

Conformally Invariant Processes in the Plane

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1 Mathematical problems from critical phenomena

These lectures will focus on recent rigorous work on continuum limits of planar lattice models from statistical physics at criticality. For an introduction, I would like to discuss the general problem of critical exponents and scaling limits for lattice models in equilibrium statistical mechanics. There are a number of models, [e.g., self-avoiding walk (polymers), percolation, loop-erased random walk (uniform spanning trees, domino tilings), Ising model, Potts model, nonintersecting simple random walks] that fall under this general framework.

As an example, we consider the self-avoiding walk (SAW). A SAW of length n (starting at the origin) in the integer lattice \mathbb{Z}^d is a finite sequence of points $\omega = [\omega_0 = 0, \omega_1, \dots, \omega_n]$ in \mathbb{Z}^d with $|\omega_j - \omega_{j-1}| = 1, j = 1, \dots, n$ and $\omega_j \neq \omega_k, 0 \leq j < k \leq n$. The SAW is a simple lattice model for polymers (by simple I mean simple to define, not necessarily simple to analyze!). Let C_n denote the number of SAWs of length n . It is easy to see that $C_{n+m} \leq C_n C_m$, or, in other words, $\log C_n$ is a subadditive function. Also $C_n \geq d^n$ (since any walk taking steps only of ± 1 in the d coordinate directions is a self-avoiding walk) and $C_n \leq (2d) \cdot (2d-1)^{n-1}$ (since immediate reversals are prohibited). It follows from the subadditivity of the function $\log C_n$ that

$$\lim_{n \rightarrow \infty} n^{-1} \log C_n = \inf_{n \rightarrow \infty} n^{-1} \log C_n,$$

i.e., there is a number $\beta_d \in [d, 2d-1]$ such that $C_n^{1/n} \rightarrow \beta_d$. This number is sometimes called the *connective constant*; roughly speaking, on the average, there are β_d ways to extend an n -step SAW to an $(n+1)$ -step walk. Also, $C_n \geq \beta_d^n$. The generating function for SAWs is

$$G_d(s) = \sum_{n=0}^{\infty} C_n s^n.$$

The critical value for the generating function is $s = 1/\beta_d$; this is the value at which the generating function becomes infinite. For $s < 1/\beta_d$ we can consider the probability measure on SAWs, m_{SAW}^s , that assigns measure $G_d(s)^{-1} s^n$ to each SAW of length n . One can describe the criticality problem for SAW as the problem of analyzing the measure m_{SAW}^s as $s \uparrow 1/\beta_d$. There is a very closely related problem to consider the behavior of the uniform measure on SAWs of length n as $n \rightarrow \infty$. One example of an interesting *critical*

exponent for the SAW is the root mean-square displacement exponent ν defined by $\langle |\omega_n|^2 \rangle_n \approx n^{2\nu}$. Here $\langle \cdot \rangle_n$ denotes expectation with respect to the uniform measure on n -step SAWs (we could define ν similarly in terms of the measures m_{SAW}^s). For more information on rigorous results on SAW, see [45].

The general features that are conjectured to be true for all of these critical statistical physics models are the following:

- The behavior at criticality (i.e., the values of the critical exponents and the scaling limit at criticality) depends strongly on the spatial dimension d .
- The behavior at criticality does not depend on the choice of lattice or other microscopic details.
 - This conjecture is sometimes referred to as *universality*.
 - For example, a non-nearest neighbor SAW on the integer lattice \mathbb{Z}^d is expected to have the same qualitative behavior as the nearest neighbor SAW on \mathbb{Z}^d (provided the walk is truly d -dimensional).
- There is a critical dimension d_c such that above this dimension the critical behavior is “mean-field” and easy to describe.
 - The term mean-field means different things for different models. In the case of random walks with interactions or potentials (such as SAW), mean-field means the same as non-interacting walks, i.e., the “mean-field” theory for SAWs is just simple random walk.
 - While it is conjectured that the behavior is easy to describe in high dimensions, it is by no means trivial to *prove* that this behavior holds. As an example of results in this direction see [1, 7, 20, 25, 45].
 - For SAW and most of the other models listed above $d_c = 4$. For percolation d_c is conjectured to be 6.
 - Simple random walk paths have “fractal dimension” 2. This follows from the fact that in n steps they tend to be distance $n^{1/2}$ from the origin. This can be restated by saying that the number of points in the path in a disk of radius R looks like R^2 . Two 2-dimensional sets tend to intersect in dimensions less than four and tend not to intersect in greater than four dimensions. This

is the heuristic reason why $d_c = 4$ for models like the SAW for which the intersection of random walk paths is important.

- At the critical dimension d_c there is mean-field behavior with logarithmic corrections.
 - For the SAW in $d = 4$, it is conjectured that $\langle |\omega(n)|^2 \rangle_n \sim cn(\log n)^{1/4}$. While this is still conjecture, the analogous result has been proved for SAWs on a “hierarchical lattice” [6].
 - Rigorous results at the critical dimension have been obtained for some models, see [19, 26, 25].
- Below the critical dimension, the scaling limit and critical exponents do not display mean-field behavior. However, it is believed that a limit exists that is scale invariant and also invariant under rotations.
 - For the SAW, the exponent ν is conjectured to be $3/4$ for $d = 2$ and approximately $.588\dots$ for $d = 3$.
- In $d = 2$, since the limit is scale and rotationally invariant, it is, in fact, conformally invariant. The exponents are expected to be rational numbers.
 - Theoretical physicists used nonrigorous methods from conformal field theory to give exact predictions for these exponents. See, for example, [5, 11, 21] for a discussion of these ideas. Since their predictions are very consistent with numerical simulations, it is generally believed that these predictions are correct.
 - The existence of a conformally invariant limit has not been proved for SAW. However, under the assumption of such a conformally invariant limit, the critical behavior can be determined [40].
- In dimensions strictly between 2 and d_c , there is no reason to believe that the exponents will be rational numbers or in any other way computable.
 - These dimensions are the hardest for rigorous analysis and remain open problems today.
 - It is generally difficult to show that the exponent exists. In the case of the random walk intersection exponent it can be shown

that it is the same as the Brownian motion intersection exponent and this can be shown to exist in $d = 3$ (see §4.2 for more details). However, the exact value has not been determined and perhaps no exact expression can be found for it.

These lectures will consider the case $d = 2$. Mathematicians are now starting to understand rigorously the scaling limit of two-dimensional systems. For most of these models, the general strategy can be described as:

- Construct possible continuum limits for these models. Show that there are only a limited number of such limits that are conformally invariant.
- Prove that the lattice model approaches the continuum limit.

We should think of the first step as being similar for all of these models. We will spend the next couple of lectures discussing the continuum limits. One example you should already know — the scaling limit of simple random walk is Brownian motion (which in two dimensions is conformally invariant). The important new ideas are *restriction measures* and *stochastic Loewner evolution (SLE)*. The later lectures will discuss rigorous results about lattice models approaching the continuum limit — we will discuss nonintersecting random walks (which can be shown to be equivalent to problems about exceptional sets of Brownian paths), percolation on the triangular lattice, and the loop-erased random walk. As a rule, the methods used for the second step are particular to each model.

2 Brownian motion and restriction measures

2.1 Review of complex analysis

If $D \subset \mathbb{C}$ is a domain (i.e., a nonempty, connected, open subset), a function f is *analytic* or *holomorphic* on D if the (complex) derivative exists everywhere on D . Recall that if $f = u + iv$, then f is analytic at $z = x + iy$, if and only if u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$. f is *conformal* at z if it is analytic with $f'(z) \neq 0$; an analytic function is conformal at z if and only if it is locally one-to-one. We will say that $f : D \rightarrow D'$ is a *conformal transformation* if f is analytic, one-to-one, and onto. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

A domain D is said to be *simply connected* if the complement of D is connected. The Riemann mapping theorem states that if D is any simply

connected subset of \mathbb{C} other than the entire plane \mathbb{C} and $z \in D$, then there is a unique conformal transformation $f : D \rightarrow \mathbb{D}$ such that $f(z) = 0$, $f'(z) > 0$.

2.2 Conformal invariance of Brownian motion

Let B_t be a complex Brownian motion, i.e., $B_t = B_t^1 + iB_t^2$, where B_t^1, B_t^2 are independent one dimensional standard Brownian motions. If D is a domain, let

$$\tau_D = \inf\{t : B_t \notin D\}.$$

Proposition 1 *Suppose $f : D \rightarrow \mathbb{C}$ is a nonconstant analytic function. Then $Y_t = f(B_t), 0 \leq t < \tau_D$ is a time change of a Brownian motion. More precisely, $Y_t = f(B_{\sigma_t})$ is a standard Brownian motion where σ_t satisfies*

$$\int_0^{\sigma_t} |f'(B_s)|^2 ds = t.$$

Proof (Sketch). The Cauchy-Riemann equations imply that u, v are harmonic on D , i.e., $\Delta u(z) = \Delta v(z) = 0, z \in D$. Itô's formula then gives (for $t < \tau_D$),

$$\begin{aligned} d[u(B_t)] &= u_x(B_t) dB_t^1 + u_y(B_t) dB_t^2 \\ d[v(B_t)] &= v_x(B_t) dB_t^1 + v_y(B_t) dB_t^2 = -u_y(B_t) dB_t^1 + v_x(B_t) dB_t^2. \end{aligned}$$

In particular,

$$\langle u(B) \rangle_t = \langle v(B) \rangle_t = \int_0^t [u_x(B_s)^2 + u_y(B_s)^2] ds = \int_0^t |f'(B_s)|^2 ds,$$

and $\langle u(B), v(B) \rangle_t = 0$.

If $\gamma : [0, T] \rightarrow \mathbb{C}, \gamma^* : [0, T^*] \rightarrow \mathbb{C}$ are two continuous curves, we will write $\rho(\gamma, \gamma^*) < \epsilon$ if there is an increasing homeomorphism $\phi : [0, T] \rightarrow [0, T^*]$ such that $|\gamma(t) - \gamma^*(\phi(t))| < \epsilon, 0 \leq t \leq T$. Note that ρ is a metric on the set \mathcal{C} of equivalence classes of curves γ where two curves are equivalent if one is just an increasing reparametrization of the other. When we discuss measures on \mathcal{C} we will mean measures on the Borel σ -algebra generated by this metric.

Suppose $D \subset \mathbb{C}$ is a simply connected domain other than the entire plane and suppose B_t is a Brownian motion starting at $z \in D$. With probability one, $\tau_D < \infty$. Then if we consider $B_t, 0 \leq t \leq \tau_D$, the Brownian motion generates a probability measure on \mathcal{C} that we denote $\mu_D(z)$. It is supported on curves that start at z and stay in D except for the terminal point which

lies in ∂D . Since we are considering $\mu_D(z)$ as a measure on \mathcal{C} , i.e., we are ignoring the parametrization of the curve, we get the following corollary of Proposition 1.

Proposition 2 *If D is a simply connected domain other than \mathbb{C} , $z \in D$ and $f : D \rightarrow D'$ is a conformal transformation, then*

$$f \circ \mu_D(z) = \mu_{D'}(f(z)).$$

Here we have used $f \circ \mu_D$ for the natural pullback measure satisfying $(f \circ \mu_D)\{f \circ \gamma : \gamma \in V\} = \mu_D(V)$. Suppose, for ease, that ∂D is smooth. Consider the harmonic measure of D starting at z , i.e., the distribution of $B(\tau_D)$ given $B_0 = z$. Since ∂D is smooth, we can write this distribution as $H_D(z, w) d|w|$ where $H_D(z, w)$ is the *Poisson kernel* for D . Similarly we can write

$$\mu_D(z) = \int_{\partial D} \mu_D(z, w) d|w|,$$

where $\mu_D(z, w)$ represents a measure on \mathcal{C} supported on curves starting at z leaving D at w . The total mass $|\mu_D(z, w)|$ is $H_D(z, w)$. The probability measure $\mu_D(z, w)/|\mu_D(z, w)|$ can be considered as the conditional distribution on Brownian paths starting at z conditioned to leave D at w . If $f : D \rightarrow D'$ is a conformal transformation that is C^1 up to the boundary, then $f \circ \mu_D(z, w) = |f'(w)| \mu_{D'}(f(z), f(w))$.

2.3 Hulls

If U is a simply connected domain of the Riemann sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ whose complement $H^* = \mathbb{C}^* \setminus U$ is larger than a single point, then we call $H := H^* \cap \mathbb{C}$ a *hull*. If D is a simply connected domain, Let $\mathcal{H}_0(D)$, $\mathcal{H}_1(D)$, $\mathcal{H}_2(D)$, respectively, be the set of hulls H contained in \overline{D} whose intersection with ∂D consists of exactly 0, 1, 2 points respectively. Note that if $H \in \mathcal{H}_j(D)$ then $D \setminus H$ is: conformally equivalent to an annulus if $j = 0$; simply connected if $j = 1$; and the union of two disjoint simply connected domains if $j = 2$.

If $\gamma : [0, T] \rightarrow \overline{D}$ is a continuous curve, define the hull of γ , $H(\gamma)$, to be the path of γ “filled in”. More precisely, $H(\gamma)$ is the complement of the unbounded component of $\mathbb{C} \setminus \gamma[0, T]$. We can consider $\mu_D(z)$ or $\mu_D(z, w)$ as measures on $\mathcal{H}_1(D)$. In order to be precise, we need to specify a σ -algebra on the set of all hulls. The σ -algebra we need is generated by events of the

form V_U where U is a simply connected subset of D containing z and V_U is the collections of hulls that are contained in U (except for the boundary point). A finite measure $\mu_D(z)$ is determined by giving the $\mu_D(z)$ measure of the event V_U for each U . Note that the probability that the hull $H(\gamma)$ stays in U is the same as the probability that the path γ stays in U . This probability is the measure of ∂D with respect to harmonic measure in U starting at z ; in other words, it is the probability that a Brownian motion starting at z reaches ∂D without leaving U earlier.

2.4 Brownian excursions

Let D be a simply connected domain with smooth boundary, and let $z \in \partial D$. Let \mathbf{n} be the normal pointing into D at z and let $z_\epsilon = z + \epsilon \mathbf{n}$. The *Brownian excursion measure (at z)* $\mu_D(z)$ is defined to be the infinite measure

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} \mu_D(z_\epsilon).$$

This measure is supported on curves that start at z , immediately go into D , act like Brownian motions in D , and then are killed when they hit ∂D . The scaling is such that if A is a closed subarc of ∂D not containing z , then the $\mu_D(z)$ measure of the set of curves starting at z and leaving D at A is strictly between zero and infinity. It is fairly straightforward to show that if $f : D \rightarrow D'$ is a conformal transformation and $z \in \partial D$, then

$$f \circ \mu_D(z) = |f'(z)| \mu_{D'}(f(z)).$$

The extra factor $|f'(z)|$ comes from the fact that $f(z_\epsilon) \approx f(z) + |f'(z)| \mathbf{n}^*$ where \mathbf{n}^* is the unit normal pointing into D' at $f(z)$. We can write

$$\mu_D(z) = \int_{\partial D} \mu_D(z, w) d|w|,$$

where $\mu_D(z, w)$ is a measure on curves starting at z and ending at w . The total mass $|\mu_D(z, w)|$ is strictly between zero and infinity for $z \neq w$ (here we are using the smoothness of ∂D). These measures satisfy the scaling relation

$$f \circ \mu_D(z, w) = |f'(z)| |f'(w)| \mu_{D'}(f(z), f(w)).$$

The *excursion measure* μ_D is defined by

$$\mu_D = \int_{\partial D} \mu_D(z) d|z| = \int_{\partial D} \int_{\partial D} \mu_D(z, w) |dz| |dw|.$$

Note that

$$\begin{aligned} f \circ \mu_D &= \int_{\partial D} f \circ \mu_D(z) |dz| \\ &= \int_{\partial D} |f'(z)| \mu_{D'}(f(z)) |dz| \\ &= \int_{\partial D'} \mu_{D'}(w) d|w| = \mu_{D'}. \end{aligned}$$

In other words, the excursion measure is invariant under conformal transformations.

If $z, w \in \partial D$ with $z \neq w$ and we normalize so that $\mu_D(z, w)$ is a probability measure, then we can consider this as the measure on Brownian motion starting at z conditioned to immediately enter D and then leave at w (of course, this is conditioning on an event of probability zero, but one can make rigorous sense of this). This probability measure is conformally invariant. In order to study the process, one needs only study it on the most convenient domain. One convenient domain is the upper half plane $\mathbb{H} = \{x + iy : y > 0\}$ with boundary points 0 and ∞ . Then an excursion Z_t has the same distribution as $B_t^1 + i|W_t|$, where B_t^1, W_t are independent; B_t^1 is a standard Brownian motion; and W_t is a three dimensional Brownian motion. $|W_t|$ has the same distribution as a process Y_t satisfying the Bessel equation

$$dY_t = \frac{1}{Y_t} dt + dB_t^2. \quad (1)$$

Exercise. To check that the imaginary part of a Brownian excursion in \mathbb{H} from 0 to ∞ satisfies the Bessel equation, one needs to verify that one dimensional Brownian motions Y_t conditioned to remain positive at all times (defined in a natural way) satisfy (1). Verify this using either the “Girsanov transformation” or “ h -processes.”

Exercise. Suppose A is a compact subset not containing the origin such that $\mathbb{H} \setminus A$ is simply connected. Suppose $\Phi : \mathbb{H} \setminus A \rightarrow \mathbb{H}$ is a conformal transformation with $\Phi(0) = 0, \Phi(\infty) = \infty$, and such that $\Phi(z) = z + O(1)$ as $z \rightarrow \infty$. Show that the probability that the path of a Brownian excursion from 0 to ∞ in \mathbb{H} does not intersect A is $\Phi'(0)$.

2.5 Restriction property

Suppose $D' \subset D$ are simply connected domains (other than \mathbb{C}) with smooth boundaries, $z, w \in \partial D$, and $\partial D'$ and ∂D agree near z and w . Then the

measure $\mu_{D'}(z, w)$ is the same as $\mu_D(z, w)$ restricted to hulls $H \in \mathcal{H}_2(D)$ with $H \cap D \subset D'$. We call this property the *restriction property*. Similar restriction properties hold for $\mu_D(z)$ and μ_D .

The restriction property is almost immediate from the definition and it hardly seems important to mention it. However there are many conformally invariant families of measures that do not satisfy this property, but (essentially) only a one parameter conformally invariant family that satisfies the property.

2.6 Chordal restriction measures

Suppose $m_D(z, w)$ is a family of measures on hulls, where D ranges over simply connected subsets of \mathbb{C} with (nonempty) smooth boundary and z, w are distinct points on ∂D . Assume that $m_D(z, w)$ is supported on hulls $H \in \mathcal{H}_2(D)$ with $H \cap \partial D = \{z, w\}$. We call such a family a (*chordal restriction family with exponent a*) if

- For each D, z, w , $|m_D(z, w)| \in (0, \infty)$;
- Restriction property: if $D' \subset D$, and $\partial D'$ agrees with ∂D near z, w , then $m_{D'}(z, w)$ is $m_D(z, w)$ restricted to hulls H with $H \cap D \subset D'$;
- Conformal covariance: if $f : D \rightarrow D'$ is a conformal transformation, then

$$f \circ m_D(z, w) = |f'(z)|^a |f'(w)|^a m_{D'}(f(z), f(w)). \tag{2}$$

Remarks

1. The term conformal *covariance* is used to indicate conformal invariance up to a scalar correction factor.
2. We could consider restriction measures with other scaling factors

$$f \circ m_D(z, w) = C(z, w, D, f) m_{D'}(f(z), f(w)).$$

However, it can be shown that the only possibility for $C(z, w, D, f)$ is $|f'(z)|^a |f'(w)|^a$ for some a .

3. **Exercise.** If D, D' are simply connected domains with smooth boundaries and $z, w \in \partial D, z', w' \in \partial D'$ are distinct boundary points, then there are infinitely many conformal transformations $f : D \rightarrow D'$ with $f(z) = z', f(w) = w'$. However, the quantity $|f'(z)| |f'(w)|$ is the same for all of these transformations.

4. We could also consider the probability measures $m_D^\#(z, w) = m_D(z, w) / |m_D(z, w)|$. This family of measures satisfies conformal *invariance*

$$f \circ m_D^\#(z, w) = m_{D'}^\#(f(z), f(w)).$$

The restriction property can be stated in terms of conditional distributions of the $m_D^\#(z, w)$. These probability measures are well defined even if ∂D is not smooth.

5. The chordal restriction family with exponent a is determined by $m_{\mathbb{D}}(-1, 1)$ since $m_D(z, w)$ for other D, z, w can be determined from (2).
6. Let $m_{\mathbb{D}}(-1, 1) = \mu_{\mathbb{D}}(-1, 1)$ where μ denotes the Brownian excursion measure. For other D, z, w , define $m_{\mathbb{D}}$ by (2). Then this gives a restriction family with exponent $a = 1$.
7. **Exercise.** Suppose $m_D^1(z, w), m_D^2(z, w)$ are two restriction families with exponents a^1, a^2 , respectively. Define $m_D(z, w) = m_D^1(z, w) \times m_D^2(z, w)$, considered as a measure on hulls by the following procedure: if H^1, H^2 are nondisjoint hulls, let H be the hull generated by $H^1 \cup H^2$; if H^1, H^2 are bounded, then H is the complement of the unbounded component of $H^1 \cup H^2$. Equivalently, if V_U is the event that the intersection of the hull with D is contained in a simply connected U , then

$$m_D(z, w)[V_U] = m_D^1(z, w)[V_U] m_D^2(z, w)[V_U].$$

Then $m_D(z, w)$ is a restriction family with exponent $a^1 + a^2$. In particular, for positive integer a , the union of a Brownian excursions gives a restriction family with exponent a .

8. **Universality.** For each a , there is at most one restriction measure with exponent a satisfying $|m_{\mathbb{D}}(-1, 1)| = 1$. We can see this since $m_{\mathbb{D}}(-1, 1)$ determines the family, but if U is a simply connected subset of \mathbb{D} whose boundary contains neighborhoods of -1 and 1 in $\partial\mathbb{D}$, then the $m_{\mathbb{D}}(-1, 1)$ probability that the hull stays in U can be determined using (2). For this reason we talk of *the* restriction family of exponent a (which is really defined only up to a multiplicative constant).
9. We have considered restriction families of measures on *hulls*. One could also define restriction families of measure on *curves*. However, the universality result does not hold in this case. This is why we consider the hulls generated by curves rather than the paths of the curves.

10. For small $a > 0$, no restriction family exists. To see this assume $m_D(z, w)$ is such a family with $|m_{\mathbb{D}}(-1, 1)| = 1$. Consider the slit domains,

$$D^1 = \mathbb{D} \setminus [0, i], \quad D^2 = \mathbb{D} \setminus [-i, 0].$$

Then the $m_{\mathbb{D}}(-1, 1)$ probability that a hull stays in D^1 is $|f'(-1)|^a |f'(1)|^a$ where $f : D^1 \rightarrow \mathbb{D}$ is a conformal transformation with $f(-1) = -1, f(1) = 1$. Similarly, the $m_{\mathbb{D}}(-1, 1)$ probability that a hull stays in D^2 is also $|f'(-1)|^a |f'(1)|^a$. However, these events are disjoint; hence we must have $2 |f'(-1)|^a |f'(1)|^a \leq 1$. By finding f explicitly, we see that we need $a \geq 1/2$. (Of course, this argument does not show that families exist when $a \geq 1/2$.)

The term “universality” may seem a little strange for the uniqueness result. However, another way to think of it is that any lattice model whose continuum limit gives a chordal restriction family (i.e., satisfies restriction and conformal covariance) must lie in a one parameter family of measures. The parameter a can be considered (roughly) as the “number” of Brownian excursions. Of course, this only makes precise sense for positive integer a . We finish this section by stating a theorem without proof that tells us when restriction families exist.

Theorem 3 [41] *Chordal restriction families $m_D(z, w)$ exist for all $a \geq 5/8$ and no other values of a . For $a = 5/8$, the measures are supported on simple (non-self-intersecting) curves. For $a > 5/8$, the measures are not supported on simple curves. If $a = 5/8$ and $|m_{\mathbb{D}}(-1, 1)| = 1$, then $m_{\mathbb{D}}(-1, 1)$ is the same as the distribution of chordal $SLE_{8/3}$ from -1 to 1 in \mathbb{D} .*

In the next section, we will define $SLE_{8/3}$.

3 Stochastic Loewner evolution (SLE)

3.1 Motivation

Suppose D is a simply connected domain and z, w are distinct points in ∂D . We will be interested in probability measures on continuous curves $\gamma : [0, T] \rightarrow \overline{D}$ with $\gamma(0) = z, \gamma(T) = w$. We will be interested in curves modulo parametrization, i.e., if $\gamma^1[0, T^1] \rightarrow \overline{D}$ is another curve, we will say that γ and γ^1 are equivalent if there is an increasing homeomorphism

$\phi : [0, T] \rightarrow [0, T^1]$ such that $\gamma(t) = \gamma^1(\phi(t))$, $0 \leq t \leq T$. We will allow curves to have self-intersections but we do not allow them to have self-crossings. This means that the curve can hit its past, but if it does it always immediately reflects into the unbounded component of the complement of the path. We will call such curves *non-crossing*. Note that Brownian excursions in D do *not* have this property.

We will consider probability measures $q_D(z, w)$ on non-crossing curves (modulo reparameterization) connecting z to w . We will choose parametrizations so that $T = \infty$. We will be interested in measures satisfying two conditions:

- Conformal invariance: if $f : D \rightarrow D'$ is a conformal transformation that is continuous up to the boundary, then

$$f \circ q_D(z, w) = q_{D'}(f(z), f(w)).$$

- “Markovian” property: suppose we know the curve γ up to time t . Then conditioned on $\gamma[0, t]$ the distribution of $\gamma[t, \infty)$ is the same as $q_{D_t}(\gamma(t), w)$. Here, D_t is the connected component of $D \setminus \gamma[0, t]$ whose boundary includes w .

We will also consider probability measures $\tilde{q}_D(z, w)$ on non-crossing curves in \overline{D} from a boundary point z to an interior point w . We will be interested in curves modulo parametrization and we will choose a parametrization so that $\gamma(0) = z$, $w \notin \gamma[0, \infty)$, $\lim_{t \rightarrow \infty} \gamma(t) = w$. In this situation we let D_t be the component of $D \setminus \gamma[0, t]$ which contains w . We make the same assumptions (conformal invariance and “Markovian” property) as above. Since we are assuming conformal invariance, it suffices to define the measures $q_D(z, w)$, $\tilde{q}_D(z, w)$ for particularly nice domains. We will choose $q_{\mathbb{H}}(0, \infty)$ and $\tilde{q}_{\mathbb{D}}(w, 0)$, where \mathbb{H} is the upper half plane, \mathbb{D} is the unit disk, and $|w| = 1$.

3.2 Loewner differential equation

3.2.1 An exercise

In the following sections we will need some properties about maps from the a subset of \mathbb{H} onto \mathbb{H} . We will list these properties here, but we leave the verification as an exercise. Suppose A is a compact subset of $\overline{\mathbb{H}}$ such that $\mathbb{H}_A = \mathbb{H} \setminus A$ is simply connected. Then the following hold.

- There is a unique conformal transformation $g_A : \mathbb{H}_A \rightarrow \mathbb{H}$ such that

$$\lim_{z \rightarrow \infty} g_A(z) - z = 0.$$

- There is a number $a(A) \geq 0$ such that as $z \rightarrow \infty$,

$$g_A(z) = z + \frac{a(A)}{z} + O(|z|^{-2}).$$

- Let B_s be a standard Brownian motion starting at $z \in \mathbb{H}_A$ and let

$$\tau = \tau_A = \inf\{s : B_s \notin \mathbb{H}_A\}.$$

Then

$$\operatorname{Im}[g_A(z)] = \operatorname{Im}(z) - \mathbf{E}^z[\operatorname{Im}(B_\tau)].$$

In particular,

$$\lim_{y \rightarrow \infty} y \mathbf{E}^{iy}[\operatorname{Im}(B_\tau)] = a(A).$$

(Hint: consider $h(z) = \operatorname{Im}[z - g_A(z)]$ which is bounded, harmonic on \mathbb{H}_A .)

- If $r > 0$, then $a(rA) = r^2 a(A)$
- There exist constants c_1, c_2 such that for all A , if

$$V_A = \bigcup_{x+iy \in A} \mathcal{B}(x+iy, y),$$

(where $\mathcal{B}(z, \epsilon)$ is the closed disc of radius ϵ about z), then $c_1 a(A) \leq \operatorname{area}(V_A) \leq c_2 a(A)$.

3.2.2 Chordal

Suppose $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ is a continuous non-crossing curve with $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$. For each t , let \mathbb{H}_t be the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$, and let g_t be the unique conformal transformation of \mathbb{H}_t onto \mathbb{H} such that $g_t(z) - z = o(1)$ as $z \rightarrow \infty$. Then g_t has an expansion at infinity,

$$g_t(z) = z + \frac{a(t)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \rightarrow \infty.$$

Exercise. Show that $a(t)$ is continuous and nondecreasing in t . Moreover, if $s < t$ and $\mathbb{H}_t \neq \mathbb{H}_s$, then $a(t) > a(s)$.

With the aid of the exercise, we see that if we make the (only slightly) stronger assumption that $a(t) \rightarrow \infty$ as $t \rightarrow \infty$, then we can reparametrize γ so that $a(t) = 2t$. We call this *parametrization by capacity (from ∞ in \mathbb{H})*. For the remainder of this section we assume that γ has been parametrized this way. Given this, it can be shown (see, e.g., [32, Proposition 2.1]), that g_t satisfies the *chordal Loewner differential equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \quad (3)$$

where $U_t = g_t(\gamma(t))$. This equation is sometime written in terms of the inverse transformation $f_t(z) = g_t^{-1}(z)$,

$$\partial_t f_t(z) = -\frac{2 f_t'(z)}{z - U_t}.$$

Conversely, if $U : [0, \infty) \rightarrow \mathbb{R}$, is a continuous function, then for each $z \in \mathbb{H}$, we can solve the initial value problem (3). This solution is well defined up to a time $T_z \in (0, \infty]$. For fixed t , g_t is the conformal transformation of $\mathbb{H}_t := \{z : T_z > t\}$ onto \mathbb{H} with $g_t(z) - z = o(1)$ as $z \rightarrow \infty$. (This takes a little work; see [32] for a proof.) It is not always the case, however, that there is a curve γ so that \mathbb{H}_t is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ and $U_t = g_t(\gamma(t))$. One sufficient (but not necessary) condition is that U_t has sufficiently small Hölder-(1/2) norm [47]; in this case, γ is a simple curve.

Exercise. Find g_t and $\gamma(t)$ if $U_t = 0$ for all t .

3.2.3 Behavior under conformal maps

Suppose γ is a curve as in the previous subsection, and suppose $\Phi : \mathcal{N} \rightarrow \mathbb{H}$ is a conformal transformation defined in a neighborhood \mathcal{N} of 0 in \mathbb{H} containing $\gamma[0, t]$ sending $\partial\mathcal{N} \cap \mathbb{R}$ to \mathbb{R} . Let $\tilde{\gamma}(t) = \Phi \circ \gamma(t)$, and let \tilde{g}_t denote the conformal transformation of the unbounded component of $\mathbb{H} \setminus \tilde{\gamma}[0, t]$ onto \mathbb{H} with $\tilde{g}_t(z) - z = o(1)$ as $z \rightarrow \infty$. Let $h_t = \tilde{g}_t \circ \Phi \circ g_t^{-1}$.

Exercise. Let $\tilde{a}(t) = a[\tilde{\gamma}[0, t]]$. Under the assumptions above, show that $\partial_t \tilde{a}(t) = h_t'(U_t)^2$. (Recall from §3.2.1 that $a(rA) = r^2 a(A)$.)

Exercise. Show that $\tilde{g}_t(z)$ satisfies the modified Loewner equation

$$\partial_t \tilde{g}_t(z) = \frac{2 h_t'(U_t)^2}{\tilde{g}_t(z) - \tilde{U}_t}, \quad \tilde{g}_0(z) = z,$$

where $\tilde{U}_t = \tilde{g}_t \circ \Phi \circ \gamma(t) = \tilde{g}_t \circ \Phi \circ g_t^{-1}(U_t)$.

With the aid of the exercise and the chain rule, we can see that

$$\partial_t h_t(z) = \frac{2 h_t'(U_t)^2}{h_t(z) - \tilde{U}_t} - \frac{2 h_t'(z)}{z - U_t},$$

at least for z near U_t . Also,

$$\partial_t h_t(U_t) = \lim_{z \rightarrow U_t} \frac{2 h_t'(U_t)^2}{h_t(z) - \tilde{U}_t} - \frac{2 h_t'(z)}{z - U_t} = -3h_t''(U_t). \quad (4)$$

Similarly, one can show that

$$\partial_t h_t'(z) = \frac{2h_t'(z)}{(z - U_t)^2} - \frac{2h_t''(z)}{z - U_t} - \frac{2h_t'(U_t)^2 h_t'(z)}{(h_t(z) - \tilde{U}_t)^2},$$

and

$$\partial_t h_t'(U_t) = \lim_{z \rightarrow U_t} \partial_t h_t'(z) = \frac{h_t''(U_t)^2}{2h_t'(U_t)} - \frac{4h_t'''(U_t)}{3}.$$

Here we are using ' to denote z -derivatives.

3.2.4 Radial

Suppose $\gamma : [0, \infty) \rightarrow \overline{\mathbb{D}}$ is a non-crossing curve with $\gamma(0) = w \in \partial\mathbb{D}$, $0 \notin \gamma[0, \infty)$ and $\gamma(t) \rightarrow 0$ as $t \rightarrow \infty$. Let D_t be the component of $\mathbb{D} \setminus \gamma[0, t]$ containing the origin and let \tilde{g}_t be the unique conformal transformation of D_t onto \mathbb{D} with $\tilde{g}_t'(0) > 0$. If $s < t$ and $D_s \neq D_t$, then we can see that $\tilde{g}_t'(0) > \tilde{g}_s'(0)$; hence we can parametrize γ by *capacity (from 0 in \mathbb{D})* so that $\tilde{g}_t'(0) = e^t$. In this case, \tilde{g}_t satisfies the *radial Loewner differential equation*

$$\partial_t \tilde{g}_t(z) = -\tilde{g}_t(z) \frac{\tilde{g}_t(z) + e^{iU_t}}{\tilde{g}_t(z) - e^{iU_t}}, \quad \tilde{g}_0(z) = z, \quad (5)$$

where $U : [0, \infty) \rightarrow \mathbb{R}$ is continuous and $\tilde{g}_t(\gamma(t)) = e^{iU_t}$. If $\tilde{g}_t(z) = e^{ih_t(z)}$ (locally) we can write the differential equation as

$$\partial_t h_t(z) = \cot\left[\frac{h_t(z) - U_t}{2}\right].$$

3.3 Stochastic Loewner evolution

3.3.1 Chordal

Suppose we have a probability distribution on non-crossing curves $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ with $\gamma(0) = 0$, that are parametrized by capacity satisfying the conditions of §3.1. Then, the maps g_t are random and satisfy (3). This gives conditions on the functions U_t : in fact, we can see that for $s < t$, $U_t - U_s$ is independent of $U_r, 0 \leq r \leq s$, and has the same distribution as U_{t-s} . It is well known that the only process with continuous paths satisfying this is Brownian motion. There are two parameters we can choose: the drift μ and the variance σ^2 . If we impose a symmetry condition, that the distribution $q_{\mathbb{H}}(0, \infty)$ is symmetric about the imaginary axis, then we will know that $\mu = 0$. It is standard to use κ for the variance.

Definition. The (*chordal*) *stochastic Loewner evolution with parameter* $\kappa > 0$, denoted by SLE_{κ} , is the random collection of maps g_t generated by the initial value problem (3) where $U_t = \sqrt{\kappa} B_t$ and B_t is a standard (one-dimensional) Brownian motion.

It is not obvious, but has been proved [48, 39], that there exist random curves γ that generate g_t ; in other words, g_t is a conformal transformation of the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ onto \mathbb{H} and $g_t(\gamma(t)) = \sqrt{\kappa} B_t$. The curves are also referred to as (chordal) SLE_{κ} .

Exercise (Scaling Law). Suppose $\gamma(t)$ is an SLE_{κ} path, $r > 0$, and $\gamma^*(t) = r\gamma(t)$. Show that γ^* has the distribution of a time change of SLE_{κ} . Give the time change explicitly.

Properties of SLE_{κ} paths

The following properties hold with probability one.

- The Hausdorff dimension of the paths is conjectured to be $\min\{1 + (\kappa/8), 2\}$. This has been proved in the cases $\kappa = 8/3, 6$ (see [4, 37, 53]) and the upper bound is known for all κ [48]. (See §4.1 for a review of Hausdorff dimension.)
- If $\kappa \leq 4$, then γ is a simple path avoiding the boundary, i.e., if $0 < s < t$, then $\gamma(s), \gamma(t) \in \mathbb{H}$ and $\gamma(s) \neq \gamma(t)$.
- If $4 < \kappa < 8$, then the paths of SLE have self-intersections. The paths also hit the real line, and $\cap_{t>0} \mathbb{H}_t = \emptyset$.

- $\kappa \geq 8$, the paths are space-filling. (The upper bound on the dimension shows that the paths are not space-filling for $\kappa < 8$.)

The way to derive such geometric and fractal properties of the curve γ is by analyzing one dimensional stochastic differential equations. Let us consider the case of double points—how do we determine for which κ the paths have double points? Consider what would happen if the curve had double points. Since the “past” of the curve is conformally equivalent to a segment of the real line, it would mean that the curve had a chance to hit the real line. So we can ask a similar question: consider the hull $K_t := \overline{\mathbb{H} \setminus \mathbb{H}_t}$. What is $K_t \cap \mathbb{R}$? For topological reasons we know that $K_t \cap \mathbb{R}$ is a compact interval containing 0, perhaps the trivial one-point interval.

Let $x > 0$ and consider the probability that $x \in K_t$. Note that if $x \notin K_t$, then $\partial\mathbb{H}_t$ contains an open interval in \mathbb{R} about x . In this case, $g_t(x)$ is well defined and satisfies the Loewner equation (3) with $U_t = \sqrt{\kappa} B_t$. The first time t at which $x \in K_t$ can be characterized as the first time that $g_t(x) = U_t$. Let $Y_t = (g_t(x) - U_t)/\sqrt{\kappa}$. Then Y_t satisfies the stochastic differential equation

$$dY_t = \frac{\alpha}{Y_t} dt + d\tilde{B}_t,$$

where $\alpha = 2/\kappa$ and $\tilde{B}_t = -B_t$. The solution to this equation is a Bessel process; if $\alpha = (d - 1)/2$, then the process has the same distribution as the absolute value of a d -dimensional Brownian motion. It is well known that Bessel processes hit the origin if and only if $\alpha < 1/2$.

3.3.2 Locality for $\kappa = 6$

Suppose A is a compact subset of $\overline{\mathbb{H}}$ not containing the origin such that $A = \overline{A \cap \mathbb{H}}$ and $\mathbb{H} \setminus A$ is simply connected. Let $\Phi = \Phi_A$ be the conformal transformation of $\mathbb{H} \setminus A$ onto \mathbb{H} with $\Phi(0) = 0, \Phi(\infty) = \infty$ and $\Phi(z) \sim z$ as $z \rightarrow \infty$. Let γ be a chordal SLE_κ path starting at the origin and let T be the first time t that $A \cap \mathbb{H} \not\subset \mathbb{H}_t$. Let $\tilde{\gamma}(t) = \Phi \circ \gamma(t)$.

Proposition 4 (Locality) *If $\kappa = 6$, then $\tilde{\gamma}(t), 0 \leq t < T$, has the same distribution as a time change of SLE_κ .*

The proposition is proved using Itô’s formula and (4). We get

$$\begin{aligned} d\tilde{U}_t = d[h_t(U_t)] &= \sqrt{\kappa} h'_t(U_t) dB_t + [\partial_t h_t(U_t) + \frac{\kappa}{2} h''_t(U_t)] dt \\ &= \sqrt{\kappa} h'_t(U_t) dB_t + [(\kappa/2) - 3] h''_t(U_t) dt. \end{aligned}$$

The Itô's lemma that we are using is a slight generalization of the usual formula since the function h_t depends on the Brownian motion $B_s, 0 \leq s \leq t$. However, since the distribution of h_t does not depend on the future and h_t is C^1 in t and C^2 in the space variable, the usual formula is still valid. When $\kappa = 6$, the “ dt ” term drops out and \tilde{U}_t is a martingale. If $\kappa \neq 6$, the distribution of $\tilde{\gamma}$ is absolutely continuous with respect to (a time change of) SLE_κ .

3.3.3 Radial

Radial SLE_κ is defined as the solution to the radial Loewner equation (5) where $U_t = \sqrt{\kappa} B_t$. Radial SLE_κ is closely related to chordal SLE_κ with the same κ . Let γ denote a chordal SLE_κ path starting at the origin driven by the $U_t = \sqrt{\kappa} B_t$. Let $\tilde{\gamma}(t) = \exp\{i\gamma(t)\}$ and let T be the first time that $\tilde{\gamma}$ forms a closed loop about the origin, i.e., the first time t such that there is an $s < t$ with $\gamma(t) - \gamma(s) = \pm 2\pi$. Let $\Phi(z) = e^{iz}$. If $t < T$, then there is a neighborhood \mathcal{N} of 0 in \mathbb{H} containing $\gamma[0, t]$ such that Φ maps \mathcal{N} conformally and one-to-one into \mathbb{D} . Let ψ denote the branch of $-i \log(z)$ on $\Phi(\mathcal{N})$ with $\psi(1) = 0$. Let \tilde{g}_t be the conformal transformation of the connected component of $\mathbb{D} \setminus \Phi \circ \gamma[0, t]$ containing the origin onto \mathbb{D} with $\tilde{g}_t(0) = 0, \tilde{g}'_t(0) > 0$. Then

$$\partial_t[\psi \circ \tilde{g}_t(z)] = \log \tilde{g}'_t(0) \cot\left[\frac{\psi \circ g_t(z) - \tilde{U}_t}{2}\right],$$

where $\tilde{U}_t = \psi \circ \tilde{g}_t(\Phi \circ \gamma(t))$. In particular, $\tilde{U}_t = \psi \circ \tilde{g}_t \circ \Phi \circ g_t^{-1}(U_t)$.

Using these ideas, one can show that the distribution of a radial SLE_κ path is absolutely continuous with the distribution of $\exp\{i\gamma(t)\}$ where γ is a chordal SLE_κ path, at least up to the time where the path $\exp\{i\gamma(t)\}$ forms a closed loop about the origin. For $\kappa = 6$, they have the same distribution (modulo time change) up to the first time the loop is formed. One thing this tells us is that the qualitative behavior of radial SLE_κ paths (such as Hausdorff dimension) is the same as for chordal SLE_κ .

3.3.4 Restriction property for $\kappa = 8/3$

Consider chordal SLE_κ for $\kappa \leq 4$, so that the path $\gamma = \gamma[0, \infty)$ is a simple path with $\gamma(0, \infty) \subset \mathbb{H}$. Let A be a compact subset of $\overline{\mathbb{H}}$ not containing the origin such that $A = \overline{A} \cap \overline{\mathbb{H}}$ and $\mathbb{H} \setminus A$ is simply connected and let Φ_A be

the transformation as in §3.3.2. For $\kappa < 4$, $\mathbf{P}\{\gamma \cap A = \emptyset\} > 0$. For $\kappa = 8/3$, the formula is particularly nice.

Proposition 5 *If $\gamma = \gamma[0, \infty)$ is the path of chordal $SLE_{8/3}$, then*

$$\mathbf{P}\{\gamma \cap A = \emptyset\} = \Phi'_A(0)^{5/8}. \tag{6}$$

A corollary of the proposition is the fact that $SLE_{8/3}$ satisfies the following *restriction property*: the conditional distribution of the path $\Phi_A \circ \gamma[0, \infty)$, considered as a measure on paths modulo reparameterization, conditioned on the event $\{\gamma[0, \infty) \cap A = \emptyset\}$ is the same as the distribution of $SLE_{8/3}$. For $\kappa \leq 4$ other than $\kappa = 8/3$, the probability in (6) takes a more complicated form than just $\Phi'_A(0)^a$. This restriction property does not hold for these values. The proof of (6) is similar to that of the proof of locality for $\kappa = 6$ although the calculation is a little more complicated. Instead of studying $h_t(U_t)$, one needs to use Itô's formula to find the semimartingale representation of $M_t := h'_t(U_t)^{5/8}$ and show that this is, in fact, a martingale. As a corollary of the calculation we can see that we can define the chordal restriction measure with exponent $5/8$ using $SLE_{8/3}$.

Exercise. Explain why $h'_t(U_t)$ is the probability that a Brownian excursion in \mathbb{H} going from U_t to infinity never intersects the set $g_t(A)$.

There is a similar formula for radial $SLE_{8/3}$. Suppose A is a compact subset of \mathbb{D} with $0, 1 \notin A$, $A = \overline{A \cap \mathbb{D}}$ and such that $\mathbb{D} \setminus A$ is a simply connected domain containing the origin. Let Ψ_A be the conformal transformation of $\mathbb{D} \setminus A$ onto \mathbb{D} with $\Psi_A(0) = 0, \Psi'_A(0) > 0$.

Proposition 6 *If $\tilde{\gamma} = \tilde{\gamma}[0, \infty)$ is the path of radial $SLE_{8/3}$ going from 1 to 0 in the unit disk, then*

$$\mathbf{P}\{\tilde{\gamma} \cap A = \emptyset\} = \Psi'_A(0)^{5/48} |\Psi'_A(1)|^{5/8}.$$

Radial $SLE_{8/3}$ also satisfies the restriction property: the conditional distribution of the $\Psi_A \circ \tilde{\gamma}$ conditioned on $\mathbf{P}\{\tilde{\gamma} \cap A = \emptyset\}$ is the same as that of radial $SLE_{8/3}$ (modulo time change).

4 Applications to Brownian paths

The first major application of the stochastic Loewner evolution was in the computation of the intersection exponents for planar Brownian motion from

which one can derive the Hausdorff dimension of certain exceptional sets of a Brownian path. The proof combines the idea of universality from the first section (which states roughly that the exponents for any conformally invariant process satisfying the restriction property are the same as the union of Brownian paths) and the fact that exponents for *SLE* can be computed. In this section, we let $B_t = B_t^1 + iB_t^2$ denote a standard Brownian motion in \mathbb{C} .

4.1 Hausdorff dimension

Here we review the definition and some basic facts about Hausdorff dimension. Suppose A is a bounded, Borel subset in \mathbb{R}^d . For each $\alpha > 0$ and any $\epsilon > 0$, let

$$\mathcal{H}_\epsilon^\alpha(A) = \inf \sum_{j=1}^{\infty} [\text{diam}(\mathcal{U}_j)]^\alpha,$$

where the infimum is over all covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of A by sets of diameter less than ϵ . The Hausdorff α -measure of A is defined by

$$\mathcal{H}^\alpha(A) = \lim_{\epsilon \rightarrow 0^+} \mathcal{H}_\epsilon^\alpha(A).$$

The *Hausdorff dimension* of A , which we denote by $\dim_h(A)$, is the unique $\alpha_0 \in [0, d]$ such that $\mathcal{H}^\alpha(A) = \infty$ for $\alpha < \alpha_0$ and $\mathcal{H}^\alpha(A) = 0$ for $\alpha > \alpha_0$. Here we give two standard ways to estimate Hausdorff dimensions, one which gives upper bounds and one that gives lower bounds. For proofs and more details about Hausdorff measure and dimension see, e.g., [17].

Proposition 7 *Suppose that for all ϵ sufficiently small, A can be covered by the union of $\epsilon^{-\alpha}$ sets of diameter ϵ . Then $\dim_h(A) \leq \alpha$.*

Proposition 8 *Suppose there is a positive measure μ with $0 < \mu(A) < \infty$, $\mu(\mathbb{R}^d \setminus A) = 0$, and such that*

$$\int_A \int_A \frac{\mu(dx)\mu(dy)}{|x-y|^\alpha} < \infty.$$

Then $\dim_h(A) \geq \alpha$.

Exercise. Using the two propositions, find the Hausdorff dimension of the Cantor set A . Recall that $A = \bigcap_n A_n$ where $A_0 = [0, 1]$, $A_1 = [0, 1/3] \cup [2/3, 1]$

and recursively A_{n+1} is obtained from A_n by removing the open “middle third” from each interval. (Hint: the measure μ to consider is the measure that gives measure 2^{-n} to each of the 2^n intervals in A_n .)

Kaufman [22] was the first to prove the following fact about complex Brownian motions. Note the order of quantifiers in the proposition: there is a single null event such that off that event, the dimension doubles for all Borel sets A .

Proposition 9 *If B_t is a complex Brownian motion, then with probability one, for all Borel $A \subset [0, 1]$, $\dim_h[B(A)] = 2\dim_h(A)$.*

Exercise. Explain why the following one dimensional version of Kaufman’s theorem is *false*. Let B_t be a standard one-dimensional Brownian motion. Then with probability one, for every Borel $A \subset [0, 1]$,

$$\dim_h[B(A)] = \min\{1, 2\dim_h(A)\}.$$

4.2 Cut/frontier/pioneer points for Brownian paths

A time $t \in [0, 1]$ is called a *cut time* and B_t is a *cut point* for Brownian motion on $[0, 1]$ if $B[0, t] \cap B(t, 1] = \emptyset$. It is not obvious that a planar Brownian path has cut times; Burdzy [8] was the first to show this.

Exercise. Show that for each $t \in [0, 1]$, with probability one t is not a cut time. Conclude that with probability one, the (one dimensional) Lebesgue measure of the set of cut times is zero.

Exercise. Show that with probability one, the set of cut times $t \in [0, 1]$ is a closed, nowhere dense set.

In [27], it was shown that one can give the Hausdorff dimension of the set of cut times and cut points in terms of a particular *intersection exponent* for Brownian motion. The basic idea is simple. Let I denote the set of cut times in $[1/4, 3/4]$, say. Call $I(k, n) := [(k - 1)2^{-n}, k2^{-n}]$ an “approximate cut interval” if $B[0, (k - 1)2^{-n}] \cap B[k2^{-n}, 1] = \emptyset$ and let I_n be the union of the approximate cut intervals $I(k, n)$ for $1/4 \leq k2^{-n} \leq 3/4$. Then $I_1 \supset I_2 \supset \dots$ and $I = \cap I_n$. If $J(k, n)$ denotes the event that $I(k, n)$ is an approximate cut interval, subadditivity arguments can be used to show that there is a ξ such that $\mathbf{P}[J(k, n)] \approx 2^{-n\xi/2}$ as $n \rightarrow \infty$ and relatively easy estimates show that $\xi < 2$. Roughly speaking this says that it takes on the order of $2^{n(2-\xi)/2}$ intervals of length 2^{-n} to cover I and hence one would guess that the fractal

dimension of I is $(2 - \xi)/2$. In order to make this a rigorous argument for the Hausdorff dimension, it was necessary to improve the convergence rate given by the subadditivity argument. In particular it was shown that there were constants c_1, c_2 such that

$$c_1 2^{-n\xi/2} \leq \mathbf{P}[J(k, n)] \leq c_2 2^{-n\xi/2}, \quad (7)$$

$$c_1 2^{-n\xi/2} |k - j|^{-\xi/2} \leq \mathbf{P}[J(j, n) \cap J(k, n)] \leq c_2 2^{-n\xi/2} |k - j|^{-\xi/2}, \quad j \neq k.$$

These estimates were established without knowing the exact value of ξ . Combining these estimates with the basic propositions in §4.1, one can show that (with positive probability) the Hausdorff dimension of I is $(2 - \xi)/2$ and, using a result of Kaufman, we get that the set of cut points, i.e., $B(I)$, has Hausdorff dimension $2 - \xi$. The same argument was used for a number of other exceptional sets (see [30] for a discussion of this method for finding Hausdorff dimensions of sets).

Exercise. Prove the following fact that is used in the derivation of (7): suppose $f(n)$ is a function from $\{1, 2, \dots, n\}$ into $(0, \infty)$ such that there exist $c_1, c_2, \in (0, \infty)$ such that for all n, m ,

$$c_1 f(n) f(m) \leq f(n + m) \leq c_2 f(n) f(m).$$

Then there exist $C_1, C_2, \alpha \in (-\infty, \infty)$ such that for all n

$$C_1 e^{\alpha n} \leq f(n) \leq C_2 e^{\alpha n}.$$

Theorem 10

- A time $t < t_0$ is called a cut time and B_t is called a cut point for $B[0, t_0]$ if

$$B[0, t) \cap B(t, t_0] = \emptyset.$$

- The frontier or outer boundary F_t of $B[0, t]$ is the set of points “connected to infinity”; more precisely, it is the boundary of the unbounded component of $\mathbb{C} \setminus B[0, t]$.
- A time t is called a pioneer time and B_t is called a pioneer point if $B_t \in F_t$.

Then with probability one the cut points, frontier, and pioneer points have Hausdorff dimension $2 - \xi(1, 1), 2 - \xi(2, 0), 2 - \xi(1, 0)$, respectively, where these exponents are defined by

$$\mathbf{P}\{B[0, \frac{1}{2} - \epsilon^2] \cap B[\frac{1}{2} + \epsilon^2, 1] = \emptyset\} \approx \epsilon^{\xi(1,1)},$$

$$\mathbf{P}\{B[0, \frac{1}{2} - \epsilon^2] \cup B[\frac{1}{2} + \epsilon^2, 1] \text{ does not disconnect}$$

$$B_{1/2} \text{ from infinity}\} \approx \epsilon^{\xi(2,0)},$$

$$\mathbf{P}\{B[\epsilon^2, 1] \text{ does not disconnect } 0 \text{ from infinity}\} \approx \epsilon^{\xi(1,0)}, \quad \epsilon \rightarrow 0+$$

As one might guess from the notation there is a whole family of intersection exponents $\xi(j, k)$ (and, in fact, $\xi(j_1, \dots, j_k)$, see [42]). Theorem 10 was proved without knowing the exact values of the exponents. However, *SLE* and “universality” has allowed us to establish all the values of the intersection exponent; in particular

$$\xi(1, 1) = 5/4, \quad \xi(2, 0) = 2/3, \quad \xi(1, 0) = 1/4.$$

These values had been conjectured using nonrigorous arguments; see [13] for $\xi(1, 1)$ and [46] for $\xi(2, 0)$.

Remark. The definition of cut points also makes sense for d -dimensional Brownian motion. For $d = 1$, there are no cut points [16] and for $d = 4$ all points are cut points [15]. The interesting open dimension is $d = 3$; here [27] we can show that the Hausdorff dimension of the set of cut points is $2 - \xi_3(1, 1)$ where $\xi_3(1, 1)$ is defined as $\xi(1, 1)$, except using a 3-dimensional Brownian motion. While we know that $\xi_3(1, 1)$ exists, we do not know its exact value nor do we even have any reason to believe that this number is rational. Rigorously we know that $1/2 < \xi_3(1, 1) < 1$ [10, 29] and numerical simulations suggest $\xi_3(1, 1) \approx .58$.

Remark. One can state analagous problems for simple random walks. Let S^1, S^2 be independent simple (nearest neighbor) random walks starting at the origin and $(1, 0)$ respectively in the integer lattice \mathbb{Z}^2 . Let

$$S^j[0, n] = \{S_k^j : k = 0, 1, \dots, n^2\}$$

be the set of points visited by the j th walker in the first n steps. Define the simple random walk intersection exponents $\zeta(1, 1), \zeta(2, 0), \zeta(1, 0)$ by the relations

$$\mathbf{P}\{S^1[0, n] \cap S^2[0, n] = \emptyset\} \asymp n^{-\zeta(1,1)},$$

$$\mathbf{P}\{S^1[0, n] \cup S^2[0, n] \text{ does not disconnect } (0, 1) \text{ from infinity}\} \asymp n^{-\zeta(2,0)},$$

$$\mathbf{P}\{S^1[0, n] \text{ does not disconnect } (0, 1) \text{ from infinity}\} \asymp n^{-\zeta(1,0)}.$$

Here \asymp means that the ratio of both sides is bounded away from 0 and infinity. Then it can be shown [9, 33] that these exponents exist and are the same as $\xi(1, 1)/2, \xi(2, 0)/2, \xi(1, 0)/2$. (The factor of 2 comes from the fact that it takes about n^2 steps of a random walk to go distance n .) In fact, the only known proofs that the random walk exponents exist show existence by establishing that the random walk exponents are the same as the Brownian exponents. (The Brownian exponents can be shown to exist using a subadditivity argument.) The proofs that the exponents are the same do not use the exact value of the exponents and are also valid for $d = 3$. The numerical estimates for $\xi_3(1, 1)$ have actually been estimates of $\zeta_3(1, 1)$, i.e., they have been done by Monte Carlo simulations of simple random walks. The rigorous relations $\zeta_3(1, 1) = \xi_3(1, 1)/2$ makes this approach valid.

4.3 A half plane exponent

We are left with the problem of finding the Brownian intersection exponents. Because the proof is slightly similar (but has all of the major new ideas), we will discuss a similar family of exponents, the *half-plane* (also called *rectangle* or *chordal*) *Brownian intersection exponents*. These can be defined in terms of the Brownian excursion measure. Let \mathcal{R}_L be the open rectangle,

$$\mathcal{R}_L = \{x + iy : 0 < x < L, 0 < y < \pi\},$$

and let $\partial_1 = [0, \pi i]$ and $\partial_2 = \partial_{2,L} = [L, L + \pi i]$ denote the vertical boundaries. Let $\mu_{\mathcal{R}_L}$ denote the Brownian excursion measure on \mathcal{R}_L restricted to excursions that go from ∂_1 to ∂_2 .

Exercise. Show that there exist constants c_1, c_2 , such that for all $L \geq 1$,

$$c_1 e^{-L} \leq |\mu_{\mathcal{R}_L}| \leq c_2 e^{-L}.$$

(You may wish to start by establishing the following estimate for Brownian motion: suppose B_t is a complex Brownian motion starting at $z = 1 + (\pi/2)i$

and let τ be the first time that the Brownian motion reaches $\partial\mathcal{R}_L$. Show that there exist constants c_1, c_2 such that

$$c_1 e^{-L} \leq \mathbf{P}^z\{B_\tau \in \partial_2\} \leq c_2 e^{-L}.$$

If we had chosen $\mathcal{R}_{L,a} = (0, L) \times (0, a)$ and started the Brownian motion at $z = 1 + (a/2)i$, then a similar estimate holds with e^{-L} replaced by $e^{-L/a}$.

Definition. If D is a simply connected domain and A_1, A_2 are nontrivial disjoint arcs on ∂D , then the π -extremal distance between A_1 and A_2 in D , denoted $L(A_1, A_2; D)$ is the unique L such that there is a conformal transformation of D onto \mathcal{R}_L such that A_1, A_2 are mapped onto ∂_1 and ∂_2 .

The π -extremal distance is π times the extremal length or extremal distance as defined in complex function theory (see, e.g., [2]). We choose to use π -extremal distance rather than the usual extremal distance because \mathcal{R}_L has the simplest crossing exponent, 1 (see the exercise above). From the definition, it is obvious that this is a conformal invariant.

Suppose γ is an excursion from ∂_1 to ∂_2 in \mathcal{R}_L . Let $D_+ = D_+(\gamma)$ be the connected component of $\mathcal{R}_L \setminus \gamma$ whose boundary includes the upper horizontal boundary $[\pi i, L + \pi i]$. The half-plane intersection exponent $\tilde{\xi}(1, \lambda)$ is defined for $\lambda > 0$ by the relation

$$\mu_{\mathcal{R}_L} [\exp\{-\lambda L(\partial_1 \cap \partial D_+, \partial_2 \cap \partial D_+; D_+)\}] \asymp e^{-L\tilde{\xi}(1, \lambda)}, \quad L \rightarrow \infty. \quad (8)$$

Here we have used $\mu_{\mathcal{R}_L}[F]$ for the integral of F with respect to the measure $\mu_{\mathcal{R}_L}$. We use \asymp to indicate that both sides are bounded by constants times the other side. One needs to show such a $\tilde{\xi}(1, \lambda)$ exists and that one can bound both sides up to multiplicative constants; this is done in a way similar to the derivation of (7). In particular, this argument does not give the value of the exponent $\tilde{\xi}(1, \lambda)$.

Exercise. Consider the $\mu_{\mathcal{R}_L} \times \mu_{\mathcal{R}_L}$ measure of pairs of excursions (γ^1, γ^2) connecting ∂_1 and ∂_2 . Let $V = V_L$ be the event that the paths of the excursions do not intersect. Show that

$$\mu_{\mathcal{R}_L} \times \mu_{\mathcal{R}_L}(V) \approx e^{-\tilde{\xi}(1,1)L}, \quad L \rightarrow \infty.$$

You may assume that (8) holds.

Exercise. Consider the $\mu_{\mathcal{R}_L} \times \mu_{\mathcal{R}_L} \times \mu_{\mathcal{R}_L}$ measure of triples of excursions connecting ∂_1 and ∂_2 . Let $V = V_L$ be the event that the paths of the

excursions are mutually disjoint. . Show that

$$\mu_{\mathcal{R}_L} \times \mu_{\mathcal{R}_L} \times \mu_{\mathcal{R}_L}(V) \approx e^{-\tilde{\xi}(1, \tilde{\xi}(1, 1))L}, \quad L \rightarrow \infty.$$

[This result is sometimes called the *cascade relation* and can be written as $\tilde{\xi}(1, 1, 1) = \tilde{\xi}(1, \tilde{\xi}(1, 1))$.]

4.4 Crossing exponent for chordal SLE_6

Here we discuss a similar crossing exponent for SLE_6 . Suppose γ is the path of SLE_6 in \mathcal{R}_L starting at $i\pi$ and ending at $L + i\pi$. (Recall that this is the same as the image of SLE_6 in the upper half plane under a conformal transformation of \mathbb{H} onto \mathbb{R}_L sending 0 to $i\pi$ and ∞ to $L + i\pi$.) Let \hat{T} be the first time t that $\gamma(t) \in \partial_2$, and let $V = V_L$ be the event that $\gamma[0, \hat{T}] \cap [0, L] = \emptyset$. On the event V , let \mathcal{D} be the connected component of $\mathcal{R}_L \setminus \gamma[0, \hat{T}]$ whose boundary includes the lower horizontal boundary $[0, L]$, and let $\hat{\mathcal{L}} = L(\partial_1 \cap \partial\mathcal{D}, \partial_2 \cap \partial\mathcal{D}; \mathcal{D})$. On the event V^c , let $\hat{\mathcal{L}} = \infty$. For $\beta \geq 0$, we define the *chordal crossing exponent* $\nu = \nu(\beta)$ by the relation

$$\mathbf{E}[\exp\{-\beta\hat{\mathcal{L}}\}] \approx e^{-\nu L}, \quad L \rightarrow \infty.$$

(Here we set $e^{-\beta(\infty)} = 0$ even for $\beta = 0$, i.e., $\mathbf{E}[\exp\{-\beta\hat{\mathcal{L}}\}] = \mathbf{E}[\exp\{-\beta\hat{\mathcal{L}}\} 1_V]$.) This is the SLE_6 analogue of $\tilde{\xi}(1, \beta)$. Here we have used \approx to indicate that the logarithms are asymptotic; in fact the derivation of the exponent ν shows that $\mathbf{E}[\exp\{-\beta\hat{\mathcal{L}}\}] \asymp e^{-\nu L}$.

The first step is to reduce this problem to an easier computation for SLE_6 . Here we give the main idea of the reduction but leave out a number of details that need to be verified to give a complete proof. The first step is to move the problem to the upper half plane. Let Φ be the conformal transformation of \mathcal{R}_L onto \mathbb{H} with $\Phi(\pi i) = 0$, $\Phi(0) = 1$, $\Phi(L + \pi i) = \infty$. This uniquely specifies Φ and it is easy to show that $\Phi(L) \asymp e^L$. Let γ be an SLE_6 path in \mathbb{H} starting at the origin going to infinity, let

$$T = \inf\{t : \gamma(t) \in (e^L, \infty)\};$$

note that $\mathbf{P}\{T < \infty\} = 1$ since $6 > 4$. Let $\tilde{\mathcal{L}}$ be the π -extremal distance between $[0, 1]$ and $[e^L, \gamma(T)]$ in $\mathbb{H} \setminus \gamma[0, T]$. Then by conformal invariance of π -extremal distance,

$$\mathbf{E}[\exp\{-\beta\tilde{\mathcal{L}}\}] \approx e^{-\nu L} = (e^L)^{-\nu}.$$

Scaling can be used to show that the left hand side is comparable to the quantity with $\tilde{\mathcal{L}}$ replaced by \mathcal{L} , the π -extremal distance between $[x_0, 1]$ and $[t, \infty]$ in $\mathbb{H} \setminus \gamma[0, t^2]$, where $t = e^L$ and $x_0 > 0$ is such that $\partial(\mathbb{H} \setminus \gamma[0, t^2]) \cap [0, 1] = [x_0, 1]$. By conformal invariance, \mathcal{L} is the π -extremal distance between $[g_{t^2}(0), g_{t^2}(1)]$ and $[g_{t^2}(t), \infty)$ in \mathbb{H} . In the terms that dominate, $g_{t^2}(t) - g_{t^2}(1)$ is of order t . Hence, we can see that

$$e^{\mathcal{L}} \approx t [g_{t^2}(1) - g_{t^2}(x_0)]^{-1}.$$

Our final approximation is $[g_{t^2}(1) - g_{t^2}(x_0)] \approx g'_{t^2}(1)$. If we accept all these approximations (they can be verified, but it takes some work), we get the relation

$$\mathbf{E}[g'_{t^2}(1)^\beta] = \mathbf{E}[\exp\{\beta \log g'_{t^2}(1)\}] \approx t^{-(\nu-\beta)}, \quad t \rightarrow \infty.$$

The quantity $\mathbf{E}[g'_{t^2}(1)^\beta]$ can be estimated using the Loewner equation.

For $t \geq 0, x > 0$, let

$$h(t, x) = h_{\kappa, \beta}(t, x) = \mathbf{E}[\exp\{\beta \log g'_t(x)\}].$$

We need to find the asymptotics of $h(t^2, 1)$ as $t \rightarrow \infty$. The scaling law for *SLE* implies that $h(t, x) = h(1, x/\sqrt{t}) = \phi(x/\sqrt{t})$ for some function ϕ of one variable, and (if ν exists), $\phi(y) \approx y^{\nu-\beta}$ as $y \rightarrow 0+$. Differentiating the Loewner equation (3) gives

$$\partial_t \log[[g_t(x) - U_t]'] = \partial_t \log g'_t(x) = -\frac{2}{(g_t(x) - U_t)^2},$$

or, if $Y_t = g_t(x) - U_t$,

$$\exp\{\beta \log g'_t(x)\} = \exp\{-\beta \int_0^t \frac{2}{Y_s^2} ds\}.$$

Since Y_t satisfies the stochastic differential equation

$$dY_t = \frac{2}{Y_t} dt - \sqrt{\kappa} dB_t,$$

with $\kappa = 6$, the Feynman - Kac formula tells us that $h(t, x)$ satisfies the PDE

$$\partial_t h = 3h'' + \frac{2}{x}h' - \frac{2\beta}{x^2}h,$$

and hence ϕ satisfies the second order ODE

$$\phi''(y) + \left(\frac{2}{3y} + \frac{2y}{3}\right) \phi'(y) - \frac{2\beta}{3y^2} \phi(y) = 0. \quad (9)$$

The two linear independent solutions can be given in terms of hypergeometric functions. However, since we are only interested in the behavior as $y \rightarrow 0+$, let us assume that we have a solution of the form $\phi(y) = y^{\nu-\beta}v(y)$ where v is smooth and nonzero at the origin and $\nu > \beta$. Then by plugging in and considering the lowest order term, we see that

$$(\nu - \beta)(\nu - \beta - 1) + \frac{2}{3}(\nu - \beta) - \frac{2\beta}{3} = 0,$$

i.e.,

$$\nu(\beta) = \beta + \frac{1 + \sqrt{1 + 24\beta}}{6}. \quad (10)$$

If V is the event defined earlier in this subsection, then $\mathbf{P}(V)$ decays like $e^{-\nu L} = e^{-L/3}$ as $L \rightarrow \infty$.

We have done calculations for $\kappa = 6$, but similar calculations can be done for other values of κ . However, the next corollary uses the locality property for $\kappa = 6$.

Corollary 11 *Suppose γ is a chordal SLE_6 path from πi to $L + \pi i$ in \mathcal{R}_L ($L \geq 1$). Let $D \subset \mathcal{R}_L$ be a simply connected subdomain with $[\pi i, L + \pi i] \subset \partial D$. Then*

$$\mathbf{P}\{\gamma[0, \infty) \cap R_L \subset D\} \asymp e^{-\mathcal{L}/3},$$

where \mathcal{L} is the π -extremal distance between $\partial D \cap [0, \pi i]$ and $\partial D \cap [L, L + \pi i]$ in D .

Exercise. Explain why the corollary follows from the exponent calculation for $\kappa = 6$. The argument should use the locality property for SLE_6 .

4.5 Using $\nu(\beta)$ to compute $\tilde{\xi}(1, \lambda)$

Here we will show how to find $\tilde{\xi}(1, \lambda)$ given $\nu(\beta)$. The same basic strategy is used to find $\xi(1, 1), \xi(2, 0), \xi(1, 0)$ from SLE_6 exponents, but for ease we will restrict our discussion to $\tilde{\xi}(1, \lambda)$.

Let γ be an SLE_6 from πi to $L + \pi i$ in \mathcal{R}_L , with the driving Brownian motion defined with respect to a probability \mathbf{P} . Let γ' be a Brownian excursion

from ∂_1 to ∂_2 in \mathcal{R}_L defined with respect to $\mu_{\mathcal{R}_L}$. Let D_+, \mathcal{D} be the domains as defined in the previous two subsections and let \mathcal{E} denote the event that $\gamma \cap \gamma' = \emptyset$. On the event \mathcal{E} , let $D = D_+ \cap \mathcal{D}$ which is a simply connected domain whose boundary includes nontrivial arcs $\partial'_1 := \partial_1 \cap \partial D, \partial'_2 := \partial_2 \cap \partial D$. For $\alpha > 0$, define the exponent ρ by

$$\mathbf{P} \otimes \mu_{\mathcal{R}_L} [1_{\mathcal{E}} \exp\{-\alpha L(\partial'_1, \partial'_2; D)\}] \approx \exp\{-\rho(\alpha) L\}, \quad L \rightarrow \infty.$$

We find ρ by doing the double integral two ways: once integrating with respect to \mathbf{P} first and the other integrating with respect to $\mu_{\mathcal{R}_L}$ first.

Suppose γ is given. Then conditioned on γ , conformal invariance tells us that

$$\begin{aligned} \mu_{\mathcal{R}_L} [1_{\mathcal{E}} \exp\{-\alpha L(\partial'_1, \partial'_2; D)\}] &\approx \mu_{\mathcal{R}_L} [\exp\{-\alpha L(\partial_1 \cap \partial D_+, \partial_2 \cap \partial D_+; D_+)\}] \\ &\asymp \exp\{-\tilde{\xi}(1, \alpha) \mathcal{L}\}. \end{aligned}$$

Here \mathcal{L} is as in the previous subsection. Therefore,

$$\begin{aligned} \mathbf{P} \otimes \mu_{\mathcal{R}_L} [1_{\mathcal{E}} \exp\{-\alpha L(\partial'_1, \partial'_2; D)\}] &\approx \mathbf{P} [\exp\{-\tilde{\xi}(1, \alpha) \mathcal{L}\}] \\ &\approx \exp\{-\nu(\tilde{\xi}(1, \alpha)) L\}, \end{aligned}$$

and $\rho(\alpha) = \nu(\tilde{\xi}(1, \alpha))$. Suppose γ' is given. Then conditioned on γ' , conformal invariance and the locality property for SLE_6 tell us

$$\mathbf{P} [1_{\mathcal{E}} \exp\{-\alpha L(\partial'_1, \partial'_2; D)\}] \approx \exp\{-\nu(\alpha) L(\partial_1 \cap \partial D_+, \partial_2 \cap \partial D_+; D)\}.$$

Therefore,

$$\begin{aligned} \mathbf{P} \otimes \mu_{\mathcal{R}_L} [1_{\mathcal{E}} \exp\{-\alpha L(\partial'_1, \partial'_2; D)\}] &\approx \mu_{\mathcal{R}_L} [\exp\{-\nu(\alpha) L(\partial_1 \cap \partial D_+, \partial_2 \cap \partial D_+; D)\}] \\ &\approx \exp\{-\tilde{\xi}(1, \nu(\alpha))\}, \end{aligned}$$

giving $\rho(\alpha) = \tilde{\xi}(1, \nu(\alpha))$. We therefore know that $\nu(\tilde{\xi}(1, \alpha)) = \tilde{\xi}(1, \nu(\alpha))$ for all $\alpha \geq 0$.

Suppose $\alpha = 0$. Since $\nu(0) = 1/3$ and $\tilde{\xi}(1, 0) = 1$, we get $\tilde{\xi}(1, 1/3) = \nu(1) = 2$. Since $\nu(1) = 2, \tilde{\xi}(1, 1) = \nu(2) = 10/3$. Continuing, we get $\tilde{\xi}(1, \lambda)$ for a countable collection of λ . A similar argument (using two domains D and D_- rather than just one), can be used to find $\tilde{\xi}(1, \lambda)$ for all $\lambda \geq 0$,

$$\tilde{\xi}(1, \lambda) = \lambda + \frac{2}{3} + \frac{1}{3}\sqrt{24\lambda + 1}.$$

See [34, 35, 36, 37] for other values of $\tilde{\xi}$ and ξ

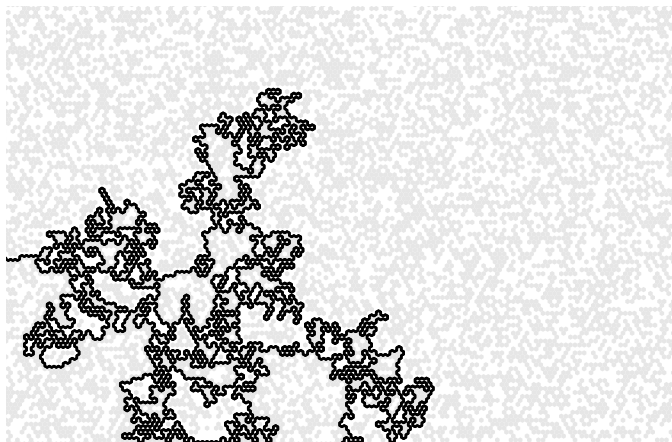


Figure 1: The boundary between black and white clusters in percolation on the triangular lattice (picture produced by Oded Schramm).

5 Critical Percolation

Let \mathbb{T} denote the planar triangular lattice. This is a graph whose vertices (considered as elements of \mathbb{C}) are

$$\{j + ik\sqrt{3} : j, k \in \mathbb{Z}\} \cup \{(j + \frac{1}{2}) + ik\frac{\sqrt{3}}{2} : j, k \in \mathbb{Z}\},$$

and whose edges are the line segments of length 1 connecting vertices. Each vertex has six nearest neighbors. In a natural way we can associate to each vertex in \mathbb{T} a hexagon whose center is the vertex and whose vertices are the midpoints of the edges incident to that vertex. These vertices (with the line segments of the hexagon) form the dual lattice which is called the hexagonal lattice. In the dual lattice each vertex has degree 3 and the faces are hexagons. Critical percolation on \mathbb{T} is the process that colors each vertex in the triangular lattice (or, equivalently, each face in the hexagonal lattice) independently black or white with the probability of white being $1/2$. (See Figure 1).

Suppose D is a simply connected domain with smooth boundary. Let A_1, A_2 be nontrivial disjoint arcs on ∂D . Suppose we consider the lattice $\delta\mathbb{T}$ and study $p(A_1, A_2, D; \delta)$, the probability that there is a connected set of white hexagons in D connecting A_1, A_2 in the lattice $\delta\mathbb{T}$. Topological considerations show that either A_1 and A_2 are connected by white hexagons or the complementary arcs are connected by black hexagons. It has been

believed for a long time that the limit

$$\lim_{\delta \rightarrow 0} p(A_1, A_2, D; \delta) = p(A_1, A_2, D)$$

exists and is in $(0, 1)$. By symmetry, if D is a square and A_1, A_2 are opposite sides, then $p(A_1, A_2; D) = 1/2$. Other properties have also been conjectured:

- The value is the same for critical percolation on other planar lattices. For example, if one considered bond percolation on the square lattice \mathbb{Z}^2 (each bond is open or closed independently with probability $1/2$), then the analogous limit would be the same.
- The quantity $p(A_1, A_2; D)$ is a conformal invariant, i.e., if $f : D \rightarrow D'$ is a conformal transformation, then $p(f(A_1), f(A_2); D') = p(A_1, A_2; D)$ (see [24]).

Assuming conformal invariance and using techniques of conformal field theory, Cardy [12] gave a prediction for $p(A_1, A_2; D)$ that is now called Cardy's formula. His techniques assumed that the scaling limit produced a conformally invariant "field"; he did not use any specific lattice model of percolation (he was using the universality hypothesis that all planar models of independent percolation have the same scaling limit at criticality). Since he was assuming conformal invariance of $p(A_1, A_2; D)$, he could use any domain; he chose the upper half plane \mathbb{H} for D and A_1, A_2 were intervals on the real line. He found the probability as the solution of a differential equation, and the formula was given in terms of hypergeometric functions. Carleson noted that the function Cardy produced took a particularly nice form if one chooses D to be an equilateral triangle of side length 1, A_1 one of the sides, and A_2 an interval with one endpoint on the vertex opposite A_1 . In this case, Cardy's prediction for $p(A_1, A_2, D)$ is just the length of A_2 .

Schramm [49], also assuming conformal invariance of the scaling limit, showed that the boundary between percolation clusters was given by SLE_6 . More precisely, consider the path in Figure 1. It is obtained by considering critical percolation on the triangular lattice in the half-plane where hexagons are colored black or white independently with probability $1/2$ for each *except* that the lower boundary consists of all black hexagons to the left of the origin and all white hexagons to the right. There is a curve starting at the origin that forms a boundary between black and white hexagons. This discrete curve moves according to a simple rule. At each step there are two possible bonds to choose from (since the boundary lies on the hexagonal

lattice and each vertex has degree three on this lattice) — if the hexagon whose boundary includes those two bonds is already known, this determines the next move. Otherwise, one looks at this hexagon to determine which way to move. This process is sometimes called the *percolation exploration process*. Schramm showed that, under the assumption of a conformally invariant limit, the limit of this process is SLE_6 . (The nature of the process shows that it is “local” in nature.)

Smirnov [51] recently proved Cardy’s formula for the percolation exploration process on the triangular lattice. This has made the identification of the boundary SLE_6 rigorous. As an example let D be the equilateral triangle as above and consider the set of sites connected to A_3 by white hexagons. The boundary of the “hull” formed by these open sites is the same as the hull of an SLE_6 path from the vertex $A_1 \cap A_3$ to the vertex $A_2 \cap A_3$. This process can also be given in terms of Brownian motion in D reflected at an oblique angle (see [53]).

6 Loop-erased random walk

The *loop-erased random walk (LERW)* is obtained from simple random walk by erasing loops. There are a number of equivalent ways to define this measure. In fact, one of the reasons that the process is interesting is that it arises in a number of different ways.

6.1 Definitions

Let S denote simple random walk in \mathbb{Z}^d . If A is a finite subset of \mathbb{Z}^d we write

$$\partial A = \{z \in \mathbb{Z}^d \setminus A : \text{dist}(z, A) = 1\}, \quad \tau_A = \inf\{j \geq 0 : S(j) \notin A\}.$$

If $x, y \in A, z \in \partial A$, let

$$G_A(x, y) = \mathbf{E}^x \left[\sum_{j=0}^{\tau_A-1} 1_{\{S(j)=y\}} \right], \quad H_A(x, z) = \mathbf{P}^x \{S(\tau_A) = z\}.$$

6.1.1 Loop erasure

If $\omega = [\omega_0, \omega_1, \dots, \omega_n]$ is a nearest neighbor path in \mathbb{Z}^d , we define its (chronological) loop-erasure $L(\omega)$ as follows. Let

$$j_0 = \max\{j : \omega_j = \omega_0\}.$$

If $j_0 = n$, we set $l = 0$. Inductively, if $j_k < n$, we define

$$j_{k+1} = \max\{j : \omega_j = \omega_{j_{k+1}}\},$$

and let l be the first k such that $j_k = n$. Then,

$$L(\omega) = [\omega_{j_0}, \omega_{j_1}, \dots, \omega_{j_l}].$$

Note that $L(\omega)$ is a self-avoiding subpath of ω from ω_0 to ω_n .

If A is a finite subset (or, in fact, any subset with the property that $\tau_A < \infty$ with probability one), the *loop-erased random walk (LERW)* starting at x in A (stopped upon leaving A) is the measure on paths given by

$$L [S(0), \dots, S(\tau_A)],$$

where S is a simple random walk with $S(0) = x$. This measure is supported on self-avoiding paths starting at x that lie in A except for the terminal point which is in ∂A .

Exercise. Suppose A is a finite subset of \mathbb{Z}^d and $x \in A$. Let $\omega = [\omega_0, \dots, \omega_n]$ be a nearest neighbor self-avoiding path with $\omega_0 = x$, $\{\omega_0, \dots, \omega_{n-1}\} \subset A$ and $\omega_n \notin A$. Then the probability that the LERW starting at x in A produces the path ω is $(2d)^{-n} Q(\omega_0, \omega_1, \dots, \omega_{n-1}; A)$ where

$$Q(\omega_0, \omega_1, \dots, \omega_{n-1}; A) = \prod_{j=0}^{n-1} G_{A_j}(\omega_j, \omega_j).$$

Here $A_0 = A$ and $A_j = A \cup \{\omega_0, \dots, \omega_{j-1}\}$ for $j > 0$.

The way to solve the previous exercise is to consider how many ways one can add loops back onto a self-avoiding path to get a simple path. The order in which one adds the loops comes from the chronological nature of the definition of the LERW. However, as the next exercise demonstrates, we can give any order to the vertices and get the same distribution when one adds loops.

Exercise. Show that if A, ω are as in the previous exercise and $\sigma : \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$ is a permutation, then

$$Q(\omega_{\sigma(0)}, \omega_{\sigma(1)}, \dots, \omega_{\sigma(n-1)}; A) = Q(\omega_0, \omega_1, \dots, \omega_{n-1}; A). \quad (11)$$

(Hint: it suffices to show that for each k this holds for the permutation that transposes k and $k + 1$.)

6.1.2 Laplacian random walk

If A is a finite subset of \mathbb{Z}^d and $V \subset \partial A$, we let

$$p(x; A, V) = \mathbf{P}^x\{S(\tau_A) \in V\} = \sum_{z \in V} H_A(x, z).$$

Note that $p(\cdot; A, V)$ is discrete harmonic in A with boundary value 1 on V and 0 on $\partial A \setminus V$. (A function $G : \mathbb{Z}^d \rightarrow \mathbb{R}$ is discrete harmonic at z if

$$G(z) = \frac{1}{4} \sum_{|e|=1} G(z+e). \quad)$$

If $a \in \mathbb{R}$, the *Laplacian random walk with exponent a* in A starting at x is the process $\hat{S}(j)$ with $\hat{S}(0) = x$ and (if $\omega_0, \dots, \omega_{k-1} \in A$ and $|\omega_k - \omega_{k-1}| = 1$),

$$\mathbf{P}\{\hat{S}(k) = \omega_k \mid [\hat{S}(0), \dots, \hat{S}(k-1)] = [\omega_0, \dots, \omega_{k-1}]\} =$$

$$\frac{p(\omega_k, A \setminus \{\omega_0, \dots, \omega_{k-1}\}, \partial A)^a}{\sum_{|x - \omega_{k-1}|=1} p(x, A \setminus \{\omega_0, \dots, \omega_{k-1}\}, \partial A)^a}.$$

This walk was introduced in [44]; however, the case $a = 1$ was already known to be the same as the loop-erased walk. We leave this as an exercise.

Exercise. The Laplacian random walk with exponent $a = 1$ starting at x in A is the same as the LERW starting at x in A .

6.1.3 Uniform spanning trees

There is a close relationship between LERW and spanning trees chosen from the uniform distribution. The nicest description of this is due to Wilson [55] and is referred to as *Wilson's algorithm*. Let A be a finite subset of \mathbb{Z}^d . We will “wire” the boundary, i.e., we will consider the graph $\mathcal{G}(A)$ whose vertices are A plus a single extra vertex called ∂A . Two vertices in A are adjacent if they are nearest neighbors and $x \in A$ is adjacent to ∂A if $\text{dist}(x, \mathbb{Z}^d \setminus A) = 1$. A spanning tree of $\mathcal{G}(A)$ is a connected subgraph of $\mathcal{G}(A)$ with no cycles containing all vertices. Wilson's algorithm to generate a spanning tree of $\mathcal{G}(A)$ goes as follows:

- Choose some $x \in A$ and do LERW from x in A . Put all edges of the LERW in the tree. This gives a tree. If this is a spanning tree, stop.

- Otherwise, choose a vertex that is not in the tree, and run LERW in the set of vertices not in the tree. (In other words, start a simple random walk at that vertex, let it run until it leaves A or reaches a vertex that has already been added to the tree, and then erase loops from this path.) Add all edges from this path to the tree — note that we still have a tree.
- Continue until we have a spanning tree.

Exercise. Show that Wilson’s algorithm chooses a tree from the uniform distribution of all spanning trees. (Hint: For a given spanning tree \mathcal{T} , write down an expression for the probability that it is chosen. Use (11) to show this is the same for all trees.)

In particular, if $x \in A$, then the distribution of the unique self-avoiding path from x to ∂A in a uniform spanning tree is the same as that of LERW starting at x in A .

6.2 LERW in the plane

Let A denote the set of finite, simply connected subsets of \mathbb{Z}^2 containing the origin (we say that A is simply connected if A and $\mathbb{Z}^2 \setminus A$ are both connected). Let $\text{inrad}(A) = \min\{|z| : z \in \partial A\}$. Suppose $z \in \partial A$ and consider LERW conditioned so that it leaves A at z (which is the same as the loop erasure of a simple random walk S with $S(0) = 0$ conditioned so that $S(\tau_A) = z$).

Now suppose D is a bounded simply connected domain in \mathbb{C} containing the origin and z is a point on ∂D . Consider a grid $\delta\mathbb{Z}^2$ and on this grid take the LERW from 0 to (a lattice point near z). One would expect that there should be a limit distribution on curves connecting 0 to z (or z to 0). Since the scaling limit of random walk (Brownian motion) is conformally invariant and the order of points (which is all that is relevant for the loop-erasing procedure) does not change under conformal transformation, one could conjecture that the limit is conformally invariant. In fact, Schramm [49] first developed the stochastic Loewner evolution in order to understand the limit of LERW — under the assumption of a conformally invariant limit, he showed that the limit must be (radial) SLE_2 . The choice $\kappa = 2$ was determined by a certain exponent for loop-erased walk that Kenyon had recently established rigorously [23] (also using a conformal invariance argument). Recently [39], Schramm, Werner, and I proved that the scaling limit of planar LERW is

SLE_2 . In this subsection I will discuss this theorem, although I will not give the proof.

6.2.1 LERW as a Markov chain on domains

If $A \in \mathcal{A}$ and $z \in \partial A$, we can consider LERW as a measure on self-avoiding paths $\omega = [\omega_0, \dots, \omega_l]$ with $\omega_0 = z, \omega_l = 0$, and $\{\omega_1, \dots, \omega_l\} \subset A$ (this is the time reversal of the LERW starting at the origin conditioned to leave A at z). We can also look at this as a Markov chain on ordered pairs (A', w) where $A' \in \mathcal{A}$ and $w \in \partial A'$,

$$(A_0, \omega_0), (A_1, \omega_1), (A_2, \omega_2), \dots, (A_l, \omega_l),$$

where $A_0 = A$; for $j = 1, 2, \dots, l-1$, A_j is the connected component of $A \setminus \{\omega_0, \dots, \omega_j\}$ containing the origin; and $A_l = \emptyset$ (in this setup, the “terminal point” $(\emptyset, 0)$ should be added to the state space as an absorbing state). Suppose $A \in \mathcal{A}$, $z \in \partial A$, $w \in A$ with $|w - z| = 1$, and A' denotes the connected component of $A \setminus \{w\}$ containing the origin. Then the probability of the transition $(A, z) \mapsto (A', w)$ is the probability that a LERW starting at the origin in A conditioned to leave A at z visits w as its last visit in A before hitting z (or, equivalently, the first step of the reversed walk starting at z is to w , or, equivalently, that the last step of the simple random walk from 0 to z is to w).

As we have defined it, the time to make the transition $(A, z) \mapsto (A', w)$ is 1. This is not the most convenient parametrization from the perspective of scaling limits. We will instead use *discrete parameterization by capacity*. If $A \in \mathcal{A}$, define the (discrete, simple random walk) capacity of A to be $G_A(0, 0)$.

Exercise. If $A \in \mathcal{A}$ and $x \in A$, $x \neq 0$, let A_x denote the connected component of $A \setminus \{x\}$ containing the origin.

- If $\text{dist}(x, \partial A) = 1$, then $A_x \in \mathcal{A}$. (Is this true if $\text{dist}(x, \partial A) > 1$?)
-

$$G_A(0, 0) - G_{A_x}(0, 0) = H_{A_x}(0, x)G_A(x, 0) = H_{A_x}(0, x)^2 G_A(x, x).$$

Parametrization by discrete capacity will be the parametrization that says that the time to make the transition $(A, z) \mapsto (A', w)$ is $G_A(0, 0) - G_{A'}(0, 0)$.

Remark. Everything we have done so far in this subsection applies to arbitrary graphs. However, the fact below only holds for the square lattice.

Fact. (see, e.g., [25]) There is a positive constant c such that for all $A \in \mathcal{A}$,

$$| G_A(0, 0) - \frac{2}{\pi} \log[\text{inrad}(A)] | \leq c.$$

Moreover, if $C_r = \{z \in \mathbb{Z}^2 : |z| < r\}$,

$$| G_{C_r}(0, 0) - \frac{2}{\pi} \log r | \leq c/r.$$

6.2.2 LERW as a Markov chain on conformal maps

For any $A \in \mathcal{A}$, let \tilde{A} be the simply connected domain obtained from A by replacing each lattice point with the square of side length 1 centered at the point. Let ϕ_A be the unique conformal transformation of \tilde{A} onto the unit disk \mathbb{D} with $\phi_A(0) = 0, \phi'_A(0) > 0$. If $z \in \partial A$, We will abuse notation slightly and write $\phi_A(z)$. This should be interpreted as $\phi_A(z')$ where z' is the point of $\partial \tilde{A}$ closest to z (if there is more than one closest point, we can make any choice, it will not matter for what we discuss here). Note that $|\phi_A(z)| = 1$. We call $-\log \phi'_A(0)$ the *continuous capacity of A (with respect to the origin)*. A standard estimate shows that there is a $c > 0$ such that

$$| \log \text{inrad}(A) + \log \phi'_A(0) | \leq c.$$

In fact, one can show that there is a function $\delta_r \rightarrow 0$ as $r \rightarrow \infty$ such that

$$| \frac{\pi}{2} G_A(0, 0) + \log \phi'_A(0) | \leq \delta_r, \tag{12}$$

provided that $\text{inrad}(A) \geq r$. (δ_r can be chosen to be $cr^{-\alpha}$ for some $\alpha > 0$.)

A realization of the LERW, $(A_0, \omega_0), (A_1, \omega_1), \dots$, gives a sequence of conformal maps ϕ_0, ϕ_1, \dots where $\phi_j = \phi_{A_j}$. Let $\psi_j = \phi_j \circ \phi_0^{-1}$. Then ψ_j is the unique conformal transformation of $\phi_0(\tilde{A}_j)$ onto \mathbb{D} with $\psi_j(0) = 0$ and $\psi'_j(0) > 0$. In fact, $\log \psi'_j(0)$ is the difference between the continuous capacities of \tilde{A} and \tilde{A}_j . Let $w_j = \phi_j(z_j)$, so that w_0, w_1, \dots is a process taking values on the unit circle. Define a process Y_t on the unit circle by $Y_{\log \psi'_j(0)} = w_j$ and by linear interpolation for other values of t . The main result in [39] can be stated as follows.

Theorem. *For each $T < \infty$, there is a $\delta_r = \delta_{r,T}$ with $\delta_r \rightarrow 0$ as $r \rightarrow \infty$ such that if $A \in \mathcal{A}$ with $\text{inrad}(A) > r$ and Y_t is defined as above, then Y_t*

and a standard one dimensional Brownian motion B_t can be defined on the same probability space so that, except for an event of probability δ_r ,

$$\sup_{0 \leq t \leq T} |Y_t - \exp\{i\sqrt{2}B_t\}| \leq \delta_r.$$

Remarks.

- We chose to define Y_t using continuous capacity. We could have similarly defined Y_t using discrete capacity, i.e., choosing $Y_{s(j)} = w_j$ where $s(j) = (\pi/2) [G_A(0,0) - G_{A_j}(0,0)]$. The result would still hold. An important part of the proof is the fact that discrete and continuous capacities are very close, see (12).
- Let \tilde{g}_t be the solution of the radial Loewner equation (5) with $U_t = \sqrt{2}B_t$. Then \tilde{g}_t gives the conformal maps for radial SLE_2 . Let $f_t = \tilde{g}_t^{-1}$. From the theorem and properties of the (deterministic) Loewner equation, we can show that for every T there is a $\delta_r = \delta_{r,T}$ as above such that, except perhaps on an event of probability δ_r ,

$$\sup_{0 \leq t \leq T} \sup_{|z| \leq 1 - \delta_r} |f_t(z) - \phi_t^{-1}(z)| \leq \delta_r.$$

(Here ϕ_j has been extended in a natural way to a ϕ_t parameterized by capacity.) This gives a (weak) form of the convergence of LERW to SLE_2 .

7 Self-Avoiding Walk

At the moment we do not know how to prove that the scaling limit of self-avoiding walks exists. However, it is believed that the limit should be in some sense a conformally invariant (or conformally covariant) measure on simple paths. The nature of the limit, if it exists, shows that the limiting measure should also satisfy the restriction property. This makes $SLE_{8/3}$ the *only* possible limit. Nonrigorous predictions about critical exponents for self-avoiding walks can be reinterpreted as rigorous scaling exponents for $SLE_{8/3}$. See [40] for details.

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