# Topics in Random Walks in Random Environment

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**Abstract**

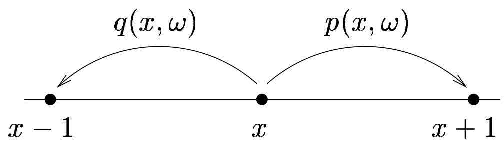
Over the last twenty-five years random motions in random media have been intensively investigated and some new general methods and paradigms have by now emerged. *Random walks in random environment* constitute one of the canonical models of the field. However in dimension bigger than one they are still poorly understood and many of the basic issues remain to this day unresolved. The present series of lectures attempt to give an account of the progresses which have been made over the last few years, especially in the study of multi-dimensional random walks in random environment with ballistic behavior.

# Contents

# 1 Presentation of the model-dimension one

In this first lecture we will introduce the subject in a one-dimensional context. There are many different ways to "inject" randomness in a medium, where a stochastic motion is performed. Here are two emblematic examples:

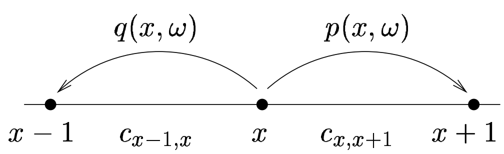
**a) Site randomness:**




one chooses i.i.d. variables $p(x,\omega)$, $x \in \mathbb{Z}$, with values in $[0,1]$, $q(x,\omega) = 1 - p(x,\omega)$, and for a given realization $\omega$ of the environment, one considers the Markov chain $(X_n)$ on $\mathbb{Z}$, which has probability $p(x,\omega)$ of jumping to the right neighbor $x+1$ and $q(x,\omega)$ of jumping to the left neighbor $x-1$, given it is located in $x$. This is the so-called "**random walk in random environment**".

The model goes back to Chernov [8], Temkin [44], and was at first introduced as a toy-model for the replication of DNA-chains.

**b) Bond randomness:**




one now chooses i.i.d. variables $c_{x,x+1}(\omega)$, $x \in \mathbb{Z}$, with values in $(0,\infty)$, and for a given realization of the environment, $(X_n)$ is a Markov chain on $\mathbb{Z}$ performing jumps to nearest neighbors with a transition kernel now determined by

$$p(x,\omega) = \frac{c_{x,x+1}(\omega)}{c_{x-1,x}(\omega) + c_{x,x+1}(\omega)}\,.$$

This is essentially a random conductivity model of the type introduced in disordered media physics going at least back to Fatt [15] or Kirkpatrick [22], (see also the book of Hughes [16]).

Both models in a), b) are easily generalized to higher dimension. It is striking that they turn out to exhibit quite different behaviors. We will now discuss in the context of model a) and b) one of the general techniques of random motions in random media.

## A) The environment viewed from the particle

This technique has been quite powerful in the study of various kinds of random motions in random media. It focuses on the investigation of the environment as viewed from the current location of the walker. More specifically in the case of b), for $0 < a < b < \infty$,

$$\Omega = [a,b]^{\mathcal{E}} \text{ with } \mathcal{E} = \big\{\{x,x+1\}, x \in \mathbb{Z}\big\}, \text{ the set of nearest-neighbor bonds on } \mathbb{Z}, \text{ endowed with the canonical product } \sigma\text{-field } \mathcal{B} \tag{1}$$

$$\mathbb{P} = \text{a product measure on } \Omega, \text{ making the canonical coordinates i.i.d.} \tag{2}$$

$$t_x, x \in \mathbb{Z}, \text{ the canonical translations on } \Omega\text{: } (t_y\omega)(\{x,x+1\}) = \omega(\{x+y,x+y+1\}) \tag{3}$$

$$P_{x,\omega}, x \in \mathbb{Z}, \text{ the canonical law of the Markov chain on } \mathbb{Z} \text{ with transition probability described in b), with } c_{x,x+1}(\omega) = \omega(\{x,x+1\})\,. \tag{4}$$

The **environment viewed from the particle** is the $\omega$-valued process:

$$\overline{\omega}_n = t_{X_n}\omega,\ n \geq 0\,. \tag{5}$$

**Fact:** (see Lecture 1 of [5]).

Under $P_{0,\omega}$, $\omega \in \Omega$, or under $P_0 = \mathbb{P} \times P_{0,\omega}$, $\overline{\omega}_n$ is a Markov chain with state space $\Omega$, transition kernel:

$$Rf(\omega) = p(0,\omega)\, f \circ t_1(\omega) + q(0,\omega)\, f \circ t_{-1}(\omega),\ f \text{ bounded measurable on } \Omega\,, \tag{6}$$

and respective initial laws $\delta_\omega$ (Dirac mass at $\omega$) and $\mathbb{P}$.

The above fact is a-priori of little use because this Markov chain has a huge state space (in particular it accommodates simultaneously all possible static laws $\mathbb{P}$ of the environment!). However what makes it useful is that one has in case b) an explicit invariant probability $\mathbb{Q}$ for the kernel $R$ (i.e. $\mathbb{Q}R = \mathbb{Q}$), which is absolutely continuous with respect to $\mathbb{P}$. Namely define:

$$\mathbb{Q} = \frac{1}{Z}\,\big(\omega(\{-1,0\}) + \omega(\{0,1\})\big)\,\mathbb{P} \quad (Z\text{: normalization constant}) \tag{7}$$

then

$$\begin{aligned}
&\int hRg\,d\mathbb{Q} \overset{(6)}{=} \int h(p(0,\omega)\,g\circ t_1 + q(0,\omega)\,g\circ t_{-1})\,d\mathbb{Q} \overset{(7)}{=} \\
&\tfrac{1}{Z}\int h\,\omega(\{0,1\})\,g\circ t_1\,d\mathbb{P} + \tfrac{1}{Z}\int h\,\omega(\{-1,0\})\,g\circ t_{-1}\,d\mathbb{P} \overset{\substack{\text{translation}\\\text{invariance}}}{=} \\
&\tfrac{1}{Z}\int h\circ t_{-1}\,\omega(\{-1,0\})\,g\,d\mathbb{P} + \tfrac{1}{Z}\int h\circ t_1\,\omega(\{0,1\})\,g\,d\mathbb{P} \overset{\substack{\text{similar}\\\text{calculation}}}{=} \int (Rh)\,g\,d\mathbb{Q}.
\end{aligned}$$

In other words:

$$\mathbb{Q} \text{ is reversible for } R \text{ (and hence invariant for } R). \tag{8}$$

Using general arguments (see Lecture 1 of [5] or Kozlov [24]), one has:

$$\begin{array}{l}\mathbb{Q} \text{ is an ergodic invariant probability of the Markov chain on}\\ (\Omega,\mathcal{B}) \text{ attached to } R.\end{array} \tag{9}$$

It is in fact **the only invariant measure of $R$ absolutely continuous with respect to $\mathbb{P}$** (from general arguments, one knows that it is also equivalent to $\mathbb{P}$, here this is obvious by direct inspection of (7)). The fact that $\mathbb{P}$ was an i.i.d. measure does not play an important role and a translation invariant ergodic measure $\mathbb{P}$ would work as well (i.e. $t_x\mathbb{P} = \mathbb{P}$ for all $x$, and $A \in \mathcal{B}$ with $t_x^{-1}(A) = A$ for all $x \Longrightarrow \mathbb{P}(A) = 0$ or 1).

The measure $\mathbb{Q}$ plays an important role. In particular it enables to apply the ergodic theorem (cf. [13], Chapter 6 §2) to the Markov chain $\overline{\omega}_n$. For instance consider

$$\begin{array}{l} p(x,\omega) - q(x,\omega) = E_{x,\omega}[X_1 - X_0] \overset{\text{def}}{=} d(x,\omega)\\ \textbf{the local drift} \text{ at } x \text{ in the environment } \omega. \end{array} \tag{10}$$

If $\mathcal{F}_n = \sigma(X_\sigma, \dots, X_n)$ is the filtration generated by $(X_.)$, then

$$\begin{array}{l} M_n = X_n - X_0 - \sum\limits_0^{n-1} d(X_k,\omega) \text{ is an } (\mathcal{F}_n)\text{-martingale under } P_{0,\omega}\\ \text{with increments bounded by } c = 2\,. \end{array} \tag{11}$$

As a result of Azuma's inequality (cf. [1]):

$$P_{0,\omega}[\pm M_n > \lambda c\sqrt{n}] \le \exp\Big\{-\frac{\lambda^2}{2}\Big\}, \text{ for } \lambda > 0,\ n \ge 0. \tag{12}$$

Choosing $0 < \epsilon < \frac{1}{2}$, $\lambda = n^\epsilon$, we see from Borel-Cantelli's lemma that

$$\text{for } \omega \in \Omega,\ P_{0,\omega}\text{-a.s.},\ \frac{M_n}{n} \to 0, \tag{13}$$

and hence questions on the limit of $\frac{X_n}{n}$ are transferred to questions on the limit of $\frac{1}{n}\sum_0^{n-1} d(X_k,\omega) = \frac{1}{n}\sum_0^{n-1} d(0,\overline{\omega}_k)$. As a result of Birkhoff's ergodic theorem, (cf. [13]), we conclude that:

$$\text{for } \mathbb{Q}\text{-a.e. } \omega, \text{ (or } \mathbb{P}\text{-a.e. } \omega),\ P_{0,\omega}\text{-a.s.},\ \frac{1}{n}\sum_0^{n-1} d(0,\overline{\omega}_k) \to \int_\Omega d(0,\omega)\, d\mathbb{Q} \tag{14}$$

and specifically,

$$\int d(0,\omega) d\mathbb{Q} \underset{(10)}{\overset{(7)}{=}} \frac{1}{Z}\int [\omega(\{0,1\}) - \omega(\{-1,0\})]\, d\mathbb{P} = 0,$$

so that

$$P_0\text{-a.s. (or equivalently for } \mathbb{P}\text{-a.e. } \omega,\ P_{0,\omega}\text{-a.s.)}\ \frac{X_n}{n} \to 0, \tag{15}$$

(incidentally, one can also argue that under the semi-product measure $\mathbb{Q} \times P_{0,\omega}$, the sequence $(X_{k+1} - X_k)_{k\ge 0}$ is stationary, ergodic, and from this fact recover (15)).

We thus conclude that the walk has a vanishing limiting velocity in the case of model b). For much more on this model, in arbitrary dimension, we refer to [2], [3], [21], [27], [28].

As we will now see things run quite differently in the case of model a) i.e. for random walks in random environment. We now choose

$$\Omega = [a,b]^{\mathbb{Z}} \text{ with } 0 < a < \tfrac{1}{2} < b < 1, \tag{16}$$

and keep similar notations as in (1) - (5), with obvious modifications, we also set $p(x,\omega) = \omega(x)$. An important role is played by the quantity

$$\rho(x,\omega) = \frac{q(x,\omega)}{p(x,\omega)} \text{ (we will write } \rho \text{ for } \rho(0,\omega)). \tag{17}$$

It will be instructive to discuss the following result of Solomon [36]:

**Theorem 1.1.** 1) *Depending on whether* $\mathbb{E}[\log \rho] < 0, > 0, = 0,$

$$\begin{aligned} P_0\text{-}a.s. \quad & \lim_n X_n = +\infty, \ or \\ & \lim_n X_n = -\infty, \ or \\ & \limsup_n X_n = +\infty, \ \liminf_n X_n = -\infty\,. \end{aligned} \tag{18}$$

2) *Moreover,* $P_0$*-a.s.* $\frac{X_n}{n} \to v$*, where in case*

$$\begin{array}{lll} \text{i)} & \mathbb{E}[\rho] < 1\,, & v = \dfrac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]} > 0\,, \\ \text{ii)} & \dfrac{1}{\mathbb{E}[\rho^{-1}]} \le 1 \le \mathbb{E}[\rho], & v = 0\,, \\ \text{iii)} & 1 < \dfrac{1}{\mathbb{E}[\rho^{-1}]}\,, & v = \dfrac{\mathbb{E}[\rho^{-1}]-1}{1+\mathbb{E}[\rho^{-1}]} < 0\,. \end{array} \tag{19}$$

**Remarks:** 1) Since $\frac{1}{\mathbb{E}[\rho^{-1}]} \overset{\text{Jensen}}{\le} \mathbb{E}[\rho]$, i), ii), iii) cover the only possible positions of 1 relative to $\mathbb{E}[\rho]$ and $1/\mathbb{E}[\rho^{-1}]$.

2) As a result of Jensen's inequality in i) of 2),

$$\mathbb{E}[\rho] = \mathbb{E}\Big[\frac{1}{p} - 1\Big] = \mathbb{E}\Big[\frac{1}{p}\Big] - 1 \ge \frac{1}{\mathbb{E}[p]} - 1\,,$$

and $0 < u \to \frac{1-u}{1+u}$ is a decreasing function, we thus see that

$$0 < v \le \frac{2 - \frac{1}{\mathbb{E}[p]}}{\frac{1}{\mathbb{E}[p]}} = 2\mathbb{E}[p] - 1 = \mathbb{E}[p] - \mathbb{E}[q] = \mathbb{E}[d(0,\omega)]\,, \tag{20}$$

and the right inequality is strict when the law of $p$ is not concentrated on one point. Analogously exchanging the role of "left" and "right", in case iii)

$$0 > v \ge \mathbb{E}[p] - \mathbb{E}[q] = \mathbb{E}[d(0,\omega)]\,, \tag{21}$$

with a strict inequality when the law of $p$ is not concentrated on one point.

$\mathbb{E}[p] - \mathbb{E}[q]$ can be viewed as the naive (but wrong) guess for the asymptotic velocity. Hence from (20), (21), one already senses that some slowdown of the walk occurs.

3) In case (19) ii), the particle can move very slowly, see B) further below. In particular it has been shown by Sinai [35], that when $0 < \mathbb{E}[(\log \rho)^2] < \infty$ and $\mathbb{E}[\log \rho] = 0$, $X_n$ has typical size $\sim (\log n)^2$ under $P_0$, see also Kesten [19], Revesz [32]. $\square$

*Sketch of proof of Theorem* 1.1*:*

• **(1.18)**: One defines for $a \le b$:

$$\Pi_{a,b} = \Pi_{a<y\le b}\, \rho(y,\omega) \quad (= 1, \text{ when } a = b)\,, \tag{22}$$

as well as:

$$\begin{aligned} f(x,\omega) = -\sum_{0\le z<x} \Pi_{0,z},\ &\text{ for } x \ge 0\,, \\ \sum_{x\le z<0} \Pi^{-1}_{z,0},\ &\text{ for } x < 0\,, \end{aligned} \tag{23}$$

(so $f(0,\omega) = 0$, $f(1,\omega) = -1$ and for $x \in \mathbb{Z}$, $p(x,\omega)\, f(x+1,\omega) + q(x,\omega)\, f(x-1,\omega) = f(x,\omega)$).

As a result of the law of large numbers:

$$\begin{aligned} \mathbb{P}\text{-a.s.:}\quad \Pi_{0,z} &= \exp\{z(\mathbb{E}[\log \rho] + o(1))\}, \text{ as } z \to \infty\,, \\ \Pi^{-1}_{z,0} &= \exp\{z(\mathbb{E}[\log \rho] + o(1))\}, \text{ as } z \to -\infty\,. \end{aligned} \tag{24}$$

So for instance when $\mathbf{\mathbb{E}[\log \rho] < 0}$:

$$\mathbb{P}\text{-a.s.,}\ \lim_{x\to\infty} f(x,\omega) = -c < 0,\ \lim_{x\to-\infty} f(x,\omega) = +\infty\,. \tag{25}$$

On the other hand using the Markov property:

$$f(X_n,\omega) \text{ is a martingale under } P_{0,\omega}\,. \tag{26}$$

Using the martingale convergence theorem $P_{0,\omega}$-a.s., $f(X_n,\omega)$ converges to a finite limit, which is necessarily $-c$. Hence

$$\text{for } \mathbb{P}\text{-a.e. } \omega,\ P_{0,\omega}\text{-a.s.: } \lim_n X_n = +\infty\,. \tag{27}$$

Analogously we obtain the statement in (18) concerning the case $\mathbf{\mathbb{E}[\log \rho] > 0}$.

In the case $\mathbb{E}[\log \rho] = 0$: One can now show that

$$\mathbb{P}\text{-a.s.: } \lim_{x\to\infty} f(x,\omega) = -\infty, \ \lim_{x\to-\infty} f(x,\omega) = +\infty\,. \tag{28}$$

Defining for $A \geq 0$,

$$T = \inf\{k \geq 0,\ X_k \geq A\} \text{ and } S = \inf\{k \geq 0, X_k \leq -A\}\,,$$

one argues by looking at the martingales $f(X_{n\wedge T},\omega)$ $f(X_{n\wedge S},\omega)$, and the martingale convergence theorem that

$$\mathbb{P}\text{-a.s., } P_{0,\omega}[T < \infty \text{ and } S < \infty] = 1\,. \tag{29}$$

Letting $A \to \infty$, we obtain the statement (18).

• **(1.19):** We first look at **case i):**

The method of the environment viewed from the particle applies. Indeed, if we define:

$$\mathbb{Q} = f(\omega)\,\mathbb{P}, \text{ with } f(\omega) = \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]}\,(1+\rho(0,\omega))\Big(\sum_{y\geq 0}\Pi_{0,y}\Big) \tag{30}$$

(note that $f \geq 0$, and $\int f(\omega)\,d\mathbb{P} \overset{\textbf{independence}}{=} \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]}\,(1+\mathbb{E}[\rho])(\sum_{n\geq 0}\mathbb{E}[\rho]^n) =$ $1$ because $\mathbb{E}[\rho] \overset{\text{i)}}{<} 1$). Moreover:

$$\begin{aligned}
&\mathbb{Q}R = (p(-1,\omega)\,f\circ t_{-1} + q(1,\omega)\,f\circ t_1)\,\mathbb{P} \\
&(\text{because } \mathbb{Q}(Rg) = \int f(\omega)\big(p(0,\omega)\,g\circ t_1 + q(0,\omega)\,g\circ t_{-1}\big)\,d\mathbb{P} \\
&\overset{\substack{\text{translation}\\ \text{invariance}}}{=} \int \big(p(-1,\omega)\,f\circ t_{-1} + q(1,\omega)\,f\circ t_1)\,g\,d\mathbb{P}\big)\,,
\end{aligned} \tag{31}$$

and as we now see $\mathbb{Q}R = \mathbb{Q}$, because:

$$\begin{aligned}
&\frac{1+\mathbb{E}[\rho]}{1-\mathbb{E}[\rho]}\,(p(-1,\omega)\,f\circ t_{-1} + q(1,\omega)\,f\circ t_1) = \\
&\underbrace{p(-1,\omega)(1+\rho(-1,\omega))}_{\overset{\shortparallel}{1}}\Big(\sum_{y\geq -1}\Pi_{-1,y}\Big) + \overbrace{q(1,\omega)(1+\rho(1,\omega))}^{\underset{\shortparallel}{\rho(1,\omega)}}\Big(\sum_{y\geq 1}\Pi_{1,y}\Big) = \\
&1 + \rho(0,\omega) + \rho(0,\omega)\,\rho(1,\omega) + \cdots + \rho(1,\omega) + \rho(1,\omega)\,\rho(2,\omega) + \cdots = \\
&(1+\rho(0,\omega))\big(1 + \rho(1,\omega) + \rho(1,\omega)\,\rho(2,\omega) + \dots\big) = \frac{1+\mathbb{E}[\rho]}{1-\mathbb{E}[\rho]}\,f(\omega)\,.
\end{aligned} \tag{32}$$

Hence $\mathbb{Q}$ is the only invariant probability absolutely continuous with respect to $\mathbb{P}$ and by similar arguments as in (15), for $\mathbb{P}$-a.e. $\omega$, $P_{0,\omega}$-a.s.,

$$\begin{aligned} &\frac{X_n}{n} \to \int d(0,\omega) d\mathbb{Q} = \\ &\frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]}\, \mathbb{E}\big[\underbrace{(p(0,\omega)-q(0,\omega))(1+\rho(0,\omega))}_{\substack{\shortparallel \\ 1-\rho(0,\omega)}}(1+\rho(1,\omega)+\dots)\big] \\ &\overset{\text{invariance}}{=} \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]} \times (1-\mathbb{E}[\rho]) \Big(\sum_{n\geq 0} \mathbb{E}[\rho^n]\Big) = \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]}\,. \end{aligned} \tag{33}$$

This proves i). The claim iii) is of course entirely analogous.

We now turn to **case ii):** We will use an argument of comparison to deduce ii) from i) and iii). To this end we observe that if $\omega, \omega' \in \Omega$ (see (1.16)) and

$$\omega(x) \leq \omega'(x), \text{ for all } x \in \mathbb{Z}\,, \tag{34}$$

then we can construct a Markov chain on $\mathbb{Z}^2_{\text{even}} = \{\overline{x} = (x,x') \in \mathbb{Z}^2;\ x - x' \in 2\,\mathbb{Z}\}$, $(X_n, X'_n)$, with law $P_{\overline{x},\omega,\omega'}$ for any $\overline{x} = (x,x') \in \mathbb{Z}^2_{\text{even}}$, such that

$$\begin{cases} \text{under } P_{\overline{x},\omega,\omega'}, (X_n) \text{ has same law as under } P_{x,\omega} \\ \text{under } P_{\overline{x},\omega,\omega'}, (X'_n) \text{ has same law as under } P_{x',\omega} \\ \text{if } x \leq x', \text{ then } P_{\overline{x},\omega,\omega'}[X_n \leq X'_n, \text{ for all } n] = 1,\ (\overline{x} = (x,x'))\,. \end{cases} \tag{35}$$

Indeed we choose the probability of the Markov chain by letting the particle jump independently when located at $\overline{x} = (x,x')$ with $x \neq x'$, with respective probabilities $p(x,\omega) = \omega(x)$, $p(x,\omega') = \omega'(x')$ of respectively moving to $x+1$ and $x'+1$, and $q(x,\omega) = 1-\omega(x)$, $q(x',\omega') = 1-\omega'(x')$ of respectively moving to $x-1$, and $x'-1$. On the other hand, when

$$\begin{aligned} \overline{x} = (x,x) \text{ the chain jumps with probability } &\omega(x):\ (x,x) \to (x+1,x+1) \\ &\omega'(x) - \omega(x):\ (x,x) \to (x-1,x+1) \\ &1-\omega'(x):\ (x,x) \to (x-1,x-1)\,. \end{aligned}$$

In particular if we define $\omega' \geq \omega$ via:

$$\omega'(x) = \omega(x)(1-\eta) + \eta b\,,\ (\text{recall } b > \tfrac{1}{2}, \text{ cf. } (16)),\ \eta \in (0,1)\,, \tag{36}$$

then by taking $\eta$ large enough $\mathbb{E}[\rho'] < 1$, and by (35) and i)

$$\text{for } \mathbb{P}\text{-a.e. } \omega,\ P_{(0,0),\omega,\omega'}\text{-a.s.},\ \overline{\lim_n} \frac{X_n}{n} \le \overline{\lim_n} \frac{X'_n}{n} = \lim_n \frac{X'_n}{n} = \frac{1-\mathbb{E}[\rho']}{1+\mathbb{E}[\rho']} \quad (37)$$

by adjusting $\eta$, we can make the right hand side arbitrarily small and hence

$$P_0\text{-a.s.},\ \overline{\lim_n} \frac{X_n}{n} \le 0\,. \quad (38)$$

By an entirely analogous argument:

$$P_0\text{-a.s.},\ \underline{\lim_n} \frac{X_n}{n} \ge 0\,, \quad (39)$$

and this finishes the proof of (19) in case ii). □

**Remarks:** 1) Note that unlike (19), (18) does not rely on the independence of the variables $p(x,\omega) = \omega(x)$, $x \in \mathbb{Z}$, but only on the fact that they are stationary and ergodic, (in the case $\mathbb{E}[\log \rho] = 0$, one can use a result of Kesten [18] on stationary sequences to conclude that $\mathbb{P}$-a.s., $\Pi_{0,z}$ does not tend to zero as $z$ tends to $\pm\infty$, so that (28) holds). For the correct generalization of (19) for a stationary ergodic environment we refer to Theorem 2.1.12 of Zeitouni [45], see also Molchanov [26], p. 277.

2) Let us point out that it is possible that

$$\mathbb{E}[d(0,\omega)] = \mathbb{E}[p(0,\omega)] - \mathbb{E}[q(0,\omega)] > 0\,, \text{ but}$$
$$\mathbb{P}_0\text{-a.s. } X_n \to -\infty \text{ (although with null limiting velocity because of (21))}\,.$$

Indeed if with probability
$$\begin{aligned} \tfrac{3}{4}:&\quad \omega = p_0 \quad \text{with } \tfrac{2}{3} < p_0 < 1 \\ \tfrac{1}{4}:&\quad \omega = \epsilon \quad\ \ \text{with } \epsilon > 0 \text{ small}, \end{aligned}$$

then $\mathbb{E}[p(0,\omega)] \ge \frac{3}{4} \times p_0 > \frac{1}{2}$ and hence $\mathbb{E}[d(0,\omega)] = 2\mathbb{E}[p(0,\omega)] - 1 > 0$. On the other hand:

$$\mathbb{E}[\log \rho] = \frac{3}{4}\ \log \frac{1-p_0}{p_0} + \frac{1}{4}\ \log\ \frac{1-\epsilon}{\epsilon} > 0 \text{ if } \epsilon \text{ is small}.$$

□

**B) The effect of traps in model a)**

The above proof of (19) however sheds little light on the nature of the phenomenon taking place and in particular leading to the fact that under (19) ii) (i.e. $\mathbb{E}[\rho^{-1}]^{-1} \le 1 \le \mathbb{E}[\rho]$) the limiting velocity $v$ vanishes. We will now see that this is related to certain **large deviation effects** leading to the presence of certain "**traps**" in the medium. To explain this we assume that

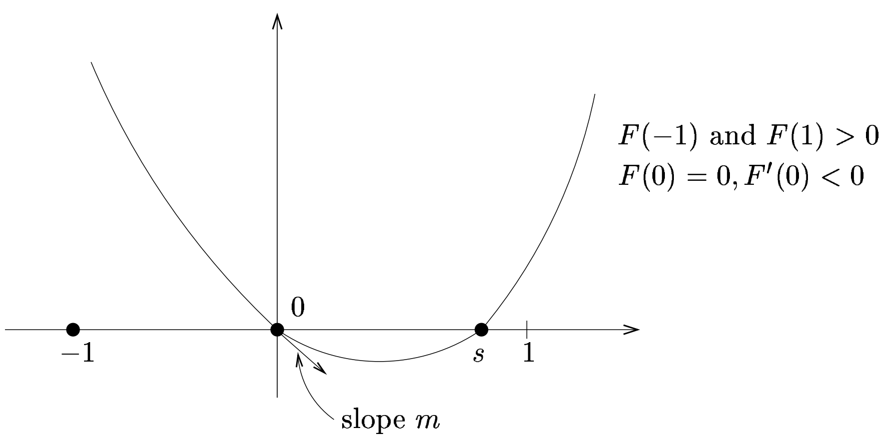
$$\mathbb{E}[\rho] > 1,\ \mathbb{E}[\rho^{-1}] > 1 \tag{40}$$

and for specificity

$$m = \mathbb{E}[\log \rho] < 0\,. \tag{41}$$

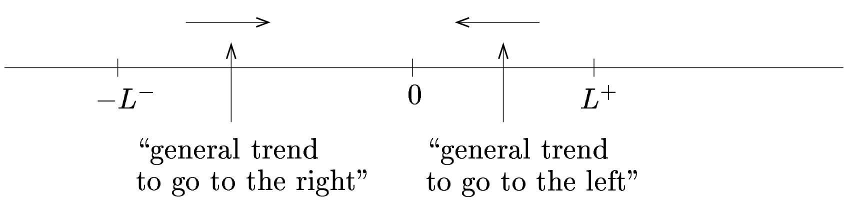
By (1.18) we know that the walk tends $P_0$-a.s. to $\infty$. We introduce the strictly convex analytic function

$$F(u) = \log(\mathbb{E}[\rho^u]),\ u \in \mathbb{R}, \tag{42}$$




"The graph of $F$"

We are going to create a type of trap in $U = [-L^-, L^+]$



To quantitatively measure the "trapping effect" taking place in $U$, we observe that from the martingale property of $f(X_n, \omega)$, with

$$H_z \stackrel{\text{def}}{=} \inf\{k \geq 0, X_k = z\}, \text{ for } z \in \mathbb{Z}, \tag{43}$$

$$\begin{aligned} P_{1,\omega}[H_0 < H_{L^+ + 1}] &= \frac{f(L^+ + 1, \omega) - f(1, \omega)}{f(L^+ + 1, \omega) - f(0, \omega)} = 1 - \frac{\overbrace{f(0,\omega) - f(1,\omega)}^{\stackrel{1}{\shortparallel}\,(23)}}{\underbrace{f(0,\omega) - f(L^+ + 1,\omega)}_{\shortparallel\,(23)\ \sum_{0 \leq z \leq L^+} \Pi_{0,z}}} \\ &\geq (1 - \exp\{-L_+ \underbrace{R_{[1,L^+]}}_{\substack{\shortparallel \text{ def} \\ R^+}}\})_+ \end{aligned} \tag{44}$$

with the notation:

$$R_J = \frac{1}{|J|} \sum_{x \in J} \log\, \rho(x, \omega), \tag{45}$$

for $J$ a non-empty finite subset of $\mathbb{Z}$ and $|J|$ = cardinality of $J$. In a similar fashion:

$$P_{-1,\omega}[H_0 < H_{-(L^- + 1)}] \geq (1 - \exp\{L_- \overbrace{R_{[L^-, -1]}}^{\substack{R^- \\ \shortparallel \text{ def}}}\})_+ \,. \tag{46}$$

Let us write $R_L$ for $R_{[1,L]}$, when $L \geq 1$. The law of large numbers yields that

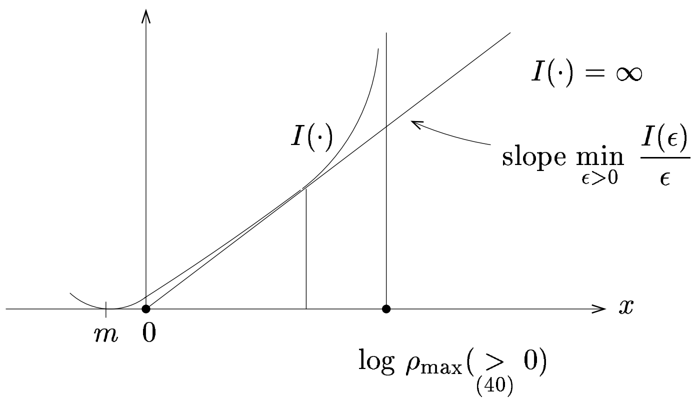
$$R_L \underset{L \to \infty}{\longrightarrow} m = \mathbb{E}[\log \rho] < 0, \;\; \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})\,. \tag{47}$$

But large deviations do occur and from Cramer's theory (see [12] or [11]), for $\epsilon > 0$,

$$P(R_L > \epsilon) = \exp\{-L(I(\epsilon) + o(1))\} \text{ as } L \to \infty\,, \tag{48}$$

where for $x \in \mathbb{R}$,

$$I(x) = \sup_{u \in \mathbb{R}} \{ux - F(u)\} \tag{49}$$




If we now set for $U = [-L^-, L^+]$

$$\gamma(U) = 1 \wedge \max(\exp\{L^- R^-\},\ \exp\{-L^+ R^+\}) \tag{50}$$

then with $T_U$ the exit time from $U$:

$$T_U = \inf\{k \geq 0,\ X_k \notin U\}\,, \tag{51}$$

we see from the strong Markov property that

$$P_{0,\omega}[T_U > n] \geq P_{0,\omega}[\widetilde{H}_0 < T_U]^n \geq (1 - \gamma(U))^n \tag{52}$$

(with $\widetilde{H}_0 = \inf\{k \geq 1,\ X_k = 0\}$ the hitting time of 0). In particular, if $\gamma(U) \leq n^{-1}$:

$$P_{0,\omega}[T_U > n] \geq (1 - \gamma(U))^n \geq c = e^{-2} > 0, \text{ for } n \text{ large}. \tag{53}$$

Note that for $L^- \geq \frac{2}{|m|} \log n, L^+, \epsilon$ with $L^+ \epsilon \geq \log n$,

$$\mathbb{P}\Big[\gamma(U) \leq \frac{1}{n}\Big] \geq \mathbb{P}\Big[R^- \leq \frac{m}{2}\,,\ R^+ \geq \epsilon\Big] \tag{54}$$

using independence, (47), (48), for $\delta > 0$ (small) and large $n$:

$$\geq \tfrac{1}{2}\ \exp\{-I(\epsilon)\, L^+\, (1 + \delta)\}\,.$$

We can now optimize $\epsilon, L^+$ by looking at:

$$\inf\{I(\epsilon)L^+;\ \epsilon L^+ \geq \log n\} = \inf\Big\{\frac{I(\epsilon)}{\epsilon} \times \epsilon L^+,\ \epsilon L^+ \geq \log\, n\Big\} = \inf_{\epsilon>0}\ \frac{I(\epsilon)}{\epsilon} \times \log n$$

and observe that $\inf_{\epsilon>0} \frac{I(\epsilon)}{\epsilon}$ is the slope of the tangent to $I(\cdot)$ which is passing through the origin of the coordinate axes. On the other hand (cf. [12], p. 55):

$$F(u) = \sup_x (xu - I(x)) \quad \text{(compare with (49))}\,. \tag{55}$$

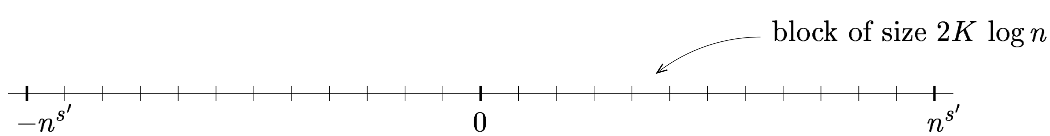
Hence if $\alpha = \min\limits_{\epsilon>0} \frac{I(\epsilon)}{\epsilon}\,(>0)$ we see that $F(\alpha) = 0$. In other words:

$$\min_{\epsilon>0} \frac{I(\epsilon)}{\epsilon} = s \ \text{ is the positive zero of the function } F(\cdot) \text{ (by assumption } s < 1)\,. \tag{56}$$

Hence for a suitable $K > 0$ and any $\delta > 0$, for large $n$:

$$\mathbb{P}[P_{0,\omega}[T_{[-K\log n, K\log n]} > n] \geq e^{-2}] \geq n^{-s(1+\delta)}\,. \tag{57}$$

With (57) we have a have bound on the probability of creating a "trap" centered at the origin of size $\sim \log n$, which typically will retain the walk for $n$ units of time. If $s' > s$, choosing $\delta$ small in (57), we can easily infer that for large $n$, there will be many analogous such "traps" both in $[-n^{s'}, 0]$ and $[0, n^{s'}]$.




These traps will induce slowdowns of the walk and will typically prevent that it moves to distances $n^{s'}$ from the origin before time $n$. More precisely for large $n$:

$$\begin{aligned} &P_0[X_n > n^{s'}] \leq P_0[T_1 + \cdots + T_M < n, \\ &X_. \text{ reaches the center of the i.th block } i = 1, \ldots, M] \end{aligned} \tag{58}$$

with $M$ the number of blocks in $[0, n^{s'}]$ (which is of order const $\frac{n^{s'}}{\log n}$), and $T_i$ the time to exit the i.th block after reaching its center. So picking $\lambda > 0$, and applying Chebychev's inequality

$$\begin{aligned} &\leq \exp \lambda n\, E_0[\exp\{-\lambda(T_1 + \cdots + T_M)\}\,, \\ &X_. \text{ reaches the center of the i.th block } i = 1, \ldots, M]\,. \end{aligned}$$

Using the strong Markov property under $P_{0,\omega}$, the independence under $\mathbb{P}$ of environments in different blocks, and the stationarity, we find

$$
\begin{aligned}
&= \quad \exp\{\lambda n\}\ E_0[e^{-\lambda T_{[-K \ \log n, K \log n]}}]^M \\
&\overset{(57)}{\leq} \quad \exp\{\lambda n\} \Big(1 - \frac{e^{-2}}{n^{s(1+\delta)}} + \frac{e^{-2-\lambda n}}{n^{s(1+\delta)}}\Big)^M, \ \text{ using } 1 - x \leq e^{-x} \\
&\leq \quad \exp\Big\{\lambda n - M \frac{e^{-2}}{n^{s(1+\delta)}}(1 - e^{-\lambda n})\Big\} \quad M \sim \text{const}\, \frac{n^{s'}}{\log n}\,,
\end{aligned}
$$

if we now choose $\delta$ small, $\lambda = \frac{1}{2}\, n^{s'-s(1+2\delta)-1}$, for $n$ large

$$
\leq \ \exp\Big\{-\frac{1}{2}\, n^{s'-s(1+2\delta)} - n^{s'-s(1+2\delta)}\Big\} = \exp\Big\{-\frac{1}{2}\, n^{s'-s(1+2\delta)}\Big\}\,.
$$

Estimating similarly $P_0[X_n < -n^{s'}]$, we find that $\sum_n P_0[|X_n| > n^{s'}] < \infty$, and from Borel Cantelli's lemma we conclude that:

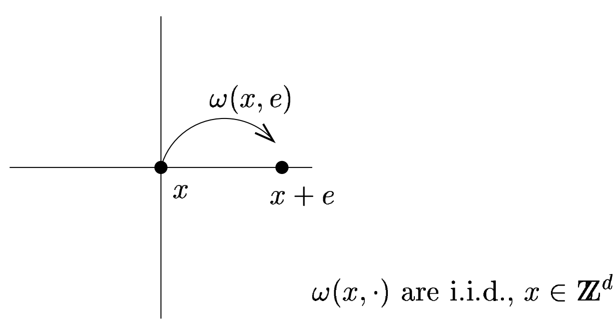
$$
P_0\text{-a.s.} \ \lim_n \frac{|X_n|}{n^{s'}} = 0, \ \text{ for } s' > s = \text{positive zero of } F \ (\text{recall that } s < 1)\,, \tag{59}
$$

(more detailed results can be found in [20]).

We see from (59) that the walk is truly moving sublinearly (in a more precise fashion than explained in (19), ii)). **The role of traps** in this **slowdown** is also brought to light, (see also [10]).

## 2 Higher dimension - traps - conditions (T) and (T')

We are now turning to **higher dimensional random walks** in **random environment**, and we have the following setting:



$$\Omega = \mathcal{P}_\kappa^{\mathbb{Z}^d} \text{ where } \kappa \in (0, \tfrac{1}{2d}] \text{ is the "ellipticity constant" and} \tag{60}$$
$\mathcal{P}_\kappa =$ the collection of $(2d)$-vectors $(p(e))_{|e|=1}$ with
$p(e) \in [\kappa, 1]$ for all $e$ and $\sum_{|e|=1} p(e) = 1$ .

$\mathbb{P} =$ product measure on $\Omega$, making the $\omega(x, \cdot)$, $x \in \mathbb{Z}^d$, (61)
i.i.d. with common distribution $\mu$ .

$P_{x,\omega}$, $x \in \mathbb{Z}^d$, the canonical law of the Markov chain on $\mathbb{Z}^d$ (62)
with transition probability $\omega(y, e)$ of jumping to $y + e$
when located in $y$ .

$P_x = \mathbb{P} \times P_{x,\omega}$, (this is a semi-direct product). The integration (63)
over $\mathbb{P}$ restores some translation invariance but typically
destroys the Markov property. One can however represent
the law of the walk under $P_x$ as a type of "edge oriented
reinforced random walk", see for instance [14] .

In the multi-dimensional context ($d \geq 2$), there is no known simple classification of the asymptotic behavior of the walk in the fashion of (18) and (19) of Lecture 1. Such a classification of possible asymptotic behaviors of the walk as far as recurrence, transience, ballistic and diffusive behavior of the walks are concerned, is still very much "under construction". The method of the environment viewed from the particle has had so far relatively little impact on the study of multi-dimensional random walks in random environment. There are no explicit formulas as (30), and the existence of the dynamic invariant measure $\mathbb{Q}$ is only known in a few cases, cf. [4], [23], [25]. One substantial difficulty for the mathematical investigation of the model stems from its genuinely non-reversible character. Until recently few references dealt with multi-dimensional random walks in random environment (see Kalikow [17], for a sufficient condition for transience, Lawler [25] for a central limit theorem for driftless walks, and Bricmont-Kupiainen [7] for a central limit theorem for small isotropic perturbations of the simple random walk, when $d \geq 3$). However over the last few years there has been progress in the understanding of the situation where the walk has a non-degenerate limiting velocity thanks to certain conditions which we describe further below. For other recent developments see [4], [6], [45], [46], [47], [48]. We further refer to [9], [31], [45] for dependent environments, and to [23], [34], for continuous time and state space.

## Traps and spectral considerations

This is maybe one of the most important effects for random walks in random environment. We have already seen the importance of traps in the one-dimensional context when discussing (19) ii) at the end of Lecture 1).

Informally traps are "pockets where the walk can spend a long time with a comparatively high probability". We are now going to define a spectral quantity to measure trapping effects.

For $\phi \neq U \subseteq \mathbb{Z}^d$, we define

$$R^U_{1,\omega} = 1_U \, P_\omega \, 1_U \text{ with } P_\omega \, f(x) = \sum_{|e|=1} \omega(x,e)\, f(x+e)\,, \tag{64}$$

in other words $R^U_{1,\omega}$ is the transition kernel of the walk killed when exiting $U$. If $R^U_{n,\omega} = (R^U_{1,\omega})^n$, $n \geq 1$, then

$$\begin{aligned} &R^U_{n,\omega}\, f(x) = E_{x,\omega}[f(X_n),\, T_U > n], \text{ with } T_U = \inf\{n \geq 0,\, X_n \notin U\}, \\ &\text{the exit time from } U\,. \end{aligned} \tag{65}$$

Observe that

$$\|R^U_{n,\omega}\|_{\infty,\infty} = \sup_x \, P_{x,\omega}[T_U > n]\,,$$

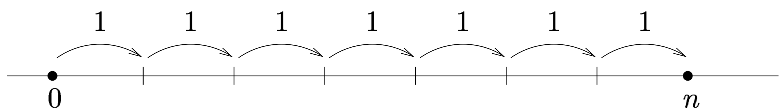
and by sub-multiplicativity ($\|R^U_{n+m,\omega}\|_{\infty,\infty} \leq \|R^U_{n,\omega}\|_{\infty,\infty}\, \|R^U_{m,\omega}\|_{\infty,\infty}$) we can define

$$\begin{aligned} \lambda_\omega(U) &= \lim_n -\frac{1}{n}\, \log \|R^U_{n,\omega}\|_{\infty,\infty} \in [0,\infty] \\ &= \sup_n -\frac{1}{n}\, \log \|R^U_{n,\omega}\|_{\infty,\infty}\,. \end{aligned} \tag{66}$$

In other words $e^{-\lambda_\omega(U)}$ is the spectral radius of $R^U_{1,\omega}$ acting on $L^\infty(U)$ (cf. Rudin [33], p. 234-325). The number $\lambda_\omega(U)$ quantifies the strength of the trap created by $\omega$ in $U$. **The smaller $\boldsymbol{\lambda_\omega(U)}$ the stronger the trapping effect**.

One should however be aware that we are dealing with a quite non-self adjoint situation, see also [5] p. 32, 33, and $\lambda_\omega(U)$ offers a quantitative lower bound on $\|R^U_{n,\omega}\|_{\infty,\infty}$ (namely as follows from the second line of (66), $\|R^U_{n,\omega}\|_{\infty,\infty} \geq e^{-n\lambda_\omega(U)}$, $n \geq 1$), but in general does not provide a quantitative upper bound on $\|R^U_{n,\omega}\|_{\infty,\infty} = \sup_x P_{x,\omega}[T_U > n]$. A trivial example to feel the problem is:

$U = \{0, \dots, n\}$ (depending on $n$), $d = 1$, $\omega$: "jump to the right"



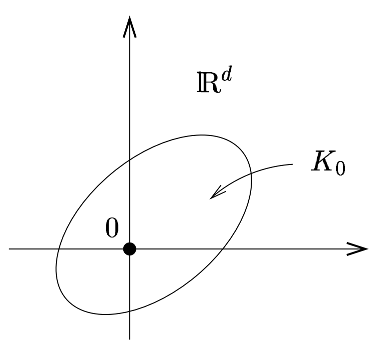

then $\lambda_\omega(U) = \infty$, but $P_{0,\omega}[T_U > n] = 1$ which is not dominated by $e^{-n\lambda_\omega(U)}$. The problem comes from the fact that **$\boldsymbol{\lambda_\omega(U)}$ only measures an asymptotic ability for survival**. However the above remark on the non-quantitative character of $\lambda_\omega(U)$ should be taken with a grain of salt, as we will explain further below, (cf. (75)).

Before expanding on this last remark, let us first discuss the fact that there is a **classification** of the **strength** of **possible traps** which can be **created by a random environment**. To this end we consider the **local drift** at $x$ in the environment $\omega$:

$$d(x,\omega) = \sum_{|e|=1} \omega(x,e)e = E_{x,\omega}[X_1 - X_0]\,, \tag{67}$$

and introduce

$$K_0 = \text{the convex hull of the support of the law of } d(0,\omega) \text{ under } \mathbb{P}\,. \tag{68}$$




"An example of $K_0$ in the plain nestling case"

The position of 0 with respect to $K_0$ plays a crucial role. Namely (cf. Theorem 1.2 in [40]), with $B_L = B(0,L) \cap \mathbb{Z}^d$, and $c_1, c_2$ two positive constants depending on $d$ and the single site distribution $\mu$:

- in the "**non-nestling case**" (i.e. $0 \notin K_0$):
  i) $\mathbb{P}$-a.s., $c_2 \le \lambda_\omega(B_L) \le c_1$, $L > 1$,

- in the "**marginal nestling case**":
  ii) $\mathbb{P}$-a.s., $\frac{c_2}{L^2} \le \lambda_\omega(B_L)$, $L > 1$, and $\mathbb{P}\Big[\lambda_\omega(B_L) \le \frac{c_1}{L^2}\Big] > 0, L > 1$,

- in the "**plain nestling case**":
  iii) $\mathbb{P}$-a.s., $e^{-c_2 L} \le \lambda_\omega(B_L)$, $L > 1$, and $\mathbb{P}[\lambda_\omega(B_L) \le e^{-c_1 L}] > 0, L$ large. (69)

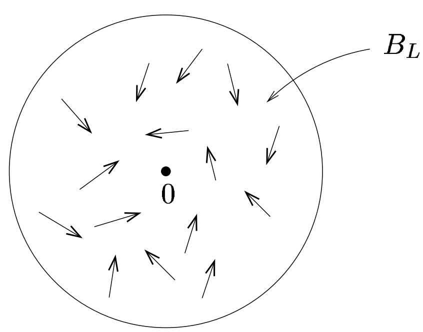
The above list corresponds to an increasing strength of the possible traps. In case iii) the rightmost control is for instance obtained by creating "**naive traps**" corresponding to the event

$$\mathcal{T}_L = \Big\{\omega : \ \forall x \in B_L \backslash \{0\}, \ d(x,\omega) \cdot \frac{x}{|x|} \le -\gamma\Big\}, \tag{70}$$

when $\gamma > 0$ is chosen such that

$$\inf_{w \in S^{d-1}} \mathbb{P}[(d(0,\omega) \cdot w)_- \ge \gamma] \ge \gamma > 0$$

(using the plain-nestling assumption and $P\big[X > \frac{1}{2}\, E[X]\big] \ge \frac{1}{4}\, \frac{E[X]^2}{E[X^2]}$, for $X$ a positive variable).




"A naive trap in the plain nestling case, cf. (70)"

As was alluded to above, the number $\lambda_\omega(U)$ can be used to produce quantitative lower bounds (i.e. non-exclusively asymptotic) on the probability of surviving in $U$. For instance

**Lemma 2.1.**

*For* $L > 0$, $n \geq 0$, $P_0[|X_n| < 2L] \geq P_0[T_{B_{2L}} > n] \geq \frac{1}{|B_L|}\,\mathbb{E}[\exp\{-n\lambda_\omega(B_L)\}]\,.$ (71)

*Proof.* For $x \in B_L$, $P_0[T_{B_{2L}} > n] \geq P_0[T_{B_L - x} > n] \overset{\text{translation invariance}}{=} P_x[T_{B_L} > n]$, so that summing over $x \in B_L$:

$$P_0[T_{B_{2L}} > n] \geq \frac{1}{|B_L|} \sum_{x \in B_L} P_x[T_{B_L} > n] = \frac{1}{|B_L|}\,\mathbb{E}\Big[\sum_{x\in B_L} P_{x,\omega}[T_{B_L} > n]\Big]$$

$$\overset{\text{(66) second line}}{\geq} \frac{1}{|B_L|}\,\mathbb{E}[\exp\{-n\lambda_\omega(B_L)\}]\,.$$

□

There is a-priori no quantitative upper-bounds in general. One can however introduce for $\phi \neq U \subseteq \mathbb{Z}^d$,

$$\overline{\lambda}_\omega(U) = \frac{\log 2}{t_\omega(U)}\,, \text{ where } t_\omega(U) = \inf\Big\{n \geq 0,\ \|R^U_{n,\omega}\|_{\infty,\infty} \leq \tfrac{1}{2}\Big\} \in \{1, \dots, \infty\}\,. \tag{72}$$

yields quantitative upper bound on decay of

From the inequality $e^{-\lambda_\omega(U)\, t_\omega(U)} \leq \frac{1}{2}$, when $t_\omega(U) < \infty$, we infer

$$\lambda_\omega(U) \geq \overline{\lambda}_\omega(\omega)\,. \tag{73}$$

Of course these numbers can be very different, for instance in the example given above (67), $\lambda_\omega(U) = \infty$, but $\overline{\lambda}_\omega(U) = \frac{\log 2}{n+1}$ !!!

However one can devise an upper bound for $\lambda_\omega(U)$ in terms of $\overline{\lambda}_\omega(U)|U|$, and when $\overline{\lambda}_\omega(U)|U|$ is small then $\lambda_\omega(U)$ is small as well:

**Lemma 2.2.** *($\phi \neq U$ finite, $\omega \in \Omega$)*

$$\begin{aligned} &\text{there exist } x_0 \in U, \text{ such that } P_{x_0,\omega}[\widetilde{H}_{x_0} > T_U] \le \frac{2|U|}{t_\omega(U)}\,, \\ &(\widetilde{H}_{x_0} = \inf\{n \ge 1, X_n = x_0\} \text{ the hitting time of } x_0) \end{aligned} \tag{74}$$

$$e^{-\lambda_\omega(U)} \ge 1 - \frac{2}{\log 2}\,\overline{\lambda}_\omega(U)|U|\,, \tag{75}$$

*or equivalently:*

$$t_\omega(U) \le 2|U| \,/\, (1 - e^{-\lambda_\omega(U)})\,.$$

*Proof.* For some $x_1 \in U$, $\frac{1}{2} < P_{x_1,\omega}[T_U \ge t_\omega(U)]$ and by a classical Markov chain calculation

$$\frac{1}{2}\,t_\omega(U) \le E_{x_1,\omega}[T_U] = \sum_{y \in U} \frac{P_{x_1,\omega}[H_y < T_U]}{P_{y,\omega}[\widetilde{H}_y > T_U]} \le \frac{|U|}{\inf\limits_{y \in U} P_{y,\omega}[\widetilde{H}_y > T_U]}\,,$$

and (74) follows. Then observe that for $y \in U$

$$P_{y,\omega}[\widetilde{H}_y < T_U]^n \le P_{y,\omega}[T_U > n] \le \|R^U_{n,\omega}\|_{\infty,\infty}$$

(this has very much the same flavor as (52)), and hence taking the $n$-th root and letting $n$ tend to infinity:

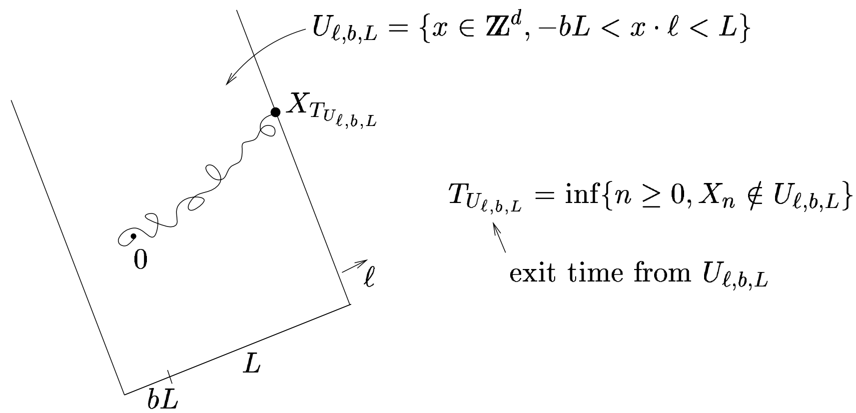
$$e^{-\lambda_\omega(U)} \ge P_{y,\omega}[\widetilde{H}_y < T_U] = 1 - P_{y,\omega}[\widetilde{H}_y > T_U]\,. \tag{76}$$

If we now choose $y = x_0$, we obtain our claim. □

Observe that (75) offers some quantitative upper-bound on the decay of $\|R^U_{n,\omega}\|_{\infty,\infty}$ (via an upper-bound on $t_\omega(U)$) from the knowledge of $\lambda_\omega(U)$ and $|U|$. This is the grain of salt alluded to after the example above (67).

We now turn to the discussion of ballistic walks, and begin with the definition of conditions which have far reaching consequences.

**Conditions (T) and (T')**:




**Definition 2.3.** *($\ell \in S^{d-1}$, $0 < \gamma \leq 1$)*
*The condition* (T)$_\gamma$ *holds relative to $\ell \in S^{d-1}$ (notation:* (T)$_\gamma|\ell$*), if*

$$\begin{aligned} &\textit{for all } \ell' \in S^{d-1} \textit{ in a neighborhood of } \ell, \\ &\overline{\lim_{L\to\infty}} L^{-\gamma} \log P_0[X_{T_{U_{\ell',b,L}}} \cdot \ell' < 0] < 0, \textit{ for all } b > 0\,. \end{aligned} \tag{77}$$

**Terminology:**

- The condition (T) relative to $\ell$ is (T)$_{\gamma=1}|\ell$
- The condition (T') relative to $\ell$ is (T)$_\gamma|\ell$ for all $0 < \gamma < 1$.

It is clear that:

$$\text{(T)}|\ell \Longrightarrow \text{(T')}|\ell \Longrightarrow \text{(T)}_\gamma|\ell, \text{ for } 0 < \gamma < 1\,. \tag{78}$$

The **belief** is that in fact the above conditions are equivalent. Some partial results in this direction are known, (for instance this is known when $d = 1$, and all conditions in (65) are equivalent to $P_0[\lim X_n \cdot \ell = \infty] = 1$, see Proposition 2.6 of [40], and when $d \geq 2$, (T')$|\ell \Longleftrightarrow$ (T)$_\gamma|\ell$ if $\frac{1}{2} < \gamma < 1$).

We will now provide an equivalent formulation of (T)$_\gamma|\ell$ in terms of certain **regeneration times**.

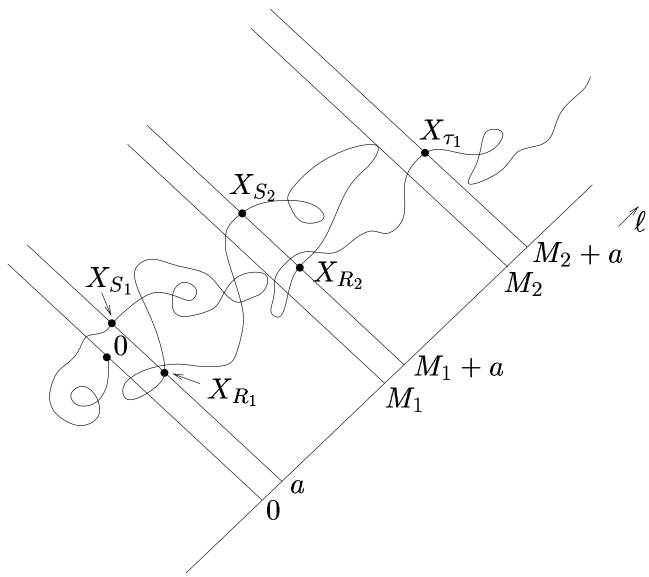
**Notation:** for $\ell \in S^{d-1}, u \in \mathbb{R}$,

$$\begin{aligned} &T_u^\ell = \inf\{n \geq 0, X_n \cdot \ell \geq u\},\ \widetilde{T}_u^\ell = \inf\{n \geq 0, X_n \cdot \ell \leq u\} \\ &D^\ell = \inf\{n \geq 0, X_n \cdot \ell < X_0 \cdot \ell\}\,. \end{aligned} \tag{79}$$

Assume that

$$P_0[\lim_n X_n \cdot \ell = \infty] = 1\,, \tag{80}$$

then for an arbitrary $a > 0$, we will define a random variable $\tau_1$, $P_0$-a.s. finite which is "the first time where $X_n \cdot \ell$ goes by an amount at least $a$ above its previous local maxima and never goes below this level from then on", (this variable will not be a stopping time relative to the natural filtration of $X_n$).




More precisely:

$$S_0 = 0,\ M_0 = X_0 \cdot \ell; \qquad (\theta\text{: canonical shift on space of trajectories})$$

$$S_1 = T^\ell_{M_0+a} \leq \infty,\ R_1 = D^\ell \circ \theta_{S_1} + S_1 \leq \infty,$$

$$M_1 = \sup\{X_n \cdot \ell,\, 0 \leq n \leq R_1\} \leq \infty$$

and by induction:

$$S_{k+1} = T^\ell_{M_k+a} \leq \infty,\ R_{k+1} = D^\ell \circ \theta_{S_{k+1}} + S_{k+1} \leq \infty,$$

$$M_{k+1} = \sup\{X_n \cdot \ell,\, 0 \leq n \leq R_{k+1}\}\,.$$

With (80) it is not hard to show (see Proposition 1.2 of [43])

$$P_0[D^\ell = \infty] > 0, \text{ and } P_0\text{-a.s. } K < \infty, \text{ provided} \\ K \stackrel{\text{def}}{=} \inf\{k \ge 1, S_k < \infty \text{ and } R_k = \infty\}\,. \tag{81}$$

We can then define the regeneration time

$$\tau_1 = S_K\,. \tag{82}$$

As we will now explain the conditions $(\mathrm{T})_\gamma|\ell$ can be rephrased in terms of an estimate on the size of the trajectory $X_k$, $0 \le k \le \tau_1$, (cf. [41], Theorem 1.1).

**Theorem 2.4.** *($\ell \in S^{d-1}$, $0 < \gamma \le 1$, $a > 0$)*

*One has the equivalence*

$$\begin{aligned} &\text{i) } (\mathrm{T})_\gamma|\ell \\ &\text{ii) } (80) \textit{ and for some } c > 0,\ E_0[\exp\{c \sup_{0 \le k \le \tau_1} |X_k|^\gamma\}] < \infty\,. \end{aligned} \tag{83}$$

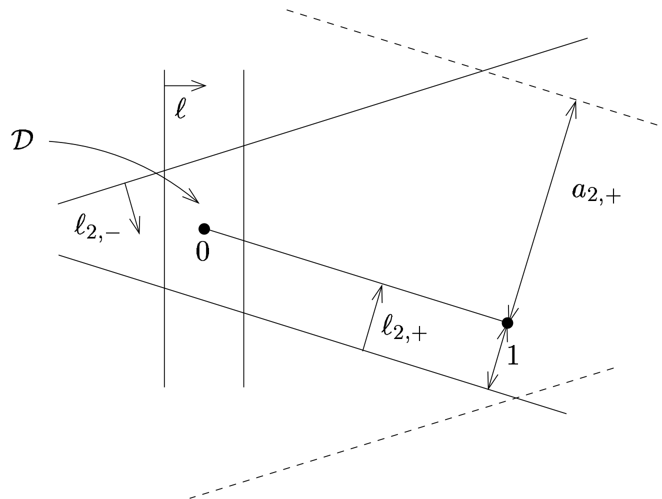
*Sketch of proof:* For simplicity we assume $\gamma = 1$.

**i) $\Longrightarrow$ ii):** We choose an orthonormal basis $(f_i)_{1 \le i \le d}$ of $\mathbb{R}^d$ with $f_1 = \ell$, and for each $2 \le i \le d$, unit vectors $\ell_{i,+}$, $\ell_{i,-}$ in $\mathbb{R} f_1 + \mathbb{R} f_i$, such that:

$$\ell_{i,\pm} \cdot f_1 > 0,\ \ell_{i,+} \cdot f_i > 0,\ \ell_{i,-} \cdot f_i < 0, \text{ and} \tag{84}$$

$$\overline{\lim_{L \to \infty}}\, L^{-1} \log P_0[X_{T_{U_{\ell',b,L}}} \cdot \ell' < 0] < 0, \text{ for } \ell' = \ell,\ \ell_{i,+},\ \ell_{i,-}, \text{ and } b > 0\,. \tag{85}$$

We can now pick numbers $a_{i,\pm} > 0$ large enough so that




$$\begin{aligned}\mathcal{D} = &\ \{x \in \mathbb{R}^d,\ |x \cdot \ell| \le 1,\ x \cdot \ell_{i,\pm} \ge -1, \ \ \text{for } i = 2, \dots, d\} \\ \subseteq &\ \{x \in \mathbb{R}^d,\ x \cdot \ell_{i,\pm} < a_{i,\pm}\} \qquad (\mathcal{D} \text{ is a compact set})\,.\end{aligned} \tag{86}$$

(Such a choice is possible because $|x \cdot \ell| = |x \cdot f_1| \le 1$ and $|x \cdot f_i|$ is bounded for $i \ge 2$, when $x \in \mathcal{D}$}). Then define for $L > 1$:

$$\begin{aligned}\Delta_L = &\ \{x \in \mathbb{Z}^d,\ \ell \cdot x \in (-L, L),\ \ell_{i,\pm} \cdot x > -L \text{ for } i = 2, \dots, d\} \\ \subseteq &\ L\mathcal{D} \cap \mathbb{Z}^d \ \ (\text{so } \Delta_L \text{ is a finite set})\,.\end{aligned} \tag{87}$$

Because of (77), we see that $X_.$ tends to exit $\Delta_L$ "through the right". Namely if $T_{\Delta_L}$ is the exit time from $\Delta_L$,

$$\overline{\lim_{L\to\infty}}\ L^{-1} \log P_0[T_{\Delta_L} < T_L^\ell] < 0\,. \tag{88}$$

Note that $P_0[T_L^\ell = \infty] \le P_0[T_{\Delta_L} < T_L^\ell] \xrightarrow{L\to\infty} 0$ and hence

$$P_0\text{-a.s. } \overline{\lim_n}\ X_n \cdot \ell = \infty\,. \tag{89}$$

As a next step we observe that

$$\overline{\lim_{L\to\infty}}\ L^{-1} \log P_0\big[\widetilde{T}^\ell_{\frac{L}{2}} \circ \theta_{T_L^\ell} < T^\ell_{\frac{4}{3}L} \circ \theta_{T_L^\ell}\big] < 0\,. \tag{90}$$

Indeed:

$$P_0\big[\widetilde{T}^\ell_{\frac{L}{2}}\circ\theta_{T^\ell_L} < T^\ell_{\frac{4}{3}L}\circ\theta_{T^\ell_L}\big] \le P_0[T_{\Delta_L} < T^\ell_L] + P_0\big[\overbrace{\widetilde{T}^\ell_{\frac{L}{2}}\circ\theta_{T^\ell_L} < T^\ell_{\frac{4}{3}L}\circ\theta_{T^\ell_L}}^{\left(T_{\Delta_L}\overset{P_0-\text{a.s.}}{\le} T^\ell_L\right)}, T_{\Delta_L} = T^\ell_L\big]$$

using translation invariance and the strong Markov property

$$\le P_0[T_{\Delta_L} < T^\ell_L] + |\partial\Delta_L|\, P_0\big[\widetilde{T}^\ell_{-\frac{L}{2}} < T^\ell_{\frac{L}{3}}\big]$$

and (90) follows from (88).

Applying Borel-Cantelli's lemma:

$$P_0\text{-a.s. for large integer } L,\; T^\ell_{\frac{4}{3}L} < T^\ell_L + \widetilde{T}^\ell_{\frac{L}{2}}\circ\theta_{T^\ell_L}\,. \tag{91}$$

So on a set of full $P_0$-measure we can construct $L_k \in \mathbb{N} \uparrow \infty$, with $L_{k+1} = [\frac{4}{3}\,L_k]$ and $T^\ell_{L_{k+1}} < T^\ell_{L_k} + \widetilde{T}^\ell_{\frac{L_k}{2}}\circ\theta_{T^\ell_{L_k}}$, for $k \ge 0$. This implies that (80) holds, i.e.

$$P_0\text{-a.s. } \lim_n\, X_n\cdot\ell = \infty\,.$$

(as noted above (81) follows). For the next lemma we use the notation

$$M = \sup\{X_k\cdot\ell - X_0\cdot\ell,\; 0 \le k \le D^\ell\}\,. \tag{92}$$

**Lemma 2.5.** *(Assuming* (80)*)*

*Under $P_0$, $X_{\tau_1}\cdot\ell$ is stochastically dominated by $a+1+G_J$, where $G_0 = 0$ and $G_n$ is the sum of i.i.d. variables $\overline{M}_1,\dots,\overline{M}_n$ distributed like $M+a+1$ under $P_0[\,\cdot\,|D^\ell<\infty]$ and $J$ is an independent geometric variable with parameter $P_0[D^\ell<\infty]$.*

(93)

*Sketch of proof of Lemma 2.5:* $F(\cdot)\ge 0$, non-decreasing, then

$$E_0[F(X_{\tau_1}\cdot\ell)] \overset{(82)}{=} \sum_{k\ge1} E_0[F(X_{s_k}\cdot\ell),\; S_k<\infty,\; D^\ell\circ\theta_{S_k}=\infty]$$

$$= \sum_{k\ge1}\sum_{x\in\mathbb{Z}^d} \mathbb{E}\big[\underbrace{E_{0,\omega}[F(X_{S_k}\cdot\ell), S_k<\infty, X_{S_k}=x]}_{\sigma(\omega(y,\cdot);\ell\cdot y<\ell\cdot x)-meas.}\;\underbrace{P_{x,\omega}[D^\ell=\infty]}_{\sigma(\omega(y,\cdot);\ell\cdot y\ge\ell\cdot x)-meas.}\big]$$

$$\overset{\text{independence}}{=} \sum_{k\ge1} E_0[F(X_{S_k}\cdot\ell),\; S_k<\infty]\,P_0[D^\ell=\infty]\,. \tag{94}$$

Then for $k \geq 2$:

$$E_0[F(X_{S_k}\cdot\ell), S_k<\infty] \leq E_0[F(M_{k-1}+a+1), S_{k-1}<\infty,\, D^\ell\circ\theta_{S_{k-1}}<\infty] = \sum_{x\in\mathbb{Z}^d} E_0[F(M_{k-1}+a+1),\, S_{k-1}<\infty, X_{S_{k-1}}=x,\, D^\ell\circ\theta_{S_{k-1}}<\infty]$$

and arguing as above

$$\begin{aligned} &\leq E_0\,\overline{E}[F(X_{S_{k-1}}\cdot\ell+\overline{M}),\, S_{k-1}<\infty]\,P_0[D^\ell<\infty] \\ &\overset{\text{by induction}}{\leq} E_0\times\overline{E}^{\otimes(k-1)}[F(a+1+\overline{M}_1\cdots+\overline{M}_{k-1})]\,P_0[D^\ell<\infty]^{(k-1)} \end{aligned} \tag{95}$$

and one checks directly that (95) also holds when $k=1$. Inserting this in (94) we find $E_0[F(X_{\tau_1}\cdot\ell)] \leq E[F(a+1+G_J)]$, and our claim follows. □

As a result for $c>0$,

$$\begin{aligned} &E_0[\exp\{cX_{\tau_1}\cdot\ell\}] \\ &\leq e^{c(a+1)}P_0[D^\ell=\infty]\sum_{k\geq1}P_0[D^\ell<\infty]^{(k-1)}\,E_0[e^{c(M+a+1)}|D^\ell<\infty]^{(k-1)} \end{aligned} \tag{96}$$

and if we show that for some $c'>0$, $E_0[e^{c'M}|D^\ell<\infty]<\infty$, it will follow that for some small $c>0$,

$$E_0[\exp\{cX_{\tau_1}\cdot\ell\}]<\infty\,. \tag{97}$$

To finish proving (97), note that for large $k$

$$\begin{aligned} &P_0\Big[\Big(\tfrac{4}{3}\Big)^k \leq M < \Big(\tfrac{4}{3}\Big)^{k+1}, D^\ell<\infty\Big] \\ &\leq P_0\Big[\widetilde{T}_0^\ell\circ\theta_{T^\ell_{(\frac{4}{3})^k}} < T^\ell_{(\frac{4}{3})^{k+1}}\circ\theta_{T^\ell_{(\frac{4}{3})^k}}\Big] \overset{(90)}{<} e^{-c''(\frac{4}{3})^k} \end{aligned}$$

and $E_0[e^{c'M}|D^\ell<\infty]<\infty$ for some $c'>0$, easily follows and (97) as well.

We can now finish the proof of i) $\Longrightarrow$ ii). We pick $r>0$ such that $\mathcal{D}\subset B(0,r)$ (see (86)) and hence $\Delta_L\subseteq B(0,rL)$. Then for large $u>0$:

$$\begin{aligned} &P_0\big[\sup_{n\leq\tau_1}|X_n|\geq u\big] \leq P_0[T_{\Delta_{\frac{u}{2r}}}<\tau_1] \leq P_0\Big[X_{\tau_1}\cdot\ell\geq\frac{u}{2r}\Big] + \\ &P_0\big[T_{\Delta_{\frac{u}{2r}}}<T^\ell_{\frac{u}{2r}}\big] \overset{(97),(88)}{\leq} \exp\{-\text{const}\,u\} \end{aligned} \tag{98}$$

and ii) follows.

**ii) $\Longrightarrow$ i):** Under (80), one can in fact use $\tau_1$ as a **regeneration time** and see that under $P_0$, $X_{\tau_1+\cdot} - X_{\tau_1}$ has same distribution as $X.$ under $P_0[\,\cdot|D^\ell = \infty]$ and then iterate:

$$\tau_2 = \tau_1(X) + \tau_1(X_{\tau_1+\cdot} - X_{\tau_1}) \quad (= \infty, \text{ if } \tau_1 = \infty), \text{ and by induction}$$

$$\tau_{k+1} = \tau_1(X) + \tau_k(X_{\tau_1+\cdot} - X_{\tau_1}), \ \ (\text{see [43]})\,.$$

One obtains the crucial **renewal property:**

$$\text{under } P_0, (X_{\tau_1\wedge\cdot}),\ (X_{(\tau_1+\cdot)\wedge\tau_2} - X_{\tau_1}), \dots, (X_{(\tau_k+\cdot)\wedge\tau_{k+1}} - X_{\tau_k}) \tag{99}$$

are independent and except for the first variable distributed like $(X_{\tau_1\wedge\cdot})$ under
$P_0[\cdot\,|D^\ell = \infty]$.

Incidentally note that $\tau_{k+1} - \tau_k$ is indistinguishable from a measurable function of
$((X_{(\tau_k+\cdot)\wedge\tau_{k+1}} - X_{\tau_k})$, (namely the first time when the trajectory remains from then on constant).

From the integrability of $\sup_{k\le\tau_1}|X_k|$ under $P_0[\,\cdot|D^\ell = \infty]$ and the law of large numbers, we easily deduce that

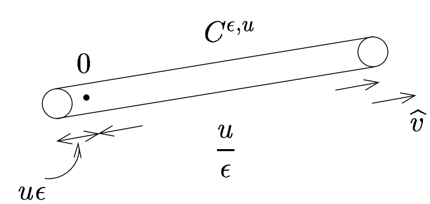
$$P_0\text{-a.s., } |X_n| \to \infty \text{ and } \frac{X_n}{|X_n|} \to \widehat{v} = \frac{E_0[X_{\tau_1}|D^\ell = \infty]}{|E_0[X_{\tau_1}|D^\ell = \infty]|} \tag{100}$$

and of course $\widehat{v}\cdot\ell > 0$.

The next step is that one considers for $\epsilon > 0, u > 0$

$$C^{\epsilon,u} \text{ the intersection of } \mathbb{Z}^d \text{ with the cylinder } u\widehat{R}\big(\big(-\epsilon,\tfrac{1}{\epsilon}\big) \times B_{d-1}\big(0,\tfrac{\epsilon}{2}\big)\big) \quad (B : (d-1)\text{-dim. ball}) \tag{101}$$

($\widehat{R}$ some rotation such that $\widehat{R}(e_1) = \widehat{v}$).



One has with the help of the exponential bound in ii), the renewal property (99), and a Cramer-type large deviation control,

$$\overline{\lim_{u\to\infty}}\, u^{-1} \log\, P_0\big[T_{C^{\epsilon,u}} < T_{\frac{u}{\epsilon}}^{\widehat{v}}\big] < 0\,, \tag{102}$$

(see Lemma 1.3 of [41] or Lemma 2.3 of [40]). But since $\widehat{v}\cdot\ell > 0$, by making $\epsilon > 0$ small in (101), i) follows immediately. □

**Remarks:** 1) Under $(\mathrm{T})_\gamma|\ell$

$$P_0\text{-a.s., } |X_n| \to \infty \text{ and } \frac{X_n}{|X_n|} \to \widehat{v} \in S^{d-1} \text{ deterministic, and} \tag{103}$$

$$(\mathrm{T})_\gamma|\ell' \text{ holds if and only if } \widehat{v}\cdot\ell' > 0\,. \tag{104}$$

2) Note that in i) we have **only used** the fact that

$$\overline{\lim_{L}}\, L^{-\gamma} \log P_0[X_{T_{U_{\ell',b,L}}} \cdot \ell' < 0] < 0, \text{ for } b > 0\,,$$

**for finitely many** $\ell'$ such that $\mathcal{D}$ in (86) is a compact subset of $\mathbb{R}^d$.

**Open problem:** Is $(\mathrm{T})_\gamma|\ell$ equivalent to

$$\overline{\lim_{L}}\, L^{-\gamma} \log P_0[X_{T_{U_{\ell,b,L}}} \cdot \ell < 0], \text{ for } b > 0\,,$$

(in other words: only $\ell' = \ell$ is needed)? □

As we will see **(83) i) and (83) ii) have different merits**. On one hand (83) i) will be more appropriate to derive sufficient criteria for checking $(\mathrm{T})_\gamma$ or analyzing possible equivalences of these conditions, cf. below (78). On the other hand (83) ii) together with the renewal property (99), will be more handy to derive consequences of (T) or (T') on the asymptotic behavior of the walk.

# 3 Checking conditions (T) and (T')

We are now going to discuss **various ways of checking conditions (T) and (T')**.

Conditions (T) and (T'), cf. (77), require a control on the large $L$ asymptotics of the exit distribution of $X_.$ from slabs $U_{\ell,b,L}$, under $P_0$. This is not a-priori an easy task, since in particular the walk $X_.$ under $P_0$ does not have the Markov property. We will now discuss two ways of handling this difficulty.

**Kalikow's approach:** The idea is to reintroduce a Markovian character to the problem by introducing certain **auxiliary Markov chains**, following [17]. Namely, for $U \subsetneq \mathbb{Z}^d$ a connected subset containing 0, one defines a Markov chain with state space $U \cup \partial U$ and transition kernel:

$$\begin{aligned} \widehat{P}_U(x, x+e) &= \frac{\mathbb{E}[\widetilde{g}_U(0,x,\omega)\, \omega(x,e)]}{\mathbb{E}[\widetilde{g}_U(0,x,\omega)]},\ x \in U,\ |e| = 1\,, \\ \widehat{P}_U(x,x) &= 1,\ \text{when } x \in \partial U\ (\stackrel{\text{def}}{=} \{z \in U^c, \exists z' \in U, |z - z'| = 1\})\,, \end{aligned} \tag{105}$$

here we use the notation $\widetilde{g}_U(\cdot,\cdot,\omega)$ for the function:

$$\widetilde{g}_U(x,y,\omega) = E_{x,\omega}\Big[\sum_0^{T_U} 1\{X_n = y\}\Big]\,, \tag{106}$$

which is a slight modification (on $\partial U$) of the Green function, cf. (164). We denote by $\widehat{P}_{x,U}$ the law of the canonical Markov chain starting at $x \in U \cup \partial U$, corresponding to the transition kernel (105). The interest of this notion introduced by Kalikow is

**Lemma 3.1.** *If $\widehat{P}_{0,U}[T_U < \infty] = 1$, then $P_0[T_U < \infty] = 1$ and $X_{T_U}$ has same law under $\widehat{P}_{0,U}$ and $P_0$.*

(This "reduces" the calculation of the exit distribution of $X$ from $U$ under $P_0$, to a Markovian problem).

*Proof.* (the i.i.d. character of $\mathbb{P}$, plays no role).

For $x \in U \cup \partial U$, $\omega \in \Omega$, we deduce from the simple Markov property the following equation:

$$\widetilde{g}_U(0,x,\omega) = \delta_{0,x} + \sum_{y \in U} \widetilde{g}_U(0,y,\omega)\, \omega(y, x-y),\ \text{with } \omega(x,z) = 0,\ \text{when } |z| \neq 1\,.$$

Integrating over $\mathbb{P}$, we obtain

$$\begin{aligned} \mathbb{E}[\widetilde{g}_U(0,x,\omega)] &= \delta_{0,x} + \sum_{y \in U} \mathbb{E}[\widetilde{g}_U(0,y,\omega)\, \omega(y, x-y)] \\ &= \delta_{0,x} + \sum_{y \in U} \mathbb{E}[\widetilde{g}_U(0,y,\omega)]\, \widehat{P}_U(y, x-y)\,. \end{aligned} \tag{107}$$

Moreover:

$$\widehat{g}_U(x) = \widehat{E}_{0,U}\Big[\sum_{k=0}^{T_U} 1\{X_k = x\}\Big],\ x \in U \cup \partial U\,, \tag{108}$$

is the minimal non-negative solution of the equation:

$$f(x) = \delta_{0,x} + \sum_{y \in U} f(y)\, \widehat{P}_U(y, x-y),\ f : U \cup \partial U \to \mathbb{R}_+\,, \tag{109}$$

(one observes that $\widehat{g}_{U,n}(x) \stackrel{\text{def}}{=} \widehat{E}_{0,U}[\sum_{k=0}^{T_U \wedge n} 1\{X_k = x\}]$ satisfies:

$$\widehat{g}_{U,n+1}(x) = \delta_{0,x} + \sum_{x \in U} \widehat{g}_{U,n}(y)\, \widehat{P}_U(y, x-y)\,,$$

and by induction one also sees that $f \geq \widehat{g}_{U,n}$, so that letting $n \to \infty$, $f \geq \widehat{g}_U$ follows).

Hence from (107), we find

$$\widehat{g}_U(x) \leq \mathbb{E}[\widetilde{g}_U(0,x,\omega)],\ x \in U \cup \partial U\,. \tag{110}$$

Moreover, for $x \in \partial U$,

$$\widehat{g}_U(x) = \widehat{P}_{0,U}[T_U < \infty,\ X_{T_U} = x],\ \mathbb{E}[\widehat{g}_U(0,x,\omega)] = P_0[T_U < \infty, X_{T_U} = x]$$

and by assumption $\sum_{x \in \partial U} \widehat{g}_U(x) = 1$, so that from (110) we obtain

$$\widehat{g}_U(x) = \mathbb{E}[\widetilde{g}_U(0,x,\omega)];\ x \in \partial U\,.$$

(It is also not hard to see that this equality in fact holds for $x \in U$ and as a by-product of summing over $x$ in $U$, one finds that $\widehat{E}_{0,U}[T_U] = E_0[T_U]$). $\square$

One way to make use of this is to introduce the **auxiliary local drift**:

$$\widehat{d}_U(x) = \widehat{E}_{x,U}[X_1 - X_0],\ x \in U \cup \partial U\,, \tag{111}$$

and introduce the **Kalikow condition** relative to $\ell \in S^{d-1}$, (Notation $(K)|\ell$):

$$\inf_{U, x \in U} \widehat{d}_U(x) \cdot \ell = \epsilon(\ell,\mu) > 0\,. \tag{112}$$

As we now see it implies condition (T) (and hence (T')).

**Proposition 3.2.** *($\ell \in S^{d-1}, d \geq 1$)*

$$(K)|\ell \Longrightarrow (\mathrm{T})|\ell \tag{113}$$

*Proof.* We can find $\gamma > 0$, such that for $\theta \in [0,1], |u| \le 1$,

$$|e^{-\theta u} - 1 + \theta u| \le \gamma \theta^2 \,.$$

Hence for $n \ge 0$, $U$ as in (105): with $(\mathcal{F}_n)_{n \ge 0}$ the filtration of $X_.$, $x \in U \cup \partial U$,

$$\begin{aligned}&\widehat{E}_{x,U}[e^{-\theta X_{n+1}\cdot \ell} | \mathcal{F}_n] = e^{-\theta X_n \cdot \ell}\big(1\{X_n \in \partial U\} + 1\{X_n \in U\} \\ &\widehat{E}_{X_n,U}[\exp\{-\theta(X_1 - X_0)\cdot \ell\}]\big), \ \widehat{P}_{x,U}\text{-a.s.}\,.\end{aligned}$$

Because of (112), for $0 \le \theta \le \theta(\epsilon)$, and $y \in U$:

$$\widehat{E}_{y,U}[e^{-\theta(X_1 - X_0)\cdot \ell}] \le 1 - \theta \widehat{d}_U(y) \cdot \ell + \gamma \theta^2 \le 1 \,. \tag{114}$$

Hence we have:

$$e^{-\theta(\epsilon) X_n \cdot \ell}, \ n \ge 0, \text{ is an } (\mathcal{F}_n)\text{-supermartingale under } \widehat{P}_{x,U} \,. \tag{115}$$

From the stopping theorem we immediately obtain:

$$\overline{\lim_{L\to\infty}} \ L^{-1} \log P_0\big[X_{T_{U_{\ell,b,L}}} \cdot \ell < 0\big] \le -b\theta(\epsilon) < 0 \tag{116}$$

(recall the notations above (77)).

Note also that $(K)|\ell \Longrightarrow (K)|\ell'$ for all $\ell'$ in some neighborhood of $\ell$, so that applying (116) to such $\ell'$, we obtain (T)$|\ell$. □

One can then provide conditions (much more explicit than (112) which imply $(K)|\ell$ and hence (T)$|\ell$.

**Proposition 3.3.** *($\ell \in S^{d-1}, d \ge 1$)*

$$\textit{If } \mathbb{E}[(d(0,\omega)\cdot \ell)_+] > \frac{1}{\kappa}\,\mathbb{E}[(d(0,\omega)\cdot\ell)_-], \textit{ then } (K)|\ell \textit{ and hence (T)}|\ell \textit{ hold}. \tag{117}$$

*($\kappa$ is defined in (60)).*

*Proof.* For $U$ as in (105), and $x \in U$, by a standard Markov chain calculation:

$$\widetilde{g}_U(0,x,\omega) = \frac{P_{0,\omega}[H_x < T_U]}{P_{x,\omega}[\widetilde{H}_x > T_U]} = \frac{P_{0,\omega}[H_x < T_U]}{\sum\limits_{|e|=1} \omega(x,e) P_{x+e,\omega}[H_x > T_U]} \tag{118}$$

($H_x$: entrance time, $\widetilde{H}_x$: hitting time). Now

$$\widehat{d}_U(x)\cdot\ell \overset{(111)}{=} \frac{1}{\mathbb{E}[\widetilde{g}_U(0,x,\omega)]}\,\mathbb{E}\Big[\frac{P_{0,\omega}[H_x<T_U]}{\sum\limits_{|e|=1}\omega(x,e)P_{x+e,\omega}[H_x>T_U]}\,d(x,\omega)\cdot\ell\Big]$$

$$\geq \frac{1}{\mathbb{E}[\widetilde{g}_U(0,x,\omega)]}\,\mathbb{E}\Big[\frac{P_{0,\omega}[H_x<T_U]}{\max\limits_{|e|=1}P_{x+e,\omega}[H_x>T_U]}\,(d(x,\omega)\cdot\ell)_+ - \frac{1}{\kappa}\,(d(x,\omega)\cdot\ell)_-\Big]$$

and using independence

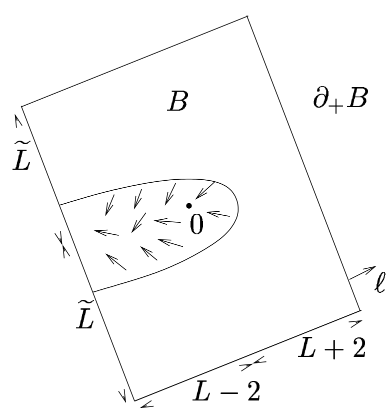
$$= \mathbb{E}\Big[(d(0,\omega)\cdot\ell)_+ - \frac{1}{\kappa}\,(d(0,\omega)\cdot\ell)_-\Big]\,\underbrace{\frac{1}{\mathbb{E}[\widetilde{g}_U(0,x,\omega)]}\,\mathbb{E}\Big[\frac{P_{0,\omega}[H_x<T_U]}{\max\limits_{|e|=1}P_{x+e,\omega}[H_x>T_U]}\Big]}_{\overset{(118)}{\geq}\,\kappa\,\widetilde{g}_U(0,x,\omega)}$$

$$\geq \kappa\,\mathbb{E}[(d(0,\omega)\cdot\ell)_+ - \frac{1}{\kappa}\,(d(0,\omega)\cdot\ell)_-]\,,$$

and our claim follows. □

With the above proposition we already have a variety of concrete situations where (T)$|\ell$ holds. We will now discuss another approach to check (T') (and possibly (T) but this is open for the time being).

**Effective Criterion:**




**Notations:** $\ell\in S^{d-1}$, $R$ is a rotation of $\mathbb{R}^d$ with $R(e_1)=\ell$, $L>2$, $\widetilde{L}>0$,

$$B = R\big((-(L-2),L+2)\times(-\widetilde{L},\widetilde{L})^{d-1}\big)\cap\mathbb{Z}^d:\ \text{a box}$$

$$\partial_+B = \partial B\cap\big\{x\in\mathbb{Z}^d,\ \ell\cdot x\geq L+2,\ |R(e_i)\cdot x|<\widetilde{L},\ \text{for each } i\geq 2\big\} \tag{119}$$

$$\rho_B = \frac{P_{0,\omega}[X_{T_B}\notin\partial_+B]}{P_{0,\omega}[X_{T_B}\in\partial_+B]}\,.$$

One has an equivalent characterization of (T')$|\ell$ in terms of moments of the $\rho_B$'s. Note that $\rho_B$ has some flavor of $\rho$ in (17).

**Theorem 3.4.** *($\ell \in S^{d-1}$)*

*(T′)$|\ell$ is equivalent to*

$$\inf_{B, 0<\alpha\le 1} \Big\{ c_1(d) \Big(\log \frac{1}{\kappa}\Big)^{3(d-1)} \widetilde{L}^{(d-1)} L^{3(d-1)+1} \, \mathbb{E}[\rho_B^\alpha] \Big\} < 1 \tag{120}$$

*where $B$ in the above infimum varies over the collection of boxes $B$ corresponding to an arbitrary rotation $R$, $L \ge c_2(d)$, and $3\sqrt{d} \le \widetilde{L} < L^3$, and $c_1(d)$, $c_2(d)$ are dimension dependent constants bigger than one.*

**Remarks:** 1) The interest of the theorem comes from the infimum in (120) and the effective character of (120). In other words it suffices to find a box $B$ and an $\alpha$ for which:

$$c_1(d) \Big(\log \frac{1}{\kappa}\Big)^{3(d-1)} \widetilde{L}^{(d-1)} L^{3(d-1)+1} \, \mathbb{E}[\rho_B^\alpha] < 1\,,$$

to know that (T')$|\ell$ holds. If this is the case the infimum in (120) is in fact 0.

2) The condition (120) has some flavor of the condition $\mathbb{E}[\rho] < 1$, in (19), in the one-dimensional case, which ensures the ballistic behavior of the walk. In particular (120) involves a **small denominator question** (namely $P_{0,\omega}[X_{T_B} \in \partial_+ B]$ may be small), which is related to the possibility of **atypical exit probabilities** of the walk, a phenomenon which is related to the **occurrence of traps** in the medium. □

**Some comments on the proof:** (for more details see Section 2 of [41])

- (120) $\Longrightarrow$ (T')$|\ell$:

The idea is to define a sequence of boxes $B_k$, which grow, but not too fast, and tend to look like "infinite slabs", and use a control on a moment of $\rho_{B_k}$ to obtain a control on a moment of $\rho_{B_{k+1}}$. This is a "**renormalization method**", where a certain induction hypothesis concerning the smallness of certain moments of $\rho_{B_k}$ is propelled "from one scale to the next scale". Specifically one defines for a suitable $u_0 \in (0,1]$:

$$L_k = \Big(\frac{\mathrm{const}(d)}{u_0}\Big)^k \, 8^{\frac{k(k-1)}{2}} \, L_0, \; \widetilde{L}_k = \Big(\frac{L_k}{L_0}\Big)^3 \widetilde{L}_0 \tag{121}$$

and one controls for a suitable $a_0 \in (0,1]$, $\mathbb{E}[\rho_{B_k}^{a_0 2^{-k}}]$, (one shows that

$$c(d)\ \widetilde{L}_{k+1}^{(d-1)} L_k\, \mathbb{E}\big[\rho_{B_k}^{a_0 2^{-k}}\big] \le \kappa^{u_0 \frac{L_k}{8^k}}\ , \tag{122}$$

by induction). Note that

$$\mathbb{E}\big[P_{0,\omega}[X_{T_{B_k}} \notin \partial_+ B_k]\big] \le \mathbb{E}\big[P_{0,\omega}[X_{T_{B_k}} \notin \partial_+ B_k]^{a_0 2^{-k}}\big] \le \mathbb{E}[\rho_{B_k}^{a_0 2^{-k}}]\,,$$

and from (122) one can prove that for some $c > 0$,

$$\overline{\lim_{L\to\infty}}\ L^{-1}\, \exp\{c(\log L)^{\frac{1}{2}}\}\, \log P_0\big[X_{T_{U_{\ell,b,L}}} \cdot \ell < 0\big] < 0, \text{ for all } b > 0\,. \tag{123}$$

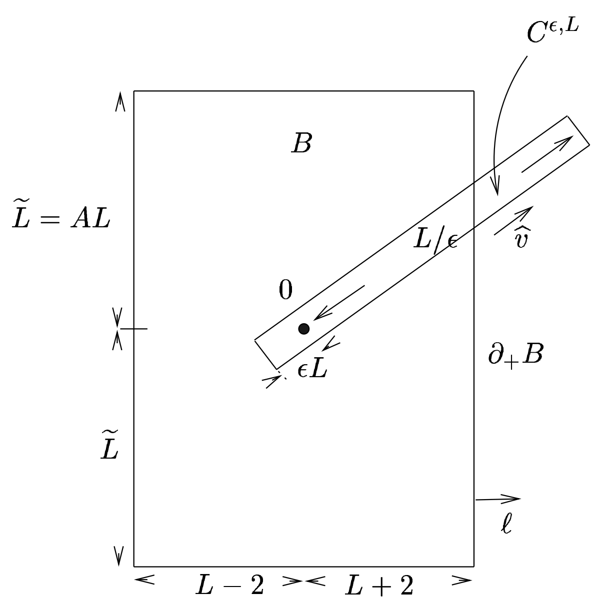
Then by a perturbation argument one shows that a similar control as (120) holds for all small enough perturbations $\ell'$ of $\ell$ and one obtains for $\ell'$ in a neighborhood of $\ell$ controls like (123), which in fact is more than enough to show (T')$|\ell$.

- (T')$|\ell \Longrightarrow$ (120):

In fact we explain $(\mathrm{T})_\gamma|\ell \Longrightarrow$ (120), when $\frac{1}{2} < \gamma < 1$, which of course implies the claim.

We choose the box $B$ of the form:

$$\begin{aligned} &L \text{ large, } \widetilde{L} = A\,L, \text{ where } A \text{ is some constant (depending on } \widehat{v}, \ell) \\ &R \text{ any rotation with } R(e_1) = \ell\,. \end{aligned} \tag{124}$$



If $C^{\epsilon,\mu}$ is defined as in (88), choosing $\epsilon > 0$ small we have for all large $L$:

$$P_{0,\omega}\big[T^{\widehat{v}}_{\frac{L}{\epsilon}} = T_{C^{\epsilon},L}\big] \le P_{0,\omega}[X_{T_B} \in \partial_+ B]\,. \tag{125}$$

Hence for $a \in (0,1]$ and $c > 0$:

$$\mathbb{E}[\rho_B^a] \le \mathbb{E}[\rho_B^a, P_{0,\omega}[X_{T_B} \in \partial_+ B] \ge e^{-cL^\gamma}] + \mathbb{E}[\rho_B^a, P_{0,\omega}[X_{T_B} \in \partial_+ B] < e^{-cL^\gamma}]$$

using the fact that $\mathbb{E}[Z^a] \le \mathbb{E}[Z]^a$, for $Z \ge 0$, since $0 < a \le 1$,

$$\le e^{acL^\gamma}\, P_0[X_{T_B} \notin \partial_+ B]^a + \underbrace{\kappa^{-c(d)aL}}_{\text{using ellipticity}}\ \mathbb{P}[P_{0,\omega}[X_{T_B} \notin \partial_+ B] \ge 1 - e^{-cL^\gamma}]$$

$$\overset{(125)}{\le} e^{acL^\gamma}\, P_0\big[T^{\widehat{v}}_{\frac{L}{\epsilon}} > T_{C^{\epsilon},L}\big]^a + \frac{\kappa^{-c(d)aL}}{1 - e^{-cL^\gamma}}\, P_0\big[T^{\widehat{v}}_{\frac{L}{\epsilon}} > T_{C^{\epsilon},L}\big]\,. \tag{126}$$

Now under $(T_\gamma)|\ell$, the control corresponding to (89) is:

$$\lim_{u\to\infty} u^{-\gamma} \log P_0[T_{C^{\epsilon},u} < T^{\widehat{v}}_{\frac{u}{\epsilon}}] < 0\,. \tag{127}$$

Hence choosing $a = L^{-\frac{1}{2}}$ and $c$ sufficiently small in (126), we obtain:

$$\overline{\lim}\, L^{-(\gamma - \frac{1}{2})} \log \mathbb{E}[\rho_B^{L^{-\frac{1}{2}}}] < 0\,, \tag{128}$$

and this is more than enough to prove (120). $\square$

We see that (120) implies already a condition which formally looks stronger than (T')$|\ell$, cf. (123), (77), (78). One has in fact the

**Open problem:** Are all $(\mathrm{T})_\gamma|\ell$, for $0 < \gamma \le 1$, equivalent? (And of course then equivalent to (120)).

The **effective criterion** is interesting from different perspectives. On the one hand it is a **theoretical tool**: for instance we have seen thanks to (121) that

$$\text{for } \frac{1}{2} < \gamma < 1,\ \ell \in S^{d-1},\ (T_\gamma)|\ell \text{ and } (\mathrm{T'})|\ell \text{ are equivalent}\,. \tag{129}$$

One can also rather easily check with the help of (120) that the set of single site distributions $\mu$ on $\mathcal{P}_\kappa$ (we have $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$) for which (T')$|\ell$ holds is open for the weak topology (i.e. (T')$|\ell$ is stable under small perturbations of the distribution). This is not so clear from the definition of (T')$|\ell$ in (77) or even from (83), on the other hand the continuity of $\mu \to \mathbb{E}[\rho_B^\alpha]$ for $B$, $\alpha$ as in (120), is straightforward. Indeed, $P_{0,\omega}[X_{T_B} \in \partial_+ B, T_B \leq M]$, for $M \to \infty$, are polynomials with positive coefficients in the variables $\omega(x,e)$, $x \in B$, $|e| = 1$, which uniformly in $\omega$ converge to $P_{0,\omega}[X_{T_B} \in \partial_+ B]$ which is uniformly positive by ellipticity. This continuity and (120) now implies the claim.

On the other hand the **effective criterion** as we later will see is also an **instrument to construct examples** where (T') holds.

# 4 Asymptotic behavior under (T) and (T')

We will now discuss some **consequences of conditions (T) and (T')**. This will explain the interest of these conditions.

**Law of large numbers and central limit theorem:**

The full strength of conditions (T) or (T') appears when $d \geq 2$. When $d = 1$, $(\mathrm{T})_\gamma|\ell$ is equivalent to $P_0[\lim X_n \cdot \ell = \infty] = 1$, for any $0 < \gamma \leq 1$, and $\ell = \pm 1$, cf. [41]. Hence in the one-dimensional context, all these conditions are equivalent to a transient behavior of the walk in the direction $\ell$, and as we have seen in (18), (19) ii) this may happen without a ballistic behavior of the walk. The multi-dimensional case turns out to be different.

**Theorem 4.1.** *($d \geq 2$, $\ell \in S^{d-1}$, under (T')$|\ell$)*

$$P_0\text{-a.s.}, \ \frac{X_n}{n} \to v \ \text{ deterministic with } v \cdot \ell > 0, \tag{130}$$

$$B^n_{\cdot} = \frac{X_{[\cdot n]} - [\cdot n]v}{\sqrt{n}} \ \text{ converges in law on the space } D(\mathbb{R}_+, \mathbb{R}^d) \text{ of right continuous } \mathbb{R}^d\text{-valued trajectories with left limits, to a Brownian motion with non-degenerate covariance matrix } A\,. \tag{131}$$

*Sketch of Proof:* With the help of the renewal property (99), the law of large numbers follows in essence from the estimate:

$$E_0[\tau_1 | D^\ell = \infty] < \infty\,, \tag{132}$$

and as a matter of fact

$$v = \frac{E_0[X_{\tau_1} | D^\ell = \infty]}{E_0[\tau_1 | D^\ell = \infty]}\,. \tag{133}$$

In a similar vein the functional central limit theorem follows from the estimate

$$E_0[\tau_1^2 | D^\ell = \infty] < \infty\,, \tag{134}$$

and the limiting covariance is then:

$$A = \frac{E_0[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t | D^\ell = \infty]}{E_0[\tau_1 | D^\ell = \infty]}\,. \tag{135}$$

For details see [43] and also [39]. In some sense proving (130) and (131) boils down to controlling the tail of $\tau_1$. We discuss these estimates further below. □

**Open problem:** When $d \geq 2$, are (T') or (T) characteristic of ballistic behavior: does a strong law of large numbers with non-vanishing limiting velocity imply (T) (and of course (T'))? More generally, does (80) imply (T)$|\ell$ ?

**Tail estimates on the regeneration times**

As we now see under (T') (and of course (T)), $\tau_1$ has **finite moments of arbitrary order** under $P_0$, **when $\boldsymbol{d \geq 2}$**. This is quite different of the one-dimensional situation, and this feature should be viewed as a reflection of the weakening effect of traps as dimension increases.

**Theorem 4.2.** *($d \geq 2$, under* (T')$|\ell, a > 0$*) (a enters the definition of* $\tau_1$ *above* (81)*)*

$$\begin{aligned} &P_0[\tau_1 > u] \leq \exp\{-(\log u)^\alpha\}, \textit{ for large } u, \\ &\textit{when } \alpha < \frac{2d}{d+1} \quad (\longleftarrow \textit{ number bigger than 1!})\,. \end{aligned} \tag{136}$$

Let us mention that in the plain nestling situation, under the above assumptions, we can use the naive traps discussed in (70) and obtain the lower bound

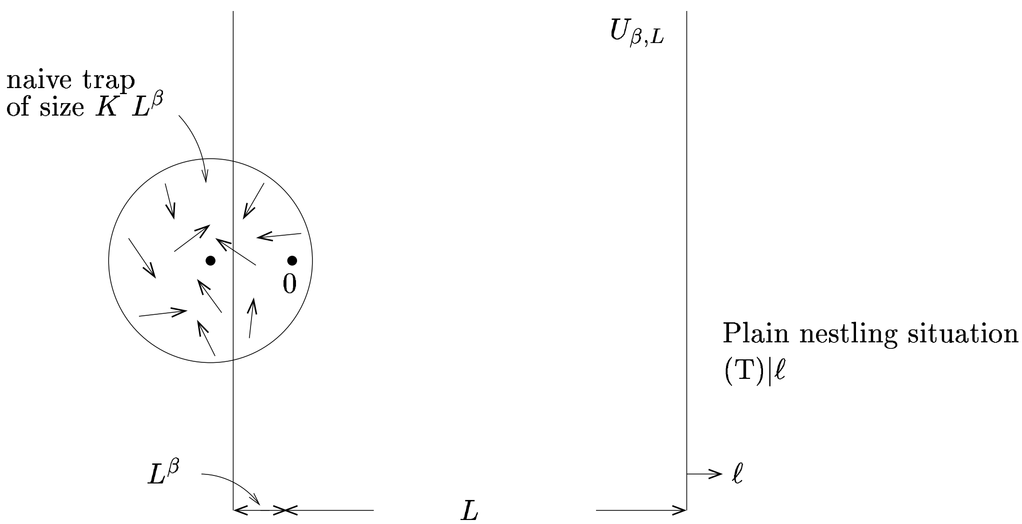
$$P_0[\tau_1 > u] \geq \exp\{-c(\log u)^d\}, \text{ for large } u\,. \tag{137}$$

Essentially one chooses $L = K \log u$, with $K$ large and one constrains the walk to remain in $B_L$ up to time $[u] + 2$ and be in 0 at time $[u] + 1$ or $[u] + 2$, depending on the parity of $[u]$, which happens with a not too small probability $\geq \exp\{-\text{const}(\log u)\}$, uniformly for $\omega$ in the trapping event $\mathcal{T}_L$ of (70), if $K$ is chosen large. On the other hand $\mathbb{P}[\mathcal{T}_L] \geq \exp\{-\text{const}\, L^d\}$ in the plain nestling situation, and (137) easily follows.

*Sketch of proof of* (136): We will explain how certain estimates on the occurrence of **atypical quenched exit distributions** of the walk from certain asymmetric large slabs are crucial in the derivation of (136). We need some notations:

$$\beta \in [0,1], \text{ and } U_{\beta,L} = \{x \in \mathbb{Z}^d,\ x \cdot \ell \in (-L^\beta, L)\}\,. \tag{138}$$

We want to control the $\mathbb{P}$-probability that $P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot \ell > 0]$ is atypically small, (this should not be confused with (77)!).




"An event ensuring that $P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot \ell > 0] \le e^{-cL^\beta}$"

The crucial role of such controls in the derivation of (136) comes from the

**Proposition 4.3.** *($d \ge 2$, (T')$|\ell$)*

*Assume $\beta \in (0,1)$ is such that for all $c > 0$,*

$$\overline{\lim_{L\to\infty}}\ \frac{1}{L}\ \log \mathbb{P}[P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot \ell > 0] \le e^{-cL^\beta}] < 0, \text{ then} \tag{139}$$

$$\overline{\lim_{u\to\infty}}\ (\log u)^{-\zeta} \log P_0[\tau_1 > u] < 0, \textit{ for any } \zeta < \frac{1}{\beta}\,, \tag{140}$$

*(when* (T)$|\ell$ *holds one can choose* $\zeta = \frac{1}{\beta}$*).*

*Proof.* We choose some $\zeta$ as in (140).

We define $\Delta u = \delta \log u$ and $L(u) = \Delta(u)^{\frac{1}{\beta}} \gg \Delta(u)$, where $\delta > 0$ is small and chosen below (143). Then for large $u$,

$$P_0[\tau_1 > u] \le P_0[\tau_1 > u, T_{C_{L(u)}} \le \tau_1] + P_0[T_{C_{L(u)}} > u]\,, \tag{141}$$

with the notation $C_r = (-\frac{r}{2}, \frac{r}{2})^d$. Since $T_{C_{L(u)}} \le \tau_1$ forces $\sup_{0 \le k \le \tau_1} |X_k| \ge \frac{L(u)}{2}$, applying Chebychev's inequality and (83) ii) with $\gamma$ close to 1, so that $\frac{\gamma}{\beta} > \zeta$, we obtain for large $u$:

$$\le \exp\{-(\log u)^{\zeta}\} + P_0[T_{C_{L(u)}} > u]\,.$$

We thus only need to estimate the last term. In the notation of (72) we have:

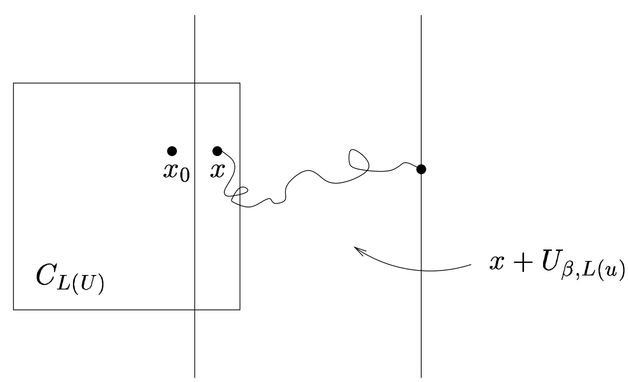
$$\begin{aligned} P_0[T_{C_{L(u)}} > u] &\le \mathbb{E}\Big[P_{0,\omega}[T_{C_{L(u)}} > u],\ t_\omega(C_{L(u)}) \le \frac{u}{(\log u)^{\frac{1}{\beta}}}\Big] + \\ \mathbb{P}\Big[t_\omega(C_{L(u)}) > \frac{u}{(\log u)^{\frac{1}{\beta}}}\Big] &\underset{(74)}{\overset{\substack{\text{Markov}\\ \text{property}}}{\le}} \Big(\frac{1}{2}\Big)^{[(\log u)^{\frac{1}{\beta}}]} + \\ \mathbb{P}[\text{for some } x_0 \in C_{L(u)},\ &P_{x_0,\omega}[\widetilde{H}_{x_0} > T_{C_{L(u)}}] \le \frac{2}{u}\,|C_{L(u)}|(\log u)^{\frac{1}{\beta}}]\,. \end{aligned} \tag{142}$$

On the event inside the above probability, for $x \neq x_0$

$$P_{x_0,\omega}[\widetilde{H}_{x_0} > T_{C_{L(u)}}] \ge P_{x_0}[\widetilde{H}_{x_0} > H_x]\, P_{x,\omega}[H_{x_0} > T_{C_{L(u)}}]\,. \tag{143}$$

Hence choosing $x \simeq x_0 + 2\Delta(u)\ell$ (recall $x \in \mathbb{Z}^d$), provided $\delta = \delta(d,\kappa)$ ($\kappa$ the ellipticity constant) is chosen small, for large $u$ we find

$$\frac{1}{\sqrt{u}} \ge P_{x,\omega}[H_{x_0} > T_{C_{L(u)}}] \tag{144}$$




"One way to exit $C_{L(u)}$ before reaching $x_0$ when starting at $x$"

Note that $\frac{1}{\sqrt{u}} = e^{-\frac{1}{2\delta}\Delta(u)} = e^{-\frac{1}{2\delta}L(u)^\beta}$, so that from (139), (142) and translation invariance, for large $u$:

$$\begin{aligned} P_0[T_{C_{L(u)}} > u] &\leq \Big(\frac{1}{2}\Big)^{[(\log u)^{\frac{1}{\beta}}]} + |C_{L(u)}|\mathbb{P}[P_{0,\omega}[X_{T_{U_{\beta,L(u)}}} \cdot \ell > 0] \leq e^{-\frac{1}{2\delta}L(u)^\beta}] \\ &\overset{(139)}{\leq} \exp\{-\text{const}\; L(u)\}\,. \end{aligned}$$

Together with (141), the claim (140) follows. □

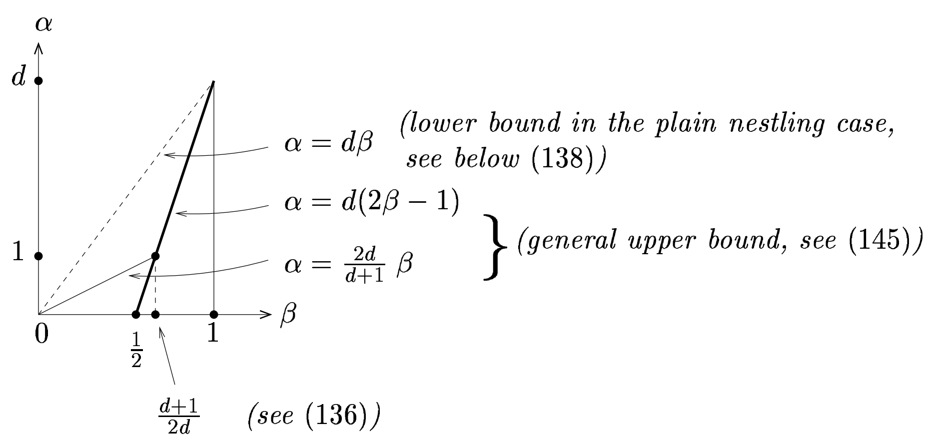
In view of the above Proposition, the heart of the matter in the proof of (136) comes from the **crucial estimate**:

**Theorem 4.4.** *($d \geq 2$, (T')$|\ell$)*

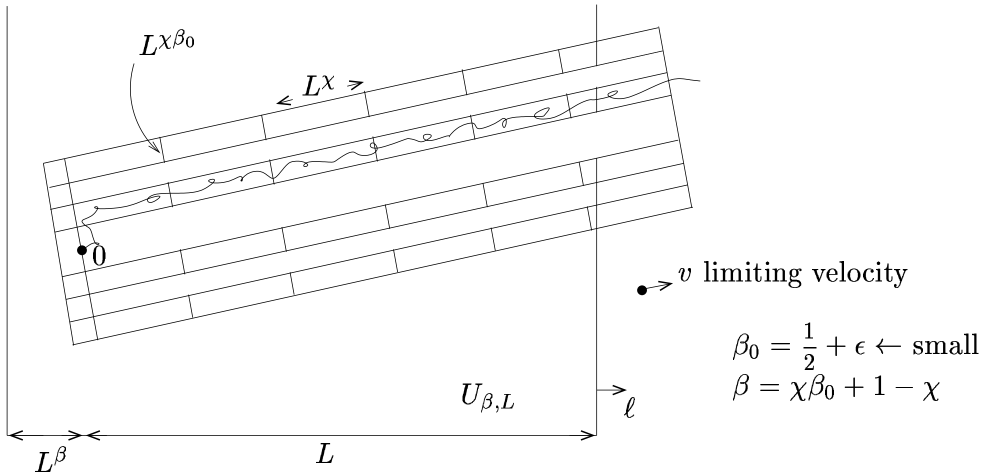
*For $c > 0$, $\beta \in (0,1)$,*

$$\overline{\lim_{L\to\infty}}\; L^{-\alpha} \log \mathbb{P}[P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot \ell > 0] \leq e^{-cL^\beta}] < 0 \;\; \textit{for}\;\; \alpha < d\Big[(2\beta-1) \vee \frac{2\beta}{d+1}\Big]\,. \tag{145}$$

*(one can replace $P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot \ell > 0]$ by $\lambda_\omega(B_L)$ in (145), see Proposition 4.7 of [40], where this is shown under (T), but also holds under (T'))*




For details about the proof of (145), we refer to Lemma 3.2, 3.3 in [41].




The idea of the proof of (145) for $\frac{1}{2} < \beta < 1$, and $\alpha < d(2\beta - 1)$, is to construct within distance $\sim L^{\beta}$ of 0 a number of potential escape routes for the walk along thin tubes in the direction of the limiting velocity $v$. These tubes are made by piling up smaller boxes of height $L^{\chi}$ and base of size $L^{\chi\beta_0}$ with $\beta_0$ close to $\frac{1}{2}$. One uses a renormalization step showing that if a certain estimate holds on the smaller boxes then the probability that none of the escape routes along the tubes yields a probability at least $e^{-cL^{\beta}}$ to exit $U_{\beta,L}$ through the "right side" is smaller than $\exp\{-L^{d(2\beta-1)-\eta)}\}$ ($\eta$ small) for $L$ large.

The case $\alpha < \alpha_0 \overset{\text{def}}{=} \frac{2d\beta}{d+1} < 1$, follows rather straightforwardly from the previous bound. One defines $U = \{x \in \mathbb{Z}^d, x \cdot \ell \in (-L^{\beta}, L^{\gamma})\}$, with $\gamma = \alpha_0 + \epsilon < 1$, ($\epsilon$ small). One notes that $P_{0,\omega}[X_{T_U} \cdot \ell > 0] = P_{0,\omega}[X_{T_{U_{\beta,L}}} \cdot \ell > 0] + \text{rem}_{.}$, where $\text{rem}_{.} = P_{0,\omega}[X_{T_U} \cdot \ell > 0, X_{T_{U_{\beta,L}}} \cdot \ell < 0]$. But from Chebychev's inequality and the control on $M$ below (97), for large $L$, $\mathbb{P}[\text{rem}_{.} \geq e^{-cL^{\beta}}] \leq e^{-L^{\alpha_0}}$. Hence one only needs to estimate $\mathbb{P}[P_{0,\omega}[X_{T_U} \cdot \ell > 0] \leq e^{-\frac{c}{2} L^{\beta}}]$, when $L$ is large. But the claim follows from the previous bound and the identity $U = U_{\frac{\beta}{\gamma}, L^{\gamma}}$, with $\frac{1}{2} < \frac{\beta}{\gamma} < 1$, ($\epsilon$ is small). $\square$

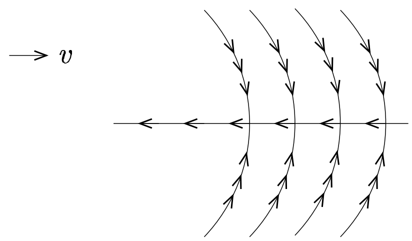
This completes our sketch of the proof of (136). $\square$

**Open problems:**

Can one pick $\alpha = d\beta$ in (145)?

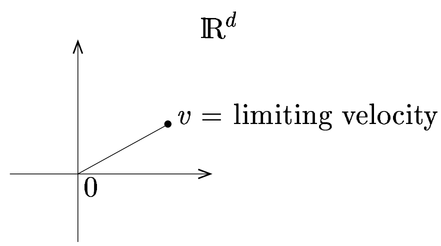
Can one have an upper bound in (136) of comparable order to the lower bound (137)?

Both issues are closely related (cf. Proposition 4.3), and the implicit question at hand is “whether there is anything truly better than naive traps”, to make $\tau_1$ big or to provoke an atypical exit distribution from slabs like $U_{\beta,L}$? Note that some traps may not be massive like the naive traps of (70), but diffuse, and functioning for instance like a fountain which works both with the help and against gravity.




a diffuse trap
“like a fountain”

**Slowdowns:** We now turn to **large deviations** of $\frac{X_n}{n}$ under $P_0$ or $P_{0,\omega}$. An **interesting effect** takes place **in the nestling case** when we assume (T) or (T’). First of all (cf. Theorem 4.1 of [40])




“The segment $[0,v]$ is critical in the nestling case”

**Theorem 4.5.** *($d \ge 2$, under* (T)$|\ell$*)*

*If $\mathcal{O}$ is a neighborhood of $[0,v]$, then*

$$\overline{\lim_n}\ \frac{1}{n}\ \log P_0\Big[\frac{X_n}{n} \notin \mathcal{O}\Big] < 0\,. \tag{146}$$

*In the non-nestling case,* (146) *holds for $\mathcal{O}$ a neighborhood of $v$. In the nestling case on the other hand, for $\mathcal{U}$ an open set with $\mathcal{U} \cap [0,v] \neq \emptyset$,*

$$\lim_n\ \frac{1}{n}\ \log P_0\Big[\frac{X_n}{n} \in \mathcal{U}\Big] = 0\,. \tag{147}$$

**Open problem:** Prove a large deviation principle for $\frac{X_n}{n}$ under $P_0$, with a rate function necessarily vanishing on $[0, v]$, in the nestling case.

On the other hand under $P_{0,\omega}$, one has in the nestling situation a large deviation principle due to Zerner [46].

**Theorem 4.6.** *(nestling case)*

*For $\mathbb{P}$-a.e. $\omega$, $\frac{X_n}{n}$ satisfies a large deviation principle on $\mathbb{R}^d$ under $P_{0,\omega}$ with speed $n$ and deterministic continuous convex rate function $I(\cdot)$ which is finite on $\{x \in \mathbb{R}^d, |x_1| + \cdots + |x_d| \leq 1\}$ and infinite outside. In addition when* (T) *holds:*

$$\{I(\cdot) = 0\} = [0, v]\,. \tag{148}$$

We thus see in view of (147) and (148) the **critical nature** of large deviations of $\frac{X_n}{n}$ in the neighborhood of $[0, v]$. Such large deviations can be viewed as **slowdowns** of the walk, where the general direction of the motion is kept unchanged but the pace is slower.

To begin the discussion of this finer effect we start with the case of **walks which are neutral or biased to the right**, where the theory is quite successful and slowdowns are linked to another well-known effect. The walks are now such that for some $\delta > 0$:

$$\begin{aligned} &\text{i)} \quad \mathbb{P}\Big[\Big\{\omega(0,\cdot) \equiv \frac{1}{2d}\Big\} \cup \{d(0,\omega)\cdot e_1 \geq \delta\}\Big] = 1 \\ &\text{ii)} \quad \mathbb{P}\Big[\Big\{\omega(0,\cdot) \equiv \frac{1}{2d}\Big\}\Big] > 0,\ \mathbb{P}[\{d(0,\omega)\cdot e_1 \geq \delta\}] > 0\,. \end{aligned} \tag{149}$$

Note that in view of (117), $(\mathrm{T})|e_1$ is satisfied. The description of the critical large deviations comes in the next theorem:

**Theorem 4.7.** *($d \geq 1$, under (149))*

*If $\mathcal{U} \cap [0, v] \neq \phi$, $\mathcal{U}$ open*

$$\varliminf_n n^{-\frac{d}{d+2}} \log P_0\Big[\frac{X_n}{n} \in \mathcal{U}\Big] > -\infty\,. \tag{150}$$

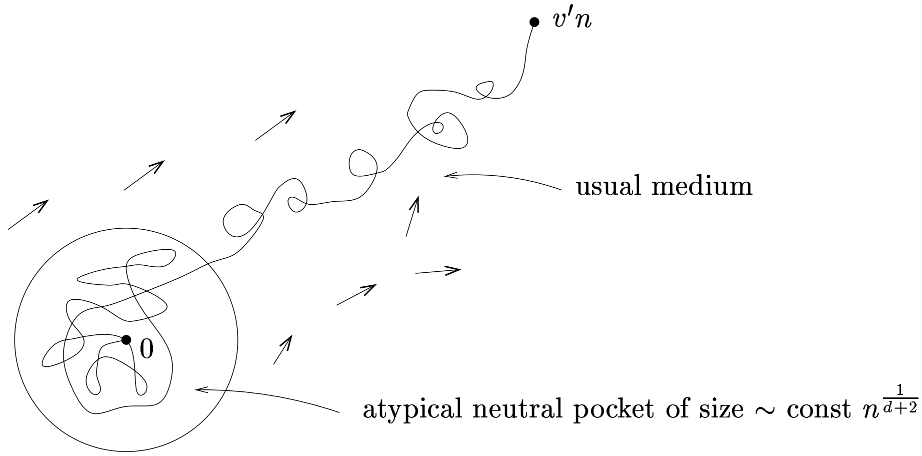
*If $\mathcal{O}$ is a neighborhood of $v$:*

$$\varlimsup_n n^{-\frac{d}{d+2}} \log P_0\Big[\frac{X_n}{n} \notin \mathcal{O}\Big] < 0\,. \tag{151}$$

*Moreover $\mathbb{P}$-a.s., for $\mathcal{U}$ and $\mathcal{O}$ as above similar estimate hold under $P_{0,\omega}$ with $n^{-\frac{d}{d+2}}$ replaced by $\frac{(\log n)^{\frac{2}{d}}}{n}$.*

The above asymptotics are strongly reminiscent both in the annealed and quenched situation to what happens in the problem of the "long time survival of Brownian motion among Poissonian obstacles", see [37], Chapter 4 §5.

**Intuitively** in the **annealed situation**: $\frac{X_n}{n} \sim v' \in [0, v]$




"The configuration presents an atypical neutral pocket where the walk spends a fraction of the time $n$ before moving close to $v'n$ at time $n$"

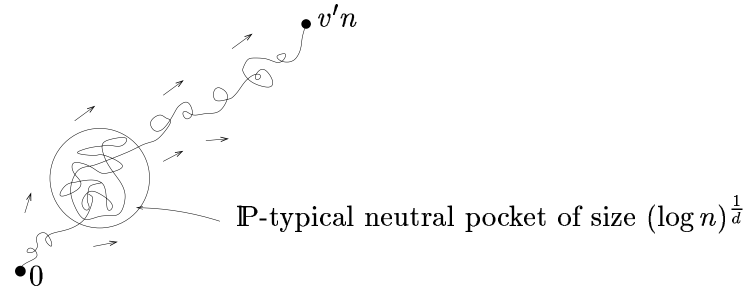
Of course the upper bound (151) is much harder to prove than the lower bound (150). The crucial control comes from an estimate (cf. [38], Theorem 2.4)

$$\overline{\lim_{n}}\, u^{-\frac{d}{d+2}} \log P_0[\tau_1 > u] < 0 \tag{152}$$

(compare with (136)), which in turn follows from controls in the case of walks neutral or biased to the right an analogous role to (145), namely (cf. [38], Proposition 3.1):

$$\begin{aligned}&\text{for some } p_0 \in (\tfrac{1}{2}, 1),\ \overline{\lim_{L \to \infty}}\, L^{-d} \log \mathbb{P}[P_{0,\omega}[X_{T_{U_L}} \cdot e_1 > 0] \le p_0] < 0,\\ &\text{with } U_L = \{x \in \mathbb{Z}^d,\ |x \cdot e_1| < L\}\,.\end{aligned} \tag{153}$$

**Intuitively** in the **quenched situation**, the walk uses neutral pockets typically present in the medium "in its way"




"The walk uses a $\mathbb{P}$-typical neutral pocket where it spends a fraction of the time $n$ before moving close to $v'n$ at time $n$"

In the case of walks neutral or biased to the right the role of neutral pockets is clearly brought to light, and roughly speaking there is nothing truly better than neutral pockets to slowdown the walk. When $d = 1$, a critical large deviation principle has even been proved by Pisztora-Povel-Zeitouni, [30] in the annealed case and by Pisztora-Povel, [29] in the quenched case.

**Open problem:** Prove a critical large deviation principle, when $d \geq 2$.

**In the general case** the estimates on slowdowns are less complete, cf. [40], Theorem 4.3, [41], Theorem 3.4.

**Theorem 4.8.** *($d \geq 2$, under* (T')*)*
*For $\mathcal{U}$ and $\mathcal{O}$ as in* (150), (151), *in the plain nestling case:*

$$\underline{\lim}_{n} (\log n)^{-d} \log P_0 \Big[\frac{X_n}{n} \in \mathcal{U}\Big] > -\infty\,, \tag{154}$$

*and in the general case:*

$$\overline{\lim}_{n} (\log n)^{-\alpha} \log \mathbb{P}_0 \Big[\frac{X_n}{n} \in \mathcal{O}\Big] < 0, \text{ for } \alpha < \frac{2d}{d+1}\,. \tag{155}$$

*(for the quenched estimates* $(\log n)^{-d}$ *is replaced by* $\frac{e^{c(\log n)^{\frac{1}{d}}}}{n}$ *and* $(\log n)^{-\alpha}$ *by* $\frac{1}{n}\, e^{c(\log n)^{\frac{1}{\alpha}}}$*).*

Of course (155) is substantially more delicate than (154) and relies in a crucial way on the renewal property (99) and (136). The rates in the plain-nestling situation lower bound and in the general upper bound do not quite match, and the leading role of traps in slowdowns is not fully demonstrated.

**Open problem:** Can one choose $\alpha = d$ in (155)? (This question is closely linked to the open problems below (145).)

## 5 Small perturbations of the simple random walk

We now discuss **some examples** of walks **satisfying condition (T')**. In (117), we have already provided with the help of Kalikow's condition (112), some concrete examples where (T) and hence (T') hold. The class we now discuss turns out to provide examples where (T') holds but Kalikow's condition breaks down.

We assume $d \geq 3$, and for $\epsilon \in (0,1)$ introduce

$$\begin{aligned} \mathcal{S}_\epsilon = \Big\{ (p(e))_{|e|=1}, \text{ with } \sum_{|e|=1} p(e) = 1 \text{ and } \big|p(e) - \tfrac{1}{2d}\big| \leq \tfrac{\epsilon}{4d} \text{ for each } e \Big\} \\ \subseteq \mathcal{P}_{\kappa = \frac{1}{4d}} \text{ in the notations of (60)}\,. \end{aligned} \tag{156}$$

The $(2d)$-vectors in $\mathcal{S}_\epsilon$ should be viewed as "small perturbations" of the simple random walk. Unlike [7], we investigate an anisotropic situation.

**Theorem 5.1.** $(d \geq 3)$

*For $\eta \in (0,1)$, there exists $\epsilon_0(d,\eta) \in (0,1)$ such that for $0 < \epsilon < \epsilon_0$, when the single site distribution $\mu$ is supported on $\mathcal{S}_\epsilon$ and*

$$\begin{aligned} \lambda \stackrel{\text{def}}{=} \mathbb{E}[d(0,\omega) \cdot e_1] > \epsilon^{\frac{5}{2}-\eta}, &\quad \text{when } d = 3\,, \\ \epsilon^{3-\eta}, &\quad \text{when } d \geq 4\,, \end{aligned} \tag{157}$$

*then* (T')$|e_1$ *holds.*

One interest of the above result is that it reads the ballistic nature of the walk directly from an expectation of the local drift with respect to the static measure $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$, and not from an expectation with respect to the so far unknown dynamic measure $\mathbb{Q}$ (cf. beginning of Lecture 1). As we will later see the fact that the exponents of $\epsilon$ which appear in (157) can exceed 2, plays an important role. It enables to construct examples of walks for which (T')$|e_1$ holds but Kalikow's condition $(K)|\ell$ is not fulfilled for any $\ell \in S^{d-1}$. Let us also mention that (157) cannot be replaced by the weaker assumption $\lambda > 0$, (for more on this see the remark following the sketch of proof of Theorem 5.1, above (185)).

*Sketch of proof*: (for more details see [42]). This is an application of the effective criterion (see (120)). One shows that for $0 < \epsilon < \epsilon_0(d,\eta)$ and $\lambda$ as in (157)

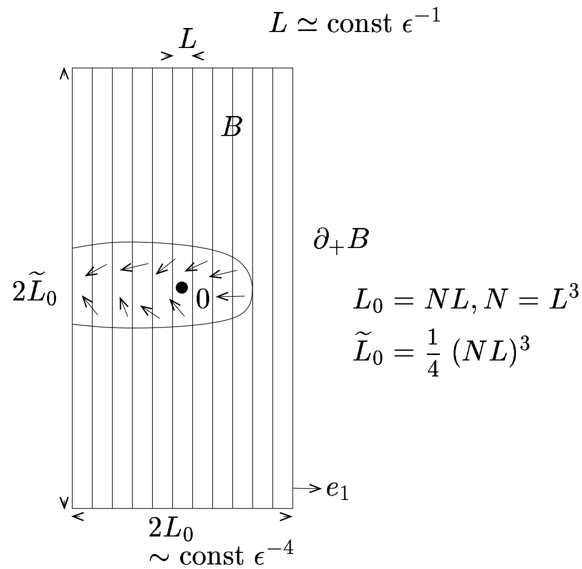
$$c_1(d)\Big(\log\frac{1}{\kappa}\Big)^{3(d-1)}\kappa^{-2}\,\widetilde{L}_0^{(d-1)}\,L_0^{3(d-1)+1}\,\mathbb{E}[\sqrt{\rho_B}]<1\,, \tag{158}$$

with

$$\rho_B=\frac{P_{0,\omega}[X_{T_B}\notin\partial_+B]}{P_{0,\omega}[X_{T_B}\in\partial_+B]}\quad\text{(cf. (119) for the notations)}$$

and $B$ is the box in the figure below.

As mentioned before in checking (158) there is a problem of a possible *small denominator* in $\rho_B$ corresponding to atypical exit distributions. To take care of this difficulty we chop $B$ in layers of thickness $L\simeq\text{const}\,\epsilon^{-1}$. Since $\mu$ is concentrated on $\mathcal{S}_\epsilon$, the walk is pretty much comparable to a simple random walk if one only considers times up to the exit time from a slab of thickness $2L$.




One shows that for $\epsilon<\epsilon_1(d,\eta)$

$$\mathbb{E}[\sqrt{\rho}_B]\le\frac{2\mathbb{E}[\widehat{\rho}(0)]^{\frac{N}{2}}}{(1-\mathbb{E}[\widehat{\rho}(0)]^{\frac{1}{2}})_+}+e^{-\text{const}\,L_0} \tag{159}$$

$e^{-\text{const}\,L_0}$: term controlling exit of $B$ through $\partial B\cap\{z:\sup_{j\ge2}|z\cdot e_j|\ge\widetilde{L}_0\}$

with

$$\widehat{\rho}(0,\omega) = \sup_x \left\{ \frac{\widehat{q}(x,\omega)}{\widehat{p}(x,\omega)},\ x \in B,\ x \cdot e_1 = 0 \right\}, \tag{160}$$

where in the notations of (66):

$$\widehat{q}(x,\omega) = 1 - \widehat{p}(x,\omega) = P_{x,\omega}[\widetilde{T}^{e_1}_{-L} < T^{e_1}_L]\,. \tag{161}$$

Define the slab $U$:

$$U = \{x \in \mathbb{Z}^d,\ |x \cdot e_1| < L\}\,, \tag{162}$$

then using the fact that $X_n \cdot e_1 - X_0 \cdot e_1 - \sum_0^{n-1} d(X_k,\omega) \cdot e_1$ is a $P_{x,\omega}$-martingale and the stopping theorem we find

$$\widehat{p}(x,\omega) = \frac{1}{2} + \frac{1}{2L}\, G_U(d \cdot e_1)(x)\,, \tag{163}$$

where

$$\begin{aligned} G_U(d \cdot e_1)(x) &= E_{x,\omega}\Big[\sum_0^{T_U - 1} d(X_k,\omega) \cdot e_1\Big] \\ &= \sum_{z \in U} g_U(x,z,\omega)\, d(z,\omega) \cdot e_1 \end{aligned} \tag{164}$$

($g_U$ stands for Green function of the walk killed outside $U$).

Now since $\omega(z,\cdot) \in \mathcal{S}_\epsilon$ for all $z$, $L \sim \text{const}\,\epsilon^{-1}$ (const: small), one sees with a comparison of $G_U$ with $G_U^0$ attached to the simple random walk that

$$\widehat{\rho}(0,\omega) \le 3 \quad \leftarrow \ \text{we do not have a small denominator problem}\,. \tag{165}$$

One now wants to see that $\mathbb{E}[\widehat{\rho}(0,\omega)] < 1$, and in essence this amounts to showing that for

$$D(\omega) \overset{\text{def}}{=} G_U(d \cdot e_1)(0)\,, \tag{166}$$

$\mathbb{E}[D]$ dominates the fluctuations of $G_U(d \cdot e_1)(x)$ one encounters when writing

$$\widehat{\rho}(0,\omega) = \sup \left\{ \frac{1 - \frac{1}{L}\, G_U(d \cdot e_1)(x)}{1 + \frac{1}{L}\, G_U(d \cdot e_1)(x)},\ x \in B, x \cdot e_1 = 0 \right\}. \tag{167}$$

**Crucial controls:** for a suitable $c(d) > 0$, when $\epsilon < \epsilon_2(d,\eta)$:

$$\mathbb{E}[D] \ge 2c\, \lambda_0\, L^2\,, \quad \text{and} \tag{168}$$

$$\mathbb{P}[|D(\omega) - \mathbb{E}[D]| > u] \le 2 \exp\Big\{ - \frac{u^2}{2v_D}\Big\}, \text{ for } u > 0\,, \tag{169}$$

$$\begin{array}{llll} \text{where} & \sqrt{v}_D \le & \epsilon^{\frac{1}{2}-\frac{\eta}{2}}, & d=3\,, \\ & & \epsilon^{1-\frac{\eta}{2}}, & d=4\,, \\ & & \epsilon, & d \ge 5\,, \\ \text{and} & \lambda_0 = & \epsilon^{\frac{5}{2}-\eta}, & d=3\,, \quad \text{(cf. (157))} \\ & = & \epsilon^{3-\eta}, & d \ge 4\,. \end{array}$$

Then with these estimates one has

$$\mathbb{E}[\widehat{\rho}(0,\omega)] \le \frac{1 - c\lambda_0 L}{1 + c\lambda_0 L} + \underbrace{3|\{x \in B, x \cdot e_1 = 0\}|}_{\le \text{const } (L^4)^{3(d-1)}} \; \mathbb{P}\underbrace{[D - \mathbb{E}[D] \le -c\lambda_0 L^2]}_{\le \text{const } \exp\left\{-c^2 \frac{\lambda_0^2 L^4}{v_D}\right\}}\,. \tag{170}$$

Now

$$\begin{aligned} \lambda_0 L^2 \ge \; & \text{const}\,(d,\eta)\; \epsilon^{\frac{1}{2}-\eta},\; d=3, \\ & \text{const}\,(d,\eta)\; \epsilon^{1-\eta},\; d \ge 4\,, \end{aligned}$$

and hence:

$$\lambda_0 L^2 \gg \sqrt{v}_D\,. \tag{171}$$

As a result for $\epsilon \le \epsilon_3(d,\eta)$

$$\mathbb{E}[\widehat{\rho}(0)] \le 1 - \frac{c}{2}\,\lambda_0 L\,,$$

and inserting this in (159), we find:

$$\mathbb{E}[\widehat{\rho}_B] \le \; 2\, \frac{\big(1 - \frac{c}{2}\,\lambda_0 L\big)^{\frac{N}{2}}}{1 - \sqrt{1 - \frac{c}{2}\,\lambda_0 L}} + e^{-\text{const}\, L_0} \le \; \frac{\text{const}}{\lambda_0 L}\; \exp\Big\{ - \frac{c}{4}\,\lambda_0 L_0\Big\} + e^{-\text{const}\, L_0}\,, \tag{172}$$

and this is more than enough to prove (158).

Let us give some comments on the proof of (168), (169) which are the crucial controls showing that the expectation of $D$ dominates the relevant fluctuations of $D$ around its mean. For instance for (168), we can write

$$\mathbb{E}[D] = \mathbb{E}[(G_U\lambda)(0)] + \mathbb{E}[G_U(d \cdot e_1 - \lambda)(0)]\,. \tag{173}$$

Now because $G_U$ is close to $G_U^0$, (see notations above (165)), it is no hard to see that:

$$(G_U\lambda)(0) \ge c'(d)\,\lambda L^2 \ge c'(d)\,\lambda_0 L^2 \tag{174}$$

so that the same holds for $\mathbb{E}[G_U(\lambda)(0)]$. One wants then to see that $\mathbb{E}[G_U(d\cdot e_1-\lambda)(0)]$ is small compared to $c'(d)\,\lambda_0 L^2$. Observe that $\mathbb{E}[d(x,\omega)\cdot e_1-\lambda]=0$, but $|d(x,\omega)\cdot e_1-\lambda|$ can a-priori be of order $\sim\epsilon\gg\lambda_0$ (see below (169)), and to prove that $\mathbb{E}[G_U(d\cdot e_1-\lambda)(0)]$ is small compared to $c'(d)\,\lambda_0 L^2$ one needs to use cancellations. One writes (see (118))

$$G_U(d\cdot e_1-\lambda)(0)=\sum_{x\in U} P_{0,\omega}[H_x<T_U]\,\frac{d(x,\omega)\cdot e_1-\lambda}{P_{x,\omega}[\widetilde{H}_x>T_U]} \tag{175}$$

and introducing

$$\overline{P}_{x,\omega}=P_{x,\overline{\omega}_x},\text{ where } \overline{\omega}_x(y,\cdot) \text{ is } \omega(y,\cdot),\text{ for } y\neq x,\text{ and } \mathbb{E}[\omega(0,\cdot)],\text{ for } y=x\,, \tag{176}$$

one can single out the effect of $\omega(x,0)$ in the denominator through:

$$P_{x,\omega}[\widetilde{H}_x>T_U]=\overline{P}_{x,\omega}[\widetilde{H}_x>T_U]+\sum_{|e|=1}\big(\underbrace{\omega(x,e)-\mathbb{E}[\omega(0,e)]}_{\overset{\text{def}}{=}\ \widetilde{\delta}(x,e)}\big)\,P_{x+e,\omega}[H_x>T_U]\,. \tag{177}$$

Coming back to (175), using an expansion and the fact that:

$$\Big|\sum_{|e|=1}\widetilde{\delta}(x,e)\,\frac{P_{x+e,\omega}[H_x>T_U]}{\overline{P}_{x,\omega}[H_x>T_U]}\Big|\le\frac{2\epsilon}{4d}\times 2d\times\frac{1}{\kappa}\le\frac{\epsilon}{\kappa}\le\frac{1}{2}$$

one finds:

$$\begin{aligned}
&G_U(d\cdot e_1-\lambda)(0)=A+B+C,\ \text{ where}\\
&A\ =\ \sum_{x\in U} P_{0,\omega}[H_x<T_U]\,\frac{(d(x,\omega)\cdot e_1-\lambda)}{\overline{P}_{x,\omega}[\widetilde{H}_x>T_U]}\,,\\
&\qquad\ \text{(observe that using independence } \mathbb{E}[A]=0)\\
&B\ =\ -\sum_{x\in U}\frac{P_{0,\omega}[H_x<T_U]}{\overline{P}_{x,\omega}[\widetilde{H}_x>T_U]}\,(d(x,\omega)\cdot e_1-\lambda)\sum_{|e|=1}\widetilde{\delta}(x,e)\,\frac{P_{x+e,\omega}[H_x>T_U]}{\overline{P}_{x,\omega}[\widetilde{H}_x>T_U]}\,,\\
&|C|\ \le\ c(d)\,\epsilon^3L^2\,.
\end{aligned} \tag{178}$$

Note that $\epsilon^3L^2\ll\lambda_0L^2$ for small $\epsilon$. However if one brutally bounds $B$, one obtains $|\mathbb{E}[B]|\le\text{const}\,\epsilon^2L^2$ and $\epsilon^2L^2\gg\lambda_0L^2$, which is a serious matter, since we plan to show that $\mathbb{E}[G_U(d\cdot\ell_1-\lambda)(0)]=\mathbb{E}[A]+\mathbb{E}[B]+\mathbb{E}[C]$ is small compared to $c'(d)\lambda_0L^2$!

To overcome this difficulty, observe that

$$\sum_{|e|=1} \widetilde{\delta}(x,e) = 0\,, \tag{179}$$

so we can introduce a counterterm in the last sum in $B$ not depending on $e$ and obtain:

$$\begin{aligned} B = -\sum_{x\in U} \frac{P_{0,\omega}[H_x < T_U]}{\overline{P}_{x,\omega}[\widetilde{H}_x > T_U]}\, (d(x,\omega)\cdot e_1 - \lambda) \\ \sum_{|e|=1} \widetilde{\delta}(x,e)\, \frac{P_{x+e,\omega}[H_x > T_U] - P_{x+e_1,\omega}[H_x > T_U]}{\overline{P}_{x,\omega}[\widetilde{H}_x > T_U]} \end{aligned} \tag{180}$$

and then one finds in a rather straightforward fashion

$$|\mathbb{E}[B]| \le \text{const}\, \epsilon^2 \sum_{x\in U} g_U(0,x,\omega)\, |g_U(x+e,x,\omega) - g_U(x+e_1,x,\omega)|$$

with the notation of (164) for the Green function.

Now the idea is that when $x$ is **in the bulk** of $U$, i.e. far from $\partial U$, $g_U(x+e,x,\omega) - g_U(x+e_1,x,\omega)$ is close to the same expression where $g_U$ is replaced by the Green function of the simple random walk, but because of the isotropy of the simple random walk, the corresponding expression vanishes. On the other hand when $x$ is “close to $\partial U$”, then $g_U(x+e,x,\omega) - g_U(x+e_1,x,\omega)$ need not be small but $\sum_{x \text{ close to } \partial U} g_U(0,x,\omega) \ll L^2$ since the walk will not stay too long near the boundary of $U$ where it can get killed. In fact one shows:

$$|\mathbb{E}[B]| \le \text{const}\, \epsilon^2 \times \left(\epsilon \log L + \frac{1}{L}\right) L^2 \ll \lambda_0 L^2\,, \tag{181}$$

and this is how the above “special cancellation of the $\epsilon^2 L^2$ term” potentially present in $B$ enables to prove (168).

The bound (169) controlling the fluctuations of $D$ is obtained by using the “martingale method” (see Proposition 3.2 of [42]), i.e. by considering the martingale

$$\begin{aligned} H_n = \mathbb{E}[G_U(d\cdot e_1)(0)|\mathcal{G}_n],\ \text{with } \mathcal{G}_n =&\sigma(\omega(x_1,\cdot),\dots,\omega(x_n,\cdot)), \\ &n \ge 0, \{\emptyset,\Omega\},\ \text{for } n=0\,, \end{aligned} \tag{182}$$

$$(\text{so } H_\infty = G_U(d\cdot e_1)(0),\ H_0 = \mathbb{E}[G_U(d\cdot e_1)(0)])\,,$$

with $x_i, i\ge 1$, an enumeration of $U$. By showing that

$$|H_n - H_{n-1}| \le \gamma_n \leftarrow \text{deterministic numbers}\,, \tag{183}$$

one finds with a slight variation on Azuma's inequality, cf. [1], that for $u > 0$,

$$\mathbb{P}[|G_U(d\cdot e_1)(0) - \mathbb{E}[G_U(d\cdot e_1)(0)]| > u] \le 2\,\exp\Big\{-\frac{u^2}{2\sum_{n\ge 1}\gamma_n^2}\Big\}. \quad (184)$$

In spirit, to prove (169), one shows that

$$\text{“}\gamma_n \le \text{const}\,\epsilon\, g_{0,U}(0,x_n)\text{''}$$

where $g_{0,U}(x,y) = (G_U^0 1_y)(x)$ is the Green function of the simple random walk killed outside $U$, (the actual control on $\gamma_n$ are slightly weaker).

Note the heuristic bound on $\sum_{n\ge 1}\gamma_n^2$:

$$\begin{aligned}\sum_{x\in U}\epsilon^2 g_{0,U}(0,x)^2 \sim \epsilon^2 L \sim \epsilon \ll (\lambda_0 L^2)^2,&\ \text{when } d=3\,,\\ \epsilon^2\log L \sim \epsilon\log\frac{1}{\epsilon} \ll (\lambda_0 L^2)^2,&\ \text{when } d=4\,,\\ \epsilon^2 \ll (\lambda_0 L^2)^2,&\ \text{when } d\ge 5\,.\end{aligned}$$

This provides some rationale for the value of $\lambda_0$, see (157) and (169). □

**Remark:** It is a natural question whether one can replace (157) with the weaker condition $\lambda > 0$, and derive a similar conclusion. This is not the case and some examples where $\mathbb{E}[d(0,\omega)] \ne 0$, and the walk exhibits a diffusive behavior can be found in the article [6]. □

**Back to Kalikow's condition:**

We now explain how the above result provides **examples** of walks **for which Kalikow's condition does not hold for any $\ell \in S^{d-1}$ but (T')$|e_1$ is satisfied**.

In particular, unlike what is known to happen when $d = 1$, cf. [43], Kalikow's condition does not characterize ballistic behavior when $d \ge 3$.

To describe the example, we consider $d \ge 3$, $0 < \epsilon < 1$, $0 < \rho < 1$ and a law $\mu$ on $\mathcal{S}_e$ (cf. (156)) which is:

$$\text{invariant under the rotations of } \mathbb{Z}^d \text{ preserving } e_1, \quad (185)$$

and such that

$$\text{var}_{\mathbb{P}}(\omega(0,e_1)) = \text{var}_{\mathbb{P}}(\omega(0,-e_1)) \ge \rho\epsilon^2\,, \quad (186)$$

$$\text{cov}_{\mathbb{P}}(\omega(0,e_1),\omega(0,-e_1)) \le (1-\rho)\,\text{var}_{\mathbb{P}}(\omega(0,e_1))\,. \quad (187)$$

One possible example of such a law corresponds to choosing an isotropic $\widetilde{\mu}$ on $\mathcal{S}_{\frac{\epsilon}{2}}$ for which:

$$\mathrm{var}_{\widetilde{\mu}}(p(e_1)) \geq \rho\epsilon^2,\ \mathrm{cov}_{\widetilde{\mu}}(p(e_1),\, p(-e_1)) \leq (1-\rho)\,\mathrm{var}_{\widetilde{\mu}}(p(e_1)) \tag{188}$$

and defining $\mu$ as the image of $\widetilde{\mu}$ under the map

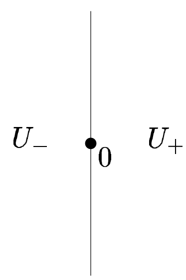
$$p(e) \to p(e) + \frac{\lambda}{2}\, e \cdot e_1,\ \text{ for } |e| = 1 \tag{189}$$

with $\lambda$ a number in $(-\frac{\epsilon}{4d}, \frac{\epsilon}{4d})$.

We now introduce

$$U_+ = \{y \in \mathbb{Z}^d, y \cdot e_1 \geq 0\},\ U_- = \{y \in \mathbb{Z}^d, y \cdot e_1 \leq 0\}\,, \tag{190}$$

and recall the definition of the auxiliary local drifts $\widehat{d}_{U_+}(\cdot)$ and $\widehat{d}_{U_-}(\cdot)$ in (111).




**Proposition 5.2.** *(under* (185), (186), (187)*)*

*With* $\lambda = \mathbb{E}[d(0,\omega) \cdot e_1]$, *provided:*

$$\epsilon < \frac{1}{4}\,\kappa^4\rho^2,\ |\lambda| \leq \frac{\kappa^2}{2}\,\rho^2\epsilon^2\,, \tag{191}$$

*then*

$$\widehat{d}_{U_+}(0) = \nu_+ e_1,\ \widehat{d}_{U_-}(0) = \nu_- e_1,\ \textit{with } \nu_+ > 0,\ \nu_- < 0\,, \tag{192}$$

*(and hence* $(K)|\ell$ *fails for every* $\ell \in S^{d-1}$*).*

*Sketch of proof:* Note that when $R$ is a rotation preserving $\mathbb{Z}^d$, $\omega \in \Omega$,

$$\begin{aligned}&\text{under } P_{x,\omega},\ (R(X_n))_{n\geq 0} \text{ is distributed as } X_n \text{ under}\\ &P_{R(x),R\omega} \text{ with } (R\omega)(y,e) \overset{\text{def}}{=} \omega(R^{-1}(y),\, R^{-1}(e))\,.\end{aligned} \tag{193}$$

Using (185) if $R$ is a rotation preserving $e_1$ and sending $e_i$ in $-e_i$, for a given $i \geq 2$, we see that for $U = U_+$ or $U_-$, with the definitions (105), (111):

$$\widehat{d}_U(0) \cdot e_i = \frac{\mathbb{E}[g_U(0,0,\omega) \cdot d(0,\omega) \cdot e_i]}{\mathbb{E}[g_U(0,0,\omega)]} = 0$$

(because $g_U(0,0,R\omega) = g_U(0,0,\omega)$ and $d(0,R\omega) \cdot e_i = -d(0,\omega) \cdot e_i$) and hence

$$\widehat{d}_U(0) \text{ is colinear to } e_1, \text{ (recall here } U \text{ is } U_+ \text{ or } U_-) \,. \tag{194}$$

Moreover by an analogous calculation as in (175) - (178)

$$\mathbb{E}[g_U(0,0,\omega)\, d(0,\omega) \cdot e_1] = \alpha_U + \beta_U + \gamma_U \,, \tag{195}$$

where with the notation (176), (177)

$$\alpha_U = \lambda \mathbb{E}\Big[\frac{1}{P_{0,\omega}[\widetilde{H}_0 > T_U]}\Big], \text{ and } |\alpha_U| \overset{(191)}{\leq} \frac{|\lambda|}{\kappa} \leq \frac{\kappa}{2}\, \rho^2\, \epsilon^2$$

$$\beta_U = \mathbb{E}\Big[\frac{(\widetilde{\delta}(0,e_1) - \widetilde{\delta}(0,-e_1))}{\overline{P}_{0,\omega}[\widetilde{H}_0 > T_U]^2} \sum_{|e|=1} \widetilde{\delta}(0,e)\, P_{e,\omega}[H_0 < T_U]\Big]$$

(where we have used (179) and $\sum_{|e|=1} \widetilde{\delta}(0,e)\, P_{e,\omega}[H_0 < T_U] = -\sum_{|e|=1} \widetilde{\delta}(0,e)\, P_{e,\omega}[H_0 > T_U]$)

$$|\gamma_U| \leq 2\Big(\frac{\epsilon}{\kappa}\Big)^3 \quad \Big(\frac{\epsilon}{\kappa} \leq \frac{1}{2} \text{ because of (191)}\Big) .$$

We will now see that "$\beta_U$ dominates". In contrast to the situation above (181), the boundary effects are now predominant. Indeed using independence:

$$\beta_U = \sum_{|e|=1} \mathbb{E}\Big[(\widetilde{\delta}(0,e_1) - \widetilde{\delta}(0,-e_1))(\widetilde{\delta}(0,e_1) + \widetilde{\delta}(0,-e_1))\Big]\, \mathbb{E}\Big[\frac{P_{e,\omega}[H_0 < T_U]}{\overline{P}_{0,\omega}[\widetilde{H}_0 > T_U]^2}\Big] .$$

Note that by (185) and (193),

$$\mathbb{E}\Big[\frac{P_{e,\omega}[H_0 < T_U]}{\overline{P}_{0,\omega}[\widetilde{H}_0 > T_U]^2}\Big] \text{ remains the same for all } e \text{ with } e \cdot e_1 = 0 \,. \tag{196}$$

So using (179), we find:

$$\beta_U = -\,\mathbb{E}\Big[(\widetilde{\delta}(0,e_1) - \widetilde{\delta}(0,-e_1))\,\underbrace{(\widetilde{\delta}(0,e_1) + \widetilde{\delta}(0,-e_1))}_{-\sum_{e\cdot e_1=0}\widetilde{\delta}(0,e)}\Big]\;\mathbb{E}\Big[\frac{P_{e_2+\omega}[H_0 < T_U]}{\overline{P}_{0,\omega}[\widetilde{H}_0 > T_U]^2}\Big]$$
$$+ \sum_{e=\pm e_1} \mathbb{E}[(\widetilde{\delta}(0,e_1) - \widetilde{\delta}(0,-e_1))\,\delta(0,e)]\,\mathbb{E}\Big[\frac{P_{e,\omega}[H_0 < T_U]}{\overline{P}_{0,\omega}[\widetilde{H}_0 > T_U]^2}\Big]\,.$$

Note that because $\mathrm{var}_{\mathbb{P}}(\omega(0,e_1)) = \mathrm{var}_{\mathbb{P}}(\omega(0,-e_1))$ the first term vanishes and we find

$$\beta_U = \overbrace{[\mathrm{var}_{\mathbb{P}}(\omega(0,e_1)) - \mathrm{cov}_{\mathbb{P}}(\omega(0,e_1),\omega(0,-e_1))]}^{(186)-(187)\geq \rho^2\epsilon^2}$$
$$\times\,\mathbb{E}\Big[\frac{P_{e_1,\omega}[H_0 < T_U] - P_{-e_1,\omega}[H_0 < T_U]}{\overline{P}_{0,\omega}[\widetilde{H} > T_U]^2}\Big]\,.$$

Therefore specifying $U = U_+$ or $U_-$ we find

$$\beta_{U_+} \geq \kappa\rho^2\epsilon^2,\ \beta_{U_-} \leq -\kappa\rho^2\epsilon^2\,, \tag{197}$$

and hence by (195) and (197)

$$\begin{aligned}\mathbb{E}[g_{U_+}(0,0,\omega)(d\cdot e_1)(0,\omega)] &\geq -\frac{\kappa}{2}\,\rho^2\epsilon^2 + \kappa\rho^2\epsilon^2 - 2\Big(\frac{\epsilon}{\kappa}\Big)^3 \\ &= \epsilon^2\Big(\frac{\kappa}{2}\,\rho^2 - \frac{2\epsilon}{\kappa^3}\Big) > 0\end{aligned} \tag{198}$$

whereas

$$\begin{aligned}\mathbb{E}[g_{U_-}(0,0,\omega)(d\cdot e_1)(0,\omega)] &\leq \frac{\kappa}{2}\,\rho^2\epsilon^2 - \kappa\rho^2\epsilon^2 + 2\Big(\frac{\epsilon}{\kappa}\Big)^3 \\ &= -\epsilon^2\Big(\frac{\kappa}{2}\,\rho^2 - \frac{2\epsilon}{\kappa^3}\Big) < 0\end{aligned} \tag{199}$$

and the claim (192) follows. □

Combining Theorem 5.1 and the above proposition, we see that for $\rho,\eta' \in (0,1)$, we can find $\widetilde{\epsilon}(d,\eta,\rho) \in (0,1)$ such that when $\mu$ concentrated on $\mathcal{S}_\epsilon$ satisfies (185), (186), (187) and

$$\begin{aligned}\epsilon^{\frac{5}{2}-\eta} &< \lambda < \frac{\kappa^2}{2}\,\rho^2\epsilon^2, \quad \text{when } d=3,\\ \epsilon^{3-\eta} &< \lambda < \frac{\kappa^2}{2}\,\rho^2\epsilon^2, \quad \text{when } d\geq 4,\end{aligned} \tag{200}$$

then $(K)|\ell$ fails for every $\ell \in S^{d-1}$, but (T')$|e_1$ satisfied (and the walk is ballistic).

This concludes this brief account of some of the recent advances concerning random walks in random environment. Some of the ideas and paradigms discussed in these notes are currently investigated for diffusions in random environment, cf. [23],[34], and for dependent environments, cf. [9],[31],[45]. Hopefully it will clear from reading these notes that although some issues are better understood, much remains to be done.

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