapsto \iota(g) = g^\iota = {}^t g^{-1}$ of GL_n. If we define the action of this automorphism on automorphic forms by setting $\widetilde{\varphi}(g) = \varphi(g^\iota) = \varphi(w_n g^\iota)$ and let $\widetilde{\mathbb{P}}^n_m = \iota \circ \mathbb{P}^n_m \circ \iota$ then our integrals naturally satisfy the functional equation

$$I(s; \varphi, \varphi') = \widetilde{I}(1-s; \widetilde{\varphi}, \widetilde{\varphi}')$$

where

$$\widetilde{I}(s; \varphi, \varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \widetilde{\mathbb{P}}^n_m \varphi \begin{pmatrix} h & \\ & 1\end{pmatrix} \varphi'(h) |\det(h)|^{s-1/2}\, dh.$$

We have established the following result.

Theorem 2.1 *Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then the family of integrals $I(s;\varphi,\varphi')$ define entire functions of s, bounded in vertical strips, and satisfy the functional equation*

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Moreover the integrals are Eulerian and if φ and φ' are factorizable, we have

$$I(s;\varphi,\varphi') = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

with convergence absolute and uniform for $\mathrm{Re}(s) >> 0$.

The integrals occurring on the right-hand side of our functional equation are again Eulerian. One can unfold the definitions to find first that

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'})$$

where the unfolded global integral is

$$\widetilde{\Psi}(s;W,W') = \int\int W\begin{pmatrix} h & & \\ x & I_{n-m-1} & \\ & & 1\end{pmatrix} dx\; W'(h)|\det(h)|^{s-(n-m)/2}\,dh$$

with the h integral over $\mathrm{N}_m(\mathbb{A})\backslash\mathrm{GL}_m(\mathbb{A})$ and the x integral over $M_{n-m-1,m}(\mathbb{A})$, the space of $(n-m-1)\times m$ matrices, ρ denoting right translation, and $w_{n,m}$ the Weyl element $w_{n,m} = \begin{pmatrix} I_m & \\ & w_{n-m}\end{pmatrix}$ with $w_{n-m} = \begin{pmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ the standard long Weyl element in GL_{n-m}. Also, for $W \in \mathcal{W}(\pi,\psi)$ we set $\widetilde{W}(g) = W(w_n g^\iota) \in \mathcal{W}(\widetilde{\pi},\psi^{-1})$. The extra unipotent integration is the remnant of $\mathbb{P}^n_m$. As before, $\widetilde{\Psi}(s;W,W')$ is absolutely convergent for $\mathrm{Re}(s) >> 0$. For φ and φ' factorizable as before, these integrals $\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'})$ will factor as well. Hence we have

$$\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

where

$$\widetilde{\Psi}_v(s;W_v,W'_v) = \int\int W_v\begin{pmatrix} h_v & & \\ x_v & I_{n-m-1} & \\ & & 1\end{pmatrix} dx_v\; W'_v(h_v)|\det(h_v)|^{s-(n-m)/2}\,dh_v$$

where now the h_v integral is over $\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)$ and the x_v integral is over the matrix space $M_{n-m-1,m}(k_v)$. Thus, coming back to our functional equation, we find that the right-hand side is Eulerian and factors as

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi},\widetilde{W}'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v}).$$

2.3 Eulerian Integrals for $\mathrm{GL}_n \times \mathrm{GL}_n$

The paradigm for integral representations of L-functions for $\mathrm{GL}_n \times \mathrm{GL}_n$ is not Hecke but rather the classical papers of Rankin [52] and Selberg [54]. These were first interpreted in the framework of automorphic representations by Jacquet for $\mathrm{GL}_2 \times \mathrm{GL}_2$ [28] and then Jacquet and Shalika in general [36].

Let (π, V_π) and $(\pi', V_{\pi'})$ be two cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ be two cusp forms. The analogue of the construction above would be simply

$$\int_{\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)|\det(g)|^s\, dg.$$

This integral is essentially the L^2-inner product of φ and φ' and is not suitable for defining an L-function, although it will occur as a residue of our integral at a pole. Instead, following Rankin and Selberg, we use an integral representation that involves a third function: an Eisenstein series on $\mathrm{GL}_n(\mathbb{A})$. This family of Eisenstein series is constructed using the mirabolic subgroup once again.

2.3.1 The mirabolic Eisenstein series

To construct our Eisenstein series we return to the observation that $\mathrm{P}_n \backslash \mathrm{GL}_n \simeq k^n - \{0\}$. If we let $\mathcal{S}(\mathbb{A}^n)$ denote the Schwartz–Bruhat functions on $\mathbb{A}^n$, then each $\Phi \in \mathcal{S}$ defines a smooth function on $\mathrm{GL}_n(\mathbb{A})$, left invariant by $\mathrm{P}_n(\mathbb{A})$, by $g \mapsto \Phi((0,\ldots,0,1)g) = \Phi(e_n g)$. Let η be a unitary idele class character. (For our application η will be determined by the central characters of π and π'.) Consider the function

$$F(g,\Phi;s,\eta) = |\det(g)|^s \int_{\mathbb{A}^\times} \Phi(ae_n g)|a|^{ns}\eta(a)\, d^\times a.$$

If we let $\mathrm{P}'_n = \mathrm{Z}_n \mathrm{P}_n$ be the parabolic of GL_n associated to the partition $(n-1,1)$ then one checks that for $p' = \begin{pmatrix} h & y \\ 0 & d \end{pmatrix} \in \mathrm{P}'_n(\mathbb{A})$ with $h \in \mathrm{GL}_{n-1}(\mathbb{A})$

and $d \in \mathbb{A}^\times$ we have,

$$\begin{aligned} F(p'g, \Phi; s, \eta) &= |\det(h)|^s |d|^{-(n-1)s} \eta(d)^{-1} F(g, \Phi; s, \eta) \\ &= \delta^s_{\mathrm{P}'_n}(p') \eta^{-1}(d) F(g, \Phi; s, \eta), \end{aligned}$$

with the integral absolutely convergent for $\mathrm{Re}(s) > 1/n$, so that if we extend η to a character of P'_n by $\eta(p') = \eta(d)$ in the above notation we have that $F(g, \Phi; s, \eta)$ is a smooth section of the normalized induced representation $\mathrm{Ind}_{\mathrm{P}'_n(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\delta^{s-1/2}_{\mathrm{P}'_n}\eta)$. Since the inducing character $\delta^{s-1/2}_{\mathrm{P}'_n}\eta$ of $\mathrm{P}'_n(\mathbb{A})$ is invariant under $\mathrm{P}'_n(k)$ we may form Eisenstein series from this family of sections by

$$E(g, \Phi; s, \eta) = \sum_{\gamma \in \mathrm{P}'_n(k) \backslash \mathrm{GL}_n(k)} F(\gamma g, \Phi; s, \eta).$$

If we replace F in this sum by its definition we can rewrite this Eisenstein series as

$$\begin{aligned} E(g, \Phi; s, \eta) &= |\det(g)|^s \int_{k^\times \backslash \mathbb{A}^\times} \sum_{\xi \in k^n - \{0\}} \Phi(a\xi g) |a|^{ns} \eta(a) \ d^\times a \\ &= |\det(g)|^s \int_{k^\times \backslash \mathbb{A}^\times} \Theta'_\Phi(a, g) |a|^{ns} \eta(a) \ d^\times a \end{aligned}$$

and this first expression is convergent absolutely for $\mathrm{Re}(s) > 1$ [36].

The second expression essentially gives the Eisenstein series as the Mellin transform of the Theta series

$$\Theta_\Phi(a, g) = \sum_{\xi \in k^n} \Phi(a\xi g),$$

where in the above we have written

$$\Theta'_\Phi(a, g) = \sum_{\xi \in k^n - \{0\}} \Phi(a\xi g) = \Theta_\Phi(a, g) - \Phi(0).$$

This allows us to obtain the analytic properties of the Eisenstein series from the Poisson summation formula for Θ_Φ, namely

$$\begin{aligned} \Theta_\Phi(a, g) &= \sum_{\xi \in k^n} \Phi(a\xi g) = \sum_{\xi \in k^n} \Phi_{a,g}(\xi) \\ &= \sum_{\xi \in k^n} \widehat{\Phi_{a,g}}(\xi) = \sum_{\xi \in k^n} |a|^{-n} |\det(g)|^{-1} \widehat{\Phi}(a^{-1} \xi {}^t g^{-1}) \\ &= |a|^{-n} |\det(g)|^{-1} \Theta_{\hat{\Phi}}(a^{-1}, {}^t g^{-1}) \end{aligned}$$

where the Fourier transform $\hat{\Phi}$ on $\mathcal{S}(\mathbb{A}^n)$ is defined by

$$\hat{\Phi}(x) = \int_{\mathbb{A}^\times} \Phi(y)\psi(y\,{}^t x)\ dy.$$

This allows us to write the Eisenstein series as

$$\begin{aligned} E(g,\Phi,s,\eta) = |\det(g)|^s & \int_{|a|\geq 1} \Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a \\ & + |\det(g)|^{s-1}\int_{|a|\geq 1} \Theta'_{\hat{\Phi}}(a,{}^t g^{-1})|a|^{n(1-s)}\eta^{-1}(a)\ d^\times a + \delta(s) \end{aligned}$$

where

$$\delta(s) = \begin{cases} 0 & \text{if } \eta \text{ is ramified} \\ -c\Phi(0)\frac{|\det(g)|^s}{s+i\sigma} + c\hat{\Phi}(0)\frac{|\det(g)|^{s-1}}{s-1+i\sigma} & \text{if } \eta(a) = |a|^{in\sigma} \text{ with } \sigma \in \mathbb{R} \end{cases}$$

with c a non-zero constant. From this we easily derive the basic properties of our Eisenstein series [36, Section 4].

Proposition 2.1 *The Eisenstein series $E(g,\Phi;s,\eta)$ has a meromorphic continuation to all of $\mathbb{C}$ with at most simple poles at $s = -i\sigma, 1-i\sigma$ when η is unramified of the form $\eta(a) = |a|^{in\sigma}$. As a function of g it is smooth of moderate growth and as a function of s it is bounded in vertical strips (away from the possible poles), uniformly for g in compact sets. Moreover, we have the functional equation*

$$E(g,\Phi;s,\eta) = E(g^\iota,\hat{\Phi};1-s,\eta^{-1})$$

where $g^\iota = {}^t g^{-1}$.

Note that under the center the Eisenstein series transforms by the central character η^{-1}.

2.3.2 The global integrals

Now let us return to our Eulerian integrals. Let π and π' be our irreducible cuspidal representations. Let their central characters be ω and ω'. Set $\eta = \omega\omega'$. Then for each pair of cusp forms $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ and each Schwartz-Bruhat function $\Phi \in \mathcal{S}(\mathbb{A}^n)$ set

$$I(s;\varphi,\varphi',\Phi) = \int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\ dg.$$

Since the two cusp forms are rapidly decreasing on $Z_n(\mathbb{A})\,GL_n(k)\backslash GL_n(\mathbb{A})$ and the Eisenstein series is only of moderate growth, we see that the integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic. It will be bounded in vertical strips away from the poles and satisfies the functional equation

$$I(s;\varphi,\varphi',\Phi)=I(1-s;\widetilde{\varphi},\widetilde{\varphi}',\hat{\Phi}),$$

coming from the functional equation of the Eisenstein series, where we still have $\widetilde{\varphi}(g)=\varphi(g^\iota)=\varphi(w_n g^\iota)\in V_{\widetilde{\pi}}$ and similarly for $\widetilde{\varphi}'$.

These integrals will be entire unless we have $\eta(a)=\omega(a)\omega'(a)=|a|^{in\sigma}$ is unramified. In that case, the residue at $s=-i\sigma$ will be

$$\operatorname*{Res}_{s=-i\sigma} I(s;\varphi,\varphi',\Phi)=-c\Phi(0)\int_{Z_n(\mathbb{A})\,GL_n(\mathbb{A})\backslash GL_n(\mathbb{A})}\varphi(g)\varphi'(g)|\det(g)|^{-i\sigma}\;dg$$

and at $s=1-i\sigma$ we can write the residue as

$$\operatorname*{Res}_{s=1-i\sigma} I(s;\varphi,\varphi',\Phi)=c\hat{\Phi}(0)\int_{Z_n(\mathbb{A})\,GL_n(k)\backslash GL_n(\mathbb{A})}\widetilde{\varphi}(g)\widetilde{\varphi}'(g)|\det(g)|^{i\sigma}\;dg.$$

Therefore these residues define $GL_n(\mathbb{A})$ invariant pairings between π and $\pi'\otimes|\det|^{-i\sigma}$ or equivalently between $\widetilde{\pi}$ and $\widetilde{\pi}'\otimes|\det|^{i\sigma}$. Hence a residues can be non-zero only if $\pi\simeq\widetilde{\pi}'\otimes|\det|^{i\sigma}$ and in this case we can find φ, φ', and Φ such that indeed the residue does not vanish.

We have yet to check that our integrals are Eulerian. To this end we take the integral, replace the Eisenstein series by its definition, and unfold:

$$\begin{aligned}
I(s;\varphi,\varphi',\Phi)&=\int_{Z_n(\mathbb{A})\,GL_n(k)\backslash GL_n(\mathbb{A})}\varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\;dg\\
&=\int_{Z_n(\mathbb{A})\,P'_n(k)\backslash GL_n(\mathbb{A})}\varphi(g)\varphi'(g)F(g,\Phi;s,\eta)\;dg\\
&=\int_{Z_n(\mathbb{A})\,P_n(k)\backslash GL_n(\mathbb{A})}\varphi(g)\varphi'(g)|\det(g)|^s\int_{\mathbb{A}^\times}\Phi(ae_ng)|a|^{ns}\eta(a)\;da\;dg\\
&=\int_{P_n(k)\backslash GL_n(\mathbb{A})}\varphi(g)\varphi'(g)\Phi(e_ng)|\det(g)|^s\;dg.
\end{aligned}$$

We next replace φ by its Fourier expansion in the form

$$\varphi(g)=\sum_{\gamma\in N_n(k)\backslash P_n(k)}W_\varphi(\gamma g)$$

and unfold to find

$$\begin{aligned} I(s;\varphi,\varphi',\Phi) &= \int_{\mathrm{N}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} W_\varphi(g)\varphi'(g)\Phi(e_ng)|\det(g)|^s\,dg \\ &= \int_{\mathrm{N}_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} W_\varphi(g)\int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi'(ng)\psi(n)\,dn\;\Phi(e_ng)|\det(g)|^s\,dg \\ &= \int_{\mathrm{N}_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} W_\varphi(g)W'_{\varphi'}(g)\Phi(e_ng)|\det(g)|^s\,dg \\ &= \Psi(s;W_\varphi,W'_{\varphi'},\Phi). \end{aligned}$$

This expression converges for $\mathrm{Re}(s) >> 0$ by the gauge estimates as before.

To continue, we assume that φ, φ' and Φ are decomposable tensors under the isomorphisms $\pi \simeq \otimes'\pi_v$, $\pi' \simeq \otimes'\pi'_v$, and $\mathcal{S}(\mathbb{A}^n) \simeq \otimes'\mathcal{S}(k_v^n)$ so that we have $W_\varphi(g) = \prod_v W_{\varphi_v}(g_v)$, $W'_{\varphi'}(g) = \prod_v W'_{\varphi'_v}(g_v)$ and $\Phi(g) = \prod_v \Phi_v(g_v)$. Then, since the domain of integration also naturally factors we can decompose this last integral into an Euler product and now write

$$\Psi(s;W_\varphi,W'_{\varphi'},\Phi) = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v),$$

where

$$\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v) = \int_{\mathrm{N}_n(k_v)\backslash \mathrm{GL}_n(k_v)} W_{\varphi_v}(g_v)W'_{\varphi'_v}(g_v)\Phi_v(e_ng_v)|\det(g_v)|^s\,dg_v,$$

still with convergence for $\mathrm{Re}(s) >> 0$ by the local gauge estimates. Once again we see that the Euler factorization is a direct consequence of the uniqueness of the Whittaker models.

Theorem 2.2 *Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ cusp forms on* $\mathrm{GL}_n(\mathbb{A})$ *and let $\Phi \in \mathcal{S}(\mathbb{A}^n)$. Then the family of integrals $I(s;\varphi,\varphi',\Phi)$ define meromorphic functions of s, bounded in vertical strips away from the poles. The only possible poles are simple and occur iff $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ with σ real and are then at $s = -i\sigma$ and $s = 1 - i\sigma$ with residues as above. They satisfy the functional equation*

$$I(s;\varphi,\varphi',\Phi) = I(1-s;\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'},\hat{\Phi}).$$

Moreover, for φ, φ', and Φ factorizable we have that the integrals are Eulerian and we have

$$I(s;\varphi,\varphi',\Phi) = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)$$

with convergence absolute and uniform for $\mathrm{Re}(s) >> 0$.

We remark in passing that the right-hand side of the functional equation also unfolds as

$$\begin{aligned} I(1-s;\widetilde{\varphi},\widetilde{\varphi}',\hat{\Phi}) &= \int_{\mathrm{N}_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \widetilde{W}_{\varphi}(g)\widetilde{W}'_{\varphi'}(g)\hat{\Phi}(e_n g)|\det(g)|^{1-s}\, dg \\ &= \prod_v \Psi_v(1-s;\widetilde{W}_{\varphi},\widetilde{W}'_{\varphi'},\hat{\Phi}) \end{aligned}$$

with convergence for $\mathrm{Re}(s) << 0$.

We note again that if these integrals are not entire, then the residues give us invariant pairings between the cuspidal representations and hence tell us structural facts about the relation between these representations.

3 Local L-functions

If (π, V_π) is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ we have associated to the pair (π,π') a family of Eulerian integrals $\{I(s;\varphi,\varphi')\}$ (or $\{I(s;\varphi,\varphi',\Phi)\}$ if $m=n$) and through the Euler factorization we have for each place v of k a family of local integrals $\{\Psi_v(s;W_v,W'_v)\}$ (or $\{\Psi_v(s;W_v,W'_v,\Phi_v)\}$) attached to the pair of local components (π_v,π'_v). In this lecture we would like to attach a local L-function (or local Euler factor) $L(s,\pi_v\times\pi'_v)$ to such a pair of local representations through the family of local integrals and analyze its basic properties, including the local functional equation. The paradigm for such an analysis of local L-functions is Tate's thesis [64]. The mechanics of the archimedean and non-archimedean theories are slightly different so we will treat them separately, beginning with the non-archimedean theory.

3.1 The Non-archimedean Local Factors

For this section we will let k denote a non-archimedean local field. We will let $\mathfrak{o}$ denote the ring of integers of k and $\mathfrak{p}$ the unique prime ideal of $\mathfrak{o}$. Fix a generator ϖ of $\mathfrak{p}$. We let q be the residue degree of k, so $q=|\mathfrak{o}/\mathfrak{p}|=|\varpi|^{-1}$. We fix a non-trivial continuous additive character ψ of k. (π,V_π) and $(\pi',V_{\pi'})$ will now be the smooth vectors in irreducible admissible unitary generic representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$ respectively, as is true for local components of cuspidal representations. We will let ω and ω' denote their central characters.

The basic reference for this section is the paper of Jacquet, Piatetski-Shapiro, and Shalika [33].

3.1.1 The local L-function

For each pair of Whittaker functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$ and in the case $n = m$ each Schwartz-Bruhat function $\Phi \in \mathcal{S}(k^n)$ we have defined local integrals

$$\Psi(s; W, W') = \int_{\mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'(h) |\det(h)|^{s-(n-m)/2} \, dh$$

$$\widetilde{\Psi}(s; W, W') = \int_{\mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} \int_{M_{n-m-1,m}(k)} W \begin{pmatrix} h & & \\ x & I_{n-m-1} & \\ & & 1 \end{pmatrix} dx \; W'(h) |\det(h)|^{s-(n-m)/2} \, dh$$

in the case $m < n$ and

$$\Psi(s; W, W', \Phi) = \int_{\mathrm{N}_n(k)\backslash \mathrm{GL}_n(k)} W(g) W'(g) \Phi(e_n g) |\det(g)|^s \, dg$$

in the case $n = m$, both convergent for $\mathrm{Re}(s) >> 0$. To make the notation more convenient for what follows, in the case $m < n$ for any $0 \leq j \leq n-m-1$ let us set

$$\Psi_j(s : W, W') = \int_{\mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} \int_{M_{j,m}(k)} W \begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} dx \; W'(h) |\det(h)|^{s-(n-m)/2} \, dh,$$

so that $\Psi(s; W, W') = \Psi_0(s; W, W')$ and $\widetilde{\Psi}(s; W, W') = \Psi_{n-m-1}(s; W, W')$, which is still absolutely convergent for $\mathrm{Re}(s) >> 0$.

We need to understand what type of functions of s these local integrals are. To this end, we need to understand the local Whittaker functions. So let $W \in \mathcal{W}(\pi, \psi)$. Since W is smooth, there is a compact open subgroup K, of finite index in the maximal compact subgroup $\mathrm{K}_n = \mathrm{GL}_n(\mathfrak{o})$, so that $W(gk) = W(g)$ for all $k \in K$. If we let $\{k_i\}$ be a set of coset representatives of $\mathrm{GL}_n(\mathfrak{o})/K$, using that W transforms on the left under $\mathrm{N}_n(k)$ via ψ and the Iwasawa decomposition on $\mathrm{GL}_n(k)$ we see that $W(g)$ is completely determined by the values of $W(ak_i) = W_i(a)$ for $a \in \mathrm{A}_n(k)$, the maximal split (diagonal) torus of $\mathrm{GL}_n(k)$. So it suffices to understand a general Whittaker function on the torus. Let α_i, $i = 1, \ldots, n-1$, denote the standard simple roots of GL_n, so that if $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{A}_n(k)$ then $\alpha_i(a) = a_i/a_{i+1}$. By a finite function on $\mathrm{A}_n(k)$ we mean a continuous function whose translates span a finite dimensional vector space [30, 31, Section 2.2]. (For the field $k^\times$ itself the finite functions are spanned by products of characters and powers of the valuation map.) The fundamental result on the asymptotics of Whittaker functions is then the following [31, Prop. 2.2].

Proposition 3.1 *Let π be a generic representation of $\mathrm{GL}_n(k)$. Then there is a finite set of finite functions $X(\pi) = \{\chi_i\}$ on $\mathrm{A}_n(k)$, depending only on π, so that for every $W \in \mathcal{W}(\pi, \psi)$ there are Schwartz–Bruhat functions $\phi_i \in \mathcal{S}(k^{n-1})$ such that for all $a \in \mathrm{A}_n(k)$ with $a_n = 1$ we have*

$$W(a) = \sum_{X(\pi)} \chi_i(a)\phi_i(\alpha_1(a), \ldots, \alpha_{n-1}(a)).$$

The finite set of finite functions $X(\pi)$ which occur in the asymptotics near 0 of the Whittaker functions come from analyzing the Jacquet module $\mathcal{W}(\pi, \psi)/\langle \pi(n)W - W | n \in \mathrm{N}_n \rangle$ which is naturally an $\mathrm{A}_n(k)$–module. Note that due to the Schwartz-Bruhat functions, the Whittaker functions vanish whenever any simple root $\alpha_i(a)$ becomes large. The gauge estimates alluded to in Section 2 are a consequence of this expansion and the one in Proposition 3.6.

Several nice consequences follow from inserting these formulas for W and W' into the local integrals $\Psi_j(s; W, W')$ or $\Psi(s; W, W', \Phi)$ [31, 33].

Proposition 3.2 *The local integrals $\Psi_j(s; W, W')$ or $\Psi(s; W, W', \Phi)$ satisfy the following properties.*

1. *Each integral converges for $\mathrm{Re}(s) >> 0$. For π and π' unitary, as we have assumed, they converge absolutely for $\mathrm{Re}(s) \geq 1$. For π and π' tempered, we have absolute convergence for $\mathrm{Re}(s) > 0$.*
2. *Each integral defines a rational function in q^{-s} and hence meromorphically extends to all of $\mathbb{C}$.*
3. *Each such rational function can be written with a common denominator which depends only on the finite functions $X(\pi)$ and $X(\pi')$ and hence only on π and π'.*

In deriving these when $m < n-1$ note that one has that

$$W\begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j-1}\end{pmatrix} \neq 0$$

implies that x lies in a compact set independent of $h \in \mathrm{GL}_m(k)$ [33].

Let $\mathcal{I}_j(\pi, \pi')$ denote the complex linear span of the local integrals $\Psi_j(s; W, W')$ if $m < n$ and $\mathcal{I}(\pi, \pi')$ the complex linear span of the $\Psi(s; W, W', \Phi)$ if $m = n$. These are then all subspaces of $\mathbb{C}(q^{-s})$ which have "bounded denominators" in the sense of (3). In fact, these subspaces have more structure – they are modules for $\mathbb{C}[q^s, q^{-s}] \subset \mathbb{C}(q^{-s})$. To see this, note that for any $h \in \mathrm{GL}_m(k)$ we have

$$\Psi_j\left(s; \pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W, \pi'(h)W'\right) = |\det(h)|^{-s-j+(n-m)/2}\Psi_j(s; W, W')$$

and

$$\Psi(s; \pi(h)W, \pi'(h)W', \rho(h)\Phi) = |\det(h)|^{-s}\Psi(s; W, W', \Phi).$$

So by varying h and multiplying by scalars, we see that each $\mathcal{I}_j(\pi, \pi')$ and $\mathcal{I}(\pi, \pi')$ is closed under multiplication by $\mathbb{C}[q^s, q^{-s}]$. Since we have bounded denominators, we can conclude:

Proposition 3.3 *Each $\mathcal{I}_j(\pi, \pi')$ and $\mathcal{I}(\pi, \pi')$ is a fractional $\mathbb{C}[q^s, q^{-s}]$–ideal of $\mathbb{C}(q^{-s})$.*

Note that $\mathbb{C}[q^s, q^{-s}]$ is a principal ideal domain, so that each fractional ideal $\mathcal{I}_j(\pi, \pi')$ has a single generator, which we call $Q_{j,\pi,\pi'}(q^{-s})$, as does $\mathcal{I}(\pi, \pi')$, which we call $Q_{\pi,\pi'}(q^{-s})$. However, we can say more. In the case $m < n$ recall that from what we have said about the Kirillov model that when we restrict Whittaker functions in $\mathcal{W}(\pi, \psi)$ to the embedded $\mathrm{GL}_m(k) \subset \mathrm{P}_n(k)$ we get all functions of compact support on $\mathrm{GL}_m(k)$ transforming by ψ. Using this freedom for our choice of $W \in \mathcal{W}(\pi, \psi)$ one can show that in fact the constant function 1 lies in $\mathcal{I}_j(\pi, \pi')$. In the case $m = n$ one can reduce to a sum of integrals over $\mathrm{P}_n(k)$ and then use the freedom one has in the Kirillov model, plus the complete freedom in the choice of Φ to show that once again $1 \in \mathcal{I}(\pi, \pi')$. The consequence of this is that our generator can be taken to be of the form $Q_{j,\pi,\pi'}(q^{-s}) = P_{j,\pi,\pi'}(q^s, q^{-s})^{-1}$ for $m < n$ or $Q_{\pi,\pi'}(q^{-s}) = P_{\pi,\pi'}(q^s, q^{-s})^{-1}$ for appropriate polynomials in $\mathbb{C}[q^s, q^{-s}]$.

Moreover, since q^s and q^{-s} are units in $\mathbb{C}[q^s, q^{-s}]$ we can always normalize our generator to be of the form $P_{j,\pi,\pi'}(q^{-s})^{-1}$ or $P_{\pi,\pi'}(q^{-s})^{-1}$ where the polynomial $P(X)$ satisfies $P(0) = 1$.

Finally, in the case $m < n$ one can show by a rather elementary although somewhat involved manipulation of the integrals that all of the ideals $\mathcal{I}_j(\pi, \pi')$ are the same [33, Section 2.7]. We will write this ideal as $\mathcal{I}(\pi, \pi')$ and its generator as $P_{\pi,\pi'}(q^{-s})^{-1}$.

This gives us the definition of our local L-function.

Definition *Let π and π' be as above. Then $L(s, \pi \times \pi') = P_{\pi,\pi'}(q^{-s})^{-1}$ is the normalized generator of the fractional ideal $\mathcal{I}(\pi, \pi')$ formed by the family of local integrals. If $\pi' = \mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(k)$ then we write $L(s, \pi) = L(s, \pi \times \mathbf{1})$.*

One can easily show that the ideal $\mathcal{I}(\pi, \pi')$ is independent of the character ψ used in defining the Whittaker models, so that $L(s, \pi \times \pi')$ is independent of the choice of ψ. So it is not included in the notation. Also, note that for $\pi' = \chi$ an automorphic representation (character) of $\mathrm{GL}_1(\mathbb{A})$ we have the identity $L(s, \pi \times \chi) = L(s, \pi \otimes \chi)$ where $\pi \otimes \chi$ is the representation of $\mathrm{GL}_n(\mathbb{A})$ on V_π given by $\pi \otimes \chi(g)\xi = \chi(\det(g))\pi(g)\xi$.

We summarize the above in the following Theorem.

Theorem 3.1 *Let π and π' be as above. The family of local integrals form a $\mathbb{C}[q^s, q^{-s}]$–fractional ideal $\mathcal{I}(\pi, \pi')$ in $\mathbb{C}(q^{-s})$ with generator the local L-function $L(s, \pi \times \pi')$.*

Another useful way of thinking of the local L-function is the following. $L(s, \pi \times \pi')$ is the minimal (in terms of degree) function of the form $P(q^{-s})^{-1}$, with $P(X)$ a polynomial satisfying $P(0) = 1$, such that the ratios $\dfrac{\Psi(s; W, W')}{L(s, \pi \times \pi')}$ (resp. $\dfrac{\Psi(s; W, W', \Phi)}{L(s, \pi \times \pi')}$) are entire for all $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$, and if necessary $\Phi \in \mathcal{S}(k^n)$. That is, $L(s, \pi \times \pi')$ is the standard Euler factor determined by the poles of the functions in $\mathcal{I}(\pi, \pi')$.

One should note that since the L-factor is a generator of the ideal $\mathcal{I}(\pi, \pi')$, then in particular it lies in $\mathcal{I}(\pi, \pi')$. Since this ideal is spanned by our local integrals, we have the following useful Corollary.

Corollary *There is a finite collection of $W_i \in \mathcal{W}(\pi, \psi)$, $W_i' \in \mathcal{W}(\pi', \psi^{-1})$, and if necessary $\Phi_i \in \mathcal{S}(k^n)$ such that*

$$L(s, \pi \times \pi') = \sum_i \Psi(s; W_i, W_i') \quad \text{or} \quad L(s, \pi \times \pi') = \sum_i \Psi(s; W_i, W_i', \Phi_i).$$

For future reference, let us set

$$\begin{aligned} e(s;W,W') &= \frac{\Psi(s;W,W')}{L(s,\pi\times\pi')}, \quad e_j(s;W,W') = \frac{\Psi_j(s;W,W')}{L(s,\pi\times\pi')}, \\ \tilde{e}(s;W,W') &= \frac{\widetilde{\Psi}(s;W,W')}{L(s,\pi\times\pi')}, \end{aligned}$$

and

$$e(s;W,W',\Phi) = \frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')}.$$

Then all of these functions are Laurent polynomials in $q^{\pm s}$, i.e., elements of $\mathbb{C}[q^s,q^{-s}]$. As such they are entire and bounded in vertical strips. As above, there are choices of W_i, W_i', and if necessary Φ_i such that $\sum e(s;W_i,W_i')\equiv 1$ or $\sum e(s;W_i,W_i',\Phi_i)\equiv 1$. In particular we have the following result.

Corollary *The functions $e(s;W,W')$ and $e(s;W,W',\Phi)$ are entire functions, bounded in vertical strips, and for each $s_0\in\mathbb{C}$ there is a choice of W, W', and if necessary Φ such that $e(s_0;W,W')\neq 0$ or $e(s_0;W,W',\Phi)\neq 0$.*

3.1.2 The local functional equation

Either by analogy with Tate's thesis or from the corresponding global statement, we would expect our local integrals to satisfy a local functional equation. From the functional equations for our global integrals, we would expect these to relate the integrals $\Psi(s;W,W')$ and $\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')$ when $m<n$ and $\Psi(s;W,W',\Phi)$ and $\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})$ when $m=n$. This will indeed be the case. These functional equations will come from interpreting the local integrals as families (in s) of quasi-invariant bilinear forms on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$ or trilinear forms on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})\times\mathcal{S}(k^n)$ depending on the case.

First, consider the case when $m<n$. In this case we have seen that

$$\Psi\left(s;\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}W,\pi'(h)W'\right) = |\det(h)|^{-s+(n-m)/2}\Psi(s;W,W')$$

and one checks that $\Psi(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')$ has the same quasi-invariance as a bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$. In addition, if we let $\mathrm{Y}_{n,m}$

denote the unipotent radical of the standard parabolic subgroup associated to the partition $(m+1, 1, \ldots, 1)$ as before then we have the quasi-invariance

$$\Psi(s; \pi(y)W, W') = \psi(y)\Psi(s; W, W')$$

for all $y \in \mathrm{Y}_{n,m}$. One again checks that $\widetilde{\Psi}(1-s; \rho(w_{n,m})\widetilde{W}, \widetilde{W'})$ satisfies the same quasi-invariance as a bilinear form on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \psi^{-1})$.

For $n = m$ we have seen that

$$\Psi(s; \pi(h)W, \pi'(h)W', \rho(h)\Phi) = |\det(h)|^{-s}\Psi(s; W, W', \Phi)$$

and it is elementary to check that $\Psi(1-s; \widetilde{W}, \widetilde{W}', \hat{\Phi})$ satisfies the same quasi-invariance as a trilinear form on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \psi^{-1}) \times \mathcal{S}(k^n)$. Our local functional equations will now follow from the following result [33, Propositions 2.10 and 2.11].

Proposition 3.4 *(i) If $m < n$, then except for a finite number of exceptional values of q^{-s} there is a unique bilinear form B_s on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \psi^{-1})$ satisfying*

$$B_s\left(\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W, \pi'(h)W'\right) = |\det(h)|^{-s+(n-m)/2} B_s(W, W')$$
$$\text{and} \quad B_s(\pi(y)W, W') = \psi(y)B_s(W, W')$$

for all $h \in \mathrm{GL}_m(k)$ and $y \in Y_{n,m}(k)$.

(ii) If $n = m$, then except for a finite number of exceptional values of q^{-s} there is a unique trilinear form T_s on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \psi^{-1}) \times \mathcal{S}(k^n)$ satisfying

$$T_s(\pi(h)W, \pi'(h)W', \rho(h)\Phi) = |\det(h)|^{-s}T_s(W, W', \Phi)$$

for all $h \in \mathrm{GL}_n(k)$.

Let us say a few words about the proof of this proposition, because it is another application of the analysis of the restriction of representations of GL_n to the mirabolic subgroup P_n [33, Sections 2.10 and 2.11]. In the case where $m < n$ the local integrals involve the restriction of the Whittaker functions in $\mathcal{W}(\pi, \psi)$ to $\mathrm{GL}_m(k) \subset \mathrm{P}_n$, that is, the Kirillov model $\mathcal{K}(\pi, \psi)$ of π. In the case $m = n$ one notes that $\mathcal{S}_0(k^n) = \{\Phi \in \mathcal{S}(k^n) \mid \Phi(0) = 0\}$, which has co-dimension one in $\mathcal{S}(k^n)$, is isomorphic to the compactly induced representation $\mathrm{ind}_{\mathrm{P}_n(k)}^{\mathrm{GL}_n(k)}(\delta_{\mathrm{P}_n}^{-1/2})$ so that by Frobenius reciprocity a $\mathrm{GL}_n(k)$ quasi-invariant trilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})\times\mathcal{S}_0(k^n)$ reduces to a $\mathrm{P}_n(k)$-quasi-invariant bilinear form on $\mathcal{K}(\pi,\psi)\times\mathcal{K}(\pi',\psi^{-1})$. So in both cases we are naturally working in the restriction to $\mathrm{P}_n(k)$. The restrictions of irreducible representations of $\mathrm{GL}_n(k)$ to $\mathrm{P}_n(k)$ are no longer irreducible, but do have composition series of finite length. One of the tools for analyzing the restrictions of representations of GL_n to P_n, or analyzing the irreducible representations of P_n, are the *derivatives* of Bernstein and Zelevinsky [2,11]. These derivatives $\pi^{(n-r)}$ are naturally representations of $\mathrm{GL}_r(k)$ for $r\leq n$. $\pi^{(0)}=\pi$ and since π is generic the highest derivative $\pi^{(n)}$ corresponds to the irreducible common submodule (τ,V_τ) of all Kirillov models, and is hence the non-zero irreducible representation of $\mathrm{GL}_0(k)$. The poles of our local integrals can be interpreted as giving quasi-invariant pairings between derivatives of π and π' [11]. The s for which such pairings exist for all but the highest derivatives are the exceptional s of the proposition. There is always a unique pairing between the highest derivatives $\pi^{(n)}$ and $\pi'^{(m)}$, which are necessarily non-zero since they correspond to the common irreducible subspace (τ,V_τ) of any Kirillov model, and this is the unique B_s or T_s of the proposition.

As a consequence of this Proposition, we can define the local γ-factor which gives the local functional equation for our integrals.

Theorem 3.2 *There is a rational function* $\gamma(s,\pi\times\pi',\psi)\in\mathbb{C}(q^{-s})$ *such that we have*

$$\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')=\omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s;W,W')\qquad \textit{if } m<n$$

or

$$\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})=\omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s;W,W',\Phi)\qquad \textit{if } m=n$$

for all $W\in\mathcal{W}(\pi,\psi)$*,* $W'\in\mathcal{W}(\pi',\psi^{-1})$*, and if necessary all* $\Phi\in\mathcal{S}(k^n)$*.*

Again, if $\pi'=\mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(k)$ we write $\gamma(s,\pi,\psi)=\gamma(s,\pi\times\mathbf{1},\psi)$. The fact that $\gamma(s,\pi\times\pi',\psi)$ is rational follows from the fact that it is a ratio of local integrals.

An equally important local factor, which occurs in the current formulations of the local Langlands correspondence [23, 26], is the local ε-factor.

Definition *The local factor* $\varepsilon(s,\pi\times\pi',\psi)$ *is defined as the ratio*

$$\varepsilon(s,\pi\times\pi',\psi)=\frac{\gamma(s,\pi\times\pi',\psi)L(s,\pi\times\pi')}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}.$$

With the local ε-factor the local functional equation can be written in the form

$$\frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}=\omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)\frac{\Psi(s;W,W')}{L(s,\pi\times\pi')}\quad \text{if } m<n$$

or

$$\frac{\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}=\omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)\frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')}\quad \text{if } m=n\ .$$

This can also be expressed in terms of the $e(s;W,W')$, etc.. In fact, since we know we can choose a finite set of W_i, W_i', and if necessary Φ_i so that

$$\sum_i \frac{\Psi(s,;W_i,W_i')}{L(s,\pi\times\pi')}=\sum_i e(s;W_i,W_i')=1$$

or

$$\sum_i \frac{\Psi(s;W_i,W_i',\Phi_i)}{L(s,\pi\times\pi')}=\sum_i e(s;W_i,W_i',\Phi_i)=1$$

we see that we can write either

$$\varepsilon(s,\pi\times\pi',\psi)=\omega'(-1)^{n-1}\sum_i \tilde{e}(1-s;\rho(w_{n,m})\widetilde{W}_i,\widetilde{W}_i')$$

or

$$\varepsilon(s,\pi\times\pi',\psi)=\omega'(-1)^{n-1}\sum_i e(1-s;\widetilde{W}_i,\widetilde{W}_i',\hat{\Phi}_i)$$

and hence $\varepsilon(s,\pi\times\pi',\psi)\in\mathbb{C}[q^s,q^{-s}]$. On the other hand, applying the functional equation twice we get

$$\varepsilon(s,\pi\times\pi',\psi)\varepsilon(1-s,\widetilde{\pi}\times\widetilde{\pi}',\psi^{-1})=1$$

so that $\varepsilon(s,\pi\times\pi',\psi)$ is a unit in $\mathbb{C}[q^s,q^{-s}]$. This can be restated as:

Proposition 3.5 $\varepsilon(s,\pi\times\pi',\psi)$ *is a monomial function of the form* cq^{-fs}.

Let us make a few remarks on the meaning of the number f occurring in the ε–factor in the case of a single representation. Assume that ψ is unramified. In this case write $\varepsilon(s,\pi,\psi) = \varepsilon(0,\pi,\psi)q^{-f(\pi)s}$. In [34] it is shown that $f(\pi)$ is a non-negative integer, $f(\pi)=0$ iff π is unramified, that in general the space of vectors in V_π which is fixed by the compact open subgroup

$$\mathrm{K}_1(\mathfrak{p}^{f(\pi)}) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) \middle| g \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}^{f(\pi)}} \right\}$$

has dimension exactly 1, and that if $t < f(\pi)$ then the dimension of the space of fixed vectors for $\mathrm{K}_1(\mathfrak{p}^t)$ is 0. Depending on the context, either the integer $f(\pi)$ or the ideal $\mathfrak{f}(\pi) = \mathfrak{p}^{f(\pi)}$ is called the *conductor of* π. Note that the analytically defined ε-factor carries structural information about π.

3.1.3 The unramified calculation

Let us now turn to the calculation of the local L-functions. The first case to consider is that where both π and π' are unramified. Since they are assumed generic, they are both full induced representations from unramified characters of the Borel subgroup [69]. So let us write $\pi \simeq \mathrm{Ind}_{\mathrm{B}_n}^{\mathrm{GL}_n}(\mu_1 \otimes \cdots \otimes \mu_n)$ and $\pi' \simeq \mathrm{Ind}_{\mathrm{B}_m}^{\mathrm{GL}_m}(\mu'_1 \otimes \cdots \otimes \mu'_m)$ with the μ_i and μ'_j unramified characters of $k^\times$. The Satake parameterization of unramified representations associates to each of these representations the semi-simple conjugacy classes $[A_\pi] \in \mathrm{GL}_n(\mathbb{C})$ and $[A_{\pi'}] \in \mathrm{GL}_m(\mathbb{C})$ given by

$$A_\pi = \begin{pmatrix} \mu_1(\varpi) & & \\ & \ddots & \\ & & \mu_n(\varpi) \end{pmatrix} \qquad A_{\pi'} = \begin{pmatrix} \mu'_1(\varpi) & & \\ & \ddots & \\ & & \mu'_m(\varpi) \end{pmatrix}.$$

(Recall that ϖ is a uniformizing parameter for k, that is, a generator of $\mathfrak{p}$.)

In the Whittaker models there will be unique normalized $\mathrm{K} = \mathrm{GL}(\mathfrak{o})$–fixed Whittaker functions, $W_\circ \in \mathcal{W}(\pi,\psi)$ and $W'_\circ \in \mathcal{W}(\pi',\psi^{-1})$, normalized by $W_\circ(e) = W'_\circ(e) = 1$. Let us concentrate on $W_\circ$ for the moment. Since this function is right K_n–invariant and transforms on the left by ψ under N_n we have that its values are completely determined by its values on diagonal matrices of the form

$$\varpi^J = \begin{pmatrix} \varpi^{j_1} & & \\ & \ddots & \\ & & \varpi^{j_n} \end{pmatrix}$$

for $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$. There is an explicit formula for $W_\circ(\varpi^J)$ in terms of the Satake parameter A_π due to Shintani [63] for GL_n and generalized to arbitrary reductive groups by Casselman and Shalika [4].

Let $T^+(n)$ be the set of n–tuples $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$ with $j_1 \geq \cdots \geq j_n$. Let ρ_J be the rational representation of $\mathrm{GL}_n(\mathbb{C})$ with dominant weight Λ_J defined by

$$\Lambda_J \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = t_1^{j_1} \cdots t_n^{j_n}.$$

Then the formula of Shintani says that

$$W_\circ(\varpi^J) = \begin{cases} 0 & \text{if } J \notin T^+(n) \\ \delta_{\mathrm{B}_n}^{1/2}(\varpi^J)\,\mathrm{tr}(\rho_J(A_\pi)) & \text{if } J \in T^+(n) \end{cases}$$

under the assumption that ψ is unramified. This is proved by analyzing the recursion relations coming from the action of the unramified Hecke algebra on $W_\circ$.

We have a similar formula for $W'_\circ(\varpi^J)$ for $J \in \mathbb{Z}^m$.

If we use these formulas in our local integrals, we find [36, I, Prop. 2.3]

$$\begin{aligned} \Psi(s; W_\circ, W'_\circ) &= \sum_{J \in T^+(m),\ j_m \geq 0} W_\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} W'_\circ(\varpi^J) \\ &\qquad \cdot |\det(\varpi^J)|^{s-(n-m)/2} \delta_{\mathrm{B}_m}^{-1}(\varpi^J) \\ &= \sum_{J \in T^+(m),\ j_m \geq 0} \mathrm{tr}(\rho_{(J,0)}(A_\pi))\,\mathrm{tr}(\rho_J(A_{\pi'}))q^{-|J|s} \\ &= \sum_{J \in T^+(m),\ j_m \geq 0} \mathrm{tr}(\rho_{(J,0)}(A_\pi) \otimes \rho_J(A_{\pi'}))q^{-|J|s} \end{aligned}$$

where we let $|J| = j_1 + \cdots + j_m$ and we embed $\mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$ by $J = (j_1, \cdots, j_m) \mapsto (J, 0) = (j_1, \cdots, j_m, 0, \cdots, 0)$. We now use the fact from invariant theory that

$$\sum_{J \in T^+(m),\ j_m \geq 0,\ |J| = r} \mathrm{tr}(\rho_{(J,0)}(A_\pi) \otimes \rho_J(A_{\pi'})) = \mathrm{tr}(S^r(A_\pi \otimes A_{\pi'})),$$

where $S^r(A)$ is the r^{th}-symmetric power of the matrix A, and

$$\sum_{r=0}^{\infty} \mathrm{tr}(S^r(A))z^r = \det(I - Az)^{-1}$$

for any matrix A. Then we quickly arrive at

$$\Psi(s; W_\circ, W'_\circ) = \det(I - q^{-s} A_\pi \otimes A_{\pi'})^{-1} = \prod_{i,j} (1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s})^{-1}$$

a standard Euler factor of degree mn. Since the L-function cancels all poles of the local integrals, we know at least that $\det(I - q^{-s}A_\pi \otimes A_{\pi'})$ divides $L(s, \pi \times \pi')^{-1}$. Either of the methods discussed below for the general calculation of local factors then shows that in fact these are equal.

There is a similar calculation when $n = m$ and $\Phi = \Phi_\circ$ is the characteristic function of the lattice $\mathfrak{o}^n \subset k^n$. Also, since π unramified implies that its contragredient $\widetilde{\pi}$ is also unramified, with $\widetilde{W_\circ}$ as its normalized unramified Whittaker function, then from the functional equation we can conclude that in this situation we have $\varepsilon(s, \pi \times \pi', \psi) \equiv 1$.

Theorem 3.3 *If* π, π', *and* ψ *are all unramified, then*

$$L(s, \pi \times \pi') = \det(I - q^{-s} A_\pi \otimes A_{\pi'})^{-1} = \begin{cases} \Psi(s; W_\circ, W'_\circ) & m < n \\ \Psi(s; W_\circ, W'_\circ, \Phi_\circ) & m = n \end{cases}$$

and $\varepsilon(s, \pi \times \pi', \psi) \equiv 1$.

For future use, let us recall a consequence of this calculation due to Jacquet and Shalika [36].

Corollary *Suppose* π *is irreducible unitary generic admissible (our usual assumptions on* π*) and unramified. Then the eigenvalues* $\mu_i(\varpi)$ *of* A_π *all satisfy* $q^{-1/2} < |\mu_i(\varpi)| < q^{1/2}$.

To see this, we apply the above calculation to the case where $\pi' = \bar{\pi}$ the complex conjugate representation. Then $A_{\pi'} = \overline{A_\pi}$, the complex conjugate matrix, and we have from the above

$$\det(I - q^{-s} A_\pi \otimes \overline{A_\pi})\Psi(s; W_\circ, \overline{W_\circ}, \Phi_\circ) = 1.$$

The local integral in this case is absolutely convergent for $\mathrm{Re}(s) \geq 1$ and so the factor $\det(I - q^{-s}A_\pi \otimes \overline{A_\pi})$ cannot vanish for $\mathrm{Re}(s) \geq 1$. If $\mu_i(\varpi)$ is an eigenvalue of A_π then we have $1 - q^{-\sigma}|\mu_i(\varpi)|^2 \neq 0$ for $\sigma \geq 1$. Hence $|\mu_i(\varpi)| < q^{1/2}$. Note that if we apply this to the contragredient representation $\widetilde{\pi}$ as well we conclude that $q^{-1/2} < |\mu_i(\varpi)| < q^{1/2}$.

3.1.4 The supercuspidal calculation

The other basic case is when both π and π' are supercuspidal. In this case the restriction of W to P_n or W' to P_m lies in the Kirillov model and is hence compactly supported mod N. In the case of $m < n$ we find that in our integral we have W evaluated along $\mathrm{GL}_m(k) \subset \mathrm{P}_n(k)$. Since W is smooth, and hence stabilized by some compact open subgroup, we find that the local integral always reduces to a finite sum and hence lies in $\mathbb{C}[q^s, q^{-s}]$. In particular it is always entire. Thus in this case $L(s, \pi \times \pi') \equiv 1$. In the case $n = m$ the calculation is a bit more involved and can be found in [11,15]. In essence, in the family of integrals $\Psi(s; W, W', \Phi)$, if $\Phi(0) = 0$ then the integral will again reduce to a finite sum and hence be entire. If $\Phi(0) \neq 0$ and if s_0 is a pole of $\Psi(s; W, W', \Phi)$ then the residue of the pole at $s = s_0$ will be of the form

$$c\Phi(0) \int_{\mathrm{Z}_n(k)\,\mathrm{N}_n(k)\backslash \mathrm{GL}_n(k)} W(g) W'(g) |\det(g)|^{s_0}\, dg$$

which is the Whittaker form of an invariant pairing between π and $\pi' \otimes |\det|^{s_0}$. Thus we must have s_0 is pure imaginary and $\widetilde{\pi} \simeq \pi' \otimes |\det|^{s_0}$ for the residue to be nonzero. This condition is also sufficient.

Theorem 3.4 *If π and π' are both (unitary) supercuspidal, then $L(s, \pi \times \pi') \equiv 1$ if $m < n$ and if $m = n$ we have*

$$L(s, \pi \times \pi') = \prod (1 - \alpha q^{-s})^{-1}$$

with the product over all $\alpha = q^{s_0}$ with $\widetilde{\pi} \simeq \pi' \otimes |\det|^{s_0}$.

3.1.5 Remarks on the general calculation

In the other cases, we must rely on the Bernstein–Zelevinsky classification of generic representations of $\mathrm{GL}_n(k)$ [69]. All generic representations can be realized as prescribed constituents of representations parabolically induced from supercuspidals. One can proceed by analyzing the Whittaker functions of induced representations in terms of Whittaker functions of the inducing data as in [33] or by analyzing the poles of the local integrals in terms of quasi invariant pairings of derivatives of π and π' as in [11] to compute $L(s, \pi \times \pi')$ in terms of L-functions of pairs of supercuspidal representations. We refer you to those papers or [42] for the explicit formulas.

3.1.6 Multiplicativity and stability of γ–factors

To conclude this section, let us mention two results on the γ-factors. One is used in the computations of L-factors in the general case. This is the *multiplicativity of γ-factors* [33]. The second is the *stability of γ-factors* [37]. Both of these results are necessary in applications of the Converse Theorem to liftings, which we discuss in Section 5.

Proposition (Multiplicativity of γ-factors) *If $\pi = \mathrm{Ind}(\pi_1 \otimes \pi_2)$, with π_i and irreducible admissible representation of $\mathrm{GL}_{r_i}(k)$, then*

$$\gamma(s, \pi \times \pi', \psi) = \gamma(s, \pi_1 \times \pi', \psi)\gamma(s, \pi_2 \times \pi', \psi)$$

and similarly for π'. Moreover $L(s, \pi \times \pi')^{-1}$ divides $[L(s, \pi_1 \times \pi')L(s, \pi_2 \times \pi')]^{-1}$.

Proposition (Stability of γ-factors) *If π_1 and π_2 are two irreducible admissible generic representations of $\mathrm{GL}_n(k)$, having the same central character, then for every sufficiently highly ramified character η of $\mathrm{GL}_1(k)$ we have*

$$\gamma(s, \pi_1 \times \eta, \psi) = \gamma(s, \pi_2 \times \eta, \psi)$$

and

$$L(s, \pi_1 \times \eta) = L(s, \pi_2 \times \eta) \equiv 1.$$

More generally, if in addition π' is an irreducible generic representation of $\mathrm{GL}_m(k)$ then for all sufficiently highly ramified characters η of $\mathrm{GL}_1(k)$ we have

$$\gamma(s, (\pi_1 \otimes \eta) \times \pi', \psi) = \gamma(s, (\pi_2 \otimes \eta) \times \pi', \psi)$$

and

$$L(s, (\pi_1 \otimes \eta) \times \pi') = L(s, (\pi_2 \otimes \eta) \times \pi') \equiv 1.$$

3.2 The Archimedean Local Factors

We now take k to be an archimedean local field, i.e., $k = \mathbb{R}$ or $\mathbb{C}$. We take (π, V_π) to be the space of smooth vectors in an irreducible admissible unitary generic representation of $\mathrm{GL}_n(k)$ and similarly for the representation

$(\pi', V_{\pi'})$ of $\mathrm{GL}_m(k)$. We take ψ a non-trivial continuous additive character of k.

The treatment of the archimedean local factors parallels that of the non-archimedean in many ways, but there are some significant differences. The major work on these factors is that of Jacquet and Shalika in [38], which we follow for the most part without further reference, and in the archimedean parts of [36].

One significant difference in the development of the archimedean theory is that the local Langlands correspondence was already in place when the theory was developed [45]. The correspondence is very explicit in terms of the usual Langlands classification. Thus to π is associated an n dimensional semi-simple representation $\tau = \tau(\pi)$ of the Weil group W_k of k and to π' an m-dimensional semi-simple representation $\tau' = \tau(\pi')$ of W_k. Then $\tau(\pi) \otimes \tau(\pi')$ is an nm dimensional representation of W_k and to this representation of the Weil group is attached Artin-Weil L– and ε–factors [65], denoted $L(s, \tau \otimes \tau')$ and $\varepsilon(s, \tau \otimes \tau', \psi)$. In essence, Jacquet and Shalika *define*

$$L(s, \pi \times \pi') = L(s, \tau(\pi) \otimes \tau(\pi')) \quad \text{and} \quad \varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \tau(\pi) \otimes \tau(\pi'), \psi)$$

and then set

$$\gamma(s, \pi \times \pi', \psi) = \frac{\varepsilon(s, \pi \times \pi', \psi) L(1 - s, \widetilde{\pi} \times \widetilde{\pi}')}{L(s, \pi \times \pi')}.$$

For example, if π is unramified, and hence of the form $\pi \simeq \mathrm{Ind}(\mu_1 \otimes \cdots \otimes \mu_n)$ with unramified characters of the form $\mu_i(x) = |x|^{r_i}$ then

$$L(s, \pi) = L(s, \tau(\pi)) = \prod_{i=1}^{n} \Gamma_v(s + r_i)$$

is a standard archimedean Euler factor of degree n, where

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2} \Gamma(\frac{s}{2}) & \text{if } k_v = \mathbb{R} \\ 2(2\pi)^{-s} \Gamma(s) & \text{if } k_v = \mathbb{C} \end{cases}.$$

They then proceed to show that these functions have the expected relation to the local integrals. Their methods of analyzing the local integrals $\Psi_j(s; W, W')$ and $\Psi(s; W, W', \Phi)$, defined as in the non-archimedean case for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in \mathcal{S}(k^n)$, are direct analogues of those used in [33] for the non-archimedean case. Once again, a most important fact is [38, Proposition 2.2]

Proposition 3.6 *Let* π *be a generic representation of* $\mathrm{GL}_n(k)$*. Then there is a finite set of finite functions* $X(\pi) = \{\chi_i\}$ *on* $\mathrm{A}_n(k)$*, depending only on* π*, so that for every* $W \in \mathcal{W}(\pi, \psi)$ *there are Schwartz functions* $\phi_i \in \mathcal{S}(k^{n-1} \times \mathrm{K}_n)$ *such that for all* $a \in \mathrm{A}_n(k)$ *with* $a_n = 1$ *we have*

$$W(nak) = \psi(n) \sum_{X(\pi)} \chi_i(a)\phi_i(\alpha_1(a), \dots, \alpha_{n-1}(a), k).$$

Now the finite functions are related to the exponents of the representation π and through the Langlands classification to the representation $\tau(\pi)$ of W_k. We retain the same convergence statements as in the non-archimedean case [36, I, Proposition 3.17; II, Proposition 2.6], [38, Proposition 5.3].

Proposition 3.7 *The integrals* $\Psi_j(s; W, W')$ *and* $\Psi(s; W, W', \Phi)$ *converge absolutely in the half plane* $\mathrm{Re}(s) \geq 1$ *under the unitarity assumption and for* $\mathrm{Re}(s) > 0$ *if* π *and* π' *are tempered.*

The meromorphic continuation and "bounded denominator" statement in the case of a non-archimedean local field is now replaced by the following. Define $\mathcal{M}(\pi \times \pi')$ to be the space of all meromorphic functions $\phi(s)$ with the property that if $P(s)$ is a polynomial function such that $P(s)L(s, \pi \times \pi')$ is holomorphic in a vertical strip $S[a, b] = \{s \ a \leq \mathrm{Re}(s) \leq b\}$ then $P(s)\phi(s)$ is bounded in $S[a, b]$. Note in particular that if $\phi \in \mathcal{M}(\pi \times \pi')$ then the quotient $\phi(s)L(s, \pi \times \pi')^{-1}$ is entire.

Theorem 3.5 *The integrals* $\Psi_j(s; W, W')$ *or* $\Psi(s; W, W', \Phi)$ *extend to meromorphic functions of* s *which lie in* $\mathcal{M}(\pi \times \pi')$*. In particular, the ratios*

$$e_j(s; W, W') = \frac{\Psi_j(s; W, W')}{L(s, \pi \times \pi')} \quad \text{or} \quad e(s; W, W', \Phi) = \frac{\Psi(s; W, W', \Phi)}{L(s, \pi \times \pi')}$$

are entire and in fact are bounded in vertical strips.

This statement has more content than just the continuation and "bounded denominator" statements in the non-archimedean case. Since it prescribes the "denominator" to be the L factor $L(s, \pi \times \pi')^{-1}$ it is bound up with the actual computation of the poles of the local integrals. In fact, a significant part of the paper of Jacquet and Shalika [38] is taken up with the simultaneous proof of this and the local functional equations:

Theorem 3.6 *We have the local functional equations*

$$\Psi_{n-m-j-1}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')=\omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi_j(s;W,W')$$

or

$$\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})=\omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s;W,W',\Phi).$$

The one fact that we are missing is the statement of "minimality" of the L-factor. That is, we know that $L(s,\pi\times\pi')$ is a standard archimedean Euler factor (i.e., a product of Γ-functions of the standard type) and has the property that the poles of all the local integrals are contained in the poles of the L-factor, even with multiplicity. But we have not established that the L-factor cannot have extraneous poles. In particular, we do know that we can achieve the local L-function as a finite linear combination of local integrals.

Towards this end, Jacquet and Shalika enlarge the allowable space of local integrals. Let Λ and Λ' be the Whittaker functionals on V_π and $V_{\pi'}$ associated with the Whittaker models $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$. Then $\hat{\Lambda}=\Lambda\otimes\Lambda'$ defines a continuous linear functional on the algebraic tensor product $V_\pi\otimes V_{\pi'}$ which extends continuously to the topological tensor product $V_{\pi\otimes\pi'}=V_\pi\hat{\otimes}V_{\pi'}$, viewed as representations of $\mathrm{GL}_n(k)\times\mathrm{GL}_m(k)$.

Before proceeding, let us make a few remarks on smooth representations. If (π,V_π) is the space of smooth vectors in an irreducible admissible unitary representation, then the underlying Harish-Chandra module is the space of K_n-finite vectors $V_{\pi,\mathrm{K}}$. V_π then corresponds to the (Casselman-Wallach) canonical completion of $V_{\pi,\mathrm{K}}$ [66]. The category of Harish-Chandra modules is appropriate for the algebraic theory of representations, but it is useful to work in the category of smooth admissible representations for automorphic forms. If in our context we take the underlying Harish-Chandra modules $V_{\pi,\mathrm{K}}$ and $V_{\pi',\mathrm{K}}$ then their algebraic tensor product is an admissible Harish-Chandra module for $\mathrm{GL}_n(k)\times\mathrm{GL}_m(k)$. The associated smooth admissible representation is the canonical completion of this tensor product, which is in fact $V_{\pi\otimes\pi'}$, the topological tensor product of the smooth representations π and π'. It is also the space of smooth vectors in the unitary tensor product of the unitary representations associated to π and π'. So this completion is a natural place to work in the category of smooth admissible representations.

Now let

$$\mathcal{W}(\pi\otimes\pi',\psi)=\{W(g,h)=\hat{\Lambda}(\pi(g)\otimes\pi'(h)\xi)|\xi\in V_{\pi\otimes\pi'}\}.$$

Then $\mathcal{W}(\pi\otimes\pi',\psi)$ contains the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ and is again equal to the topological tensor product. Now we can extend all our local integrals to the space $\mathcal{W}(\pi\otimes\pi',\psi)$ by setting

$$\Psi_j(s;W) = \int\int W\left(\begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix}, h\right)\ dx\ |\det(h)|^{s-(n-m)/2}\ dh$$

and

$$\Psi(s;W,\Phi) = \int W(g,g)\Phi(e_n g)|\det(g)|^s\ dh$$

for $W\in\mathcal{W}(\pi\otimes\pi',\psi)$. Since the local integrals are continuous with respect to the topology on the topological tensor product, all of the above facts remain true, in particular the convergence statements, the local functional equations, and the fact that these integrals extend to functions in $\mathcal{M}(\pi\times\pi')$.

At this point, let $\mathcal{I}_j(\pi,\pi') = \{\Psi_j(s;W)|W\in\mathcal{W}(\pi\otimes\pi')\}$ and let $\mathcal{I}(\pi,\pi')$ be the span of the local integrals $\{\Psi(s;W,\Phi)|W\in\mathcal{W}(\pi\otimes\pi',\psi),\ \phi\in\mathcal{S}(k^n)\}$. Once again, in the case $m<n$ we have that the space $\mathcal{I}_j(\pi,\pi')$ is independent of j and we denote it also by $\mathcal{I}(\pi,\pi')$. These are still independent of ψ. So we know from above that $\mathcal{I}(\pi,\pi')\subset\mathcal{M}(\pi\times\pi')$. The remainder of [38] is then devoted to showing the following.

Theorem 3.7 $\mathcal{I}(\pi,\pi') = \mathcal{M}(\pi\times\pi')$.

As a consequence of this, we draw the following useful Corollary.

Corollary *There is a Whittaker function W in $\mathcal{W}(\pi\otimes\pi',\psi)$ such that $L(s,\pi\times\pi') = \Psi(s;W)$ if $m<n$ or finite collectionof functions $W_i\in\mathcal{W}(\pi\otimes\pi',\psi)$ and $\Phi_i\in\mathcal{S}(k^n)$ such that $L(s,\pi\times\pi') = \sum_i\Psi(s;W_i,\Phi_i)$ if $m=n$.*

In the cases of $m=n-1$ or $m=n$, Jacquet and Shalika can indeed get the local L-function as a finite linear combination of integrals involving only K-finite functions in $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$, that is, without going to the completion of $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$, but this has not been published.

As a final result, let us note that in [12] it is established that the linear functionals $e(s;W) = \Psi(s;W)L(s,\pi\times\pi')^{-1}$ and $e(s;W,\Phi) = \Psi(s;W,\Phi)$ $L(s,\pi\times\pi')^{-1}$ are continuous on $\mathcal{W}(\pi\otimes\pi',\psi)$, uniformly for s in compact sets. Since there is a choice of $W\in\mathcal{W}(\pi\otimes\pi',\psi)$ such that $e(s;W)\equiv 1$ or $W_i\in\mathcal{W}(\pi\otimes\pi',\psi)$ and $\Phi_i\in\mathcal{S}(k^n)$ such that $\sum e(s;W_i,\Phi_i)\equiv 1$, as a result of this continuity and the fact that the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ is dense in $\mathcal{W}(\pi\otimes\pi',\psi)$ we have the following result.

Proposition 3.8 *For any* $s_0 \in \mathbb{C}$ *there are choices of* $W \in \mathcal{W}(\pi,\psi)$*,* $W' \in \mathcal{W}(\pi',\psi^{-1})$ *and if necessary* Φ *such that* $e(s_0;W,W') \neq 0$ *or* $e(s_0;W,W',\Phi) \neq 0$*.*

4 Global L-functions

Once again, we let k be a global field, $\mathbb{A}$ its ring of adeles, and fix a non-trivial continuous additive character $\psi = \otimes\psi_v$ of $\mathbb{A}$ trivial on k.

Let (π, V_π) be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ (see Section 1 for all the implied assumptions in this terminology) and $(\pi', V_{\pi'})$ a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$. Since they are irreducible we have restricted tensor product decompositions $\pi \simeq \otimes'\pi_v$ and $\pi' \simeq \otimes'\pi'_v$ with (π_v, V_{π_v}) and $(\pi'_v, V_{\pi'_v})$ irreducible admissible smooth generic unitary representations of $\mathrm{GL}_n(k_v)$ and $\mathrm{GL}_m(k_v)$ [14,18]. Let $\omega = \otimes'\omega_v$ and $\omega' = \otimes'\omega'_v$ be their central characters. These are both continuous characters of $k^\times\backslash\mathbb{A}^\times$.

Let S be the finite set of places of k, containing the archimedean places S_∞, such that for all $v \notin S$ we have that π_v, π'_v, and ψ_v are unramified.

For each place v of k we have defined the local factors $L(s,\pi_v\times\pi'_v)$ and $\varepsilon(s,\pi_v\times\pi'_v,\psi_v)$. Then we can at least formally define

$$L(s,\pi\times\pi') = \prod_v L(s,\pi_v\times\pi'_v) \quad \text{and} \quad \varepsilon(s,\pi\times\pi') = \prod_v \varepsilon(s,\pi_v\times\pi'_v,\psi_v).$$

We need to discuss convergence of these products. Let us first consider the convergence of $L(s,\pi\times\pi')$. For those $v \notin S$, so π_v, π'_v, and ψ_v are unramified, we know that $L(s,\pi_v\times\pi'_v) = \det(I - q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1}$ and that the eigenvalues of A_{π_v} and $A_{\pi'_v}$ are all of absolute value less than $q_v^{1/2}$. Thus the partial (or incomplete) L-function

$$L^S(s,\pi\times\pi') = \prod_{v\notin S} L(s,\pi_v\times\pi'_v) = \prod_{v\notin S} \det(I - q^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1}$$

is absolutely convergent for $\mathrm{Re}(s) >> 0$. Thus the same is true for $L(s,\pi\times\pi')$.

For the ε–factor, we have seen that $\varepsilon(s,\pi_v\times\pi'_v,\psi_v) \equiv 1$ for $v \notin S$ so that the product is in fact a finite product and there is no problem with convergence. The fact that $\varepsilon(s,\pi\times\pi')$ is independent of ψ can either be checked by analyzing how the local ε–factors vary as you vary ψ, as is done in [7, Lemma 2.1], or it will follow from the global functional equation presented below.

4.1 The Basic Analytic Properties

Our first goal is to show that these L-functions have nice analytic properties.

Theorem 4.1 *The global L–functions $L(s, \pi \times \pi')$ are nice in the sense that*

1. *$L(s, \pi \times \pi')$ has a meromorphic continuation to all of $\mathbb{C}$,*
2. *the extended function is bounded in vertical strips (away from its poles),*
3. *they satisfy the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

To do so, we relate the L-functions to the global integrals.

Let us begin with continuation. In the case $m < n$ for every $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ we know the integral $I(s; \varphi, \varphi')$ converges absolutely for all s. From the unfolding in Section 2 and the local calculation of Section 3 we know that for $\mathrm{Re}(s) >> 0$ and for appropriate choices of φ and φ' we have

$$\begin{aligned} I(s; \varphi, \varphi') &= \prod_v \Psi_v(s; W_{\varphi_v}, W_{\varphi'_v}) \\ &= \left(\prod_{v \in S} \Psi_v(s; W_{\varphi_v}, W_{\varphi'_v}) \right) L^S(s, \pi \times \pi') \\ &= \left(\prod_{v \in S} \frac{\Psi_v(s; W_{\varphi_v}, W_{\varphi'_v})}{L(s, \pi_v \times \pi'_v)} \right) L(s, \pi \times \pi') \\ &= \left(\prod_{v \in S} e_v(s; W_{\varphi_v}, W_{\varphi'_v}) \right) L(s, \pi \times \pi') \end{aligned}$$

We know that each $e_v(s; W_v, W'_v)$ is entire. Hence $L(s, \pi \times \pi')$ has a meromorphic continuation. If $m = n$ then for appropriate $\varphi \in V_\pi$, $\varphi' \in V_{\pi'}$, and $\Phi \in \mathcal{S}(\mathbb{A}^n)$ we again have

$$I(s; \varphi, \varphi', \Phi) = \left(\prod_{v \in S} e_v(s; W_{\varphi_v}, W'_{\varphi'_v}, \Phi_v) \right) L(s, \pi \times \pi').$$

Once again, since each $e_v(s; W_v, W'_v, \Phi_v)$ is entire, $L(s, \pi \times \pi')$ has a meromorphic continuation.

Let us next turn to the functional equation. This will follow from the functional equation for the global integrals and the local functional equations. We will consider only the case where $m < n$ since the other case is entirely analogous. The functional equation for the global integrals is simply

$$I(s;\varphi,\varphi') = \tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Once again we have for appropriate φ and φ'

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v})\right) L(s,\pi\times\pi')$$

while on the other side

$$\tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \left(\prod_{v\in S} \tilde{e}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v})\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

However, by the local functional equations, for each $v\in S$ we have

$$\begin{aligned}\tilde{e}_v(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v) &= \frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v)}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}\\ &= \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\frac{\Psi(s;W_v,W'_v)}{L(s,\pi\times\pi')}\\ &= \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)e_v(s,W_v,W'_v)\end{aligned}$$

Combining these, we have

$$L(s,\pi\times\pi') = \left(\prod_{v\in S}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Now, for $v\notin S$ we know that π'_v is unramified, so $\omega'_v(-1)=1$, and also that $\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\equiv 1$. Therefore

$$\begin{aligned}\prod_{v\in S}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v) &= \prod_{v}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\\ &= \omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi')\\ &= \varepsilon(s,\pi\times\pi')\end{aligned}$$

and we indeed have

$$L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Note that this implies that $\varepsilon(s, \pi \times \pi')$ is independent of ψ as well.

Let us now turn to the boundedness in vertical strips. For the global integrals $I(s; \varphi, \varphi')$ or $I(s; \varphi, \varphi, \Phi)$ this simply follows from the absolute convergence. For the L-function itself, the paradigm is the following. For every finite place $v \in S$ we know that there is a choice of $W_{v,i}$, $W'_{v,i}$, and $\Phi_{v,i}$ if necessary such that

$$\begin{aligned} L(s, \pi_v \times \pi'_v) &= \sum \Psi(s; W_{v,i}, W'_{v'i}) \quad \text{or} \\ L(s, \pi_v \times \pi'_v) &= \sum \Psi(s; W_{v,i}, W'_{v'i}, \Phi_{v,i}). \end{aligned}$$

If $m = n - 1$ or $m = n$ then by the unpublished work of Jacquet and Shalika mentioned toward the end of Section 3 we know that we have similar statements for $v \in S_\infty$. Hence if $m = n-1$ or $m = n$ there are global choices φ_i, φ'_i, and if necessary Φ_i such that

$$L(s, \pi \times \pi') = \sum I(s; \varphi_i, \varphi'_i) \quad \text{or} \quad L(s, \pi \times \pi') = \sum I(s; \varphi_i, \varphi'_i, \Phi_i).$$

Then the boundedness in vertical strips for the L-functions follows from that of the global integrals.

However, if $m < n - 1$ then all we know at those $v \in S_\infty$ is that there is a function $W_v \in \mathcal{W}(\pi_v \otimes \pi'_v, \psi_v) = \mathcal{W}(\pi_v, \psi_v) \hat{\otimes} \mathcal{W}(\pi'_v, \psi_v^{-1})$ or a finite collection of such functions $W_{v,i}$ and of $\Phi_{v,i}$ such that

$$L(s, \pi_v \times \pi'_v) = I(s; W_v) \quad \text{or} \quad L(s, \pi_v \times \pi'_v) = \sum I(s; W_{v,i}, \Phi_{v,i}).$$

To make the above paradigm work for $m < n - 1$ we would need to rework the theory of global Eulerian integrals for cusp forms in $V_\pi \hat{\otimes} V_{\pi'}$. This is naturally the space of smooth vectors in an irreducible unitary cuspidal representation of $\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_m(\mathbb{A})$. So we would need extend the global theory of integrals parallel to Jacquet and Shalika's extension of the local integrals in the archimedean theory. There seems to be no obstruction to carrying this out, and then we obtain boundedness in vertical strips for $L(s, \pi \times \pi')$ in general.

We should point out that if one approaches these L-functions by the method of constant terms and Fourier coefficients of Eisenstein series, then Gelbart and Shahidi have shown a wide class of automorphic L-functions, including ours, to be bounded in vertical strips [17].

4.2 Poles of L-functions

Let us determine where the global L-functions can have poles. The poles of the L-functions will be related to the poles of the global integrals. Recall from Section 2 that in the case of $m < n$ we have that the global integrals $I(s;\varphi,\varphi')$ are entire and that when $m = n$ then $I(s;\varphi,\varphi',\Phi)$ can have at most simple poles and they occur at $s = -i\sigma$ and $s = 1 - i\sigma$ for σ real when $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$. As we have noted above, the global integrals and global L-functions are related, for appropriate φ, φ', and Φ, by

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v})\right) L(s,\pi\times\pi')$$

or

$$I(s;\varphi,\varphi',\Phi) = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)\right) L(s,\pi\times\pi').$$

On the other hand, we have seen that for any $s_0 \in \mathbb{C}$ and any v there is a choice of local W_v, W'_v, and Φ_v such that the local factors $e_v(s_0;W_v,W'_v) \neq 0$ or $e_v(s_0;W_v,W'_v,\Phi_v) \neq 0$. So as we vary φ, φ' and Φ at the places $v \in S$ we see that division by these factors can introduce no extraneous poles in $L(s,\pi\times\pi')$, that is, in keeping with the local characterization of the L-factor in terms of poles of local integrals, globally the poles of $L(s,\pi\times\pi')$ are precisely the poles of the family of global integrals $\{I(s;\varphi,\varphi')\}$ or $\{I(s;\varphi,\varphi',\Phi)\}$. Hence from Theorems 2.1 and 2.2 we have.

Theorem 4.2 *If $m < n$ then $L(s,\pi\times\pi')$ is entire. If $m = n$, then $L(s,\pi\times\pi')$ has at most simple poles and they occur iff $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ with σ real and are then at $s = -i\sigma$ and $s = 1 - i\sigma$.*

If we apply this with $\pi' = \widetilde{\pi}$ we obtain the following useful corollary.

Corollary *$L(s,\pi\times\widetilde{\pi})$ has simple poles at $s = 0$ and $s = 1$.*

4.3 Strong Multiplicity One

Let us return to the Strong Multiplicity One Theorem for cuspidal representations. First, recall the statement:

Theorem (Strong Multiplicity One) *Let (π,V_π) and $(\pi',V_{\pi'})$ be two cuspidal representations of $GL_n(\mathbb{A})$. Suppose there is a finite set of places S such that for all $v \notin S$ we have $\pi_v \simeq \pi'_v$. Then $\pi = \pi'$.*

We will now present Jacquet and Shalika's proof of this statement via L-functions [36]. First note the following observation, which follows from our analysis of the location of the poles of the L-functions.

Observation *For π and π' cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$, $L(s,\pi\times\widetilde{\pi}')$ has a pole at $s=1$ iff $\pi\simeq\pi'$.*

Thus the L-function gives us an analytic method of testing when two cuspidal representations are isomorphic, and so by the Multiplicity One Theorem, the same.

Proof: If we take π and π' as in the statement of Strong Multiplicity One, we have that $\pi_v\simeq\pi'_v$ for $v\notin S$ and hence

$$L^S(s,\pi\times\widetilde{\pi})=\prod_{v\notin S}L(s,\pi_v\times\widetilde{\pi}_v)=\prod_{v\notin S}L(s,\pi_v\times\widetilde{\pi}'_v)=L^S(s,\pi\times\widetilde{\pi}')$$

Since the local L-factors never vanish and for unitary representations they have no poles in $\mathrm{Re}(s)\geq 1$ (since the local integrals have no poles in this region) we see that for $s=1$ that $L(s,\pi\times\widetilde{\pi}')$ has a pole at $s=1$ iff $L^S(s,\pi\times\widetilde{\pi}')$ does. Hence we have that since $L(s,\pi\times\widetilde{\pi})$ has a pole at $s=1$, so does $L^S(s,\pi\times\widetilde{\pi})$. But $L^S(s,\pi\times\widetilde{\pi})=L^S(s,\pi\times\widetilde{\pi}')$, so that both $L^S(s,\pi\times\widetilde{\pi}')$ and then $L(s,\pi\times\widetilde{\pi}')$ have poles at $s=1$. But then the L-function criterion above gives that $\pi\simeq\pi'$. Now apply Multiplicity One. □

In fact, Jacquet and Shalika push this method much further. If π is an irreducible automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, but not necessarily cuspidal, then it is a theorem of Langlands [44] that there are cuspidal representations, say $\tau_1,\dots,\tau_r$ of $\mathrm{GL}_{n_1},\dots,\mathrm{GL}_{n_r}$ with $n=n_1+\cdots+n_r$, such that π is a constituent of $\mathrm{Ind}(\tau_1\otimes\cdots\otimes\tau_r)$. Similarly, π' is a constituent of $\mathrm{Ind}(\tau'_1\otimes\cdots\otimes\tau'_{r'})$. Then the generalized version of the Strong Multiplicity One theorem that Jacquet and Shalika establish in [36] is the following.

Theorem (Generalized Strong Multiplicity One) *Given π and π' irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ as above, suppose that there is a finite set of places S such that $\pi_v\simeq\pi'_v$ for all $v\notin S$. Then $r=r'$ and there is a permutation σ of the set $\{1,\dots,r\}$ such that $n_i=n'_{\sigma(i)}$ and $\tau_i=\tau'_{\sigma(i)}$.*

Note, the cuspidal representations τ_i and τ'_i need not be unitary in this statement.

4.4 Non-vanishing Results

Of interest for questions from analytic number theory, for example questions of equidistribution, are results on the non-vanishing of L-functions. The fundamental non-vanishing result for GL_n is the following theorem of Jacquet and Shalika [35] and Shahidi [56, 57].

Theorem 4.3 *Let π and π' be cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A})$. Then the L-function $L(s,\pi\times\pi')$ is non-vanishing for $\mathrm{Re}(s)\geq 1$.*

The proof of non-vanishing for $\mathrm{Re}(s)>1$ is in keeping with the spirit of these notes [36, I, Theorem 5.3]. Since the local L-functions are never zero, to establish the non-vanishing of the Euler product for $\mathrm{Re}(s)>1$ it suffices to show that the Euler product is absolutely convergent for $\mathrm{Re}(s)>1$, and for this it is sufficient to work with the incomplete L-function $L^S(s,\pi\times\pi')$ where S is as at the beginning of this Section. Then we can write

$$L^S(s,\pi\times\pi')=\prod_{v\notin S}L(s,\pi_v\times\pi'_v)=\prod_{v\notin S}\det(I-q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1}$$

with absolute convergence for $\mathrm{Re}(s)>>0$.

Recall that an infinite product $\prod(1+a_n)$ is absolutely convergent iff the associated series $\sum\log(1+a_n)$ is absolutely convergent.

Let us write

$$A_{\pi_v}=\begin{pmatrix}\mu_{v,1}&&\\&\ddots&\\&&\mu_{v,n}\end{pmatrix}\quad\text{and}\quad A_{\pi'_v}=\begin{pmatrix}\mu'_{v,1}&&\\&\ddots&\\&&\mu'_{v,m}\end{pmatrix}.$$

We have seen that $|\mu_{v,i}|<q_v^{1/2}$ and $|\mu'_{v,j}|<q_v^{1/2}$. Then

$$\begin{aligned}\log L(s,\pi_v\times\pi'_v)&=-\sum_{i,j}\log(1-\mu_{v,i}\mu'_{v,j}q_v^{-s})\\&=\sum_{i,j}\sum_{d=1}^{\infty}\frac{(\mu_{v,i}\mu'_{v,j})^d}{dq_v^{ds}}=\sum_{d=1}^{\infty}\frac{\mathrm{tr}(A^d_{\pi_v})\,\mathrm{tr}(A^d_{\pi'_v})}{dq_v^{ds}}\end{aligned}$$

with the sum absolutely convergent for $\mathrm{Re}(s)>>0$. Then, still for $\mathrm{Re}(s)>>0$,

$$\log(L^S(s,\pi\times\pi'))=\sum_{v\notin S}\sum_{d=1}^{\infty}\frac{\mathrm{tr}(A^d_{\pi_v})\,\mathrm{tr}(A^d_{\pi'_v})}{dq_v^{ds}}.$$

If we apply this to $\pi' = \overline{\pi} = \widetilde{\pi}$ we find

$$\log(L^S(s,\pi\times\overline{\pi})) = \sum_{v\notin S}\sum_{d=1}^{\infty}\frac{|\operatorname{tr}(A_{\pi_v}^d)|^2}{dq_v^{ds}}$$

which is a Dirichlet series with non-negative coefficients. By Landau's Lemma this will be absolutely convergent up to the its first pole, which we know is at $s = 1$. Hence this series, and the associated Euler product $L(s,\pi\times\widetilde{\pi})$, is absolutely convergent for $\mathrm{Re}(s) > 1$.

An application of the Cauchy–Schwatrz inequality then implies that the series

$$\log(L^S(s,\pi\times\pi')) = \sum_{v\notin S}\sum_{d=1}^{\infty}\frac{\operatorname{tr}(A_{\pi_v}^d)\operatorname{tr}(A_{\pi'_v}^d)}{dq_v^{ds}}$$

is also absolutely convergent for $\mathrm{Re}(s) > 1$. Thus $L(s,\pi\times\pi')$ is absolutely convergent and hence non-vanishing for $\mathrm{Re}(s) > 1$.

To obtain the non-vanishing on the line $\mathrm{Re}(s) = 1$ requires the technique of analyzing L-functions via their occurrence in the constant terms and Fourier coefficients of Eisenstein series, which we have not discussed. They can be found in the references [35] and [56, 57] mentioned above.

4.5 The Generalized Ramanujan Conjecture (GRC)

The current version of the GRC is a conjecture about the structure of cuspidal representations.

Conjecture (GRC) *Let π be a (unitary) cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ with decomposition $\pi \simeq \otimes'\pi_v$. Then the local components π_v are tempered representations.*

However, it has an interesting interpretation in terms of L-functions which is more in keeping with the origins of the conjecture. If π is cuspidal, then at every finite place v where π_v is unramified we have associated a semisimple conjugacy class, say $A_{\pi_v} = \begin{pmatrix}\mu_{v,1} & & \\ & \ddots & \\ & & \mu_{v,n}\end{pmatrix}$ so that

$$L(s,\pi_v) = \det(I - q_v^{-s}A_{\pi_v})^{-1} = \prod_{i=1}^{n}(1-\mu_{v,i}q_v^{-s})^{-1}.$$

If v is an archimedean place where π_v is unramified, then we can similarly write

$$L(s,\pi)=\prod_{i=1}^{n}\Gamma_v(s+\mu_{v,i})$$

where

$$\Gamma_v(s)=\begin{cases}\pi^{-s/2}\Gamma(\frac{s}{2}) & \text{if } k_v\simeq\mathbb{R}\\ 2(2\pi)^{-s}\Gamma(s) & \text{if } k_v\simeq\mathbb{C}\end{cases}.$$

Then the statement of the GRC in these terms is

Conjecture (GRC for L-functions) *If π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and if v is a place where π_v is unramified, then $|\mu_{v,i}|=1$ for v non-archimedean and $\mathrm{Re}(\mu_{v,i})=0$ for v archimedean.*

Note that by including the archimedean places, this conjecture encompasses not only the classical Ramanujan conjectures but also the various versions of the Selberg eigenvalue conjecture [27].

Recall that by the Corollary to Theorem 3.3 we have the bounds $q_v^{-1/2}<|\mu_{v,i}|<q_v^{1/2}$ for v non-archimedean, and a similar local analysis for v archimedean would give $|\mathrm{Re}(\mu_{v,i})|<\frac{1}{2}$. The best bound for general GL_n is due to Luo, Rudnick, and Sarnak [46]. They are the uniform bounds

$$q_v^{-(\frac{1}{2}-\frac{1}{n^2+1})}\leq|\mu_{v,i}|\leq q_v^{\frac{1}{2}-\frac{1}{n^2+1}}\quad\text{if } v \text{ is non-archimedean}$$

and

$$|\mathrm{Re}(\mu_{v,i})|\leq\frac{1}{2}-\frac{1}{n^2+1}\quad\text{for } v \text{ archimedean.}$$

Their techniques are global and rely on the theory of Rankin–Selberg L-functions as presented here, a technique of persistence of zeros in families of L-functions, and a positivity argument.

For GL_2 there has been much recent progress. The best general estimates I am aware of at present are due to Kim and Shahidi [41], who use the holomorphy of the symmetric ninth power L-function for $\mathrm{Re}(s)>1$ to obtain

$$q_v^{-\frac{1}{9}}<|\mu_{v,i}|<q_v^{\frac{1}{9}}\quad\text{for } i=1,2, \text{ and } v \text{ non-archimedean,}$$

and Kim and Sarnak, who obtain the analogous estimate for v archimedean (with possible equality) in the appendix to [39].

For some applications, the notion of *weakly Ramanujan* [8] can replace knowing the full GRC.

Definition *A cuspidal representation* π *of* $\mathrm{GL}_n(\mathbb{A})$ *is called weakly Ramanujan if for every* $\epsilon > 0$ *there is a constant* $c_\epsilon > 0$ *and an infinite sequence of places* $\{v_m\}$ *with the property that each* π_{v_m} *is unramified and the Satake parameters* $\mu_{v_m,i}$ *satisfy*

$$c_\epsilon^{-1} q_{v_m}^{-\epsilon} < |\mu_{v_m,i}| < c_\epsilon q_{v_m}^{\epsilon}.$$

For example, if we knew that all cuspidal representations on $\mathrm{GL}_n(\mathbb{A})$ were weakly Ramanujan, then we would know that under Langlands liftings between general linear groups, the property of occurrence in the spectral decomposition is preserved [8].

For $n = 2, 3$ our techniques let us show the following.

Proposition 4.1 *For* $n = 2$ *or* $n = 3$ *all cuspidal representations are weakly Ramanujan.*

Proof: First, let π be a cuspidal representation or $\mathrm{GL}_n(\mathbb{A})$. Recall that from the absolute convergence of the Euler product for $L(s, \pi \times \overline{\pi})$ we know that the series $\displaystyle\sum_{v \notin S} \sum_d \frac{|\operatorname{tr}(A_{\pi_v}^d)|^2}{d q_v^{ds}}$ is absolutely convergent for $\mathrm{Re}(s) > 1$, so that in particular we have that $\displaystyle\sum_{v \notin S} \frac{|\operatorname{tr}(A_{\pi_v})|^2}{q_v^s}$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Thus, for a set of places of positive density, we have the estimate $|\operatorname{tr}(A_{\pi_v})|^2 < q_v^\epsilon$ for each ϵ. Since $\overline{A_{\pi_v}} = A_{\pi_v}^{-1}$ for components of cuspidal representations, we have the same estimate for $|\operatorname{tr}(A_{\pi_v}^{-1})|$.

In the case of $n = 2$ and $n = 3$, these estimates and the fact that $|\det A_{\pi_v}| = |\omega_v(\varpi_v)| = 1$ give us estimates on the coefficients of the characteristic polynomial for A_{π_v}. For example, if $n = 3$ and the characteristic polynomial of A_{π_v} is $X^3 + aX^2 + bX + c$ then we know $|a| = |\operatorname{tr}(A_{\pi_v})| < q_v^{\epsilon/2}$, $|b| = |\operatorname{tr}(A_{\pi_v}^{-1}) \det(A_{\pi_v})| < q_v^{\epsilon/2}$, and $|c| = |\det(A_{\pi_v})| = 1$. Then an application of Rouche's theorem gives that the roots of this polynomial all lie in the circle of radius q_v^ϵ as long as $q_v > 3$. Applying this to both A_{π_v} and $A_{\pi_v}^{-1}$ we find that for our set primes of positive density above we have the estimate $q_v^{-\epsilon} < |\mu_{v_m,i}| < q_v^\epsilon$. Thus we find that for $n = 2, 3$ cuspidal representations of GL_n are weakly Ramanujan. □

4.6 The Generalized Riemann Hypothesis (GRH)

This is one of the most important conjectures in the analytic theory of L-functions. Simply stated, it is

Conjecture (GRH) *For any cuspidal representation* π*, all the zeros of the* L*-function* $L(s,\pi)$ *lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

Even in the simplest case of $n = 1$ and $\pi = \mathbf{1}$ the trivial representation this reduces to the Riemann hypothesis for the Riemann zeta function!

For an interesting survey on these and other conjectures on L-functions and their relation to number theoretic problems, we refer the reader to the survey of Iwaniec and Sarnak [27].

5 Converse Theorems

Let us return first to Hecke. Recall that to a modular form

$$f(\tau) = \sum_{n-1}^{\infty} a_n e^{2\pi i n\tau}$$

for, say, $\mathrm{SL}_2(\mathbb{Z})$ Hecke attached an L function $L(s,f)$ and they were related via the Mellin transform

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^{\infty} f(iy)y^s \ d^{\times}y$$

and derived the functional equation for $L(s,f)$ from the modular transformation law for $f(\tau)$ under the modular transformation law for the transformation $\tau \mapsto -1/\tau$. In his fundamental paper [24] he inverted this process by taking a Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and assuming that it converged in a half plane, had an entire continuation to a function of finite order, and satisfied the same functional equation as the L-function of a modular form of weight k, then he could actually reconstruct a modular form from $D(s)$ by Mellin inversion

$$f(iy) = \sum_i a_n e^{-2\pi n y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} (2\pi)^{-s}\Gamma(s)D(s)y^s \ ds$$

and obtain the modular transformation law for $f(\tau)$ under $\tau \mapsto -1/\tau$ from the functional equation for $D(s)$ under $s \mapsto k - s$. This is Hecke's Converse Theorem.

In this Section we will present some analogues of Hecke's theorem in the context of L-functions for GL_n. Surprisingly, the technique is exactly the same as Hecke's, i.e., inverting the integral representation. This was first done in the context of automorphic representation for GL_2 by Jacquet and Langlands [30] and then extended and significantly strengthened for GL_3 by Jacquet, Piatetski-Shapiro, and Shalika [31]. For a more extensive bibliography and history, see [10].

This section is taken mainly from our survey [10]. Further details can be found in [7, 9].

5.1 The Results

Once again, let k be a global field, $\mathbb{A}$ its adele ring, and ψ a fixed non-trivial continuous additive character of $\mathbb{A}$ which is trivial on k. We will take $n \geq 3$ to be an integer.

To state these Converse Theorems, we begin with an irreducible admissible representation Π of $\mathrm{GL}_n(\mathbb{A})$. In keeping with the conventions of these notes, we will assume that Π is unitary and generic, but this is not necessary. It has a decomposition $\Pi = \otimes' \Pi_v$, where Π_v is an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$. By the local theory of Section 3, to each Π_v is associated a local L-function $L(s, \Pi_v)$ and a local ε-factor $\varepsilon(s, \Pi_v, \psi_v)$. Hence formally we can form

$$L(s, \Pi) = \prod_v L(s, \Pi_v) \qquad \text{and} \qquad \varepsilon(s, \Pi, \psi) = \prod_v \varepsilon(s, \Pi_v, \psi_v).$$

We will always assume the following two things about Π:

1. $L(s, \Pi)$ converges in some half plane $Re(s) >> 0$,

2. the central character ω_Π of Π is automorphic, that is, invariant under $k^\times$.

Under these assumptions, $\varepsilon(s, \Pi, \psi) = \varepsilon(s, \Pi)$ is independent of our choice of ψ [7].

Our Converse Theorems will involve twists by cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$ for certain m. For convenience, let us set $\mathcal{A}(m)$

to be the set of automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, $\mathcal{A}_0(m)$ the set of cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$, and $\mathcal{T}(m) = \coprod_{d=1}^{m} \mathcal{A}_0(d)$.

Let $\pi' = \otimes' \pi'_v$ be a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then again we can formally define

$$L(s, \Pi \times \pi') = \prod_v L(s, \Pi_v \times \pi'_v) \qquad \text{and} \qquad \varepsilon(s, \Pi \times \pi') = \prod_v \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

since again the local factors make sense whether Π is automorphic or not. A consequence of (1) and (2) above and the cuspidality of π' is that both $L(s, \Pi \times \pi')$ and $L(s, \widetilde{\Pi} \times \widetilde{\pi}')$ converge absolutely for $Re(s) >> 0$, where $\widetilde{\Pi}$ and $\widetilde{\pi}'$ are the contragredient representations, and that $\varepsilon(s, \Pi \times \pi')$ is independent of the choice of ψ.

We say that $L(s, \Pi \times \pi')$ is *nice* if it satisfies the same analytic properties it would if Π were cuspidal, i.e.,

1. $L(s, \Pi \times \pi')$ and $L(s, \widetilde{\Pi} \times \widetilde{\pi}')$ have analytic continuations to entire functions of s,

2. these entire continuations are bounded in vertical strips of finite width,

3. they satisfy the standard functional equation

$$L(s, \Pi \times \pi') = \varepsilon(s, \Pi \times \pi') L(1 - s, \widetilde{\Pi} \times \widetilde{\pi}').$$

The basic Converse Theorem for GL_n is the following.

Theorem 5.1 *Let* Π *be an irreducible admissible representation of* $\mathrm{GL}_n(\mathbb{A})$ *as above. Suppose that* $L(s, \Pi \times \pi')$ *is nice for all* $\pi' \in \mathcal{T}(n-1)$. *Then* Π *is a cuspidal automorphic representation.*

In this theorem we twist by the maximal amount and obtain the strongest possible conclusion about Π. The proof of this theorem essentially follows that of Hecke [24] and Weil [67] and Jacquet–Langlands [30]. It is of course valid for $n = 2$ as well.

For applications, it is desirable to twist by as little as possible. There are essentially two ways to restrict the twisting. One is to restrict the rank of the groups that the twisting representations live on. The other is to restrict ramification.

When we restrict the rank of our twists, we can obtain the following result.

Theorem 5.2 *Let* Π *be an irreducible admissible representation of* $\mathrm{GL}_n(\mathbb{A})$ *as above. Suppose that* $L(s, \Pi \times \pi')$ *is nice for all* $\pi' \in \mathcal{T}(n-2)$. *Then* Π *is a cuspidal automorphic representation.*

This result is stronger than Theorem 5.1, but its proof is a bit more delicate.

The theorem along these lines that is most useful for applications is one in which we also restrict the ramification at a finite number of places. Let us fix a finite set of S of finite places and let $\mathcal{T}^S(m)$ denote the subset of $\mathcal{T}(m)$ consisting of representations that are *unramified* at all places $v \in S$.

Theorem 5.3 *Let* Π *be an irreducible admissible representation of* $\mathrm{GL}_n(\mathbb{A})$ *as above. Let* S *be a finite set of finite places. Suppose that* $L(s, \Pi \times \pi')$ *is nice for all* $\pi' \in \mathcal{T}^S(n-2)$. *Then* Π *is quasi-automorphic in the sense that there is an automorphic representation* Π' *such that* $\Pi_v \simeq \Pi'_v$ *for all* $v \notin S$.

Note that as soon as we restrict the ramification of our twisting representations we lose information about Π at those places. In applications we usually choose S to contain the set of finite places v where Π_v is ramified.

The second way to restrict our twists is to restrict the ramification at all but a finite number of places. Now fix a non-empty finite set of places S which in the case of a number field contains the set S_∞ of all archimedean places. Let $\mathcal{T}_S(m)$ denote the subset consisting of all representations π' in $\mathcal{T}(m)$ which are *unramified for all* $v \notin S$. Note that we are placing a grave restriction on the ramification of these representations.

Theorem 5.4 *Let* Π *be an irreducible admissible representation of* $\mathrm{GL}_n(\mathbb{A})$ *as above. Let* S *be a non-empty finite set of places, containing* S_∞, *such that the class number of the ring* $\mathfrak{o}_S$ *of* S*-integers is one. Suppose that* $L(s, \Pi \times \pi')$ *is nice for all* $\pi' \in \mathcal{T}_S(n-1)$. *Then* Π *is quasi-automorphic in the sense that there is an automorphic representation* Π' *such that* $\Pi_v \simeq \Pi'_v$ *for all* $v \in S$ *and all* $v \notin S$ *such that both* Π_v *and* Π'_v *are unramified.*

There are several things to note here. First, there is a class number restriction. However, if $k = \mathbb{Q}$ then we may take $S = S_\infty$ and we have a Converse Theorem with "level 1" twists. As a practical consideration, if we let S_Π be the set of finite places v where Π_v is ramified, then for applications we usually take S and S_Π to be disjoint. Once again, we are losing all information at those places $v \notin S$ where we have restricted the ramification unless Π_v was already unramified there.

The proof of Theorem 5.1 essentially follows the lead of Hecke, Weil, and Jacquet–Langlands. It is based on the integral representations of L-functions, Fourier expansions, Mellin inversion, and finally a use of the weak form of Langlands spectral theory. For Theorems 5.2, 5.3, and 5.4, where we have restricted our twists, we must impose certain local conditions to compensate for our limited twists. For Theorem 5.2 and 5.3 there are a finite number of local conditions and for Theorem 5.4 an infinite number of local conditions. We must then work around these by using results on generation of congruence subgroups and either weak or strong approximation.

5.2 Inverting the Integral Representation

Let Π be as above and let $\xi \in V_\Pi$ be a decomposable vector in the space V_Π of Π. Since Π is generic, then fixing local Whittaker models $\mathcal{W}(\Pi_v, \psi_v)$ at all places, compatibly normalized at the unramified places, we can associate to ξ a non-zero function $W_\xi(g) = \prod W_{\xi_v}(g_v)$ on $\mathrm{GL}_n(\mathbb{A})$ which transforms by the global character ψ under left translation by $\mathrm{N}_n(\mathbb{A})$, i.e., $W_\xi(ng) = \psi(n) W_\xi(g)$. Since ψ is trivial on rational points, we see that $W_\xi(g)$ is left invariant under $\mathrm{N}_n(k)$. We would like to use W_ξ to construct an embedding of V_Π into the space of (smooth) automorphic forms on $\mathrm{GL}_n(\mathbb{A})$. The simplest idea is to average W_ξ over $\mathrm{N}_n(k)\backslash \mathrm{GL}_n(k)$, but this will not be convergent. However, if we average over the rational points of the mirabolic $\mathrm{P} = \mathrm{P}_n$ then the sum

$$U_\xi(g) = \sum_{\mathrm{N}_n(k)\backslash \mathrm{P}(k)} W_\xi(pg)$$

is absolutely convergent. For the relevant growth properties of U_ξ see [7]. Since Π is assumed to have automorphic central character, we see that $U_\xi(g)$ is left invariant under both $\mathrm{P}(k)$ and the center $\mathrm{Z}_n(k)$.

Suppose now that we know that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(m)$. Then we will hope to obtain the remaining invariance of U_ξ from the $\mathrm{GL}_n \times \mathrm{GL}_m$ functional equation by inverting the integral representation for $L(s, \Pi \times \pi')$. With this in mind, let $\mathrm{Q} = \mathrm{Q}_m$ be the mirabolic subgroup of GL_n which stabilizes the standard unit vector ${}^t e_{m+1}$, that is, the column vector all of whose entries are 0 except the $(m+1)^{th}$, which is 1. Note that if $m = n-1$ then Q is nothing more than the opposite mirabolic $\overline{\mathrm{P}} = {}^t\mathrm{P}^{-1}$ to P. If we let

α_m be the permutation matrix in $\mathrm{GL}_n(k)$ given by

$$\alpha_m = \begin{pmatrix} & 1 & \\ I_m & & \\ & & I_{n-m-1} \end{pmatrix}$$

then $\mathrm{Q}_m = \alpha_m^{-1}\alpha_{n-1}\overline{\mathrm{P}}\alpha_{n-1}^{-1}\alpha_m$ is a conjugate of $\overline{\mathrm{P}}$ and for any m we have that $\mathrm{P}(k)$ and $\mathrm{Q}(k)$ generate all of $\mathrm{GL}_n(k)$. So now set

$$V_\xi(g) = \sum_{\mathrm{N}'(k)\backslash\,\mathrm{Q}(k)} W_\xi(\alpha_m q g)$$

where $\mathrm{N}' = \alpha_m^{-1}\,\mathrm{N}_n\,\alpha_m \subset \mathrm{Q}$. This sum is again absolutely convergent and is invariant on the left by $\mathrm{Q}(k)$ and $\mathrm{Z}(k)$. Thus, to embed Π into the space of automorphic forms it suffices to show $U_\xi = V_\xi$, for the we get invariance of U_ξ under all of $\mathrm{GL}_n(k)$. It is this that we will attempt to do using the integral representations.

Now let $(\pi', V_{\pi'})$ be an irreducible subrepresentation of the space of automorphic forms on $\mathrm{GL}_m(\mathbb{A})$ and assume $\varphi' \in V_{\pi'}$ is also factorizable. Let

$$I(s; U_\xi, \varphi') = \int_{\mathrm{GL}_m(k)\backslash\,\mathrm{GL}_m(\mathbb{A})} \mathbb{P}_m^n U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\; dh.$$

This integral is always absolutely convergent for $Re(s) >> 0$, and for all s if π' is cuspidal. As with the usual integral representation we have that this unfolds into the Euler product

$$\begin{aligned} I(s; U_\xi, \varphi') &= \int_{\mathrm{N}_m(\mathbb{A})\backslash\,\mathrm{GL}_m(\mathbb{A})} W_\xi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-(n-m)/2}\; dh \\ &= \prod_v \int_{\mathrm{N}_m(k_v)\backslash\,\mathrm{GL}_m(k_v)} W_{\xi_v} \begin{pmatrix} h_v & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\; dh_v \\ &= \prod_v \Psi_v(s; W_{\xi_v}, W'_{\varphi'_v}). \end{aligned}$$

Note that unless π' is generic, this integral vanishes.

Assume first that π' is cuspidal. Then from the local theory of L-functions from Section 3, for almost all finite places we have $\Psi_v(s; W_{\xi_v}, W'_{\varphi'_v}) = L(s, \Pi_v \times \pi'_v)$ and for the other places $\Psi_v(s; W_{\xi_v}, W'_{\varphi'_v}) = e_v(s; W_{\xi_v}, W'_{\varphi'_v})$ $L(s, \Pi_v \times \pi'_v)$ with the $e_v(s; W_{\xi_v}, W'_{\varphi'_v})$ entire and bounded in vertical strips. So in this case we have $I(s; U_\xi, \varphi') = e(s)L(s, \Pi \times \pi')$ with $e(s)$ entire and bounded in vertical strips. Since $L(s;\Pi\times\pi')$ is assumed to be nice we may conclude that $I(s;U_\xi,\varphi')$ has an analytic continuation to an entire function which is bounded in vertical strips. When π' is not cuspidal, it is a subrepresentation of a representation that is induced from (possibly non-unitary) cuspidal representations σ_i of $\mathrm{GL}_{r_i}(\mathbb{A})$ for $r_i < m$ with $\sum r_i = m$ and is in fact, if our integral doesn't vanish, the unique generic constituent of this induced representation. Then we can make a similar argument using this induced representation and the fact that the $L(s,\Pi\times\sigma_i)$ are nice to again conclude that for all π', $I(s;U_\xi,\varphi') = e(s)L(s,\Pi\times\pi') = e'(s)\prod L(s,\Pi\times\sigma_i)$ is entire and bounded in vertical strips. (See [7] for more details on this point.)

Similarly, consider $I(s;V_\xi,\varphi')$ for $\varphi'\in V_{\pi'}$ with π' an irreducible subrepresentation of the space of automorphic forms on $\mathrm{GL}_m(\mathbb{A})$, still with

$$I(s;V_\xi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m V_\xi \begin{pmatrix} h & \\ & 1\end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\, dh.$$

Now this integral converges for $Re(s) << 0$. However, when we unfold, we find

$$I(s;V_\xi,\varphi') = \prod \widetilde{\Psi}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\xi_v},\widetilde{W}'_{\varphi'_v}) = \tilde{e}(1-s)L(1-s,\widetilde{\Pi}\times\widetilde{\pi}')$$

as above. Thus $I(s;V_\xi,\varphi')$ also has an analytic continuation to an entire function of s which is bounded in vertical strips.

Now, utilizing the assumed global functional equation for $L(s,\Pi\times\pi')$ in the case where π' is cuspidal, or for the $L(s,\Pi\times\sigma_i)$ in the case π' is not cuspidal, as well as the local functional equations at $v\in S_\infty\cup S_\Pi\cup S_{\pi'}\cup S_\psi$ as in Section 3 one finds

$$I(s;U_\xi,\varphi') = e(s)L(s,\Pi\times\pi') = \tilde{e}(1-s)L(1-s,\widetilde{\Pi}\times\widetilde{\pi}') = I(s;V_\xi,\varphi')$$

for all φ' in all irreducible subrepresentations π' of $\mathrm{GL}_m(\mathbb{A})$, in the sense of analytic continuation. This concludes our use of the L-function.

We now rewrite our integrals $I(s;U_\xi,\varphi')$ and $I(s;V_\xi,\varphi')$ as follows. We first stratify $\mathrm{GL}_m(\mathbb{A})$. For each $a\in\mathbb{A}^\times$ let $\mathrm{GL}^a_m(\mathbb{A}) = \{g\in \mathrm{GL}_m(\mathbb{A}) \mid \det(g)=a\}$. We can, and will, always take $\mathrm{GL}^a_m(\mathbb{A}) = \mathrm{SL}_m(\mathbb{A})\cdot\begin{pmatrix} a & \\ & I_{m-1}\end{pmatrix}$. Let

$$\langle \mathbb{P}^n_m U_\xi,\varphi'\rangle_a = \int_{\mathrm{SL}_m(k)\backslash \mathrm{GL}^a_m(\mathbb{A})} \mathbb{P}^n_m U_\xi \begin{pmatrix} h & \\ & 1\end{pmatrix}\varphi'(h)\, dh$$

and similarly for $\langle \mathbb{P}^n_m V_\xi, \varphi' \rangle_a$. These are both absolutely convergent for all a and define continuous functions of a on $k^\times \backslash \mathbb{A}^\times$. We now have that $I(s; U_\xi, \varphi')$ is the Mellin transform of $\langle \mathbb{P}^n_m U_\xi, \varphi' \rangle_a$,

$$I(s; U_\xi, \varphi') = \int_{k^\times \backslash \mathbb{A}^\times} \langle \mathbb{P}^n_m U_\xi, \varphi' \rangle_a \ |a|^{s-1/2} \ d^\times a,$$

similarly for $I(s; V_\xi, \varphi')$, and that these two Mellin transforms are equal in the sense of analytic continuation. By Mellin inversion as in Lemma 11.3.1 of Jacquet-Langlands [30], we have that $\langle \mathbb{P}^n_m U_\xi, \varphi' \rangle_a = \langle \mathbb{P}^n_m V_\xi, \varphi' \rangle_a$ for all a, and in particular for $a = 1$. Since this is true for all φ' in all irreducible subrepresentations of automorphic forms on $\mathrm{GL}_m(\mathbb{A})$, then by the weak form of Langlands' spectral theory for SL_m we may conclude that $\mathbb{P}^n_m U_\xi = \mathbb{P}^n_m V_\xi$ as functions on $\mathrm{P}_{m+1}(\mathbb{A})$. More specifically, we have the following result.

Proposition 5.1 *Let* Π *be an irreducible admissible representation of* $GL_n(\mathbb{A})$ *as above. Suppose that* $L(s, \Pi \times \pi')$ *is nice for all* $\pi' \in \mathcal{T}(m)$. *Then for each* $\xi \in V_\Pi$ *we have* $\mathbb{P}^n_m U_\xi(I_{m+1}) = \mathbb{P}^n_m V_\xi(I_{m+1})$.

This proposition is the key common ingredient for all our Converse Theorems.

5.3 Remarks on the Proofs

All of our Converse Theorems take Proposition 5.1 as their starting point. Theorem 5.1 follows almost immediately. In Theorems 5.2, 5.3, and 5.4 we must add local conditions to compensate for the fact that we do not have the full family of twists from Theorem 5.1 and then work around them. We will sketch these arguments here. Details for Theorems 5.1 and 5.4 can be found in [7] and for Theorems 5.2 and 5.3 can be found in [9].

5.3.1 Theorem 5.1

Let us first look at the proof of Theorem 5.1. So we now assume that Π is as above and that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-1)$. Then we have that for all $\xi \in V_\Pi$, $\mathbb{P}^n_{n-1} U_\xi(I_n) = \mathbb{P}^n_{n-1} V_\xi(I_n)$. But for $m = n-1$ the projection operator $\mathbb{P}^n_{n-1}$ is nothing more than restriction to P_n. Hence we have $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\Pi$. Then for each $g \in \mathrm{GL}_n(\mathbb{A})$, we have $U_\xi(g) = U_{\Pi(g)\xi}(I_n) = V_{\Pi(g)\xi}(I_n) = V_\xi(g)$. So the map $\xi \mapsto U_\xi(g)$ gives our embedding of Π into the space of automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, since now

U_ξ is left invariant under $\mathrm{P}(k)$, $\mathrm{Q}(k)$, and hence all of $\mathrm{GL}_n(k)$. Since we still have

$$U_\xi(g) = \sum_{\mathrm{N}_n(k)\backslash \mathrm{P}(k)} W_\xi(pg)$$

we can compute that U_ξ is cuspidal along any parabolic subgroup of GL_n. Hence Π embeds in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$ as desired.

5.3.2 Theorem 5.2

Next consider Theorem 5.2, so now suppose that $n \geq 3$, and that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-2)$. Then from Proposition 5.1 we may conclude that $\mathbb{P}^n_{n-2} U_\xi(I_{n-1}) = \mathbb{P}^n_{n-2} V_\xi(I_{n-1})$ for all $\xi \in V_\Pi$. Since the projection operator $\mathbb{P}^n_{n-2}$ now involves a non-trivial integration over $k^{n-1}\backslash\mathbb{A}^{n-1}$ we can no longer argue as in the proof of Theorem 5.1. To get to that point we will have to impose a local condition on the vector ξ at one place.

Before we place our local condition, let us write $F_\xi = U_\xi - V_\xi$. Then F_ξ is rapidly decreasing as a function on P_{n-1}. We have $\mathbb{P}^n_{n-2} F_\xi(I_{n-1}) = 0$ and we would like to have simply that $F_\xi(I_n) = 0$. Let $u = (u_1, \ldots, u_{n-1}) \in \mathbb{A}^{n-1}$ and consider the function

$$f_\xi(u) = F_\xi \begin{pmatrix} I_{n-1} & {}^t u \\ & 1 \end{pmatrix}.$$

Now $f_\xi(u)$ is a function on $k^{n-1}\backslash\mathbb{A}^{n-1}$ and as such has a Fourier expansion

$$f_\xi(u) = \sum_{\alpha \in k^{n-1}} \hat{f}_\xi(\alpha) \psi_\alpha(u)$$

where $\psi_\alpha(u) = \psi(\alpha \cdot {}^t u)$ and

$$\hat{f}_\xi(\alpha) = \int_{k^{n-1}\backslash\mathbb{A}^{n-1}} f_\xi(u) \psi_{-\alpha}(u)\ du.$$

In this language, the statement $\mathbb{P}^n_{n-2} F_\xi(I_{n-1}) = 0$ becomes $\hat{f}_\xi(e_{n-1}) = 0$, where as always, e_k is the standard unit vector with 0's in all places except the k^{th} where there is a 1.

Note that $F_\xi(g) = U_\xi(g) - V_\xi(g)$ is left invariant under $(\mathrm{Z}(k)\,\mathrm{P}(k)) \cap (\mathrm{Z}(k)\,\mathrm{Q}(k))$ where $\mathrm{Q} = \mathrm{Q}_{n-2}$. This contains the subgroup

$$\mathrm{R}(k) = \left\{ r = \begin{pmatrix} I_{n-2} & & \\ \alpha' & \alpha_{n-1} & \alpha_n \\ & & 1 \end{pmatrix} \middle| \alpha' \in k^{n-2}, \alpha_{n-1} \neq 0 \right\}.$$

Using this invariance of F_ξ, it is now elementary to compute that, with this notation, $\hat{f}_{\Pi(r)\xi}(e_{n-1}) = \hat{f}_\xi(\alpha)$ where $\alpha = (\alpha', \alpha_{n-1}) \in k^{n-1}$. Since $\hat{f}_\xi(e_{n-1}) = 0$ for all ξ, and in particular for $\Pi(r)\xi$, we see that for every ξ we have $\hat{f}_\xi(\alpha) = 0$ whenever $\alpha_{n-1} \neq 0$. Thus

$$f_\xi(u) = \sum_{\alpha \in k^{n-1}} \hat{f}_\xi(\alpha)\psi_\alpha(u) = \sum_{\alpha' \in k^{n-2}} \hat{f}_\xi(\alpha', 0)\psi_{(\alpha',0)}(u).$$

Hence $f_\xi(0, \ldots, 0, u_{n-1}) = \sum_{\alpha' \in k^{n-2}} \hat{f}_\xi(\alpha', 0)$ is constant as a function of u_{n-1}. Moreover, this constant is $f_\xi(e_{n-1}) = F_\xi(I_n)$, which we want to be 0. This is what our local condition will guarantee.

If v is a finite place of k, let $\mathfrak{o}_v$ denote the ring of integers of k_v, and let $\mathfrak{p}_v$ denote the prime ideal of $\mathfrak{o}_v$. We may assume that we have chosen v so that the local additive character ψ_v is normalized, i.e., that ψ_v is trivial on $\mathfrak{o}_v$ and non-trivial on $\mathfrak{p}_v^{-1}$. Given an integer $n_v \geq 1$ we consider the open compact group

$$\begin{aligned} \mathrm{K}_{00,v}(\mathfrak{p}_v^{n_v}) = \{g = (g_{i,j}) \in \mathrm{GL}_n(\mathfrak{o}_v) \mid &(i)\ g_{i,n-1} \in \mathfrak{p}_v^{n_v} \text{ for } 1 \leq i \leq n-2; \\ &(ii)\ g_{n,j} \in \mathfrak{p}_v^{n_v} \text{ for } 1 \leq j \leq n-2; \\ &(iii)\ g_{n,n-1} \in \mathfrak{p}_v^{2n_v}\}. \end{aligned}$$

(As usual, $g_{i,j}$ represents the entry of g in the i-th row and j-th column.)

Lemma *Let v be a finite place of k as above and let (Π_v, V_{Π_v}) be an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$. Then there is a vector $\xi'_v \in V_{\Pi_v}$ and a non-negative integer n_v such that*

1. *for any $g \in \mathrm{K}_{00,v}(\mathfrak{p}_v^{n_v})$ we have $\Pi_v(g)\xi'_v = \omega_{\Pi_v}(g_{n,n})\xi'_v$*

2. $\displaystyle\int_{\mathfrak{p}_v^{-1}} \Pi_v \begin{pmatrix} I_{n-2} & & \\ & 1 & u \\ & & 1 \end{pmatrix} \xi'_v \, du = 0.$

The proof of this Lemma is simply an exercise in the Kirillov model of Π_v and can be found in [9].

If we now fix such a place v_0 and assume that our vector ξ is chosen so that $\xi_{v_0} = \xi'_{v_0}$, then we have

$$F_\xi(I_n) = f_\xi(e_{n-1}) = \mathrm{Vol}(\mathfrak{p}_{v_0}^{-1})^{-1} \int_{\mathfrak{p}_{v_0}^{-1}} f_\xi(0,\dots,0,u_{v_0})\ du_{v_0}$$
$$= \mathrm{Vol}(\mathfrak{p}_{v_0}^{-1})^{-1} \int_{\mathfrak{p}_{v_0}^{-1}} F_\xi \begin{pmatrix} I_{n-2} & & \\ & 1 & u_{v_0} \\ & & 1 \end{pmatrix} du_{v_0} = 0$$

for such ξ.

Hence we now have $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\Pi$ such that $\xi_{v_0} = \xi'_{v_0}$ at our fixed place. If we let $\mathrm{G}' = \mathrm{K}_{00,v_0}(\mathfrak{p}_{v_0}^{n_{v_0}})\,\mathrm{G}^{v_0}$, where we set $\mathrm{G}^{v_0} = \prod'_{v\neq v_0} \mathrm{GL}_n(k_v)$, then we have that this group preserves the local component ξ'_{v_0} up to a constant factor so that for $g \in \mathrm{G}'$ we have $U_\xi(g) = U_{\Pi(g)\xi}(I_n) = V_{\Pi(g)\xi}(I_n) = V_\xi(g)$.

We now use a fact about generation of congruence type subgroups. Let $\Gamma_1 = (\mathrm{P}(k)\,\mathrm{Z}(k)) \cap \mathrm{G}'$, $\Gamma_2 = (\mathrm{Q}(k)\,\mathrm{Z}(k)) \cap \mathrm{G}'$, and $\Gamma = \mathrm{GL}_n(k) \cap \mathrm{G}'$. Then $U_\xi(g)$ is left invariant under Γ_1 and $V_\xi(g)$ is left invariant under Γ_2. It is essentially a matrix calculation that together Γ_1 and Γ_2 generate Γ. So, as a function on G', $U_\xi(g) = V_\xi(g)$ is left invariant under Γ. So if we let $\Pi^{v_0} = \otimes'_{v\neq v_0}\Pi_v$ then the map $\xi^{v_0} \mapsto U_{\xi'_{v_o}\otimes\xi^{v_0}}(g)$ embeds $V_{\Pi^{v_0}}$ into $\mathcal{A}(\Gamma\backslash\,\mathrm{G}')$, the space of automorphic forms on G' relative to Γ. Now, by weak approximation, $\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(k)\cdot\mathrm{G}'$ and $\Gamma = \mathrm{GL}_n(k) \cap \mathrm{G}'$, so we can extend Π^{v_0} to an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Let Π_0 be an irreducible component of the extended representation. Then Π_0 is automorphic and coincides with Π at all places except possible v_0.

One now repeats the entire argument using a second place $v_1 \neq v_0$. Then we have two automorphic representations Π_1 and Π_0 of $\mathrm{GL}_n(\mathbb{A})$ which agree at all places except possibly v_0 and v_1. By the generalized Strong Multiplicity One for GL_n we know that Π_0 and Π_1 are both constituents of the same induced representation $\Xi = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ where each σ_i is a cuspidal representation of some $\mathrm{GL}_{m_i}(\mathbb{A})$, each $m_i \geq 1$ and $\sum m_i = n$. We can write each $\sigma_i = \sigma_i^\circ \otimes |\det|^{t_i}$ with σ_i° *unitary* cuspidal and $t_i \in \mathbb{R}$ and assume $t_1 \geq \cdots \geq t_r$. If $r > 1$, then either $m_1 \leq n-2$ or $m_r \leq n-2$ (or both). For simplicity assume $m_r \leq n-2$. Let S be a finite set of places containing all archimedean places, v_0, v_1, S_Π, and S_{σ_i} for each i. Taking

$\pi' = \widetilde{\sigma}_r \in \mathcal{T}(n-2)$, we have the equality of partial L-functions

$$\begin{aligned} L^S(s, \Pi \times \pi') &= L^S(s, \Pi_0 \times \pi') = L^S(s, \Pi_1 \times \pi') \\ &= \prod_i L^S(s, \sigma_i \times \pi') = \prod_i L^S(s + t_i - t_r, \sigma_i^\circ \times \widetilde{\sigma}_r^\circ). \end{aligned}$$

Now $L^S(s, \sigma_r \times \widetilde{\sigma}_r)$ has a pole at $s = 1$ and all other terms are non-vanishing at $s = 1$. Hence $L(s, \Pi \times \pi')$ has a pole at $s = 1$ contradicting the fact that $L(s, \Pi \times \pi')$ is nice. If $m_1 \leq 2$, then we can make a similar argument using $L(s, \widetilde{\Pi} \times \sigma_1)$. So in fact we must have $r = 1$ and $\Pi_0 = \Pi_1 = \Xi$ is cuspidal. Since Π_0 agrees with Π at v_1 and Π_1 agrees with Π at v_0 we see that in fact $\Pi = \Pi_0 = \Pi_1$ and Π is indeed cuspidal automorphic.

5.3.3 Theorem 5.3

Now consider Theorem 5.3. Since we have restricted our ramification, we no longer know that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-2)$ and so Proposition 5.1 above is not immediately applicable. In this case, for each place $v \in S$ we fix a vector $\xi'_v \in V_{\Pi_v}$ as in the above Lemma. (So we must assume we have chosen ψ so it is unramified at the places in S.) Let $\xi'_S = \prod_{v \in S} \xi'_v \in \Pi_S$. Consider now only vectors ξ of the form $\xi^S \otimes \xi'_S$ with ξ^S arbitrary in V_{Π^S} and ξ'_S fixed. For these vectors, the functions $\mathbb{P}^n_{n-2} U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ and $\mathbb{P}^n_{n-2} V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ are unramified at the places $v \in S$, so that the integrals $I(s; U_\xi, \varphi')$ and $I(s; V_\xi, \varphi')$ vanish unless $\varphi'(h)$ is also unramified at those places in S. In particular, if $\pi' \in \mathcal{T}(n-2)$ but $\pi' \notin \mathcal{T}^S(n-2)$ these integrals will vanish for all $\varphi' \in V_{\pi'}$. So now, for this fixed class of ξ we actually have $I(s; U_\xi, \varphi') = I(s; V_\xi, \varphi')$ for all $\varphi' \in V_{\pi'}$ for all $\pi' \in \mathcal{T}(n-2)$. Hence, as before, $\mathbb{P}^n_{n-2} U_\xi(I_{n-1}) = \mathbb{P}^n_{n-2} V_\xi(I_{n-1})$ for all such ξ.

Now we proceed as before. Our Fourier expansion argument is a bit more subtle since we have to work around our local conditions, which now have been imposed before this step, but we do obtain that $U_\xi(g) = V_\xi(g)$ for all $g \in \mathrm{G}' = (\prod_{v \in S} \mathrm{K}_{00,v}(\mathfrak{p}_v^{n_v}))\, \mathrm{G}^S$. The generation of congruence subgroups goes as before. We then use weak approximation as above, but then take for Π' any constituent of the extension of Π^S to an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. There is no use of strong multiplicity one nor any further use of the L-function in this case. More details can be found in [9].

5.3.4 Theorem 5.4

Let us now sketch the proof of Theorem 5.4. We fix a non-empty finite set of places S, containing all archimedean places, such that the ring $\mathfrak{o}_S$ of S-integer has class number one. Recall that we are now twisting by all cuspidal representations $\pi' \in \mathcal{T}_S(n-1)$, that is, π' which are unramified at all places $v \notin S$. Since we have not twisted by all of $\mathcal{T}(n-1)$ we are not in a position to apply Proposition 5.1. To be able to apply that, we will have to place local conditions *at all* $v \notin S$.

We begin by recalling the definition of the conductor of a representation Π_v of $\mathrm{GL}_n(k_v)$ and the conductor (or level) of Π itself. Let $\mathrm{K}_v = \mathrm{GL}_n(\mathfrak{o}_v)$ be the standard maximal compact subgroup of $\mathrm{GL}_n(k_v)$. Let $\mathfrak{p}_v \subset \mathfrak{o}_v$ be the unique prime ideal of $\mathfrak{o}_v$ and for each integer $m_v \geq 0$ set

$$\mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v}) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}_v) \middle| g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \cdots \quad 0 & * \end{pmatrix} \pmod{\mathfrak{p}^{m_v}} \right\}$$

and $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v}) = \{g \in \mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v}) \mid g_{n,n} \equiv 1 \pmod{\mathfrak{p}_v^{m_v}})\}$. Note that for $m_v = 0$ we have $\mathrm{K}_{1,v}(\mathfrak{p}_v^0) = \mathrm{K}_{0,v}(\mathfrak{p}_v^0) = \mathrm{K}_v$. Then for each local component Π_v of Π there is a unique integer $m_v \geq 0$ such that the space of $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$–fixed vectors in Π_v is exactly one. For almost all v, $m_v = 0$. We take the ideal $\mathfrak{p}_v^{m_v} = \mathfrak{f}(\Pi_v)$ as the conductor of Π_v. Then the ideal $\mathfrak{n} = \mathfrak{f}(\Pi) = \prod_v \mathfrak{p}_v^{m_v} \subset \mathfrak{o}$ is called the conductor of Π. For each place v we fix a non-zero vector $\xi_v^\circ \in \Pi_v$ which is fixed by $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$, which at the unramified places is taken to be the vector with respect to which the restricted tensor product $\Pi = \otimes' \Pi_v$ is taken. Note that for $g \in \mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v})$ we have $\Pi_v(g)\xi_v^\circ = \omega_{\Pi_v}(g_{n,n})\xi_v^\circ$.

Now fix a non-empty finite set of places S, containing the archimedean places if there are any. As is standard, we will let $\mathrm{G}_S = \prod_{v \in S} \mathrm{GL}_n(k_v)$, $\mathrm{G}^S = \prod_{v \notin S} \mathrm{GL}_n(k_v)$, $\Pi_S = \otimes_{v \in S} \Pi_v$, $\Pi^S = \otimes'_{v \notin S} \Pi_v$, etc. Then the compact subring $\mathfrak{n}^S = \prod_{v \notin S} \mathfrak{p}_v^{m_v} \subset k^S$ or the ideal it determines $\mathfrak{n}_S = k \cap k_S \mathfrak{n}^S \subset \mathfrak{o}_S$ is called the S–conductor of Π. Let $\mathrm{K}_1^S(\mathfrak{n}) = \prod_{v \notin S} \mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$ and similarly for $\mathrm{K}_0^S(\mathfrak{n})$. Let $\xi^\circ = \otimes_{v \notin S} \xi_v^\circ \in \Pi^S$. Then this vector is fixed by $\mathrm{K}_1^S(\mathfrak{n})$ and transforms by a character under $\mathrm{K}_0^S(\mathfrak{n})$. In particular, since $\prod_{v \notin S} \mathrm{GL}_{n-1}(\mathfrak{o}_v)$ embeds in $\mathrm{K}_1^S(\mathfrak{n})$ via $h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ we see that when we restrict Π^S to GL_{n-1} the vector ξ° is *unramified.*

Now let us return to the proof of Theorem 5.4 and in particular the version of Proposition 5.1 we can salvage. For every vector $\xi_S \in \Pi_S$ consider the functions $U_{\xi_S\otimes\xi^\circ}$ and $V_{\xi_S\otimes\xi^\circ}$. When we restrict these functions to GL_{n-1} they become unramified for all places $v \notin S$. Hence we see that the integrals $I(s; U_{\xi_S\otimes\xi^\circ}, \varphi')$ and $I(s; V_{\xi_S\otimes\xi^\circ}, \varphi')$ vanish identically if the function $\varphi' \in V_{\pi'}$ is not unramified for $v \notin S$, and in particular if $\varphi' \in V_{\pi'}$ for $\pi' \in \mathcal{T}(n-1)$ but $\pi' \notin \mathcal{T}_S(n-1)$. Hence, for vectors of the form $\xi = \xi_S \otimes \xi^\circ$ we do indeed have that $I(s; U_{\xi_S\otimes\xi^\circ}, \varphi') = I(s; V_{\xi_S\otimes\xi^\circ}, \varphi')$ for all $\varphi' \in V_{\pi'}$ and all $\pi' \in \mathcal{T}(n-1)$. Hence, as in Proposition 5.1 we may conclude that $U_{\xi_S\otimes\xi^\circ}(I_n) = V_{\xi_S\otimes\xi^\circ}(I_n)$ for all $\xi_S \in V_{\Pi_S}$. Moreover, since ξ_S was arbitrary in V_{Π_S} and the fixed vector ξ° transforms by a character of $\mathrm{K}_0^S(\mathfrak{n})$ we may conclude that $U_{\xi_S\otimes\xi^\circ}(g) = V_{\xi_S\otimes\xi^\circ}(g)$ for all $\xi_S \in V_{\Pi_S}$ and all $g \in \mathrm{G}_S\,\mathrm{K}_0^S(\mathfrak{n})$.

What invariance properties of the function $U_{\xi_S\otimes\xi^\circ}$ have we gained from our equality with $V_{\xi_S\otimes\xi^\circ}$. Let us let $\Gamma_i(\mathfrak{n}_S) = \mathrm{GL}_n(k) \cap \mathrm{G}_S\,\mathrm{K}_i^S(\mathfrak{n})$ which we may view naturally as congruence subgroups of $\mathrm{GL}_n(\mathfrak{o}_S)$. Now, as a function on $\mathrm{G}_S\,\mathrm{K}_0^S(\mathfrak{n})$, $U_{\xi_S\otimes\xi^\circ}(g)$ is naturally left invariant under $\Gamma_{0,\mathrm{P}}(\mathfrak{n}_S) = \mathrm{Z}(k)\,\mathrm{P}(k) \cap \mathrm{G}_S\,\mathrm{K}_0^S(\mathfrak{n})$ while $V_{\xi_S\otimes\xi^\circ}(g)$ is naturally left invariant under $\Gamma_{0,\mathrm{Q}}(\mathfrak{n}_S) = \mathrm{Z}(k)\,\mathrm{Q}(k) \cap \mathrm{G}_S\,\mathrm{K}_0^S(\mathfrak{n})$ where $\mathrm{Q} = \mathrm{Q}_{n-1}$. Similarly we set $\Gamma_{1,\mathrm{P}}(\mathfrak{n}_S) = \mathrm{Z}(k)\,\mathrm{P}(k) \cap \mathrm{G}_S\,\mathrm{K}_1^S(\mathfrak{n})$ and $\Gamma_{1,\mathrm{Q}}(\mathfrak{n}_S) = \mathrm{Z}(k)\,\mathrm{Q}(k) \cap \mathrm{G}_S\,\mathrm{K}_1^S(\mathfrak{n})$. The crucial observation for this Theorem is the following result.

Proposition *The congruence subgroup $\Gamma_i(\mathfrak{n}_S)$ is generated by $\Gamma_{i,\mathrm{P}}(\mathfrak{n}_S)$ and $\Gamma_{i,\mathrm{Q}}(\mathfrak{n}_S)$ for $i = 0, 1$.*

This proposition is a consequence of results in the stable algebra of GL_n due to Bass which were crucial to the solution of the congruence subgroup problem for SL_n by Bass, Milnor, and Serre. This is the reason for the restriction to $n \geq 3$ in the statement of Theorem 5.4.

From this we get not an embedding of Π into a space of automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, but rather an embedding of Π_S into a space of classical automorphic forms on G_S. To this end, for each $\xi_S \in V_{\Pi_S}$ let us set

$$\Phi_{\xi_S}(g_S) = U_{\xi_S\otimes\xi^\circ}((g_S, 1^S)) = V_{\xi_S\otimes\xi^\circ}((g_S, 1^S))$$

for $g_S \in \mathrm{G}_S$. Then Φ_{ξ_S} will be left invariant under $\Gamma_1(\mathfrak{n}_S)$ and transform by a Nebentypus character χ_S under $\Gamma_0(\mathfrak{n}_S)$ determined by the central character ω_{Π^S} of Π^S. Furthermore, it will transform by a character $\omega_S = \omega_{\Pi_S}$ under the center $\mathrm{Z}(k_S)$ of G_S. The requisite growth properties are satisfied and hence the map $\xi_S \mapsto \Phi_{\xi_S}$ defines an embedding of Π_S into the space $\mathcal{A}(\Gamma_0(\mathfrak{n}_S)\backslash \mathrm{G}_S; \omega_S, \chi_S)$ of classical automorphic forms on G_S relative to the congruence subgroup $\Gamma_0(\mathfrak{n}_S)$ with Nebentypus χ_S and central character ω_S.

We now need to lift our classical automorphic representation back to an adelic one and hopefully recover the rest of Π. By strong approximation for GL_n and our class number assumption we have the isomorphism between the space of classical automorphic forms $\mathcal{A}(\Gamma_0(\mathfrak{n}_S)\backslash \mathrm{G}_S; \omega_S, \chi_S)$ and the $\mathrm{K}_1^S(\mathfrak{n})$ invariants in $\mathcal{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A}); \omega)$ where ω is the central character of Π. Hence Π_S will generate an automorphic subrepresentation of the space of automorphic forms $\mathcal{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A}); \omega)$. To compare this to our original Π, we must check that, in the space of classical forms, the $\Phi_{\xi_S\otimes\xi^\circ}$ are Hecke eigenforms for a classical Hecke algebra and that their Hecke eigenvalues agree with those from Π. We do this only for those $v \notin S$ which are unramified, where it is a rather standard calculation. As we have not talked about Hecke algebras, we refer the reader to [7] for details.

Now if we let Π' be any irreducible subrepresentation of the representation generated by the image of Π_S in $\mathcal{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A}); \omega)$, then Π' is automorphic and we have $\Pi'_v \simeq \Pi_v$ for all $v \in S$ by construction and $\Pi'_v \simeq \Pi_v$ for all $v \notin S'$ by the Hecke algebra calculation. Thus we have proven Theorem 5.4.

5.4 Converse Theorems and Liftings

In this section we would like to make some general remarks on how to apply these Converse Theorems to the problem of functorial liftings [3].

In order to apply these theorems, you must be able to control the global properties of the L-function. However, for the most part, the way we have of controlling global L-functions is to associate them to automorphic forms or representations. A minute's thought will then lead one to the conclusion that the primary application of these results will be to the lifting of automorphic representations from some group H to GL_n.

Suppose that H is a split classical group, π an automorphic representation of H, and ρ a representation of the L-group of H. Then we should be able to associate an L-function $L(s, \pi, \rho)$ to this situation [3]. Let us assume that $\rho : {}^L\mathrm{H} \to \mathrm{GL}_n(\mathbb{C})$ so that to π should be associated an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$. What should Π be and why should it be automorphic.

We can see what Π_v should be at almost all places. Since we have the (arithmetic) Langlands (or Langlands-Satake) parameterization of representations for all archimedean places and those finite places where the representations are unramified [3], we can use these to associate to π_v and the map

$\rho_v : {}^L\mathrm{H}_v \to \mathrm{GL}_n(\mathbb{C})$ a representation Π_v of $\mathrm{GL}_n(k_v)$. If H happens to be GL_m then we in principle know how to associate the representation Π_v at all places now that the local Langlands conjecture has been solved for GL_m [23, 26], but in practice this is still not feasible. For other situations, we do not know what Π_v should be at the ramified places. We will return to this difficulty momentarily. But for now, let's assume we can finesse this local problem and arrive at a representation $\Pi = \otimes' \Pi_v$ such that $L(s, \pi, \rho) = L(s, \Pi)$. Π should then be the Langlands lifting of π to GL_n associated to ρ.

For simplicity of exposition, let us now assume that ρ is simply the standard embedding of ${}^L\mathrm{H}$ into $\mathrm{GL}_n(\mathbb{C})$ and write $L(s, \pi, \rho) = L(s, \pi) = L(s, \Pi)$. We have our candidate Π for the lift of π to GL_n, but how to tell whether Π is automorphic. This is what the Converse Theorem lets us do. But to apply them we must first be able to define and control the twisted L-functions $L(s, \pi \times \pi')$ for $\pi' \in \mathcal{T}$ with an appropriate twisting set $\mathcal{T}$ from one of our Converse Theorems. This is one reason why it is always crucial to define not only the standard L-functions for H, but also the twisted versions. If we know, from the theory of L-functions of H twisted by GL_m for appropriate π', that $L(s, \pi \times \pi')$ is *nice* and $L(s, \pi \times \pi') = L(s, \Pi \times \pi')$ for twists, then we can use Theorem 5.1 or 5.2 to conclude that Π is cuspidal automorphic or Theorem 5.3 or 5.4 to conclude that Π is quasi-automorphic and at least obtain a weak automorphic lifting Π' which is verifiably the correct representation at almost all places. At this point this relies on the state of our knowledge of the theory of twisted L-functions for H.

Let us return now to the (local) problem of not knowing the appropriate local lifting $\pi_v \mapsto \Pi_v$ at the ramified places. We can circumvent this by a combination of global and local means. The global tool is simply the following observation.

Observation *Let Π be as in Theorem 5.3 or 5.4. Suppose that η is a fixed (highly ramified) character of $k^\times \backslash \mathbb{A}^\times$. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T} \otimes \eta$, where $\mathcal{T}$ is either of the twisting sets of Theorem 5.3 or 5.4. Then Π is quasi-automorphic as in those theorems.*

The only thing to observe, say by looking at the local or global integrals, is that if $\pi' \in \mathcal{T}$ then $L(s, \Pi \times (\pi' \otimes \eta)) = L(s, (\Pi \otimes \eta) \times \pi')$ so that applying the Converse Theorem for Π with twisting set $\mathcal{T} \otimes \eta$ is equivalent to applying the Converse Theorem for $\Pi \otimes \eta$ with the twisting set $\mathcal{T}$. So, by either Theorem 5.3 or 5.4, whichever is appropriate, $\Pi \otimes \eta$ is quasi-automorphic and hence Π is as well.

Now, if we begin with π automorphic on $\mathrm{H}(\mathbb{A})$, we will take T to be the set of finite places where π_v is ramified. For applying Theorem 5.3 we want $S = T$ and for Theorem 5.4 we want $S \cap T = \emptyset$. We will now take η to be highly ramified at all places $v \in T$. So at $v \in T$ our twisting representations are all locally of the form (unramified principal series)$\otimes$(highly ramified character).

We now need to know the following two local facts about the local theory of L-functions for H.

1. *Multiplicativity of γ-factors*: If ${\pi'}_v = \mathrm{Ind}({\pi'}_{1,v} \otimes {\pi'}_{2,v})$, with ${\pi'}_{i,v}$ and irreducible admissible representation of $\mathrm{GL}_{r_i}(k_v)$, then

$$\gamma(s, \pi_v \times {\pi'}_v, \psi_v) = \gamma(s, \pi_v \times {\pi'}_{1,v}, \psi_v)\gamma(s, \pi_v \times {\pi'}_{2,v}, \psi_v)$$

   and $L(s, \pi_v \times {\pi'}_v)^{-1}$ should divide $[L(s, \pi_v \times {\pi'}_{1,v})L(s, \pi \times {\pi'}_{2,v})]^{-1}$. If $\pi_v = \mathrm{Ind}(\sigma_v \otimes \pi'_v)$ with σ_v an irreducible admissible representation of $GL_r(k_v)$ and π'_v an irreducible admissible representation of $\mathrm{H}'(k_v)$ with $\mathrm{H}' \subset \mathrm{H}$ such that $\mathrm{GL}_r \times \mathrm{H}'$ is the Levi of a parabolic subgroup of H, then

$$\gamma(s, \pi_v \times {\pi'}_v, \psi_v) = \gamma(s, \sigma_v \times {\pi'}_v, \psi_v)\gamma(s, \pi'_v \times {\pi'}_v, \psi_v)\gamma(s, \widetilde{\sigma}_v \times {\pi'}_v, \psi_v).$$

2. *Stability of γ-factors*: If $\pi_{1,v}$ and $\pi_{2,v}$ are two irreducible admissible representations of $\mathrm{H}(k_v)$, then for every sufficiently highly ramified character η_v of $GL_1(k_v)$ we have

$$\gamma(s, \pi_{1,v} \times \eta_v, \psi_v) = \gamma(s, \pi_{2,v} \times \eta_v, \psi_v)$$

   and

$$L(s, \pi_{1,v} \times \eta_v) = L(s, \pi_{2,v} \times \eta_v) \equiv 1.$$

Once again, for these applications it is crucial that the local theory of L-functions is sufficiently developed to establish these results on the local γ-factors. As we have seen in Section 3, both of these facts are known for GL_n.

To utilize these local results, what one now does is the following. At the places where π_v is ramified, choose Π_v to be arbitrary, except that it should have the same central character as π_v. This is both to guarantee that the central character of Π is the same as that of π and hence automorphic and to guarantee that the stable forms of the γ–factors for π_v and Π_v agree. Now form $\Pi = \otimes' \Pi_v$. We choose our character η so that at the places $v \in T$ we have that the L– and γ–factors for both $\pi_v \otimes \eta_v$ and $\Pi_v \otimes \eta_v$ are in their stable form and agree. We then twist by $\mathcal{T} \otimes \eta$ for this *fixed* character η. If $\pi' \in \mathcal{T} \otimes \eta$, then for $v \in T$, ${\pi'}_v$ is of the form ${\pi'}_v = \mathrm{Ind}(\mu_{v,1} \otimes \cdots \otimes \mu_{v,m}) \otimes \eta_v$ with each $\mu_{v,i}$ an unramified character of $k_v^\times$. So at the places $v \in T$ we have

$$\begin{aligned} \gamma(s, \pi_v \times {\pi'}_v) &= \gamma(s, \pi_v \times (\mathrm{Ind}(\mu_{v,1} \otimes \cdots \otimes \mu_{v,m}) \otimes \eta_v)) \\ &= \prod \gamma(s, \pi_v \otimes (\mu_{v,i}\eta_v)) \text{ (by multiplicativity)} \\ &= \prod \gamma(s, \Pi_v \otimes (\mu_{v,i}\eta_v)) \text{ (by stability)} \\ &= \gamma(s, \Pi_v \times (\mathrm{Ind}(\mu_{v,1} \otimes \cdots \otimes \mu_{v,m}) \otimes \eta_v)) \text{ (by multiplicativity)} \\ &= \gamma(s, \Pi_v \times {\pi'}_v) \end{aligned}$$

and similarly for the L-factors. From this it follows that globally we will have $L(s, \pi \times \pi') = L(s, \Pi \times \pi')$ for all $\pi' \in \mathcal{T} \otimes \eta$ and the global functional equation for $L(s, \pi \times \pi')$ will yield the global functional equation for $L(s, \Pi \times \pi')$. So $L(s, \Pi \times \pi')$ is *nice* and we may proceed as before. We have, in essence, twisted away all information about π and Π at those $v \in T$. The price we pay is that we also lose this information in our conclusion since we only know that Π is quasi-automorphic. In essence, the Converse Theorem fills in a correct set of data at those places in T to make the resulting global representation automorphic.

5.5 Some Liftings

To conclude, let us make a list of some of the liftings that have been accomplished using these Converse Theorems. Some have used the above trick of multiplicativity and stability of γ–factors to handle the ramified places. Others, principally those that involve GL_2, have adopted a technique of Ramakrishnan [51] involving a sequence of base changes and descents to get a more complete handle on the ramified places.

1. The symmetric square lifting from GL_2 to GL_3 by Gelbart and Jacquet [15].

2. Non-normal cubic base change for GL_2 by Jacquet, Piatetski-Shapiro, and Shalika [32].

3. The tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 by Ramakrishnan [51].

4. The lifting of generic cusp forms from SO_{2n+1} to GL_{2n}, with Kim, Piatetski-Shapiro, and Shahidi [6].

5. The tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6 and the symmetric cube lifting from GL_2 to GL_4 by Kim and Shahidi [40].

6. The exterior square lifting from GL_4 to GL_6 and the symmetric fourth power lift from GL_2 to GL_5 by Kim [39].

For the most part, it was Theorem 5.3 that was used in each case, with the exception of (4), where a simpler variant was used requiring twists by $\mathcal{T}^S(n-1)$. For the non-normal cubic base change both Theorem 5.3 with $n = 3$ and Theorem 5.1 with $n = 2$ were used.

Acknowledgments

Most of what I know about L-functions for GL_n I have learned through my years of work with Piatetski-Shapiro. I owe him a great debt of gratitude for all that he has taught me.

For several years Piatetski-Shapiro and I have envisioned writing a book on L-functions for GL_n [13]. The contents of these notes essentially follows our outline for that book. In particular, the exposition in Sections 1, 2, and parts of 3 and 4 is drawn from drafts for this project. The exposition in Section 5 is drawn from the survey of our work on Converse Theorems [10]. I would also like to thank Piatetski-Shapiro for graciously allowing me to present part of our joint efforts in these notes.

I would also like to thank Jacquet for enlightening conversations over the years on his work on L-functions for GL_n.

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