An Introduction to $\mathbb{A}^1$-homotopy Theory

Fabien Morel*

Université Paris 7, Paris, France

(Dedicated to H. Bass on the occasion of his 70th birthday)

Lectures given at the
School on Algebraic K-Theory and its Applications
Trieste, 8 - 19 July 2002

LNS0315005

*morel@math.jussieu.fr

Contents

1 Introduction **361**

2 Recollection on simplicial homotopy theory **364**

2.1 Presheaves and sheaves 364
2.2 Simplicial (pre-)sheaves 369
2.3 The classical homotopy category [30] 373
2.4 The *simplicial* homotopy category of sheaves 378

3 Unstable $\mathbb{A}^1$-homotopy theory **383**

3.1 The $\mathbb{A}^1$-homotopy category [25] 383
3.2 $\mathbb{A}^1$-localization of connected simplicial sheaves 388
3.3 Thom spaces and homotopy purity 393

4 Stable $\mathbb{A}^1$-homotopy theory of S^1-spectra **397**

4.1 Recollection on S^1-spectra 397
4.2 $\mathbb{A}^1$-localization of S^1-spectra and the connectivity theorem . . 404
4.3 The homotopy t-structure on $\mathcal{SH}^{S^1}(k)$ 409

5 Stable $\mathbb{A}^1$-homotopy theory of $\mathbb{P}^1$-spectra **415**

5.1 $\mathbb{P}^1$-spectra 415
5.2 The homotopy t-structure 421
5.3 Examples 423

6 $\pi_0(S^0)$ and Milnor-Witt K-theory of fields **426**

6.1 The element ϵ 426
6.2 The Hopf map 428
6.3 Milnor-Witt K-theory and $\oplus_{n\in\mathbb{Z}}[S^0, \mathbb{G}_m{}^{\wedge n}]$ 431
6.4 The basic theorem 435

References **438**

1 Introduction

One of the starting points in homotopy theory is the following result. Let $d > 0$ be a natural number, S^d the d-dimensional sphere and let $i \in \mathbb{Z}$ be an integer such that $i + d > 0$. Then[1] :

$$\pi_{i+d}(S^d) = \begin{cases} 0 \; if \; i < 0 \\ \mathbb{Z} \; if \; i = 0 \end{cases}$$

Recall [1] that for d large enough the group $\pi_{i+d}(S^d)$ only depends on $i \in \mathbb{Z}$. It is called the i-th stable homotopy group of the "sphere spectrum S^0" and is denoted by $\pi_i(S^0)$. The above result implies the following one on stable homotopy groups of spheres :

$$\pi_i(S^0) = \begin{cases} 0 \; if \; i < 0 \\ \mathbb{Z} \; if \; i = 0 \end{cases}$$

In these lectures, our aim is to give an overview of the basic definitions and theorems in $\mathbb{A}^1$-homotopy theory [39, 25, 19] by addressing, among other things, the analogue of the above theorem in the $\mathbb{A}^1$-homotopy theory of smooth schemes over a fixed base field k.

The first question to answer is : what are the algebraic analogues of the spheres ?

Clearly we have two different types of smooth k-varieties which are reasonable "algebraic spheres". The first one is the projective line $\mathbb{P}^1$, that is, the one point compactification of the affine line $\mathbb{A}^1$. One would like also to consider its "smash-products" by itself $\mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1$ as higher dimensional spheres. The problem is that these smash-products are no longer smooth k-varieties and have to be performed in a larger category than just the category of smooth k-varieties. The other obvious smooth varieties which are algebraic spheres are the complement of 0 in the affine space of dimension $n \geq 0$, denoted by $\mathbb{A}^n - \{0\}$. In a sense which is made precise in [25], these two types of spheres, in fact, are related as follows : the "simplicial suspension" of $\mathbb{A}^n - \{0\}$ is homotopy equivalent to $\mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1$.

[1] where for $m > 0$ we denote by $\pi_m(X)$ the m-th homotopy group of a pointed space X

We should emphasize that our previous considerations mostly come from our intuition of the "topology" of these algebraic varieties: given any complex or real embedding of the base field, the topological spaces we get are indeed homotopy equivalent to spheres.

The second question to answer is what is the expected connectivity of these algebraic spheres ?

To illustrate this serious problem, observe that from the intuition mentioned above, we get rather confused: the complex point of $\mathbb{P}^1$ is a 2 dimensional sphere (the Riemann sphere) and is thus simply connected. But through a real embedding $\mathbb{P}^1$ gives a circle, which is not simply connected!

Curiously[2], the connectivity is given by the real topology as it is clear from the following two statements which will be made precise later. One "explanation" of that phenomenon has already been given: $\mathbb{P}^1$ which is a curve is the suspension of $\mathbb{A}^1 - \{0\} = \mathbb{G}_m$. Thus $\mathbb{G}_m$ must rather be a 0-dimensional object, like it is if one consider its real topology!

Theorem 1 *Let k be a field and let n, i be integers. Then the group of "stable $\mathbb{A}^1$-homotopy classes of maps over k"*

$$[S^i, (\mathbb{G}_m)^{\wedge n}]$$

is trivial for $i < 0$.

We shall also discuss several recent results concerning a precise description of the groups

$$[S^0, (\mathbf{G}_m)^{\wedge n}]$$

by introducing the Milnor-Witt K-theory of fields. In particular, these are all non-trivial and in fact for $n = 0$ one has:

Theorem 2 *Let k be a perfect field of characteristic $\neq 2$. Then the group of "stable $\mathbb{A}^1$-homotopy classes of maps over k"*

$$[S^0, S^0]$$

is canonically isomorphic to the Grothendieck-Witt ring of quadratic forms over k.

[2] contrary to the "motivic" intuition which rather fits with the complex topology

Everywhere in this text, k is a perfect field, $Sm(k)$ is the category of separated smooth schemes of finite type over k, called smooth k-varieties in the sequel. For simplicity, if no confusion can arise, we will just denote by $\mathcal{V}$ this category of smooth k-varieties.

For any scheme X and any point $x \in X$, $\mathcal{O}_{X,x}$ will denote the local ring of X at x, m_x its maximal ideal, and $\kappa(x)$ its residue field.

$\mathcal{S}et$ will denote the category of sets.

2 Recollection on simplicial homotopy theory

2.1 Presheaves and sheaves

Definition 2.1.1 *Let $\{f_\alpha : U_\alpha \to X\}_\alpha$ be a finite family of* Étale *morphisms in $Sm(k)$.*

1) $\{f_\alpha : U_\alpha \to X\}_\alpha$ is called a covering family *in the* étale *topology if and only if X is the union of the open subsets $f_\alpha(U_\alpha)$.*

2) $\{f_\alpha : U_\alpha \to X\}_\alpha$ is called a covering family *in the* Nisnevich *topology if and only if for any point $x \in X$ there exists an α and a point $y \in U_\alpha$ which maps to x and has the same residue field (i.e. $\kappa(x) \cong \kappa(y)$).*

3) $\{f_\alpha : U_\alpha \to X\}_\alpha$ is called a covering family *in the* Zariski *topology if each of the f_α is an open immersion and the U_α's cover X.*

Of course Zariski coverings are Nisnevich coverings, and Nisnevich coverings are étale coverings. Nisnevich topology was introduced in [26].

Let us denote by $Preshv(\mathcal{V})$ the category of *presheaves* of sets on $\mathcal{V}$, that is to say the category of functors

$$\mathcal{V}^{op} \to \mathcal{S}et$$

In the sequel, τ will always denote one of the three symbols Zar, Nis, Ét.

Definition 2.1.2 *A presheaf of sets*

$$F : (\mathcal{V})^{op} \to \mathcal{S}et$$

is a sheaf in the τ-topology if for any covering family $\{f_\alpha : U_\alpha \to X\}_\alpha$ in the τ-topology, the obvious diagram of sets:

$$F(X) \to \Pi_\alpha F(U_\alpha) \rightrightarrows \Pi_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$

defines the set $F(X)$ as the equalizer of the two maps on the right.

We shall denote by $Shv(\mathcal{V}_\tau)$ the full subcategory of $Preshv(\mathcal{V})$ consisting of sheaves of sets in the τ-topology.

Lemma 2.1.3 *[17] For any $X \in \mathcal{V}$, the presheaf of sets:*

$$Y \mapsto Hom_{\mathcal{V}}(Y, X)$$

is a sheaf in the Étale topology (and thus also in the Nisnevich and Zariski topology).

Observe that we thus have the following obvious full embeddings:

$$\mathcal{V} \subset Shv(\mathcal{V}_{\text{ét}}) \subset Shv(\mathcal{V}_{Nis}) \subset Shv(\mathcal{V}_{Zar}) \subset Preshv(\mathcal{V})$$

Recall from [25] the following

Definition 2.1.4 *A* distinguished *square in $\mathcal{V}$ is a cartesian square of the form :*

$$\begin{array}{ccc} W & \to & V \\ \downarrow & & \downarrow p \\ U & \stackrel{i}{\to} & X \end{array}$$

such that p is an étale morphism, i is an open immersion and

$$p^{-1}((X-U)_{r\acute{e}d}) \to (X-U)_{r\acute{e}d}$$

is an isomorphism of schemes (where $_{r\acute{e}d}$ means we endow the closed subset with the reduced induced structure).

We observe that for such a square, the family $\{U \stackrel{i}{\to} X, V \stackrel{p}{\to} X\}$ is a covering family in the Nisnevich topology. Observe also that if p is an open immersion, the last condition exactly means that U and V cover X. This squares will play for the Nisnevich topology the role played in the Zariski topology by the squares of the form:

$$\begin{array}{ccc} U \cap V & \to & V \\ \downarrow & & \downarrow \\ U & \to & U \cup V \end{array}$$

for open subsets U and V in X. Here is the smallest example of genuine distinguished square in the Nisnevich topology:

Example 2.1.5 Let L be a finite field extension of k and $x \in L$ a generator of L over k (recall that k is perfect so that there is always one). Denote by $x_0 : Spec(L) \to \mathbb{A}^1$ the closed embedding corresponding to x and let U be the open complement. The morphism $\mathbb{A}^1{}_L \to \mathbb{A}^1$ is étale and the pull back of x_0 by this morphism is the finite étale L-scheme $F := Spec(L \otimes_k L)$ closely embedded into $\mathbb{A}^1{}_L$. The diagonal section $y_0 : SpecL \to F$ of $F \to SpecL$ thus splits F as $F' \amalg SpecL$. Let Ω denote the open complement to the closed subscheme $F' \subset \mathbb{A}^1{}_L$. Then one checks that the square:

$$\begin{array}{ccc} \Omega - y_0 & \to & \Omega \\ \downarrow & & \downarrow \\ U & \to & \mathbb{A}^1 \end{array}$$

is a distinguished square.

It is well-known that a presheaf of sets F is a sheaf in the Zariski topology if for any $X \in \mathcal{V}$ and any pair of open subsets U and V covering X the map

$$F(X) \to F(U) \times_{F(U \cap V)} F(V)$$

is bijective. The following lemma makes precise that to some extent the Nisnevich topology behaves very closely to the Zariski topology.

Lemma 2.1.6 *[25] A presheaf of sets F is a sheaf in the Nisnevich topology (on $Sm(k)$) if and only if for any distinguished square in $\mathcal{V}$ the map*

$$F(X) \to F(U) \times_{F(U \times_X V)} F(V)$$

is bijective.

The following is well known:

Lemma 2.1.7 *[3] For any τ, the inclusion $Shv(\mathcal{V}_\tau) \subset Preshv(\mathcal{V})$ admits as left adjoint $a_\tau : Preshv(\mathcal{V}) \to Shv(\mathcal{V}_\tau)$ called the "associated sheaf" functor.*

Definition 2.1.8 *A τ-point in $\mathcal{V}$ is a morphism of schemes*

$$x : Spec(K) \to X$$

such that

1) K is a separably closed field in case $\tau =$ ét;

2) K is the residue field of the image (also denoted x) of the point of $SpecK$ in X in case τ is either Nis or Zar.

We are now in position to define the category of neighborhoods of a τ-point.

Definition 2.1.9 *Let $x : SpecK \to X$ be a τ-point $\in \mathcal{V}$. The category $Neib_\tau^x$ of neighborhoods of x is*

1) the category of pairs $(f : U \to X, y : Spec(K) \to U)$ with f an étale morphism with U irreducible and y a τ-point of U with the same field K, which lifts x in case τ is ét or Nis,

2) the category of open subsets of X which contain x in case τ is Zar.

That category is left filtering (and essentially small).

Definition 2.1.10 *Let $x : SpecK \to X$ be a τ-point $\in \mathcal{V}$.*

1) for any presheaf F, the fiber of F over[3] *x is the set :*

$$F_x := colim_{(U \to X, y) \in Neib_\tau^x} F(U)$$

2) the fiber functor *associated to x, is the functor*

$$Preshv(\mathcal{V}) \to \mathcal{S}et, F \mapsto F_x$$

Example 2.1.11 The fiber of the affine line $\mathbb{A}^1 \in Preshv(\mathcal{V})$ over x is[4]:

1) the *strict henselization* $\mathcal{O}_{X,x}^{sh}$ of the local ring $\mathcal{O}_{X,x}$ of x in X in case $\tau =$ ét.

2) the *henselization* $\mathcal{O}_{X,x}^{h}$ of the local ring $\mathcal{O}_{X,x}$ of x in X in case $\tau =$ Nis.

3) the local ring $\mathcal{O}_{X,x}$ of x in X in case $\tau =$ Zar.

Lemma 2.1.12 *[3] 1) For any presheaf F the canonical map*

$$F_x \to (a_\tau F)_x$$

is a bijection.

[3] or "at"

[4] We refer the reader to [32] for the notions of henselian rings and henselization

2) The family of these fiber functors form a conservative set of functors in the sense that a morphism in $Shv(\mathcal{V}_\tau)$ is an isomorphism (resp. a monomorphism, an epimorphism) if and only if each of its fibers are bijections (resp. monomorphisms, surjections).

Lemma 2.1.13 *1) For any distinguished square*

$$\begin{array}{ccc} W & \rightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

the corresponding square in $Shv(\mathcal{V}_{Nis})$ is both cartesian and cocartesian. As a corollary, one gets a canonical isomorphism of sheaves of sets[5] (in the Nisnevich topology)

$$V/W \cong X/U$$

2) For any closed point $x \in X \in \mathcal{V}$, with residue field L, the sheaf $X/X - \{x\}$ is isomorphic to the sheaf $\mathbb{A}^n_L/(\mathbb{A}^n_L - \{0\})$ with $n = dim_x(X)$.

3) For any closed point $x \in X \in \mathcal{V}$, with residue field L, the henselian local ring $\mathcal{O}^h_{x,X}$ is isomorphic to the local henselian ring $\mathcal{O}^h_{0,\mathbb{A}^n_L}$ with $n = dim_x(X)$.

The first part is rather formal (and its Zariski analogue also holds). To prove 2) and 3) one observes that because X is smooth over k, there is an open subset U of X containing x and an étale morphism $f : U \to \mathbb{A}^n$. Let then $U_L \to U$ be the obvious étale morphism (U_L denoting $U \times Spec(L)$. The pull back of x in U_L is a finite étale L-scheme F which admits (obviously) a closed point y with residue field L. Write $F = \{y\} \amalg F'$ and $\Omega := U_L - \{F'\}$ (the open complement of the finite closed subscheme $F' \subset U_L$).

Then $y : Spec(L) \to \Omega \to U$ is a Nisnevich neighborhood of x and thus: $\Omega/\Omega - \{y\} \cong U/U - \{x\} \cong X/X - \{x\}$ by 1) (applied twice) and $\mathcal{O}^h_{x,X} = \mathcal{O}^h_{x,U} \cong \mathcal{O}^h_{y,\Omega}$. We conclude using now the étale morphism $\Omega \subset U_L \to \mathbb{A}^n_L$ induced by f_L and the image z of y in $\mathbb{A}^n_L$, which is a closed point in $\mathbb{A}^n_L$ with residue field L (thus after some translation one can assume $z = 0$). We thus get in much the same way isomorphisms $X/X - \{x\} \cong \Omega/\Omega - \{y\} \cong \mathbb{A}^n_L/(\mathbb{A}^n_L - \{0\})$ and $\mathcal{O}^h_{x,X} \cong \mathcal{O}^h_{y,\Omega} \cong \mathcal{O}^h_{0,\mathbb{A}^n_L}$.

[5] when $F \to G$ is a monomorphism of sheaves of sets, G/F means the categorical quotient

2.2 Simplicial (pre-)sheaves

Recollection on simplicial objects Recall that Δ is the *simplicial* category, i.e. the category of finite ordered sets $\underline{n} := \{0, \dots, n\}$ and order preserving maps. For any $i \in \underline{n}$, we denote as usual by

$$d^i : \underline{n-1} \to \underline{n}$$

(assuming here $n \geq 1$) the i-th coface map, i.e. the unique injective (increasing) map which avoids the value $i \in \underline{n}$ and

$$s^i : \underline{n+1} \to \underline{n}$$

the i-th codegeneracy map, i.e. the unique surjective (increasing) map which takes twice the value i. These cofaces and codegeneracies maps satisfies altogether the usual cosimplicial identities [16].

If $\mathcal{C}$ is a category, a *simplicial object* (resp. a *cosimplicial* object) in $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$ (resp. $\Delta \to \mathcal{C}$). Let's denote by $\Delta^{op}\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$ (resp. $\Delta\mathcal{C}$ that of cosimplicial objects).

For instance we will denote by $\Delta^{op}\mathcal{S}et$ the category of simplicial sets.

Example 2.2.1 For any integer $n \geq 0$, we denote by Δ^n the n-th standard simplex $\Delta^n \in \Delta^{op}\mathcal{S}et$ by which we mean the functor

$$\Delta^{op} \to \mathcal{S}et, \underline{m} \mapsto Hom_{\Delta}(\underline{m}, \underline{n})$$

and observe, by the Yoneda lemma, that for any simplicial set X the obvious map

$$Hom_{\Delta^{op}\mathcal{S}et}(\Delta^n, X) \to X_n$$

is a bijection.

Example 2.2.2 Let Top denote the category of topological spaces. For each integer $n \geq 0$ we denote as usual by Δ^n_{top} the standard *topological n-simplex* whose value for each $\underline{n}$ is the topological space

$$\Delta^n_{top} = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} / \Sigma_i t_i = 1 \ \&\ t_i \geq 0 \text{ for each } i \in \underline{n}\}$$

For each $i \in \underline{n}$, we let $e_i \in \Delta^n_{top}$ be the point with coordinates all 0 but $t_i = 1$.

We turn these topological simplices altogether into a cosimplicial topological space

$$\Delta^\bullet_{top} : \Delta \to Top$$

by the sending the increasing map $\phi : \underline{n} \to \underline{m}$ to the only affine (and continuous) map $\Delta^n_{top} \to \Delta^m_{top}$ which maps $e_i \in \Delta^n_{top}$ to $e_{\phi(i)} \in \Delta^m_{top}$.

Example 2.2.3 Given a topological space X, its *singular simplicial set*, denoted by $S(X)$, is the simplicial set

$$\Delta^{op} \to \mathcal{S}et, \underline{n} \mapsto Hom_{Top}(\Delta^n_{top}, X) =: S_n(X)$$

We observe that the functor

$$Top \to \Delta^{op}\mathcal{S}et, X \mapsto S(X)$$

admits as left adjoint the *realization functor*

$$|\,| : \Delta^{op}\mathcal{S}et \to Top, Y \mapsto |Y|$$

where $|Y|$ is the *realization* of Y, i.e. the topological space obtained as the quotient of

$$\amalg_n Y_n \times \Delta^n_{top}$$

by (the equivalence relation generated by) the relations

$$(y, \phi_{top}(t)) \sim (\phi^*(y), t)$$

for each $(\phi : \underline{n} \to \underline{m}, y \in Y_m, t \in \Delta^n_{top})$.

Equivalently, it is the coequaliser in Top of the obvious diagram:

$$\amalg_{n,m} Y_n \times \Delta^m_{top} \rightrightarrows \amalg_n Y_n \times \Delta^n_{top}$$

For any integer $n \geq 0$ the boundary $\partial\Delta^n$ of the n-standard simplex $\Delta^n \in \Delta^{op}Shv(\mathcal{V}_\tau)$, is the union over all $i \in \underline{n}$ of all the images[6] $d^i(\Delta^{n-1}) \subset \Delta^n$ (with the convention that $\partial\Delta^0 = \emptyset$). We observe that the realization $|\partial\Delta^n|$ is indeed the boundary of the topological standard n-simplex.

[6]also called the i-th face of Δ^n

Simplicial sheaves More generally, we will be very much interested in the sequel in the category $\Delta^{op}Shv(\mathcal{V}_\tau)$ of simplicial sheaves in the τ-topology. We recall here some basic notions and constructions that can be performed in that category.

For any set E we still denote by E its associated sheaf in the Zariski topology. It is clearly the sheaf of locally constant functions to the discrete set E and is in fact a sheaf in the étale (and Nisnevich) topology as well. We thus get a functor $\mathcal{S}et \to Shv(\mathcal{V}_\tau)$ and extend it to a functor $\Delta^{op}\mathcal{S}et \to \Delta^{op}Shv(\mathcal{V}_\tau)$. We still denote by the same letter both a simplicial set K and its associated simplicial sheaf in $\Delta^{op}Shv(\mathcal{V}_\tau)$. For example, for any integer $n \geq 0$ one has the n-standard simplex $\Delta^n \in \Delta^{op}Shv(\mathcal{V}_\tau)$, its boundary $\partial\Delta^n$, *etc...* For each integer $n \geq 0$ we will set

$$S^n := \Delta^n/\partial\Delta^n$$

and will call it the n-sphere. Observe with our convention that

$$S^0$$

is just the sum of two points.

The functor $\Delta^{op}\mathcal{S}et \to \Delta^{op}Shv(\mathcal{V}_\tau)$ always commutes with colimits (for instance, one can check it by computing the fibers at each τ-point).

We will always denote by $\emptyset$ the initial object of $\Delta^{op}Shv(\mathcal{V}_\tau)$ (this convention is compatible with the previous because the sheaf associated with the empty set is the initial object) and by $*$ the final object, i.e. the simplicial sheaf (associated to) Δ^0 which is also the final object in $\Delta^{op}Shv(\mathcal{V}_\tau)$ also called *the point*.

We let $\Delta^{op}Shv_\bullet(\mathcal{V}_\tau)$ denote the category of *pointed simplicial sheaves* in the τ-topology. As usual, if $\mathcal{X}$ and $\mathcal{Y}$ denote pointed simplicial sheaves, their *wedge* is the pointed simplicial sheaf colimit of the (obvious) diagram

$$\begin{array}{ccc} * & \to & \mathcal{Y} \\ \downarrow & & \\ \mathcal{X} & & \end{array}$$

and is denoted $\mathcal{X} \vee \mathcal{Y}$. And the (pointed) quotient simplicial sheaf

$$\mathcal{X} \times \mathcal{Y}/\mathcal{X} \vee \mathcal{Y}$$

is denoted $\mathcal{X} \wedge \mathcal{Y}$ and is called the *smash-product* of $\mathcal{X}$ and $\mathcal{Y}$.

Remark 2.2.4 We observe that for $n > 1$ the two simplicial sets

$$S^n$$

and

$$S^1 \wedge \cdots \wedge S^1 \text{ (n copies)}$$

are *not* isomorphic.

Let $\mathcal{X}$ be a pointed simplicial sheaf. The cone of $\mathcal{X}$ is the pointed simplicial sheaf $\mathcal{X} \wedge \Delta^1$, where Δ^1 is pointed by its 0-vertex $d^1 : * = \Delta^0 \to \Delta^1$. The cone of $\mathcal{X}$ is denoted by $C(\mathcal{X})$. The 0-coface $d^0 : * \to \Delta^1$ induces a canonical monomorphism $\mathcal{X} \to C(\mathcal{X})$. The quotient simplicial sheaf $C(\mathcal{X})/\mathcal{X}$ is isomorphic to the smash product

$$\mathcal{X} \wedge S^1$$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a pointed morphism of pointed simplicial sheaves. The cone of f, denoted by $C(f)$, is the colimit in the category of pointed simplicial sheaves of the diagram :

$$\begin{array}{ccc} \mathcal{X} & \to & \mathcal{Y} \\ \downarrow & & \\ C(\mathcal{X}) & & \end{array}$$

There is thus a canonical monomorphism $\mathcal{Y} \to C(f)$ whose cofiber $C(f)/\mathcal{Y}$ is canonically isomorphic to $\mathcal{X} \wedge S^1$.

Function objects For any simplicial sheaf $\mathcal{Y} \in \Delta^{op}Shv(\mathcal{V}_\tau)$, the functor

$$\Delta^{op}Shv(\mathcal{V}_\tau) \to \Delta^{op}Shv(\mathcal{V}_\tau), \mathcal{X} \mapsto \mathcal{X} \times \mathcal{Y}$$

has a right adjoint which we denote by $\mathcal{Z} \mapsto \underline{Hom}(\mathcal{Y}, \mathcal{Z})$.

If now $\mathcal{Y}$ is a pointed simplicial sheaf, the functor

$$\Delta^{op}Shv_\bullet(\mathcal{V}_\tau) \to \Delta^{op}Shv_\bullet(\mathcal{V}_\tau), \mathcal{X} \mapsto \mathcal{X} \wedge \mathcal{Y}$$

has a right adjoint which we denote by $\mathcal{Z} \mapsto \underline{Hom}_\bullet(\mathcal{Y}, \mathcal{Z})$.

Clearly, $\underline{Hom}_\bullet(\mathcal{Y}, \mathcal{Z})$ is the fiber at the base point of $\mathcal{Z}$ of the evaluation morphism (at the base point of $\mathcal{Y}$)

$$\underline{Hom}(\mathcal{Y}, \mathcal{Z}) \to \mathcal{Z}$$

2.3 The classical homotopy category [30]

Recall that for a topological space X, $\pi_0(X)$ denote the set of its arcwise connected components and that for each $x \in X$ and each integer $n > 0$

$$\pi_n(X, x)$$

denotes the n-th homotopy group of X at x.

Definition 2.3.1 *1) A continuous map $f : X \to Y$ in Top is called a* weak equivalence *if and only if $\pi_0(f)$ is bijective and for each $x \in X$ and each $n > 0$ the homomorphism $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.*

We denote by W the class of all weak equivalences.

2) A morphism f in $\Delta^{op}\mathcal{S}et$ is called a simplicial weak equivalence *if and only if the realization $|f|$ is a weak equivalence.*

We denote by W_s the class of simplicial weak equivalences.

Remark 2.3.2 Using the interval $[0,1]$ which is homeomorphic to Δ^1_{top} one defines the notion of *homotopy equivalences* between topological spaces as usual. Obviously a homotopy equivalence is a weak equivalence. By the Whitehead theorem, a continuous map between C.W.-complexes is a weak equivalence if and only if it is a homotopy equivalence. But this is not true in general.

As for any simplicial set X the space $|X|$ has a canonical structure of C.W.-complex, we see that a map between simplicial sets $X \to Y$ is a simplicial weak equivalence if and only if the realization

$$|X| \to |Y|$$

is a homotopy equivalence.

Using the 1-simplex $\Delta^1 \in \Delta^{op}\mathcal{S}et$ and its two vertices $d^1 : * \to \Delta^1$ and $d^0 : * \to \Delta^1$ one can define in the obvious way the notion of *simplicial homotopy equivalence* (in $\Delta^{op}\mathcal{S}et$). Obviously such a simplicial homotopy equivalence is a simplicial weak equivalence, but, again, the converse is not true in general.

Definition 2.3.3 *[30] We denote by $\mathcal{H}$, the category $Top[W^{-1}]$ formally obtained from Top by inverting the weak equivalences. It is called the homotopy category (of topological spaces).*

In much the same way, we denote by $\mathcal{H}_s$, the category $\Delta^{op}\mathcal{S}et[W_s^{-1}]$ formally obtained from $\Delta^{op}\mathcal{S}et$ by inverting the simplicial weak equivalences. It is called the homotopy category of simplicial sets.

The remark above implies that the realization functor

$$\Delta^{op}\mathcal{S}et \to Top, X \mapsto |X|$$

maps W_s to W. This implies (by the very definition of both categories) that $|-|$ induces a functor

$$\mathcal{H}_s \to \mathcal{H}$$

still called the *realization functor*.

Theorem 2.3.4 *[30],[16, page 65] For any topological space X, the natural map (given by the adjunction)*

$$|S(X)| \to X$$

is a weak equivalence. For any simplicial set Y the natural map (given by the adjunction)

$$Y \to S(|Y|)$$

is a simplicial weak equivalence. Thus the functor S maps weak equivalences to simplicial weak equivalences and the induced functors

$$|-| : \mathcal{H}_s \to \mathcal{H} \text{ and } S : \mathcal{H} \to \mathcal{H}_s$$

are equivalences of categories inverse to each other.

Quillen's homotopical algebra [30] We will now recall the basic notions of Quillen's notion of *model category*. Indeed we will in fact use the following notion of model category whose main difference with Quillen's original notion in [30] is that one requires existence all limits and colimits as well as of functorial factorizations.

Definition 2.3.5 *[30] Let $\mathcal{C}$ be a category and $i : X \to Y$ and $p : E \to B$ be morphisms in $\mathcal{C}$. We say that i has the left lifting property (LLP for short)* *with respect to p or, equivalently, that p has the right lifting property (RLP for short) if for any commutative square of the form*

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ \downarrow i & & p \downarrow \\ Y & \xrightarrow{g} & B \end{array}$$

there exists a morphism $h : Y \to E$ *which keeps the diagram commutative, i.e. such that* $p \circ h = g$ *and* $h \circ i = f$.

Definition 2.3.6 *Let* $\mathcal{C}$ *be a category equipped with three classes of morphisms* (W, C, F) *respectively called the* weak equivalences, *the* cofibrations *and the* fibrations. *We say that* $(\mathcal{C}, W, C, F)$ *is a* model category *(or that* (W, C, F) *is a model category structure on* $\mathcal{C}$*) if the following axioms hold :*

- **MC1** $\mathcal{C}$ *has all small limits and colimits*
- **MC2** *If* f *and* g *are two composable morphisms and two of* f, g *or* $g \circ f$ *are weak equivalences, then so is the third*
- **MC3** *If the morphism* f *is retract of* g *and* g *is a weak-equivalence, cofibration or fibration then so is* f
- **MC4** *Any fibration has the right lifting property with respect to* trivial cofibrations[7] *and any* trivial fibration[8] *has the right lifting property with respect to cofibrations*
- **MC5** *Any morphism* f *can be functorially (in* f*) factorized as a composition* $p \circ i$ *where* p *is a fibration and* i *a trivial cofibration and as a composition* $q \circ j$ *where* q *is an trivial fibration and* j *a cofibration.*

The associated *homotopy category* is the category $\mathcal{C}[W^{-1}]$ obtained by formally inverting W in $\mathcal{C}$ [11]. One can check [30] using the previous axioms that this category is well defined.

Remark 2.3.7 One should observe that in a model category structure, any two of the three classes (W, C, F) determines the third. For instance, F is exactly the class of morphism with the RLP with respect to all trivial cofibrations and C that of morphisms with the LLP with respect to all

[7] i.e. cofibrations which are also weak equivalences
[8] i.e. a fibration which is also a weak equivalence

trivial fibrations. One can also check that $F \cap W$ is exactly the class of morphisms having the RLP with respect to all cofibrations, that $C \cap W$ is exactly the class of morphisms having the LLP with respect to all fibrations, and thus W is the class of morphisms which can be written as a composition of a trivial cofibration followed by a trivial fibration.

Here are the two fundamental examples of model category structures.

Example 2.3.8 [30] The 4-tuple (Top, W, C, F) is a model category structure, in which W is the class of weak equivalences defined above, the class F of fibrations is that of *Serre fibrations*, i.e. morphisms with the RLP with respect to each inclusion $I^{n-1} \subset I^n$, $n > 0$, I denoting the unit interval in $\mathbb{R}$, the inclusion being induced by the inclusion $\{0\} \subset I$, and C is what it has to be : that of morphisms with the LLP with respect to all trivial fibrations. We observe that the inclusion $X \to Y$ of a relative C.W.-pair (Y, X) is a cofibration.

Example 2.3.9 [30] The 4-tuple $(\Delta^{op}\mathcal{S}et, W_s, C, F)$ is a model category structure, in which W_s is the class of simplicial weak equivalences defined above, the class C of cofibrations is that of degreewise inclusions, and F is what it has to be : that of morphisms with the RLP with respect to all trivial cofibrations. It can be shown [11] that F indeed coincides with the class of Kan fibrations.

Let $\mathcal{C}$ be a category and let

$$\Delta^\bullet : \Delta \to \mathcal{C}$$

be a cosimplicial object.

If all the colimits exist in $\mathcal{C}$ the previous functor induces a realization functor

$$\Delta^{op}\mathcal{S}et \to \mathcal{C}, K \mapsto |K|$$

where $|K|$ is the coequaliser in $\mathcal{C}$ of the obvious diagram:

$$\amalg_{n,m} K_n \times \Delta^m \rightrightarrows \amalg_n K_n \times \Delta^n$$

For instance one has tautologically $|\Delta^n| = \Delta^n$. Beware that in general the realization functor $K \mapsto |K|$ doesn't commutes to finite products.

If $\mathcal{C}$ has moreover all the finite products, then for any pair (X, Y) of objects in $\mathcal{C}$ one define the *functional simplicial set* $S(X, Y)$ as the simplicial set

$$\underline{n} \mapsto Hom_{\mathcal{C}}(X \times \Delta^n, Y)$$

Of course the two previous constructions are related by the following natural bijection in $X, Y \in \mathcal{C}$ and $K \in \Delta^{op}\mathcal{S}et$:

$$Hom_{\mathcal{C}}(X \times |K|, Y) \cong Hom_{\Delta^{op}\mathcal{S}et}(K, S(X, Y))$$

One says that two morphisms $f : X \to Y$ and $g : X \to Y$ are *simplicially homotopic* if there is a morphism

$$H : X \times \Delta^1 \to Y$$

such that $H \circ d^0 = g$ and $H \circ d^1 = f$ (here d^0 and d^1 are the obvious cofaces map $* = \Delta^0 \to \Delta^1$). We denote by

$$\pi(X, Y)$$

the quotient of the set of morphism $Hom_{\mathcal{C}}(X, Y)$ by the equivalence relation generated by the above simplicial homotopy relation. This set is of course identical to the set $\pi_0(S(X, Y))$.

Definition 2.3.10 *Given a model category $(\mathcal{C}, W, C, F)$ and*

$$\Delta^\bullet : \Delta \to \mathcal{C}$$

a cosimplicial object of $\mathcal{C}$, we say that $\Delta^\bullet$ is compatible *with the model category structure if the following axiom holds*

- **SM7'** *For any cofibration $i : X \to Y$ and any cofibration $j : K \to L \in \Delta^{op}\mathcal{S}et$ the obvious morphism*

  $$X \times |L| \amalg_{X \times |K|} Y \times |K| \to Y \times |L|$$

  is a cofibration which is moreover trivial if either i or j is.

This axiom immediately implies the following property[9] that for any cofibration $i : X \to Y$ and any fibration $E \to B$, the obvious map of simplicial sets

$$S(Y, E) \to S(X, E) \times_{S(X,B)} S(Y, B)$$

[9] See Axiom **SM7** of simplicial model categories [30]

is a Kan fibration which is moreover trivial if either i or p is.

In a model category, call *cofibrant* an object X such that the canonical morphism $\emptyset \to X$ from the initial object to X is a cofibration, and *fibrant* an object Y such that the canonical morphism $Y \to *$ from Y to the final object is a fibration.

Then Quillen proves the following principle of the *homotopical algebra* :

Theorem 2.3.11 *Given a simplicial model category $(\mathcal{C}, \Delta^\bullet, W, C, F)$ with associated homotopy category $\mathcal{H}$ then for pair (X, Y) of a cofibrant object X and a fibrant object Y the natural map $Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{H}}(X, Y)$ induces a bijection*

$$\pi(X, Y) \cong Hom_{\mathcal{H}}(X, Y)$$

To compute the set $Hom_{\mathcal{H}}(X, Y)$ for general X and Y, the principle of homotopical algebra answers: choose a trivial fibration $X_c \to X$ with X_c cofibrant (such an X_c is called a *cofibrant resolution* of X), then choose a trivial cofibration $Y \to Y_f$ with Y_f fibrant (such a Y_f is called a fibrant resolution of Y) and then observe the following sequences of bijections

$$Hom_{\mathcal{H}}(X, Y) \cong Hom_{\mathcal{H}}(X_c, Y) \cong Hom_{\mathcal{H}}(X_c, Y_f) \cong \pi(X_c, Y_f)$$

The first two bijections being completely formal, the last one being a particular case of the previous theorem.

2.4 The *simplicial* homotopy category of sheaves

Definition 2.4.1 *[13, 14] A morphism in $\Delta^{op}Shv(\mathcal{V}_\tau)$ is called a* τ-simplicial weak equivalence *(or for short if no confusion can arise a simplicial weak equivalence) if and only if each of its fibers at τ-points are weak equivalences of simplicial sets.*

We denote by W_s^τ (or simply W_s) the class of simplicial weak equivalences.

It follows from [14] that endowed with the class W_s of simplicial weak equivalences, the class C of monomorphisms as cofibrations, the class F_s of morphisms with the RLP with respect to morphisms in $C \cap W_s$ as fibrations and the standard *cosimplicial simplicial sheaf*

$$\Delta^\bullet : \Delta \to \Delta^{op}Shv(\mathcal{V}_\tau)$$

that the category $\Delta^{op}Shv(\mathcal{V}_\tau)$ is simplicial model category.

Let us denote by $\mathcal{H}_s^\tau(\mathcal{V}_\tau)$, the associated homotopy category. That category is called the *simplicial homotopy category of sheaves in the τ-topology.*

The associated sheaf functors induced obvious functors

$$\mathcal{H}_s^\tau(\mathcal{V}_{Zar}) \to \mathcal{H}_s^\tau(\mathcal{V}_{Nis}) \to \mathcal{H}_s^\tau(\mathcal{V}_{\text{ét}})$$

between these categories. Each of these functors admits a right adjoint which is induced by the forgetful functors (in the derived sense).

For any simplicial sheaf $\mathcal{Y}$ the functor

$$\Delta^{op}Shv(\mathcal{V}_\tau) \to \Delta^{op}Shv(\mathcal{V}_\tau), \mathcal{X} \mapsto \mathcal{X} \times \mathcal{Y}$$

preserves simplicial weak equivalences and thus induces a functor still denoted

$$\mathcal{H}_s(\mathcal{V}_\tau) \to \mathcal{H}_s(\mathcal{V}_\tau), \mathcal{X} \mapsto \mathcal{X} \times \mathcal{Y}$$

It has as right adjoint the functor

$$R\underline{Hom}(\mathcal{Y}, -) : \mathcal{Z} \mapsto \underline{Hom}(\mathcal{Y}, \mathcal{Z}_f)$$

(where $\mathcal{Z} \mapsto \mathcal{Z}_f$ denotes a chosen functorial fibrant resolution).

The pointed homotopy category We say that a morphism of pointed simplicial sheaves is a *pointed simplicial weak equivalence* if its underlying morphism of (unpointed) simplicial sheaves is a simplicial weak equivalence. We let W_s denote as well the class of pointed simplicial weak equivalences (in the category $\Delta^{op}Shv_\bullet(\mathcal{V}_\tau)$ of pointed simplicial sheaves). Endowed with the notions of pointed simplicial weak equivalences and pointed monomorphisms as cofibrations, the category $\Delta^{op}Shv_\bullet(\mathcal{V}_\tau)$ becomes a model category, whose homotopy category is denoted by $\mathcal{H}_{s,\bullet}(\mathcal{V}_\tau)$.

For any simplicial sheaf $\mathcal{X} \in \Delta^{op}Shv(\mathcal{V}_\tau)$ we denote $\mathcal{X}_+$ the pointed simplicial sheaf obtained by adding a base point to $\mathcal{X}$. The functor

$$\Delta^{op}Shv(\mathcal{V}_\tau) \to \Delta^{op}Shv_\bullet(\mathcal{V}_\tau), \mathcal{X} \to \mathcal{X}_+$$

so obtained is left adjoint to the forgetful functor

$$\Delta^{op}Shv_\bullet(\mathcal{V}_\tau) \to \Delta^{op}Shv(\mathcal{V}_\tau)$$

For any pointed simplicial sheaf $\mathcal{Y}$ the functor

$$\Delta^{op}Shv_{\bullet}(\mathcal{V}_{\tau}) \to \Delta^{op}Shv_{\bullet}(\mathcal{V}_{\tau}), \mathcal{X} \mapsto \mathcal{X} \wedge \mathcal{Y}$$

preserves pointed simplicial weak equivalences and thus induces a functor still denoted

$$\mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}), \mathcal{X} \mapsto \mathcal{X} \wedge \mathcal{Y}$$

It has as right adjoint the functor

$$R\underline{Hom}_{\bullet}(\mathcal{Y}, -) : \mathcal{Z} \mapsto \underline{Hom}_{\bullet}(\mathcal{Y}, \mathbb{Z}_f)$$

(where $\mathbb{Z} \mapsto \mathbb{Z}_f$ denotes a chosen functorial fibrant resolution).

Example 2.4.2 When $\mathcal{Y}$ is the simplicial circle $S^1 := \Delta^1/\partial\Delta^1$ the functor

$$\mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}), \mathcal{X} \mapsto \mathcal{X} \wedge S^1$$

is denoted

$$\Sigma : \mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}), \mathcal{X} \mapsto \Sigma(\mathcal{X})$$

and called the (simplicial) suspension. Its right adjoint is denoted

$$\Omega : \mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{\tau}), \mathcal{Z} \mapsto \Omega(\mathcal{Z})$$

and called the *loop space functor.*

The B.G.-property in the Nisnevich topology Here we recall the notion of simplicial presheaves on $\mathcal{V}$ with the B.G. property [25]. This notion is directly inspired from the corresponding notion in the Zariski topology introduced and studied by Brown and Gersten in [8].

Definition 2.4.3 *[25] Let $\mathcal{X} \in \Delta^{op}Preshv(\mathcal{V})$ be a simplicial presheaf. One says that $\mathcal{X}$ has the* B.G. property *if and only if for any distinguished square as in 2.1.6 the commutative square (of simplicial sets)*

$$\begin{array}{ccc} \mathcal{X}(X) & \to & \mathcal{X}(V) \\ \downarrow & & \downarrow \\ \mathcal{X}(U) & \to & \mathcal{X}(W) \end{array}$$

is homotopy cartesian.

Example 2.4.4 Any simplicial sheaf in the Nisnevich topology $\mathcal{X}$ which is fibrant for the corresponding simplicial model category structure has the B.G.-property : this is Remark 3.1.15 of [25].

Remark 2.4.5 Of course this notion is just the analogue for sheaves in the Nisnevich topology of the property introduced in [8] in the Zariski topology. The following theorem is proven in [25, Lemma 3.1.18] and is the analogue of the main theorem of [8].

Theorem 2.4.6 *[25] Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial presheaves. Assume that $\mathcal{X}$ and $\mathcal{Y}$ have the B.G.-property and that $a_{Nis}(f) : a_{Nis}(\mathcal{X}) \to a_{Nis}(\mathcal{Y})$ is a simplicial weak equivalence (in the Nisnevich topology). Then for any $U \in \mathcal{V}$ the map of simplicial sets :*

$$\mathcal{X}(U) \to \mathcal{Y}(U)$$

is a simplicial weak equivalence.

Corollary 2.4.7 *[25] Let $\mathcal{X}$ be a simplicial presheaf with the B.G. property. Then for any $U \in \mathcal{V}$ the obvious map*

$$\pi_0(\mathcal{X}(U)) \to Hom_{\mathcal{H}_s^{Nis}(k)}(U, a_{Nis}(\mathcal{X}))$$

is a bijection. If moreover $\mathcal{X}$ is a pointed simplicial presheaf with the B.G. property, then for any integer $n \geq 0$ and any $U \in \mathcal{V}$ the obvious map

$$\pi_n(\mathcal{X}(U)) \to Hom_{\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})}((U_+) \wedge S^n, a_{Nis}(\mathcal{X}))$$

is a bijection.

Indeed choose a fibrant resolution $a_{Nis}(\mathcal{X})_f$ of $a_{Nis}(\mathcal{X})$ and apply the principle of homotopical algebra to compute $Hom_{\mathcal{H}_s^{Nis}(k)}(U, a_{Nis}(\mathcal{X}))$ as

$$\pi_0(a_{Nis}(\mathcal{X})_f(U))$$

Then conclude by theorem 2.4.6 taking into account that $a_{Nis}(\mathcal{X})_f$ also has the B.G. property and that $\mathcal{X} \to a_{Nis}(\mathcal{X})_f$ induces a simplicial weak equivalence $a_{Nis}(\mathcal{X}) \to a_{Nis}(\mathcal{X})_f$.

Example 2.4.8 Our fundamental example is the following. Let $X \mapsto K(X)$ the presheaf of pointed simplicial sets defined in [36] such that

$$\pi_n(K(X)) = K_n^Q(X)$$

is Quillen n-th higher K-group. By the above corollary, we see that for any integer $n \geq 0$ and any $X \in \mathcal{V}$ the obvious map

$$K_n^Q(X) \to Hom_{\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})}((X_+) \wedge S^n, a_{Nis}(K))$$

is a bijection. We observe that the analogue statement holds in Zariski topology but not in the étale topology (because algebraic K-theory has not, in general, descent in the étale topology).

3 Unstable $\mathbb{A}^1$-homotopy theory

From now on, unless otherwise stated, we will always work in the Nisnevich topology.

3.1 The $\mathbb{A}^1$-homotopy category [25]

$\mathbb{A}^1$-local objects and $\mathbb{A}^1$-weak equivalences

Definition 3.1.1 *[25] 1) An object $\mathcal{Z} \in \Delta^{op}Shv(\mathcal{V}_{Nis})$ is called* $\mathbb{A}^1$-local *if and only if for any $\mathcal{X} \in \Delta^{op}Shv(\mathcal{V}_{Nis})$, the projection $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ induces a bijection :*

$$Hom_{\mathcal{H}_s(k)}(\mathcal{X}, \mathcal{Z}) \to Hom_{\mathcal{H}_s(k)}(\mathcal{X} \times \mathbb{A}^1, \mathcal{Z})$$

2) A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\Delta^{op}Shv(\mathcal{V}_{Nis})$ is called an $\mathbb{A}^1$-weak equivalence *if and only if for any $\mathbb{A}^1$-local $\mathcal{Z}$, the map :*

$$Hom_{\mathcal{H}_s(k)}(\mathcal{Y}, \mathcal{Z}) \to Hom_{\mathcal{H}_s(k)}(\mathcal{X}, \mathcal{Z})$$

is bijective.

Together with V. Voevodsky, we proved in [25] that endowed with the class $W_{\mathbb{A}^1}$ of $\mathbb{A}^1$-weak equivalences, the class C of monomorphisms as cofibrations and the class $F_{\mathbb{A}^1}$ of $\mathbb{A}^1$-fibrations[10] the category $\Delta^{op}Shv(\mathcal{V}_{Nis})$ becomes a model category. The associated homotopy category obtained from $\Delta^{op}Shv(\mathcal{V}_{Nis})$ by inverting the $\mathbb{A}^1$-weak equivalences is denoted $\mathcal{H}(k)$ and called *the homotopy category of smooth k-schemes.*

Remark 3.1.2 Any simplicial weak equivalence is of course an $\mathbb{A}^1$-weak equivalence. Also, by definition the projection

$$\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$$

is an $\mathbb{A}^1$-weak equivalence (but not a simplicial weak equivalence).

A simplicially fibrant simplicial sheaf $\mathcal{X}$ is $\mathbb{A}^1$-fibrant if and only if it is $\mathbb{A}^1$-local.

Remark 3.1.3 There are two different cosimplicial objects which are compatible with the model category $(\Delta^{op}Shv(\mathcal{V}_{Nis}), W_{\mathbb{A}^1}, C, F_{\mathbb{A}^1})$. The first one is of course the standard cosimplicial simplicial simplex

$$\underline{n} \mapsto \Delta^n$$

[10] as opposed to simplicial fibrations

The second one is the following *algebraic standard cosimplicial simplicial simplex*

$$\underline{n} \mapsto \Delta^n_{\mathbb{A}^1} := Spec(k[T_0, \ldots, T_n]/\Sigma T_i = 1)$$

(observe that for any n, $\Delta^n \cong \mathbb{A}^n$, but not functorialy). In particular, given an $\mathbb{A}^1$-fibrant simplicial sheaf $\mathcal{Y}$ and a simplicial sheaf $\mathcal{X}$ (which is always cofibrant), on the set

$$Hom_{\Delta^{op}Shv(\mathcal{V}_{Nis})}(\mathcal{X}, \mathcal{Y})$$

the simplicial homotopy equivalence relation coincides with the $\mathbb{A}^1$-homotopy relation and the quotient set is the set of morphisms

$$Hom_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y})$$

Remark 3.1.4 Let's denote by $\mathcal{H}_{s,\mathbb{A}^1}(\mathcal{V}_{Nis}) \subset \mathcal{H}_s(\mathcal{V}_{Nis})$ the full subcategory consisting of $\mathbb{A}^1$-local simplicial sheaves. The inclusion $\mathcal{H}_{s,\mathbb{A}^1}(k) \subset \mathcal{H}_s(\mathcal{V}_{Nis})$ admits[11] a left adjoint which we denote $L_{\mathbb{A}^1}(-) : \mathcal{H}_s(\mathcal{V}_{Nis}) \to \mathcal{H}_{s,\mathbb{A}^1}(k)$, and which is called the $\mathbb{A}^1$-localization functor. This functor sends by definition $\mathbb{A}^1$-weak equivalences to isomorphisms, and thus induces a functor $\mathcal{H}(k) \to \mathcal{H}_{s,\mathbb{A}^1}(\mathcal{V}_{Nis})$ which is an equivalence of categories.

In particular, if $\mathcal{X}$ is a simplicial sheaf and $\mathcal{Y}$ an $\mathbb{A}^1$-local simplicial sheaf the canonical map

$$Hom_{\mathcal{H}_s(\mathcal{V}_{Nis})}(\mathcal{X}, \mathcal{Y}) \to Hom_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y})$$

is a bijection.

Example 3.1.5 The functor $\Delta^{op}\mathcal{S}et \to \Delta^{op}Shv(\mathcal{V}_{Nis})$ obviously sends a weak equivalence of simplicial sets to a simplicial weak equivalence and thus also to an $\mathbb{A}^1$-weak equivalence. We thus get a canonical functor

$$\mathcal{H}_s\mathcal{H}(k)$$

Given a complex embedding $\sigma : k \to \mathbb{C}$ one gets a functor

$$\rho_\sigma : \mathcal{H}(k) \to \mathcal{H}$$

induced by taking the "realization of the simplicial topological space of complex points". The composition

$$\mathcal{H}_s \to \mathcal{H}$$

[11] See [25]

is of course the equivalence of categories considered above.

If $\sigma : k \to \mathbb{R}$ is a real embedding we have two induced functors (which are both left inverse to $\mathcal{H}_s \to \mathcal{H}(k)$)

$$\mathcal{H}(k) \to \mathcal{H}$$

The first one is the previous one and the second one is induced by taking the "realization of the simplicial topological space of real points". Beware that these are distinct (not isomorphic functors). For instance the topological space of complex points of the affine line minus 0 is $\mathbb{C}^\times \cong S^1$ (in $\mathcal{H}$). On the other hand the topological space of real points of the affine line minus 0 is $\mathbb{R}^\times \cong S^0$.

Example 3.1.6 Let $\mathbb{P}^\infty$ be the colimit of the projective spaces $\mathbb{P}^n$, $\mathbb{P}^n$ being included (in the obvious way) into $\mathbb{P}^{n+1}$. Then it is proven [25, 19] that the Picard group of X is naturally isomorphic to $Hom_{\mathcal{H}(k)}(X, \mathbb{P}^\infty)$.

Example 3.1.7 Recall from example 2.4.8 that the pointed simplicial sheaf $\underline{K} := a_{Nis}(K(-))$ represents in $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$ Quillen's higher K-groups K_n functor in the sense that for any smooth k-variety X one has a canonical isomorphism

$$K_n^Q(X) \cong Hom_{\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})}((X_+) \wedge S^n, \underline{K})$$

By Quillen's theorem [31, 36], the groups homomorphisms

$$K_n^Q(X) \to K_n^Q(X \times \mathbb{A}^1)$$

are isomorphisms. This can be shown to imply (because $\underline{K}$ is a group object in $\mathcal{H}_s(\mathcal{V}_{Nis})$) that the simplicial sheaf $\underline{K}$ is $\mathbb{A}^1$-local. Thus one has a canonical isomorphism, for any $X \in \mathcal{V}$

$$K_0(X) \cong Hom_{\mathcal{H}(k)}(X, \underline{K})$$

In [25] it is proven that $\underline{K}$ is isomorphic in $\mathcal{H}(k)$ to both $\mathbb{Z} \times \mathbb{G}r$ (where $\mathbb{G}r$ means the infinite algebraic grassmanian) and $\mathbb{Z} \times B\mathbb{G}l$ (where $B\mathbb{G}l$ is the classifying simplicial sheaf of the group sheaf $\mathbb{G}l$ (infinite general linear group).

In general it is very difficult to compute $Hom_{\mathcal{H}(k)}(X, Y)$ when X and Y are of finite type, for instance if X and Y are smooth k-varieties. Surprisingly however, in the few cases where this set can be computed and X is affine, this set coincides with the set of naive $\mathbb{A}^1$-homotopy classes of morphisms from X to Y. That might be always the case. The first non trivial case of that guess is when $X = Spec(k)$. Our conjecture now takes the form:

If X is a smooth k-variety, the natural map

$$X(k) \to Hom_{\mathcal{H}(k)}(Spec(k), X)$$

is surjective and identifies the right hand side with the quotient of the set $X(k)$ of k-rational points by the equivalence relation generated by:

$$(x \sim y) \Leftrightarrow \exists h : \mathbb{A}^1 \to X/h(0) = x \text{ and } h(1) = y$$

This is true[12] when X is $\mathbb{A}^1$-rigid in the sense that $\forall U \in Sm(k)$ the map $Hom_{Sm(k)}(U, X) \to Hom_{Sm(k)}(U \times \mathbb{A}^1, X)$ is bijective. For instance for $\mathbb{G}_m$, for curves of genus > 0, abelian varieties, products of those, subschemes of those, etc...

Remark 3.1.8 The surjectivity always holds by [25]: indeed for any $\mathcal{X} \in \Delta^{op}Shv(\mathcal{V}_{Nis})$, the map

$$\mathcal{X}_0(k) \to Hom_{\mathcal{H}(k)}(Spec(k), \mathcal{X})$$

is surjective.

In particular, the notion of having a k-rational point is invariant under $\mathbb{A}^1$-weak equivalences. This means that if $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic in $\mathcal{H}$ then $\mathcal{X}(k) \neq \emptyset \Leftrightarrow \mathcal{Y}(k) \neq \emptyset$.

Pointed $\mathbb{A}^1$-homotopy category A morphism in $\Delta^{op}Shv_\bullet(\mathcal{V}_{Nis})$ is said to be a pointed $\mathbb{A}^1$-weak equivalence if it is an $\mathbb{A}^1$-weak equivalence after forgetting the base point. Again by [25], endowed with the notions of pointed $\mathbb{A}^1$-weak equivalences and pointed monomorphisms as cofibrations, the category $\Delta^{op}Shv_\bullet(\mathcal{V}_{Nis})$ becomes a model category. The associated homotopy category is denoted $\mathcal{H}_\bullet(k)$ and called the pointed homotopy category of smooth k-schemes.

The forgetful functor $\Delta^{op}Shv_\bullet(\mathcal{V}_{Nis}) \to \Delta^{op}Shv(\mathcal{V}_{Nis})$ and its left adjoint $\Delta^{op}Shv(\mathcal{V}_{Nis}) \to \Delta^{op}Shv_\bullet(\mathcal{V}_{Nis}), \mathcal{X} \mapsto \mathcal{X}_+$ both preserves $\mathbb{A}^1$-weak

[12] *cf* [25]

equivalences and the induced functors on homotopy category are still adjoint to each other.

For any fixed $\mathcal{Y} \in \Delta^{op}Shv_\bullet(\mathcal{V}_{Nis})$, the smash-product by $\mathcal{Y}$ preserves $\mathbb{A}^1$-weak equivalences, inducing a functor

$$\mathcal{H}_\bullet(k) \to \mathcal{H}_\bullet(k), \mathcal{X} \mapsto \mathcal{X} \wedge \mathcal{Y}$$

This functor admits a right adjoint which we denote by

$$R_{\mathbb{A}^1}\underline{Hom}_\bullet(\mathcal{Y}, -) : \mathcal{H}_\bullet(k) \to \mathcal{H}_\bullet(k)$$

Example 3.1.9 The suspension functor $\Sigma : \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$, $\mathcal{X} \mapsto \mathcal{X} \wedge S^1$ preserves $\mathbb{A}^1$-weak equivalences and induces a functor still called the suspension functor and denoted by

$$\Sigma : \mathcal{H}_\bullet(k) \to \mathcal{H}_\bullet(k)$$

Its right adjoint

$$\Omega : \mathcal{H}_\bullet(k) \to \mathcal{H}_\bullet(k)$$

is induced by $\Omega : \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$ which is easily seen to preserve $\mathbb{A}^1$-local objects.

Example 3.1.10 The cocartesian square, corresponding to the open covering of $\mathbb{P}^1$ by the two standard affine line $\mathbb{A}^1$ and $\mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{G}_m & \to & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \to & \mathbb{P}^1 \end{array}$$

defines an isomorphism in $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$

$$T := \mathbb{A}^1/\mathbb{G}_m \cong \mathbb{P}^1/\mathbb{A}^1$$

But in $\mathcal{H}_\bullet(k)$, T is clearly weakly equivalent to $\Sigma(\mathbb{G}_m)$ (because $\mathbb{A}^1 \to *$ is an $\mathbb{A}^1$-weak equivalence) and $\mathbb{P}^1 \to \mathbb{P}^1/\mathbb{A}^1$ is clearly an $\mathbb{A}^1$-weak equivalence as well. Thus we get canonical $\mathcal{H}_\bullet(k)$-isomorphisms

$$\mathbb{P}^1 \cong T \cong S^1 \wedge \mathbb{G}_m$$

Example 3.1.11 With this notions introduced, the example 2.4.8 easily implies (as in 3.1.7) that for any smooth k-variety X one has canonical isomorphisms

$$K_n^Q(X) \cong Hom_{\mathcal{H}_\bullet(k)}((X_+) \wedge S^n, \underline{K})$$

Thus (*cf* 3.1.7) one has natural isomorphisms

$$K_n^Q(X) \cong Hom_{\mathcal{H}_\bullet(k)}((X_+) \wedge S^n, \mathbb{Z} \times B\mathbb{G}l) \cong Hom_{\mathcal{H}_\bullet(k)}((X_+) \wedge S^n, \mathbb{Z} \times \mathbb{G}r)$$

It was observed in [19, 39] that Quillen's computation of the K-groups of $X \times \mathbb{P}^1$ implies that the adjoint $\tilde{\beta}$ of the Bott morphism

$$(\mathbb{Z} \times \mathbb{G}r) \wedge \mathbb{P}^1 \to \mathbb{Z} \times \mathbb{G}r$$

which classifies the exterior tensor product by the virtual vector bundle of rank zero $1 - [\eta]$ (where η is the canonical line bundle on $\mathbb{P}^1$), is an $\mathbb{A}^1$-weak equivalence

$$\mathbb{Z} \times \mathbb{G}r \overset{\cong}{\to} R_{\mathbb{A}^1} \underline{Hom}_\bullet(\mathbb{P}^1, \mathbb{Z} \times \mathbb{G}r)$$

(Bott periodicity.)

3.2 $\mathbb{A}^1$-localization of connected simplicial sheaves

In what follows, we consider the affine line $\mathbb{A}^1$ as pointed by 0. For any $\mathcal{X} \in \Delta^{op}Shv_\bullet(\mathcal{V}_{Nis})$, let's denote by $ev_1 : \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}) \to \mathcal{X}$ the evaluation at 1

Lemma 3.2.1 *Let $\mathcal{X}$ be a simplicially fibrant simplicial sheaf. The following conditions are equivalent :*

(i) *$\mathcal{X}$ is $\mathbb{A}^1$-local ;*

(ii) *the morphism of simplicial sheaves*

$$\mathcal{X} \to \underline{Hom}(\mathbb{A}^1, \mathcal{X})$$

adjoint to the projection $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ is a simplicial weak equivalence ;

(iii) *the morphism of simplicial sheaves*

$$ev_0 : \underline{Hom}(\mathbb{A}^1, \mathcal{X}) \to \mathcal{X}$$

"evaluation at 0" is a simplicial weak equivalence.

Assume moreover that $\mathcal{X}$ is 0-connected[13] *and pointed. Then the above conditions are equivalent to the following ones :*

(iv) *the functional object $\underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X})$ is weakly contractible*[14] *;*

(v) *for any smooth k-scheme U, any integer $n \in \mathbb{N}$, the map*

$$Hom_{\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})}((U_+)\wedge S^n, \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X})) \to Hom_{\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})}((U_+)\wedge S^n, \mathcal{X})$$

is trivial (as a map of pointed sets).

The first three conditions are easily seen to be equivalent. But $\mathcal{X}$ being simplicially fibrant pointed and 0-connected, for the evaluation at 0

$$\underline{Hom}(\mathbb{A}^1, \mathcal{X}) \to \mathcal{X}$$

to a simplicial weak equivalence it is necessary and sufficient that the fiber be weakly contractible. This proves the equivalence of the first 4 conditions because that fibre is the pointed function object $\underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X})$.

The implication **(iv)** $\Rightarrow$ **(v)** is trivial. Let's prove the converse. Assume (**v**). By lemma 3.2.2 below, it is sufficient to show that for any smooth k-scheme U, any integer $n \geq 0$, any $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$-morphism $f : \mathbb{A}^1 \wedge (U_+) \wedge S^n \to \mathcal{X}$ is trivial.

But for any morphism of pointed simplicial sheaves

$$f : \mathcal{Y} \wedge \mathbb{A}^1 \to \mathcal{X}$$

let $\tilde{f} : \mathcal{Y} \wedge \mathbb{A}^1 \to \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X})$ be the adjoint of the composition

$$\mathcal{Y} \wedge (\mathbb{A}^1 \wedge \mathbb{A}^1) \stackrel{Id_{\mathcal{Y}} \wedge \mu}{\to} \mathcal{Y} \wedge \mathbb{A}^1 \stackrel{f}{\to} \mathcal{X}$$

where $\mu : \mathbb{A}^1 \wedge \mathbb{A}^1 \to \mathbb{A}^1$ denote the product of the ringed object $\mathbb{A}^1$. Then the following diagram is commutative (in $\Delta^{op}Shv_\bullet(\mathcal{V}_{Nis})$) :

$$\begin{array}{ccc} \mathcal{Y} \wedge \mathbb{A}^1 & \stackrel{\tilde{f}}{\to} & \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}) \\ || & & \downarrow ev_1 \\ \mathcal{Y} \wedge \mathbb{A}^1 & \stackrel{f}{\to} & \mathcal{X} \end{array}$$

This easily allows one to finish the proof of the lemma.

[13] which means each of its fibers are 0-connected, or equivalently, that the coequaliser of d_0 and d_1, $\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$ is the point

[14] i.e. the map to the point is a simplicial weak equivalence

Lemma 3.2.2 *A pointed simplicial sheaf $\mathcal{X}$ is weakly contractible if and only if for any $n \geq 0$, any $U \in Sm(k)$, any $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$-morphism $f : S^n \wedge (U_+) \to \mathcal{X}$ is trivial.*

Indeed, we may assume $\mathcal{X}$ is simplicially fibrant. If it is weakly contractible the conclusion is clear. Conversely, under the assumption of the lemma, the pointed fibrant simplicial set $\mathcal{X}(U)$ has all its homotopy groups trivial (at the base point) and is thus contractible. This clearly implies that $\mathcal{X}$ is weakly contractible.

Construction of the $\mathbb{A}^1$-localization of a pointed connected simplicial sheaf Let $\mathcal{Y} \mapsto \mathcal{Y}_f$ be a functorial simplicial fibrant resolution. Let $\mathcal{X}$ be a pointed simplicial sheaf. We let $L^{(1)}(\mathcal{X})$ be the cone of the obvious morphism

$$ev_1 : \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f) \to \mathcal{X}_f$$

Let $L^{(1)}_f(\mathcal{X})$ be $L^{(1)}(\mathcal{X})_f$. We let $\mathcal{X} \to L^{(1)}_f(\mathcal{X})$ be the obvious morphism of pointed simplicial sheaves. Define by induction on $n \geq 0$, $L^{(n)} := L^{(1)}_f \circ L^{(n-1)}_f$. We have natural morphisms, for any $\mathcal{X}$, $L^{(n-1)}_f(\mathcal{X}) \to L^{(n)}_f(\mathcal{X})$ and we set $L^\infty(\mathcal{X}) = colim_{n \in \mathbb{N}} L^{(n)}_f(\mathcal{X})$.

Proposition 3.2.3 *Let $\mathcal{X}$ be a pointed connected simplicial sheaf. Then the simplicial sheaf $L^\infty(\mathcal{X})$ is $\mathbb{A}^1$-local and the morphism*

$$\mathcal{X} \to L^\infty(\mathcal{X})$$

is an $\mathbb{A}^1$-weak equivalence.

The product μ of $\mathbb{A}^1$ considered above induces a morphism

$$\underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f) \wedge \mathbb{A}^1 \to \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f)$$

which is left inverse to the section at 1

$$\underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f) \to \underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f) \wedge \mathbb{A}^1$$

Thus $\underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f)$ being a retract of $\underline{Hom}_\bullet(\mathbb{A}^1, \mathcal{X}_f) \wedge \mathbb{A}^1$ is $\mathbb{A}^1$-weakly contractible (i.e. isomorphic to $*$ in $\mathcal{H}_\bullet(k)$). So we conclude that each morphism $\mathcal{X} \to L^{(1)}(\mathcal{X})$ is an $\mathbb{A}^1$-weak equivalence, and thus the composition $\mathcal{X} \to L^\infty(\mathcal{X})$ as well.

The fact that $L^\infty(\mathcal{X})$ is $\mathbb{A}^1$-local follows then formally from lemma 3.2.1 and the following lemma.

Lemma 3.2.4 *Let $\mathcal{X}^0 \to \mathcal{X}^1 \to \cdots \to \mathcal{X}^m \to \dots$ be a direct system of pointed simplicial sheaves. Then for any integer n and any $U \in Sm(k)$ the map*

$$colim_m \, Hom_{\mathcal{H}_s(\mathcal{V}_{Nis})}(S^n \wedge (U_+), \mathcal{X}^n) \to Hom_{\mathcal{H}_s(\mathcal{V}_{Nis})}(S^n \wedge (U_+), colim_m \mathcal{X}^m)$$

is a bijection.

Connectivity and $\mathbb{A}^1$-localization Let $\mathcal{X}$ be a simplicial sheaf. The colimit of the diagram (in $Shv(\mathcal{V}_{Nis})$) :

$$\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$$

is denoted by $\pi_0(\mathcal{X})$. It is the sheaf associated to the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$. We say that $\mathcal{X}$ is 0-*connected* if and only if $\pi_0(\mathcal{X})$ is the trivial sheaf.

We first deduce the following corollary from the description we gave above of the $\mathbb{A}^1$-localization of a 0-connected, pointed simplicial sheaf.

Corollary 3.2.5 *(see [25]) Let $\mathcal{X}$ be a 0-connected simplicial sheaf. Then its $\mathbb{A}^1$-localization is 0-connected.*

Indeed, $\mathcal{X}$ can be pointed (indeed the map $\mathcal{X}_0(k) \to *$ is surjective because $F \mapsto F(k)$ is a fiber functor in the Nisnevich topology). But then clearly the simplicial sheaf $L^\infty(\mathcal{X})$ constructed above is still 0-connected.

Assume moreover that $\mathcal{X}$ is a pointed simplicial sheaf. Then for any $n \geq 1$ one denote by $\pi_n(\mathcal{X}, x)$ the sheaf of groups (abelian groups if $n \geq 2$) associated to the presheaf $U \mapsto \pi_n(\mathcal{X}(U), x)$.

Definition 3.2.6 *For any $n \geq 1$ we will say that $\mathcal{X}$ is n-connected if it is 0-connected and if for any $i \in \{1, \dots, n\}$ the sheaf $\pi_n(\mathcal{X}, x)$ is trivial.*

Example 3.2.7 Let $\mathcal{A}b(\mathcal{V}_{Nis})$ be the (abelian) category of sheaves of abelian groups on $\mathcal{V}$. For any simplicial sheaf $\mathcal{X}$ one defines the chain complex[15] $C_*(\mathcal{X})$ as follows. Let $\mathbb{Z}[\mathcal{X}]$ be the free simplicial sheaf of abelian groups generated by $\mathcal{X}$. Then $C_*(\mathcal{X})$ is the normalized chain complex[16] in $\mathcal{A}b(\mathcal{V}_{Nis})$ associated with $\mathbb{Z}[\mathcal{X}]$.

[15] "chain complex" means the differential has degree -1, "cochain complex" means the differential has degree $+1$

[16] See [16] for the definition of the normalized chain complex

Let $M \in \mathcal{A}b(\mathcal{V}_{Nis})$ be a sheaf of abelian groups and let $n \geq 0$ be an integer. Let $M[n]$ denotes the chain complex in $\mathcal{A}b(\mathcal{V}_{Nis})$ concentrated in degree n where it equals the sheaf M. The functor

$$\mathcal{X} \mapsto Hom_{C_*(\mathcal{A}b(\mathcal{V}_{Nis}))}(C_*(\mathcal{X}), M[n]) =: Z^n(\mathcal{X}; M)$$

is representable by a simplicial sheaf (of abelian groups) denoted $K(M, n)$ and called the Eilenberg-MacLane simplicial sheaf of type (M, n). This simplicial sheaf has the following homotopy sheaves

$$\pi_m(K(M,n), 0) = \begin{cases} 0 \text{ if } m \neq n \\ M \text{ if } m = n \end{cases}$$

and is thus $(n-1)$-connected.

Remark 3.2.8 One can show [14, 25] that for any $X \in \mathcal{V}$, any $M \in \mathcal{A}b(\mathcal{V}_{Nis})$ and any $n \geq 0$ the group

$$Hom_{\mathcal{H}_s(\mathcal{V}_{Nis})}(X, K(M, n))$$

is canonically isomorphic to

$$H^n_{Nis}(X; M)$$

Definition 3.2.9 *Let $n \geq 0$ be an integer. We say that $M \in \mathcal{A}b(\mathcal{V}_{Nis})$ is* n-strictly $\mathbb{A}^1$-invariant *if and only if for any smooth k-scheme X and any integer $i \in \{0, \ldots, n\}$ the obvious homomorphism*

$$H^i_{Nis}(X; M) \to H^i_{Nis}(X \times \mathbb{A}^1; M)$$

is an isomorphism.

Observe that for $n = 1$ this definition obviously extends to sheaves of non-commutative groups, and for $n = 0$ to sheaves of sets. Using the previous remark, one easily checks the following :

Lemma 3.2.10 *Let $n \geq 0$ be an integer. Then for any sheaf M (which is to be a sheaf of groups for $n = 1$ and of abelian groups for $n > 1$), the simplicial sheaf $K(M, n)$ is $\mathbb{A}^1$-local if and only if M is n-strictly $\mathbb{A}^1$-invariant.*

It seems reasonable to us to conjecture

Conjecture 1 *A 0-connected, pointed simplicial sheaf $\mathcal{X}$ is $\mathbb{A}^1$-local if and only if for any $n \geq 1$, the sheaf*

$$\pi_n(\mathcal{X}; x)$$

is n-strictly $\mathbb{A}^1$-invariant.

This conjecture easily implies[17], using an obvious adjunction argument, the following other conjecture (which should generalize Corollary 3.2.5) :

Conjecture 2 *The $\mathbb{A}^1$-localization of an n-connected simplicial sheaf $\mathcal{X}$ is still n-connected.*

Our main result in these notes will be to prove a "stable" analogue of that conjecture.

3.3 Thom spaces and homotopy purity

Thom spaces of closed immersions

Definition 3.3.1 *Let $i : X \to Y$ be a closed immersion of smooth k-varieties whose open complement is $U \subset Y$. The* Thom space $Th(i)$ *of i is the pointed sheaf of sets*

$$Y/U$$

Example 3.3.2 The usual example is that of vector bundles [25]. If ξ is a vector bundle over $X \in \mathcal{V}$ with total space $E(\xi)$, the Thom space of the zero section

$$s_0 : X \to E(\xi)$$

is also called the Thom space of ξ.

We observe that if ξ is the trivial bundle of rank n, $\mathcal{O}^n$, over X then $Th(\mathcal{O}^n)$ is canonically isomorphic to the smash-product

$$\mathbb{A}^n/(\mathbb{A}^n - \{0\}) \wedge (X_+)$$

and, more generally, if ξ and η are respectively vector bundles over $X \in \mathcal{V}$ and $Y \in \mathcal{V}$ the Tom space of their external Whitney sum $\xi \Box \eta$ (over $X \times Y$) is the smash-product

$$Th(\xi \Box \eta) \cong Th(\xi) \wedge Th(\eta)$$

[17] In fact, using the trick we will use in section 4.2 it might be the case that this two conjectures are equivalent

Example 3.3.3 For any closed point $x \in X \in \mathcal{V}$, with residue field L the Thom space of the closed immersion

$$x : Spec(L) \to X$$

is isomorphic (non canonically) to the sheaf $\mathbb{A}^n/(\mathbb{A}^n - \{0\}) \wedge (Spec(L)_+)$. This follows from Lemma 2.1.13.

The homotopy purity theorem [25] The next theorem generalizes much further the previous example. This is one of the most important property of our homotopy category $\mathcal{H}(k)$[18].

Theorem 3.3.4 *[25] For any closed immersion*

$$i : X \to Y$$

with normal vector bundle ν_i over X there is a canonical isomorphism in the pointed homotopy category $\mathcal{H}_\bullet(k)$ of the form

$$Th(i) \cong Th(\nu_i)$$

In standard differential topology, this lemma would follow from the existence of tubular neighborhoods. Here, instead, we have to use a deformation to the normal bundle process [25].

An application of the homotopy purity theorem

Definition 3.3.5 *Let $\mathcal{X}$ be a simplicial sheaf and $n \geq 0$ an integer.*

We say that $\mathcal{X}$ is weakly n-connected *if and only if for any irreducible smooth k-scheme X with field of fractions F, the simplicial set $\mathcal{X}(F)$ is n-connected.*

We recall that $\mathcal{X}(F)$ is the fiber[19] at the generic point of X.

Our application of Theorem 3.3.4 is the following:

Lemma 3.3.6 *Let $\mathcal{X}$ be an $\mathbb{A}^1$-local simplicial sheaf and $n \geq 0$ an integer. Then the following conditions are equivalent :*

[18] Together with the representability of algebraic K-theory we saw previously, it justifies our choice of the Nisnevich topology

[19] In the Nisnevich topology of course

(i) $\mathcal{X}$ is weakly n-connected ;

(ii) $\mathcal{X}$ is n-connected.

The implication $(ii) \Rightarrow (i)$ is clear. Let's prove the implication $(i) \Rightarrow (ii)$. We may of course assume $\mathcal{X}$ is pointed.

Assume first $n = 0$. We must prove that the sheaf $\pi_0(\mathcal{X})$ is the trivial sheaf if we assume that $\mathcal{X}(F)$ is connected for any finite type field extension $k \subset F$. For a given smooth k-scheme U we must then show that the (pointed) set $\pi_0(\mathcal{X})(U)$ is trivial. It is sufficient to show that for any morphism $U \to \pi_0(\mathcal{X})$ there is a covering $V \to U$ such that the composition $V \to U \to \pi_0(\mathcal{X})$ is trivial[20].

As the morphism of simplicial sheaves $\mathcal{X} \to \pi_0(\mathcal{X})$ is an epimorphism, there is a covering $V_\alpha \to U$, with the V_α irreducible smooth k-schemes, such that each composite $V_\alpha \to U \to \pi_0(\mathcal{X})$ lifts to a morphism $V_\alpha \to \mathcal{X}$.

It is thus sufficient to prove that for any irreducible smooth k-scheme V and any morphism $\phi : V \to \mathcal{X}$ the composition $V \to \mathcal{X}\pi_0(\mathcal{X})$ is trivial.

By assumption, $colim_{W\subset V}\pi_0(\mathcal{X}(W))$ is the trivial set, where W runs over the ordered set of non-empty open subsets in V. This implies that there is a dense open subset $W \subset V$ such that the composition $W \to V \to \mathcal{X}$ is simplicially homotopic to the trivial morphism. We may of course assume $\mathcal{X}$ to be simplicially fibrant. Using the right lifting property of the projection $\mathcal{X} \to *$ with respect to simplicially trivial cofibrations, we see that our previous morphism $\phi : V \to \mathcal{X}$ is simplicially homotopic to a morphism $\phi' : V \to \mathcal{X}$ whose restriction to W is trivial. Of course the composite $V \to \mathcal{X} \to \pi_0(\mathcal{X})$ using ϕ' is thus the same as the one using ϕ.

It thus remains to show that the morphism $V/W \to \mathcal{X}$ induced by ϕ' induces the trivial morphism $V/W \to \pi_0(\mathcal{X})$. This clearly follows from :

Lemma 3.3.7 *For any irreducible smooth k-scheme V and any dense open subset $W \subset V$, the $\mathbb{A}^1$-localization of V/U is 0-connected.*

Let F be the closed complement of W with the reduced induced structure. Because k is perfect, there is an increasing sequence of reduced closed subschemes :

$$\emptyset = F_{-1} \subset F_0 \subset \dots F_d = F$$

[20] Remember $\mathcal{X}$ is pointed so that to be trivial means to be the base point

such that each k-scheme $F_s - F_{s-1}$ is smooth and $dim(F_i) = i$. Then in the increasing sequence of pointed sheaves :

$$* = (V - F_d)/W \to (V - F_{d-1})/W \to \dots (V - F_{-1})/W = V/W$$

the homotopy cofibers[21] at each step are of the form $V - F_{s-1}/V - F_s$, but $F_s - F_{s-1}$ is a smooth closed subscheme of $V - F_{s-1}$ of strictly positive codimension.

To achieve the proof of Lemma 3.3.7, it thus suffices to check the following two lemmas :

Lemma 3.3.8 *Let $X \to Y$ be a closed immersion between two smooth k-schemes. Assume that the codimension of X in Y is everywhere strictly positive. Then $L_{\mathbb{A}^1}(Y/(Y-X))$ is 0-connected.*

That cofiber $Y/(Y-X)$ is $\mathbb{A}^1$-homotopy equivalent to the Thom space of the normal bundle of the immersion $X \to Y$ by Theorem 3.3.4. Using the next lemma, we can easily reduce to the case where the normal bundle is trivial. In which case the Thom space is $(\mathbb{A}^1/\mathbb{A}^1 - \{0\}) \wedge (X_+)$. But $(\mathbb{A}^1/\mathbb{A}^1 - \{0\})$ is isomorphic to the suspension of $\mathbb{A}^1 - \{0\}$ whose $\mathbb{A}^1$-localization is 0-connected by corollary 3.2.5.

Lemma 3.3.9 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial presheaves. Then the $\mathbb{A}^1$-localization of the cone of f is canonically isomorphic to the $\mathbb{A}^1$-localization of the cone of $L_{\mathbb{A}^1}(f) : L_{\mathbb{A}^1}(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{Y})$.*

We proceed now to the general case $n \geq 1$. From the previous case, we know that $\pi_0(\mathcal{X})$ is trivial, so that $\mathcal{X}$ can be assumed pointed and $\mathbb{A}^1$-fibrant. But now, $\Omega^n(\mathcal{X})$ is $\mathbb{A}^1$-fibrant (thus $\mathbb{A}^1$-local) and satisfies the hypothesis for $n = 0$. This case thus implies $\Omega^n_s(\mathcal{X})$ is 0-connected, and thus we get the result.$\square$

[21] i.e. the quotients sheaves

4 Stable $\mathbb{A}^1$-homotopy theory of S^1-spectra

4.1 Recollection on S^1-spectra

Recall that S^1 denote *the simplicial circle*, that is to say the pointed simplicial sheaf $\Delta^1/\partial\Delta^1$. Here, we set $S^n := (S^1)^{\wedge n}$ for any integer $n \geq 0$.

Definition 4.1.1 *An* S^1-spectrum E *over* k *is a collection* $\{E_n, \sigma_n\}_{n\in\mathbb{N}}$ *consisting, for each integer* $n \geq 0$, *of a pointed simplicial sheaf* E_n *and a morphism* $\sigma_n : E_n \wedge S^1 \to E_{n+1}$ *of pointed simplicial sheaves. Morphisms of* S^1*-spectra are collections of morphisms of pointed simplicial sheaves which satisfy the obvious conditions. We thus obtain the category* $Sp^{S^1}(k)$ *of* S^1*-spectra over* k.

Example 4.1.2 1) For any pointed simplicial sheaf $\mathcal{X}$, it suspension spectrum $\Sigma^\infty(\mathcal{X})$ has n-th term $\mathcal{X} \wedge S^n$ and identities as structure morphisms. This construction gives a functor

$$\Sigma^\infty : \Delta^{op}Shv_\bullet(\mathcal{V}_{Nis}) \to Sp^{S^1}(k)$$

2) Let E be an S^1-spectrum in the classical sense of Bousfield-Friedlander [7]. Then using the functor which to a set associates its constant sheaf on $\mathcal{V} = Sm(k)$ on gets a functor

$$Sp \to Sp^{S^1}(k)$$

Definition 4.1.3 *1) Let* E *be an* S^1*-spectrum. Let* $n \in \mathbb{Z}$ *be an integer. We define the* n*-th stable homotopy sheaf of* E *to be the sheaf of abelian groups*

$$\pi_n(E) := colim_{r>>0}\pi_{n+r}(E_r) \in \mathcal{A}b(\mathcal{V}_{Nis})$$

where the transition morphisms $\pi_{n+r}(E_r) \to \pi_{n+r+1}(E_{r+1})$ *are induced by the structure morphisms* σ_n.

2) A morphism of S^1*-spectra* $f : E \to F$ *is called a stable simplicial weak equivalence if it induces an isomorphism*

$$\pi_n(E) \cong \pi_n(F)$$

for all integer $n \in \mathbb{Z}$.

3) A morphism of S^1-spectra $f : E \to F$ is called a stable cofibration if and only if the morphisms

$$E_0 \to F_0$$

and for each $n \geq 0$

$$E_{n+1} \amalg_{E_n \wedge S^1} F_n \wedge S^1 \to F_{n+1}$$

are cofibrations (i.e. monomorphisms of pointed simplicial sheaves).

Using the results of [25] and standard techniques *cf* [7, 15], one can show that the category of S^1-spectra endowed with the notions of stable simplicial weak equivalences as weak equivalences and of stable cofibrations as cofibrations is a model category, which we will refer to as the *stable model category structure.* We denote by $\mathcal{SH}_s^{S^1}(k)$ the associated homotopy category. The set of morphism in that category between E and F will be denoted

$$[E, F]_s^{S^1}$$

The stable fibrations are not easy to describe (see [7]) but one has

Lemma 4.1.4 *An S^1-spectrum E is stably fibrant if and only if for each $n \geq 0$ E_n is a fibrant pointed simplicial sheaf and the adjoint to σ_n :*

$$\tilde{\sigma_n} : E_n \to \underline{Hom}_\bullet(S^1, E_{n+1}) = \Omega(E_{n+1})$$

is a weak equivalence (of pointed simplicial sheaves).

Example 4.1.5 The functor $Sp \to Sp^{S^1}(k)$ considered above sends a (classical) stable weak equivalence to a stable simplicial weak equivalence so that we get an induced functor $\mathcal{SH} \to \mathcal{SH}_s^{S^1}(k)$.

The functor

$$\Sigma^\infty : \begin{cases} \Delta^{op} Shv_\bullet(\mathcal{V}_{Nis}) \to Sp^{S^1}(k) \\ \mathcal{X} \mapsto \Sigma^\infty(\mathcal{X}) \end{cases}$$

clearly sends simplicial weak equivalences to stable simplicial weak equivalences and thus induces a functor :

$$\Sigma^\infty : \begin{cases} \mathcal{H}_\bullet(k) \to \mathcal{SH}_s^{S^1}(k) \\ \mathcal{X} \mapsto \Sigma^\infty(\mathcal{X}) \end{cases}$$

That functor admits as right adjoint the functor

$$\Omega^\infty : \begin{cases} \mathcal{SH}_s^{S^1}(k) \to \mathcal{H}_\bullet(k) \\ E \mapsto \Omega^\infty(E) \end{cases}$$

which to a simplicially fibrant S^1-spectrum E associates its infinite loop space

$$\Omega^\infty(E) := colim_{r\geq 0}\underline{Hom}_\bullet(S^r, E_r)$$

Recall from that the suspension functor $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}), \mathcal{X} \mapsto \mathcal{X}\wedge S^1$ has a right adjoint which we denoted by

$$\Omega^1 : \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}) \to \mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}), \mathcal{Y} \mapsto \Omega(\mathcal{Y})$$

Definition 4.1.6 *A S^1-spectrum E is said to be an* Ω-spectrum *if and only if for any $n \geq 0$ the adjoint $E_n \to \Omega^1(E_{n+1})$ to the structural morphism σ_n is a simplicial weak equivalence.*

Thus, in particular such a stable fibrant spectrum E is an Ω-spectrum, and any S^1-spectrum admits a stable trivial cofibration $E \to E_f$ to such a stably fibrant spectrum.

Corollary 4.1.7 *Let E be a fibrant S^1-spectrum. For any smooth k-scheme U the canonical map $Hom_{\mathcal{SH}_s^{S^1}(k)}(\Sigma^\infty(U_+), E) \to Hom_{\mathcal{H}_\bullet(k)}(U, E_0)$ is bijective.*

Indeed, one applies the principle of Quillen homotopical algebra, observing that the infinite suspension spectra $\Sigma^\infty(U_+)$ is stably cofibrant.

Corollary 4.1.8 *Let E be an S^1-spectrum. Then the sheaf $\pi_0(E)$ is the associated sheaf to the presheaf*

$$U \mapsto Hom_{\mathcal{SH}_s^{S^1}(k)}(\Sigma^\infty(U_+), E)$$

One easily proves that corollary from the previous one by assuming that E is a stably fibrant S^1-spectrum.

Let E be an S^1-spectrum and $\mathcal{X}$ be a pointed simplicial sheaf. One defines the smash-product $\mathcal{X}\wedge E$ as the S^1-spectrum whose n-th term is

$\mathcal{X} \wedge E_n$ and with structure morphisms $\mathcal{X} \wedge \sigma(E)_n$. This functor induces a functor

$$\Delta^{op} Shv_\bullet(\mathcal{V}_{Nis}) \times Sp^{S^1}(k) \to Sp^{S^1}(k), (\mathcal{X}, E) \mapsto \mathcal{X} \wedge E$$

which preserves weak equivalences and thus induces a functor

$$\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}) \times \mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k)$$

We can also define the smash-product on the right

$$Sp^{S^1}(k) \times \Delta^{op} Shv_\bullet(\mathcal{V}_{Nis}) \to Sp^{S^1}(k), (E, \mathcal{X}) \mapsto E \wedge \mathcal{X}$$

with terms $E_n \wedge \mathcal{X}$, but of course this functor is canonically isomorphic to the previous one. We observe also that the smash product by some fixed $\mathcal{X} \in \Delta^{op} Shv_\bullet(\mathcal{V}_{Nis})$

$$Sp^{S^1}(k) \to Sp^{S^1}(k), E \mapsto E \wedge \mathcal{X}$$

admits a right adjoint

$$Sp^{S^1}(k) \to Sp^{S^1}(k), F \mapsto \underline{Hom}(\mathcal{X}, F)$$

Given a functorial stable fibrant resolution $E \mapsto E_f$ the functor

$$\mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k), F \mapsto R\underline{Hom}(\mathcal{X}, F) := \underline{Hom}(\mathcal{X}, F_f)$$

is easily seen to be a right adjoint to

$$\mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k), E \mapsto E \wedge \mathcal{X}$$

The triangulated structure on $\mathcal{SH}_s^{S^1}(k)$ The category $\mathcal{SH}_s^{S^1}(k)$ admits arbitrary sums : the sum of a family of S^1-spectra is the S^1-spectrum whose n-th term is the wedge of the n-terms in that family.

Lemma 4.1.9 *The smash-product by S^1*

$$\mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k), E \mapsto S^1 \wedge E$$

is an equivalence of categories.

Indeed, choose a functorial stable fibrant resolution

$$E \mapsto E_f$$

Then the functor $E \mapsto \Omega(E_f)$, where $\Omega(E_f)_n := \Omega((E_f)_n)$ is easily seen to be an inverse to $E \mapsto S^1 \wedge E$ using the two stable weak equivalences[22]

$$E \to \Omega((S^1 \wedge E)_f)$$

and

$$S^1 \wedge \Omega(E_f) \to E_f$$

As S^1 is a cogroup object in $\mathcal{H}_{s,\bullet}$, its codiagonal $\phi : S^1 \to S^1 \vee S^1$ induces for any spectrum E, a morphism $E \to E \vee E$ (induced by $\phi \wedge Id_E : S^1 \wedge E \to S^1 \wedge E \vee S^1 \wedge E$ using the above lemma), which induces a natural structure of cogroup object on E. Being natural in E, this structure has to be abelian. Thus the category $\mathcal{SH}_s^{S^1}(k)$ is additive.

In fact, it extends to a natural triangulated structure on $\mathcal{SH}_s^{S^1}(k)$ where the shift functor $E \mapsto E[1]$ is $E \mapsto S^1 \wedge E$ and where an exact triangle in that structure is one which is isomorphic to some shift of the following triangle :

$$E \to F \to Cone(f) \to S^1 \wedge E$$

where $f : E \to F$ is any morphism of S^1-spectra and $Cone(f)_n := Cone(f_n)$.

Remark 4.1.10 The triangulated category $\mathcal{SH}_s^{S^1}(k)$ is generated by spectra of the form $\Sigma^\infty(U_+)$ with $U \in \mathcal{V}$ in the sense that $E \in \mathcal{SH}_s^{S^1}(k)$ is trivial if and only if any morphism (in $\mathcal{SH}_s^{S^1}(k)$) from any shift of those spectra to E is trivial.

Example 4.1.11 For each distinguished square as in 2.1.4 the following triangle

$$\Sigma^\infty(W_+) \to \Sigma^\infty(U_+) \vee \Sigma^\infty(V_+) \to \Sigma^\infty(X_+) \to \Sigma^\infty(W_+)[1]$$

where the first morphism is the difference of the two obvious morphisms and where the second one is the sum of the two obvious morphisms.

Example 4.1.12 From Theorem 3.3.4 it follows that for any closed immersion $i : X \to Y$ in $\mathcal{V}$ with normal bundle ν_i and open complement $U \subset Y$ there is an exact triangle

$$\Sigma^\infty(U_+) \to \Sigma^\infty(Y_+) \to \Sigma^\infty(Th(\nu_i)) \to \Sigma^\infty(U_+)[1]$$

[22] Check it on each fibers

Remark 4.1.13 Following [28, 14] the category $\mathcal{SH}_s^{S^1}(k)$ admits in fact a structure of symmetric monoidal category whose monoidal structure

$$(-)\wedge(-):\mathcal{SH}_s^{S^1}(k)\times\mathcal{SH}_s^{S^1}(k)\to\mathcal{SH}_s^{S^1}(k)$$

$$(E,F)\mapsto E\wedge F$$

is induced by the smash product in the sense that the composition of the functor $\wedge : \mathcal{SH}_s^{S^1}(k) \times \mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k)$ by the functor $\Sigma^\infty \times Id$: $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis}) \times \mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k)$ is the functor already considered above.

That symmetric monoidal structure is compatible in the obvious sense with triangulated structure. Moreover it complete in the sense that for any S^1-spectrum F, the smash-product by F, $\mathcal{SH}_s^{S^1}(k)\to\mathcal{SH}_s^{S^1}(k)$, $E\mapsto E\wedge F$ has a right adjoint denoted by :

$$\underline{Hom}(F,-):\mathcal{SH}_s^{S^1}(k)\to\mathcal{SH}_s^{S^1}(k), G\mapsto \underline{Hom}(F,G)$$

and for a pointed simplicial sheaf $\mathcal{X}$ one has of course a canonical isomorphism

$$\underline{Hom}(\Sigma^\infty(\mathcal{X}),G)\cong\underline{Hom}(\mathcal{X},G)$$

The t-structure on $\mathcal{SH}_s^{S^1}(k)$

Definition 4.1.14 *A S^1-spectrum E is said to be* non negative *if and only if for any integer $n<0$ one has*

$$\pi_n(E)=0$$

We denote $\mathcal{SH}_s^{S^1}(k)_{\geq 0}\subset\mathcal{SH}_s^{S^1}(k)$ the full subcategory whose objects are non-negative. We say that E is non positive *if for any integer $n>0$ one has*

$$\pi_n(E)=0$$

We denote $\mathcal{SH}_s^{S^1}(k)_{\leq 0}\subset\mathcal{SH}_s^{S^1}(k)$ the full subcategory whose objects are non positive.

Lemma 4.1.15 *An S^1-spectrum F is non positive if and only if for any $n>0$ and $U\in Sm(k)$ the group*

$$[\Sigma^\infty(U_+)[n],F]_s^{S^1}$$

vanishes.

This easily follows from Corollary 4.1.8.

Example 4.1.16 Spectra of the form $\Sigma^\infty(U_+)[n] = S^n \wedge \Sigma^\infty(U_+)$, $n \geq 0$ and $U \in Sm(k)$ are obviously non negative. This comes from the obvious fact that the simplicial pointed sheaf $(U_+) \wedge S^n$ is $(n-1)$-connected.

Theorem 4.1.17 *The triple*

$$(\mathcal{SH}_s^{S^1}(k), \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \mathcal{SH}_s^{S^1}(k)_{\leq 0})$$

is a t-structure[23] *[6] on $\mathcal{SH}_s^{S^1}(k)$.*

We will not recall the definition of a t-structure but we just remind the reader that the inclusion $\mathcal{SH}_s^{S^1}(\mathcal{V}_{Nis})_{\geq 0} \subset \mathcal{SH}_s^{S^1}(k)$ has a left adjoint $E \mapsto E_{\geq 0}$ that the inclusion $\mathcal{SH}_s^{S^1}(k)_{\leq 0} \subset \mathcal{SH}_s^{S^1}(k)$ has a right adjoint denoted by $E \mapsto E_{\leq 0}$, that

$$\forall E \in \mathcal{SH}_s^{S^1}(k)_{\geq 0}, \forall F \in \mathcal{SH}_s^{S^1}(k)_{\leq 0}, [E[1], F]_s^{S^1} = 0$$

and that for any spectrum E there is a unique exact triangle

$$E_{\geq 0} \to E \to E_{\leq (-1)} \to E_{\geq 0}[1]$$

Once one has such a t-structure, one easily gets for each integer $n \in \mathbb{Z}$ a $\geq n$ truncation $E_{\geq n} \to E$ (with $E_{\geq n} = (E[-n])_{\geq 0}[n]$), a $\leq n$ truncation $E \to E_{\leq n}$ so that there is an exact triangle

$$E_{\geq n} \to E \to E_{\leq (n-1)} \to E_{\geq n}[1]$$

The tower $\{E_{\leq n}\}_{n \in \mathbb{Z}}$ is usually refereed to as the "Postnikov" tower for E.

This t-structure is "non-degenerate" in the sense that for any $U \in \mathcal{V}$ and for any $E \in \mathcal{SH}_s^{S^1}(k)$, the morphism :

$$[\Sigma^\infty(U_+), E_{\geq n}]_s^{S^1} \to [\Sigma^\infty(U_+), E]_s^{S^1}$$

is an isomorphism for $n < 0$ and the group :

$$[\Sigma^\infty(U_+), E_{\geq n}]_s^{S^1}$$

[23] Here we are using an homological indexation of t-structures: thus our $\mathcal{SH}_s^{S^1}(k)_{\geq 0}$ should be understood as $\mathcal{SH}_s^{S^1}(k)^{\leq 0}$ in [6], etc...

vanishes[24] for $n > dim(U)$. As a consequence the morphism :

$$[\Sigma^\infty(U_+), E]_s^{S^1} \to [\Sigma^\infty(U_+), E_{\leq n}]_s^{S^1}$$

is an isomorphism for $n \geq dim(U)$.

The functor $\mathcal{SH}_s^{S^1}(k) \to \mathcal{A}b(\mathcal{V}_{Nis}), E \mapsto \pi_0(E)$ clearly induces an equivalence of categories from the heart[25] of that t-structure with the category $\mathcal{A}b(\mathcal{V}_{Nis})$ of sheaves of abelian groups. The inverse functor is the functor

$$H : \mathcal{A}b(\mathcal{V}_{Nis}) \to \mathcal{SH}_s^{S^1}(k)$$

which sends M to the following S^1-spectrum HM : its n-th term is the simplicial sheaf (of abelian groups) $K(M,n)$ which has only one non-trivial homotopy sheaf isomorphic to M in degree n, and whose structure morphisms are the obvious morphisms.

4.2 $\mathbb{A}^1$-localization of S^1-spectra and the connectivity theorem

In the sequel, when we consider $\mathbb{A}^1$ as a pointed scheme, we will always consider 0 as the base point and for any spectrum E, we denote by $ev_1(E)$ the evaluation at one :

$$\underline{Hom}(\mathbb{A}^1, E) \to E$$

Proposition 4.2.1 *Let E be an S^1-spectrum. The following conditions are equivalent :*

(i) *for any $X \in Sp^{S^1}(k)$, the projection*

$$X \wedge \Sigma^\infty(\mathbb{A}^1{}_+) \to X \wedge \Sigma^\infty(Spec(k)_+) = X$$

induces a bijection :

$$[X, E]_s^{S^1} \to [X \wedge \Sigma^\infty(\mathbb{A}^1{}_+), E]_s^{S^1}$$

[24] this follows from the fact that the Nisnevich cohomological dimension is bounded by the Krull dimension [36, 25]

[25] The heart of a t-structure is the full subcategory consisting of objects both non-negative and non-positive. By [6] it is always an abelian category

(ii) *for any $X \in Sp^{S^1}(k)$, the group*

$$[X \wedge \Sigma^\infty(\mathbb{A}^1), E]_s^{S^1}$$

vanishes ;

(iii) *the S^1-spectrum*

$$\underline{Hom}(\mathbb{A}^1, E)$$

is trivial ;

(iv) *for any smooth k-scheme U, any integer $n \in \mathbb{Z}$, the group homomorphism induced by $ev_1(E)$:*

$$[\Sigma^\infty(U_+)[n], \underline{Hom}(\mathbb{A}^1, E)]_s^{S^1} \to [\Sigma^\infty(U_+)[n], E]_s^{S^1}$$

is trivial.

Assume moreover that E is an Ω-spectrum. Then the previous conditions are also equivalent to the following one :

(v) *each of the pointed simplicial sheaves E_n is $\mathbb{A}^1$-local.*

The proof is of course the same as the proof of lemma 3.2.1.

Definition 4.2.2 *1) An S^1-spectrum E is called* $\mathbb{A}^1$-local *if it satisfies the equivalent conditions of the proposition.*

2) A morphism $f : X \to Y$ in $Sp^{S^1}(k)$ is called a stable $\mathbb{A}^1$-weak equivalence *if and only if for any $\mathbb{A}^1$-local E, the map :*

$$[Y, E]_s^{S^1} \to [X, E]_s^{S^1}$$

is bijective.

It is possible to show that the notion of stable $\mathbb{A}^1$-weak equivalences, and of stable cofibrations define a model category structure on $Sp^{S^1}(k)$. We denote by

$$\mathcal{SH}^{S^1}(k)$$

the associated homotopy category and call it the *stable $\mathbb{A}^1$-homotopy category* of S^1-spectra. As any stable weak equivalence is a stable $\mathbb{A}^1$-weak equivalence, the category $\mathcal{SH}^{S^1}(k)$ is a localization of $\mathcal{SH}_s^{S^1}(k)$ and thus get most of its structures[26]. Given two S^1-spectra E and F, we will simply denote by

$$[E,F]^{S^1}$$

the abelian group of morphisms between E and F in $\mathcal{SH}^{S^1}(k)$.

Remark 4.2.3 The triangulated category $\mathcal{SH}^{S^1}(k)$ is generated by spectra of the form $\Sigma^\infty(U_+)$ with $U \in Sm(k)$. This follows from the fact that these spectra generate[27] the triangulated category $\mathcal{SH}_s^{S^1}(k)$ and by the fact that the right adjoint to the obvious functor $\mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}^{S^1}(k)$ is a full embedding.

Let $F \mapsto F_f$ be a functorial (simplicial) fibrant model for $\mathcal{Y}$. Let E be an S^1-spectrum. We let $L^{(1)}(E)$ be the cone of the obvious morphism

$$ev_1 : \underline{Hom}(\mathbb{A}^1, E_f) \to E_f$$

Let $L_f^{(1)}(E)$ be $L^{(1)}(E)_f$. We let $E \to L_f^{(1)}(E)$ be the obvious morphism of pointed simplicial sheaves. Define by induction on $n \geq 0$, $L^{(n)} := L_f^{(1)} \circ L_f^{(n-1)}$. We have natural morphisms, for any E, $L_f^{(n-1)}(E) \to L_f^{(n)}(E)$ and we set $L^\infty(E) = colim_{n\in\mathbb{N}} L_f^{(n)}(E)$.

Lemma 4.2.4 *Let E be an S^1-spectrum. Then the S^1-spectrum $L^\infty(E)$ is $\mathbb{A}^1$-local and the morphism*

$$E \to L^\infty(E)$$

is a stable $\mathbb{A}^1$-weak equivalence.

The proof is the same as the one of 3.2.3.

Let's denote by $\mathcal{SH}_{s,\mathbb{A}^1}^{S^1}(k) \subset \mathcal{SH}_s^{S^1}(k)$ the full subcategory consisting of $\mathbb{A}^1$-local S^1-spectra. We observe it is stable under the operation of taking cones and as such is a sub-triangulated category.

The inclusion $\mathcal{SH}_{s,\mathbb{A}^1}^{S^1}(k) \subset \mathcal{SH}_s^{S^1}(k)$ thus admits as left adjoint the functor $L^\infty(-) : \mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_{s,\mathbb{A}^1}^{S^1}(k)$, which is called the $\mathbb{A}^1$-localization functor. It sends stable $\mathbb{A}^1$-weak equivalences to isomorphisms, and thus induces a functor $\mathcal{SH}^{S^1}(k) \to \mathcal{SH}_{s,\mathbb{A}^1}^{S^1}(k)$ which is an equivalence of categories.

[26] Triangulated, symmetric monoidal,etc..

[27] see 4.1.10

Definition 4.2.5 *Let E be an S^1-spectrum and $n \in \mathbb{Z}$ an integer. We say that E is* weakly n-connected *if and only if for any $i \leq n$*

$$\pi_i(E)(F) = 0$$

for any field of rational functions F of any irreducible smooth k-scheme[28].

For instance, any n-connected S^1-spectrum is weakly n-connected.

Remark 4.2.6 If E is an Ω-spectrum, its is weakly n-connected if and only if for each $m \geq 0$ the simplicial sheaf E_m is weakly $n+m$-connected.

Lemma 4.2.7 *Let E be an $\mathbb{A}^1$-local S^1-spectrum and $n \in \mathbb{Z}$ an integer. Then the following conditions are equivalent :*

(i) E is weakly n-connected ;

(ii) E is n-connected.

This easily follows from 3.3.6 and 4.2.1 (v).

Corollary 4.2.8 *Let $f : E \to F$ be a morphism of $\mathbb{A}^1$-local S^1-spectra. Assume that f induces an isomorphism on the sections over any field extension of finite type of k. Then f is a stable weak equivalence.*

Just apply Lemma 4.2.7 to the cone of f (which is $\mathbb{A}^1$-local and weakly n-connected for any integer n).

The following, now, is the main result of this section :

Theorem 4.2.9 *Let $n \in \mathbb{Z}$ and let E be an n-connected S^1-spectrum. Then its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(E)$ is weakly n-connected.*

Combining Theorem 4.2.9 and Lemma 4.2.7 we immediately get :

Theorem 4.2.10 *Let $n \in \mathbb{Z}$ an integer and let E be an n-connected S^1-spectrum. Then its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(E)$ is still n-connected.*

Remark 4.2.11 This theorem is the stable version of our conjecture 2.

[28] or equivalently the fiber $E(F)$ at such an F is an n-connected spectrum

Sketched proof of theorem 4.2.9 By Lemma 4.2.7 and an appropriate "base change argument"[29] we are reduced to proving the following

Lemma 4.2.12 *Let k be a field. Let $n \in \mathbb{Z}$ and let E be a (-1)-connected S^1-spectrum over k. Then for any integer $n < 0$ the group*

$$Hom_{\mathcal{SH}_s^{S^1}(k)}(S^n, L_{\mathbb{A}^1}(E))$$

is trivial.

To prove this lemma let us denote by σ the S^1-spectrum which is the fiber of the obvious morphism (corresponding to $1 \in \mathbb{A}^1$) $S^0 \to \Sigma^\infty(\mathbb{A}^1))$ so that we have an exact triangle

$$\sigma \to S^0 \to \Sigma^\infty(\mathbb{A}^1) \to \sigma[1]$$

Clearly, for any S^1-spectrum E one has a triangle of the form :

$$\underline{Hom}(\mathbb{A}^1, E) \to E \to \underline{Hom}(\sigma, E)$$

so that the function spectrum $\underline{Hom}(\sigma, E)$ is isomorphic to $L_f^{(1)}(E)$ in the category $\mathcal{SH}_s^{S^1}(k)$. Iterating this remark, we see that, for any integer $n \geq 0$, the function spectrum $\underline{Hom}(\sigma^{\wedge n}, E)$ is isomorphic in $\mathcal{SH}_s^{S^1}(k)$ to $L_f^{(n)}(E)$. So that the $\mathbb{A}^1$-localization of E is isomorphic to the telescope of the diagram :

$$E \to \underline{Hom}(\sigma, E) \to \ldots \to \underline{Hom}(\sigma^{\wedge n}, E) \to \ldots$$

As the spectra S^n are finitely presented[30], any morphism

$$S^n \to L_{\mathbb{A}^1}(E)$$

factors through some $\underline{Hom}(\sigma^{\wedge n}, E) \to L_{\mathbb{A}^1}(E)$.

Now the Lemma 4.2.12 and thus the Theorem 4.2.9 is a consequence of[31] :

Lemma 4.2.13 *Let k be a field. Let $m < 0$ be an integer and let E be a -1-connected S^1-spectrum over k. Then for any integer $n \geq 0$ the group*

$$[S^m, \underline{Hom}(\sigma^{\wedge n}, E)]_s^{S^1} \cong [\sigma^{\wedge n}, E[-m]]_s^{S^1}$$

is trivial.

[29] In the following statement, the fact that the base field is perfect is NOT necessary

[30] or "compact"

[31] the following Lemma is a reformulation of an idea of Voevodsky [39]

This lemma can be reformulated by saying that for any $n \geq 0$ the S^1-spectrum $\sigma^{\wedge n}$ has cohomological dimension 0 in the sense that for any 0-connected S^1-spectrum E the group

$$[\sigma^n, E]_s^{S^1}$$

is trivial. In fact we prove by induction on $n \geq 0$ that for any $i \in \mathbb{N}$ the spectrum $(\Sigma^\infty(\mathbb{A}^1)[-1])^{\wedge i} \wedge \sigma^{\wedge n}$ is of cohomological dimension 0, that is, for any 0-connected S^1-spectrum E the group

$$[(\Sigma^\infty(\mathbb{A}^1)[-1])^{\wedge i} \wedge \sigma^{\wedge n}, E]_s^{S^1}$$

is trivial. This is true for $n = 0$ because $\mathbb{A}^i$ is a smooth variety of Krull dimension i and that the Nisnevich dimension is less or equal to the Krull dimension [26, 36, 25]. Assume the inductive hypothesis for $n-1$, with $n > 0$. Let $i \in \mathbb{N}$. Using the triangle which defines σ we get a triangle

$$(\Sigma^\infty(\mathbb{A}^1)[-1])^{\wedge i(+1)} \wedge \sigma^{\wedge(n-1)} \to (\Sigma^\infty(\mathbb{A}^1)[-1])^{\wedge i} \wedge \sigma^{\wedge n}$$
$$\to (\Sigma^\infty(\mathbb{A}^1)[-1])^{\wedge i} \wedge \sigma^{\wedge(n-1)}$$

which easily implies the result.

4.3 The homotopy t-structure on $\mathcal{SH}^{S^1}(k)$

Definition 4.3.1 *1) An S^1-spectrum F is said to be* $\mathbb{A}^1$-non positive *if and only if for any integer $n > 0$, any smooth k-scheme U the group*

$$[\Sigma^\infty(U_+)[n], F]^{S^1}$$

is trivial. We denote by $\mathcal{SH}^{S^1}_{\leq 0}(k) \subset \mathcal{SH}^{S^1}(k)$ the full subcategory whose objects are $\mathbb{A}^1$-non positive.

2) We say that an S^1-spectrum E is $\mathbb{A}^1$-non negative *if for any $F \in \mathcal{SH}^{S^1}_{\leq 0}(k)$ one has*

$$[E, F[-1]] = 0$$

We denote $\mathcal{SH}^{S^1}_{\geq 0}(k) \subset \mathcal{SH}^{S^1}(k)$ the full subcategory whose objects are $\mathbb{A}^1$-non negative.

Example 4.3.2 Spectra of the form $\Sigma^\infty(U_+)[n]$, with $n \geq 0$ and $U \in Sm(k)$ are obviously $\mathbb{A}^1$-non negative.

Lemma 4.3.3 *1) An S^1-spectrum F is $\mathbb{A}^1$-non positive if and only its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(F)$ is non positive.*

2) An S^1-spectrum E is $\mathbb{A}^1$-non negative if and only its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(E)$ is non positive.

The first part follows from the fact that, for any integer n, any smooth k-scheme U one has an isomorphism

$$[\Sigma^\infty(U_+)[n], F]^{S^1} \cong [\Sigma^\infty(U_+)[n], L_{\mathbb{A}^1}(F)]_s^{S^1}$$

The second part is an immediate consequence of the first.

Theorem 4.3.4 *1) Let E is an $\mathbb{A}^1$-local S^1-spectrum, then its non negative part*

$$E_{\geq 0}$$

is still an $\mathbb{A}^1$-local S^1-spectrum. As a consequence, the triangle[32] *of S^1-spectra*

$$E_{\geq 0} \to E \to E_{\leq -1}$$

consists of $\mathbb{A}^1$-local S^1-spectra.

2) The pair $(\mathcal{SH}^{S^1}_{\geq 0}(k), \mathcal{SH}^{S^1}_{\leq 0}(k))$ is a t-structure [6] on $\mathcal{SH}^{S^1}(k)$.

This t-structure on $\mathcal{SH}^{S^1}(k)$ is called the *homotopy t-structure*[33].

To prove 1) apply the $\mathbb{A}^1$-localization functor to $E_{\geq 0}$. By theorem 4.2.10, the S^1-spectrum $L_{\mathbb{A}^1}(E_{\geq 0})$ is still -1-connected. By functoriality, $L_{\mathbb{A}^1}(E_{\geq 0})$ maps into $L_{\mathbb{A}^1}(E) = E$ because E is $\mathbb{A}^1$-local. By the universal property of the non-negative part $E_{\geq 0}$, we thus get a canonical factorization

$$L_{\mathbb{A}^1}(E_{\geq 0}) \to E_{\geq 0} \to E$$

whose composition by the morphism $E_{\geq 0} \to L_{\mathbb{A}^1}(E_{\geq 0})$ is clearly the canonical one. Thus, $E_{\geq 0}$ is a direct summand in $L_{\mathbb{A}^1}(E_{\geq 0})$ and is also $\mathbb{A}^1$-local.

The second part follows easily from the first.□

[32] in $\mathcal{SH}^{S^1}_s(k)$

[33] It is compatible to the homotopy t-structure defined by Voevodsky on $DM^{eff}(k)$ [38], see also the discussion in section 5.3

It follows from the corresponding fact in $\mathcal{SH}_s^{S^1}(k)$ that this is non-degenerate in the sense that for any $U \in \mathcal{V}$ and for any $E \in \mathcal{SH}_s^{S^1}(k)$, the morphism :

$$[\Sigma^\infty(U_+), E_{\geq n}]^{S^1} \to [\Sigma^\infty(U_+), E]^{S^1}$$

is an isomorphism for $n < 0$ and the group :

$$[\Sigma^\infty(U_+), E_{\geq n}]_s^{S^1}$$

vanishes for $n > dim(U)$. As a consequence the morphism :

$$[\Sigma^\infty(U_+), E]_s^{S^1} \to [\Sigma^\infty(U_+), E_{\leq n}]_s^{S^1}$$

is an isomorphism for $n \geq dim(U)$.

The heart of the homotopy t-structure The heart[34] of that t-structure is denoted $\pi^{\mathbb{A}^1}(k)$. Using theorem 4.3.4, we see that the obvious functor

$$\pi^{\mathbb{A}^1}(k) \to \mathcal{SH}_{s,\geq 0}^{S^1}(k) \cap \mathcal{SH}_{s,\leq 0}^{S^1}(k)$$

is an *exact full embedding.*

As we already know that $\mathcal{SH}_{s,\geq 0}^{S^1}(k) \cap \mathcal{SH}_{s,\leq 0}^{S^1}(k)$ is canonically equivalent to $\mathcal{A}b(\mathcal{V}_{Nis})$ by the functor which maps E to the associated sheaf to the presheaf

$$U \mapsto [\Sigma^\infty(U_+), E]_s^{S^1}$$

it remains for us to identify $\pi^{\mathbb{A}^1}(k)$ as a subcategory of $\mathcal{A}b(\mathcal{V}_{Nis})$.

We first extend definition 3.2.9:

Definition 4.3.5 *We say that a sheaf of abelian groups $M \in \mathcal{A}b(\mathcal{V}_{Nis})$ is* strictly $\mathbb{A}^1$-invariant *if and only if for any smooth k-scheme U, the homomorphism*

$$H^n_{Nis}(U; M) \to H^n_{Nis}(U \times \mathbb{A}^1; M)$$

is an isomorphism.

We denote by $\mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis})$ the full subcategory of $\mathcal{A}b(\mathcal{V}_{Nis})$ consisting of strictly $\mathbb{A}^1$-invariant sheaves.

This a quite natural because of the following

[34] recall it is just $\mathcal{SH}_{\geq 0}^{S^1}(k) \cap \mathcal{SH}_{\leq 0}^{S^1}(k)$

Lemma 4.3.6 *Given a sheaf of abelian groups $M \in \mathcal{A}b(\mathcal{V}_{Nis})$, the S^1-spectrum HM is $\mathbb{A}^1$-local if and only if M is strictly $\mathbb{A}^1$-invariant.*

This Lemma immediately follows from remark 3.2.8.

Lemma 4.3.7 *1) For any S^1-spectrum E the associated (Nisnevich) sheaf to the presheaf*

$$U \mapsto [\Sigma^\infty(U_+), E]^{S^1} = [\Sigma^\infty(U_+), L_{\mathbb{A}^1}(E)]_s^{S^1}$$

is a strictly $\mathbb{A}^1$-invariant sheaf which we denote $\pi_0^{\mathbb{A}^1}(E)$.

2) The functor

$$\pi_0^{\mathbb{A}^1} : \mathcal{SH}^{S^1}(k) \subset \mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis}), E \mapsto \pi_0^{\mathbb{A}^1}(E)$$

induces an equivalence of categories

$$\pi^{\mathbb{A}^1}(k) \cong \mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis})$$

Indeed, if E is ≥ 1, or if $E \leq (-1)$ the associated sheaf is trivial. Using our homotopy t-structure, we thus reduce to the case $E = HM$ which follows from Lemma 4.3.6.

Remark 4.3.8 As a consequence the category $\mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis})$ is abelian and the inclusion $\mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis}) \subset \mathcal{A}b(\mathcal{V}_{Nis})$ admits as left adjoint the functor denoted $M \mapsto M_{st}^{\mathbb{A}^1}$ which maps M to $\pi_0^{\mathbb{A}^1}(HM)$. It also gets a symmetric monoidal structure by the formula (for M and N in the category $\mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis})$)

$$M \otimes^{\mathbb{A}^1} N := (M \otimes N)_{st}^{\mathbb{A}^1}$$

Remark 4.3.9 Given any S^1-spectrum E, its Postnikov tower in $SH^{S^1}(k)$ gives for any $X \in Sm(k)$ a spectral sequence of the form

$$H_{Nis}^p(X; \pi_{-q}^{\mathbb{A}^1}(E)) \Rightarrow [\Sigma^\infty(X_+), E[p+q]]^{S^1}$$

The Quillen-Gersten spectral sequence [31, 8] can be seen to be a particular example of such a spectral sequence, applied to the S^1 spectrum $\underline{K}$ with n-th term

$$\mathbb{G}r \wedge \mathbb{G}_m{}^{\wedge n}$$

and with structure morphisms induced by the Bott map 3.1.7 and 3.1.11. This spectrum represents algebraic K-theory in the sense that for any $X \in Sm(k)$ and any integer $n \in \mathbb{Z}$ one has a canonical isomorphism

$$K_n^Q(X) \cong [\Sigma^\infty(X_+)[n], \underline{K}]^{S^1}$$

and one can indeed show that the sheaves $\pi_q^{\mathbb{A}^1}(\underline{K})$ are the usual ones $\mathcal{K}_q$ appearing in Quillen-Gersten spectral sequence.

Homotopy groups of $\underline{Hom}(\mathbb{G}_m, E)$

Definition 4.3.10 *Let $F : Sm(k)^{op} \to Set_\bullet$ be a presheaf of pointed sets. Denote by $F_{-1} : Sm(k)^{op} \to Set_\bullet$ the presheaf of sets which maps $U \in Sm(k)$ to the kernel (in the category of pointed sets) of the evaluation at 1 : $F(U \times \mathbf{G}_m) \to F(U)$.*

We observe that if F is a sheaf of pointed sets so is F_{-1} and in fact F_{-1} is the internal pointed function sheaf : $\underline{Hom_\bullet}(\mathbb{G}_m, F)$.

Let E be an S^1-spectrum. The obvious natural transformation (in $U \in Sm(k)$) :

$$[\Sigma^\infty(U_+) \wedge \Sigma^\infty(\mathbf{G}_m), E]_s^{S^1} \wedge Hom_{Shv(\mathcal{V}_{Nis})}(U, \mathbf{G}_m) \to \pi_0(E)(U)$$

induces a morphism of sheaves of pointed sets

$$\pi_0(\underline{Hom}(\mathbb{G}_m, E) \wedge \mathbb{G}_m \to \pi_0(E)$$

and thus a morphism of sheaves of abelian groups

$$\pi_0(\underline{Hom}(\mathbb{G}_m, E) \to \pi_0(E)_{-1}$$

More generally, for any integer $n \in \mathbb{Z}$ the above construction yields a canonical morphism

$$\pi_n(\underline{Hom}(\mathbb{G}_m, E) \to \pi_n(E)_{-1}$$

Lemma 4.3.11 *Let $E \in Sp^{S^1}(k)$ be an $\mathbb{A}^1$-local S^1-spectrum. Then for any $n \in \mathbb{Z}$, the canonical morphism*

$$\pi_n(\underline{Hom}(\mathbb{G}_m, E)) \to \pi_n(E)_{-1}$$

is an isomorphism.

Remark 4.3.12 As a consequence, we observe that if $E \in \pi(k)$ is an $\mathbb{A}^1$-local S^1-spectrum which is in the heart of the homotopy t-structure on $\mathcal{SH}^{S^1}(k)$, then the function spectrum $\underline{Hom}(\mathbb{G}_m, E)$ is still in the heart. More precisely, if $E = HM$ then $\underline{Hom}(\mathbb{G}_m, HM) \cong H(M_{-1})$.

It is clear that to prove the lemma, it is sufficient to establish the formula in the previous remark, that is to say that for any $M \in \mathcal{A}b_{st\mathbb{A}^1}(\mathcal{V}_{Nis})$ the obvious morphism of S^1-spectra :

$$\underline{Hom}(\mathbb{G}_m, HM) \to H(M_{-1})$$

is an isomorphism. To do this it is sufficient by corollary 4.2.8 to check it on fields, and by a base change argument to check that this morphism induces an isomorphism for each $n \in \mathbb{Z}$:

$$[S^0, \underline{Hom}(\mathbb{G}_m, HM)[n]]_s^{S^1} \to [S^0, H(M_{-1})[n]]_s^{S^1}$$

which can be reformulated, using adjunction, as :

1) $[\Sigma^\infty(\mathbb{G}_m, HM)[n]]_s^{S^1} = 0$ for $n \neq 0$.

2) $[\Sigma^\infty(\mathbb{G}_m, HM)]_s^{S^1} \to [S^0, H(M_{-1})]_s^{S^1}$.

The second point is obvious by definition. To prove the first one[35] one simply observes that $\Sigma^\infty(\mathbf{G}_m) \cong \Sigma^\infty(\mathbb{P}^1)[-1]$, that for any smooth variety X the group $[\Sigma^\infty(X_+), HM[n])]_s^{S^1}$ is canonically isomorphic to $H^n_{Nis}(X, M)$ and that $\mathbb{P}^1$ has cohomological dimension ≤ 1.

[35] I learned this simple argument from Mike Hopkins

5 Stable $\mathbb{A}^1$-homotopy theory of $\mathbb{P}^1$-spectra

5.1 $\mathbb{P}^1$-spectra

Definition 5.1.1 *A $\mathbb{P}^1$-spectrum E over k is a collection $\{E_n, \sigma_n\}_{n\in\mathbb{N}}$ consisting, for each integer $n \geq 0$, of a pointed simplicial sheaf E_n and a morphism $\sigma_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}$. Morphisms of $\mathbb{P}^1$-spectra are collections of morphisms of pointed simplicial sheaves with the obvious conditions. We thus obtain the category $Sp^{\mathbb{P}^1}(k)$ of $\mathbb{P}^1$-spectra (over k).*

Example 5.1.2 1) The basic example is the suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X})$ of a pointed simplicial sheaf $\mathcal{X}$. Its n-th term is $\mathcal{X} \wedge {\mathbb{P}^1}^{\wedge n}$ and the structure morphisms are just the canonical isomorphisms.

2) We may define a $\mathbb{P}^1$-spectrum $\mathbb{K}$ with n-th term $\mathbb{Z} \times \mathbb{G}r$ and with σ_n equal to the Bott map (see 3.1.11)

$$(\mathbb{Z} \times \mathbb{G}r) \wedge \mathbb{P}^1 \to \mathbb{Z} \times \mathbb{G}r$$

This $\mathbb{P}^1$-spectrum is called the algebraic K-theory spectrum.

3) *T-spectra.* Recall from [25, 39] that $T := \mathbb{A}^1/\mathbb{A}^1 - \{0\}$ is the Thom space of the trivial line bundle on $Spec(k)$. The cocartesian square (of sheaves of sets)

$$\begin{array}{ccc} \mathbb{G}_m & \to & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \to & \mathbb{P}^1 \end{array}$$

defines a canonical isomorphism $\mathbb{P}^1/\mathbb{A}^1 \cong T$ which is by the way an $\mathbb{A}^1$-weak equivalence.

A *T-spectrum* E over k is a collection $\{E_n, \sigma_n\}_{n\in\mathbb{N}}$ consisting, for each integer $n \geq 0$, of a pointed simplicial sheaf E_n and a morphism $\sigma_n : E_n \wedge T \to E_{n+1}$. Using the previous morphism $\mathbb{P}^1 \to T$ one see that any T-spectrum defines a $\mathbb{P}^1$-spectrum.

4) The Thom spectrum $M\mathbb{G}l$ has n-th term the Thom space $Th(\gamma_n)$ of the canonical rank n vector bundle on the infinite grassmanian $\mathbb{G}r_n = \bigcup_r \mathbb{G}r_{n,r}$ of n-dimensional plans in the infinite affine space.

The morphism

$$\mathbb{G}r_n \times \mathbb{G}r_m \to \mathbb{G}r_{n+m}$$

"classifying" the external sum $pr_1^*\gamma_n \Box pr_2^*\gamma_m$ induces with the isomorphism of Thom spaces given in 3.3.2 a morphism

$$Th(\gamma_n) \wedge Th(\gamma_m) \to Th(\gamma_{n+m})$$

These morphisms and the obvious map $T = Th(\mathcal{O}) \to Th(\gamma_1)$ corresponding to the Thom space of the trivial line bundle over the point $0 \in \mathbb{P}^\infty$ allows one to define a T-spectrum (and thus a $\mathbb{P}^1$-spectrum) with n-th term $Th(\gamma_n)$, denoted by $M\mathbb{G}l$ and called the Thom spectrum or the $\mathbb{P}^1$-spectrum of algebraic cobordism.

5) The motivic cohomology spectrum [39, 40] $\mathbb{H}$ has n-th term $K_n := L[\mathbb{A}^n]/L[\mathbb{A}^n - \{0\}]$ where the quotient is computed in the category $\mathcal{A}b(\mathcal{V}_{Nis})$ and where, for $X \in Sm(k)$ the motivic Eilenberg-MacLane "space" $L[X]$ is the sheaf of abelian groups $U \mapsto c(U, X)$ where $c(U, X)$ denotes the group of finite correspondences from U to X (see [38, 21]).

Remark 5.1.3 We will usually just use the term "spectrum" to mean $\mathbb{P}^1$-spectrum. Also, we will simply denote $Sp(k)$ the category of $\mathbb{P}^1$-spectrum if there is no possible confusion.

Let E be a $\mathbb{P}^1$-spectrum, $U \in Sm(k)$ and $(n, m) \in (\mathbb{Z})^2$. We set

$$\tilde{\pi}_n(E)_m(U) := colim_{r>>0} Hom_{\mathcal{H}_\bullet(k)}(S^{n+m} \wedge (U_+) \wedge (\mathbb{P}^1)^{r-m}, E_r)$$

This pointed set has a canonical structure of abelian group because $\mathbb{P}^1$ is isomorphic to $S^1 \wedge \mathbf{G}_m$ in $\mathcal{H}_\bullet(k)$.

Definition 5.1.4 *1) A morphism $f : E \to F$ of $\mathbb{P}^1$-spectra is called an $\mathbb{A}^1$-stable weak equivalence if and only if for any $U \in Sm(k)$ and any pair $(n, m) \in (\mathbb{Z})^2$ the homomorphism :*

$$\tilde{\pi}_n(E)_n(U) \to \tilde{\pi}_n(F)_m(U)$$

is an isomorphism.

2) A morphism $f : E \to F$ of $\mathbb{P}^1$-spectra is called a cofibration if and only if the morphism of pointed simplicial sheaves $E_0 \to F_0$ is a cofibration and if for any $n \geq 0$, the morphism pointed simplicial sheaves

$$E_{n+1} \vee_{(E_n \wedge \mathbb{P}^1)} (F_n \wedge \mathbb{P}^1) \to F_{n+1}$$

is a cofibration.

One can show that the category of $\mathbb{P}^1$-spectra endowed with the notions of stable $\mathbb{A}^1$-weak equivalences and of morphisms which are at each level monomorphisms as cofibrations is a model category. We denote by $\mathcal{SH}^{\mathbb{P}^1}(k)$ the associated homotopy category, and call it the stable homotopy category of $\mathbb{P}^1$-spectra. The morphisms in that category are denoted as usual by $[E,F]^{\mathbb{P}^1}$.

In fact, if no confusion can arise, we will usually denote the previous category simply by $\mathcal{SH}(k)$, will call it the stable homotopy category of smooth k-schemes and will simply denote the morphisms by

$$[E,F]$$

This category is naturally a triangulated category[36] and the suspension spectrum functor induces a functor

$$\Sigma^{\infty}_{\mathbb{P}^1} : \mathcal{H}_{\bullet}(k) \to \mathcal{SH}(k)$$

In fact there is an obvious extension of that functor to a smash-product functor

$$\mathcal{SH}(k) \times \mathcal{H}_{\bullet}(k) \to \mathcal{SH}(k), (E,\mathcal{X}) \mapsto E \wedge \mathcal{X}$$

The unit for that functor is the sphere spectrum

$$S^0 := \Sigma^{\infty}_{\mathbb{P}^1}(S^0)$$

Remark 5.1.5 As in the case of S^1-spectra [15, 28] show that the previous functor extends to a symmetric monoidal structure induced by the smash-product, which is compatible with the triangulated structure.

Of course, almost by construction, the smash product by $\mathbb{P}^1$

$$\mathcal{SH}(k) \to \mathcal{SH}(k), E \mapsto E \wedge \mathbb{P}^1$$

is an equivalence of categories, so that $\mathbb{P}^1$ is invertible.

The isomorphism[37](in $\mathcal{H}_{\bullet}(k)$) $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$ shows that in $\mathcal{SH}(k)$, the spectra $S^{1,0} := \Sigma^{\infty}(S^1)$ and $S^{1,1} := \Sigma(\mathbb{G}_m)$ are both invertible as well.

Moreover for any pair $(n,i) \in \mathbb{Z}^2$ we set

$$S^{n,i} := (S^{1,0})^{\wedge n} \wedge (S^{1,1})^{\wedge(i-n)}$$

Observe that $S^{2,1}$ is isomorphic to $\Sigma^{\infty}(\mathbb{P}^1)$.

---

[36] like for the case of S^1-spectra, this follows from the fact that $\mathbb{P}^1$ is a suspension in $\mathcal{H}_{\bullet}(k)$

[37] *cf* 3.1.10

Definition 5.1.6 *For any spectrum E, and for any integers $n, i \in \mathbb{Z}$ set*

$$E(i)[n] := E \wedge S^{n,i}$$

For any $\mathcal{X} \in \Delta^{op}Shv_{\bullet}(\mathcal{V}_{Nis})$, set

$$\tilde{E}^{n,i}(\mathcal{X}) := [\Sigma^{\infty}(\mathcal{X}), E(i)[n]]$$

For any $\mathcal{X} \in \Delta^{op}Shv(\mathcal{V}_{Nis})$, set

$$E^{n,i}(\mathcal{X}) := [\Sigma^{\infty}(\mathcal{X}_+), E(i)[n]] \cong \tilde{E}^{n,i}(\mathcal{X}_+)$$

The functor $\Delta^{op}Shv(\mathcal{V}_{Nis}) \to \mathcal{A}b^{*,*}$, $\mathcal{X} \mapsto E^{*,*}(\mathcal{X})$, where $\mathcal{A}b^{*,*}$ denotes the category of bigraded abelian groups, is called the cohomology theory *on the category of simplicial smooth k-schemes associated to E.*

Example 5.1.7 1) The cohomology theory associated with the Algebraic K-theory spectrum

$$X \mapsto \mathbb{K}^{*,*}(X)$$

is $(2,1)$-periodic because, by construction, the Bott morphism induces an isomorphism

$$\mathbb{K} \wedge \mathbb{P}^1 \cong \mathbb{K}$$

But by Bott periodicity 3.1.11, this gives isomorphisms for any $X \in Sm(k)$ any $(n,i) \in \mathbb{Z}^2$

$$\mathbb{K}^{n,i}(X) \cong K^Q_{2i-n}(X)$$

2) The cohomology groups

$$\mathbb{H}^{n,i}(X)$$

associated with the motivic cohomology spectrum are the Suslin-Voevodsky motivic cohomology groups $H^n(X;\mathbb{Z}(i))$ [38, 39, 35] of $X \in Sm(k)$ with integral coefficients.

Example 5.1.8 One can easily show that for any $\mathbb{P}^1$-spectrum E and any integers $(n,m) \in \mathbb{Z}^2$ the presheaf $\tilde{\pi}_n(E)_m$ defined above is in fact isomorphic to the presheaf

$$U \mapsto [\Sigma^{\infty}_{\mathbb{P}^1}(U_+) \wedge S^n \wedge {\mathbb{G}_m}^{\wedge(-m)}, E] \cong [\Sigma^{\infty}_{\mathbb{P}^1}(U_+)[n], E \wedge \wedge {\mathbb{G}_m}^{\wedge m}]$$
$$\cong [\Sigma^{\infty}_{\mathbb{P}^1}(U_+)[n], E(m)[m]]$$

This fact explains, a posteriori, the definition we gave of stable $\mathbb{A}^1$-weak equivalence and shows that the spectra $\Sigma^{\infty}_{\mathbb{P}^1}(U_+) \wedge {\mathbb{G}_m}^{\wedge m}$, $m \in \mathbb{Z}$ (in fact $m \leq 0$ suffices) are generators of the triangulated category $\mathcal{SH}(k)$.

Remark 5.1.9 Let $Sm'(k) \subset Shv(\mathcal{V})$ denote the full subcategory consisting of sheaves which are (categorical) sums of smooth k-varieties[38]. From [25] we know that there is a functorial weak equivalence

$$\mathcal{X} \mapsto (\mathcal{X}' \to \mathcal{X})$$

with $\mathcal{X}' \in \Delta^{op} Sm'(k)$. This implies formally that the categories $\mathcal{H}_s(\mathcal{V}_{Nis})$ and $\mathcal{H}(k)$ are localizations of $\Delta^{op} Sm'(k)$.

It is then possible to write down a list of axioms on functors

$$(\Delta^{op} Sm'(k))^{op} \to \mathcal{A}b^{*,*}$$

which characterizes those of the form $E^{*,*}(-)$ for $E \in \mathcal{SH}(k)$ as in classical topology.

Remark 5.1.10 As is clear from the definition previously given, for any pointed simplicial sheaf $\mathbb{S}$, one can define the category $Sp^{\mathbb{S}}(k)$ of $\mathbb{S}$-spectra over k, the corresponding presheaves for $(n, m) \in \mathbb{Z}^2$

$$U \mapsto \tilde{\pi}_m^{\mathbb{S}}(E)_n(U) := colim_{r>>0} Hom_{\mathcal{H}_\bullet(k)}(S^n \wedge (U_+) \wedge (\mathbb{S})^{r-m}, E_r)$$

and thus the corresponding notion of stable $\mathbb{A}^1$-weak equivalences of $\mathbb{S}$-spectra and thus the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}^{\mathbb{S}}(k)$. Using [15] one sees that when $\mathbb{S}$ is "reasonable", i.e.:

1)$\mathbb{S}$ is of finite presentation (or "compact" in [15]);

2) $\mathbb{S}$ is isomorphic in $\mathcal{H}_\bullet(k)$ to a suspension;

3) the cyclic permutation on the three variables

$$\gamma : \mathbb{S} \wedge \mathbb{S} \wedge \mathbb{S} \cong \mathbb{S} \wedge \mathbb{S} \wedge \mathbb{S}$$

is the identity morphism in $\mathcal{H}_\bullet(k)$;

then $\mathcal{SH}^{\mathbb{S}}(k)$ gets in a natural way a structure of triangulated, symmetric monoidal category in which the suspension $\mathbb{S}$-spectrum of $\mathbb{S}$ is invertible.

Moreover, it is clear that a morphism[39] $\mathbb{S}' \to \mathbb{S}$ induces a triangulated monoidal functor

$$\mathcal{SH}^{\mathbb{S}'}(k) \to \mathcal{SH}^{\mathbb{S}}(k)$$

[38] $Sm'(k)$ is thus also equivalent to the category of k-schemes which are disjoint unions of finite type smooth k-schemes

[39] in $\mathcal{H}_\bullet(k)$

which is an equivalence of categories if $\mathbb{S}' \to \mathbb{S}$ is an $\mathbb{A}^1$-weak equivalence.

Finally, given two reasonable such objects $\mathbb{S}$ and $\mathbb{S}$ the smash-product by $\mathbb{S}'$ defines a canonical triangulated and symmetric monoidal functor

$$\sigma^{\mathbb{S}'} : \mathcal{SH}^{\mathbb{S}}(k) \to \mathcal{SH}^{\mathbb{S}\wedge\mathbb{S}'}(k)$$

We have in mind the following examples:

1) The canonical isomorphisms $\mathbb{P}^1 \cong T := \mathbb{A}^1/\mathbb{A}^1 - \{0\} \cong S^1 \wedge \mathbb{G}_m$ gives equivalences

$$\mathcal{SH}^{\mathbb{P}^1}(k) \cong \mathcal{SH}^{T}(k) \cong \mathcal{SH}^{S^1\wedge\mathbb{G}_m}(k)$$

2) The canonical functor thus induced by the smash-product by $\mathbb{G}_m$

$$\sigma^{\mathbb{G}_m} : \mathcal{SH}^{S^1}(k) \to \mathcal{SH}^{S^1\wedge\mathbb{G}_m}(k) \cong \mathcal{SH}^{T}(k) \cong \mathcal{SH}^{\mathbb{P}^1}(k)$$

Remark 5.1.11 We observe that the previous functor $\sigma^{\mathbb{G}_m}$ admits a right adjoint

$$\omega^{\mathbb{G}_m} : \mathcal{SH}^{S^1\wedge\mathbb{G}_m}(k) \cong \mathcal{SH}^{T}(k) \cong \mathcal{SH}^{\mathbb{P}^1}(k) \to \mathcal{SH}^{S^1}(k)$$

As a consequence, using 5.1.8, for any $\mathbb{P}^1$-spectrum E and any integers $(n,m) \in \mathbb{Z}^2$ the presheaf $\tilde{\pi}_n(E)_m$ is isomorphic to the presheaf

$$U \mapsto [\Sigma^{\infty}_{S^1}(U_+)[n], \omega^{\mathbb{G}_m}(E(m)[m])]^{S^1}$$

Definition 5.1.12 *Let E be a $\mathbb{P}^1$-spectrum and $(n,m) \in \mathbb{Z}^2$ be integers. We denote by*

$$\pi_n(E)_m \in \mathcal{A}b(\mathcal{V}_{Nis})$$

the sheaf (of abelian groups) associated to the presheaf $\tilde{\pi}_n(E)_m$.

With the notations of Section 4.3, taking into account 5.1.11, this means that

$$\pi_n(E)_m = \pi_n(\omega^{\mathbb{G}_m}(E(m)[m]))$$

Remark 5.1.13 By the previous Remark 5.1.11 and by Lemma 4.3.7 we see that the sheaves $\pi_n(E)_m$ are each strictly $\mathbb{A}^1$-invariant.

Proposition 5.1.14 *Let E be a $\mathbb{P}^1$-spectrum. Then E is trivial in $\mathcal{SH}(k)$ if and only if the sheaves $\pi_n(E)_m$ are trivial for each $(n,m) \in \mathbb{Z}^2$.*

Indeed, the vanishing for a fixed m of all $\pi_n(E)_m$ means the vanishing of all $\pi_n(\omega^{\mathbb{G}_m}(E(m)[m]))$ and from what we know on S^1-spectra this implies that the S^1-spectrum $\omega^{\mathbb{G}_m}(E(m)[m])$ is trivial. Consequently, this implies that the presheaves $\tilde{\pi}_n(E)_m$ also vanish.

5.2 The homotopy t-structure

Definition 5.2.1 *We denote $\mathcal{SH}(k)_{\geq 0} \subset \mathcal{SH}(k)$ the full subcategory consisting of $\mathbb{P}^1$-spectra E with*

$$\pi_n(E)_m = 0$$

for each $m \in \mathbb{Z}$ and each $n < 0$.

We denote $\mathcal{SH}(k)_{\leq 0} \subset \mathcal{SH}(k)$ the full subcategory consisting of $\mathbb{P}^1$-spectra F with

$$\pi_n(F)_m = 0$$

for each $m \in \mathbb{Z}$ and each $n > 0$.

Example 5.2.2 For $U \in Sm(k)$, spectra of the form

$$\Sigma^\infty_{\mathbb{P}^1}(U_+)(i)[m] \cong \Sigma^\infty_{\mathbb{P}^1}(U_+) \wedge S^{(m-i)} \wedge {\mathbb{G}_m}^m$$

are non-negative[40] if $m - i \geq 0$.

Using our previous results on the homotopy t-structure for S^1-spectra, the computation of Lemma 4.3.11 and the pair of adjoint functors $(\sigma^{\mathbb{G}_m}, \omega^{\mathbb{G}_m})$ one easily proves

Theorem 5.2.3 *The triple $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is a t-structure [6] on $\mathcal{SH}(k)$.*

This t-structure is called the *homotopy t-structure* on $\mathcal{SH}(k)$.

It is non-degenerate in the sense that for any $E \in \mathcal{SH}(k)$ and any $U \in Sm(k)$, the morphism :

$$[\Sigma^\infty_{\mathbb{P}^1}(U_+), E_{\geq n}] \to [\Sigma^\infty_{\mathbb{P}^1}(U_+), E]$$

is an isomorphism for $n \leq 0$ and the morphism :

$$[\Sigma^\infty_{\mathbb{P}^1}(U_+), E] \to [\Sigma^\infty_{\mathbb{P}^1}(U_+), E_{\leq n}]$$

is an isomorphism for $n > dim(U)$.

[40] in $\mathcal{SH}(k)_{\geq 0}$

A description of the heart The heart of the homotopy t-structure is denoted $\pi_*^{\mathbb{A}^1}(k)$. We will use the construction described in Definition 4.3.10.

Definition 5.2.4 *A* homotopy module *over k is a pair (M_*, μ_*) consisting of a $\mathbb{Z}$-graded strictly homotopy invariant sheaf M_* together with, for each $n \in \mathbb{Z}$, an isomorphism of abelian sheaves :*

$$M_n \cong (M_{n+1})_{-1},$$

Let's start with the following lemma:

Lemma 5.2.5 *For any $\mathbb{P}^1$-spectrum F the canonical morphism*

$$\omega^{\mathbb{G}_m}(F) \to \underline{Hom}(\mathbb{G}_m, \omega^{\mathbb{G}_m}(F \wedge \mathbb{G}_m))$$

is an isomorphism.

Indeed for any S^1-spectra F one obtains by adjunction and invertibility of the smash-product by $\mathbb{G}_m$ the following sequence of isomorphisms :

$$[F \wedge \mathbb{G}_m, \omega(G \wedge \mathbb{G}_m)]^{S^1} \cong [\sigma^{\mathbb{G}_m}(F \wedge \mathbb{G}_m), G \wedge \mathbb{G}_m] \cong [\sigma^{\mathbb{G}_m}(F) \wedge \mathbb{G}_m, G \wedge \mathbb{G}_m]$$
$$\cong [\sigma^{\mathbb{G}_m}(F), G] \cong [F, \omega(G)]^{S^1}$$

Let E be a spectrum. We have seen in the previous section that there are natural isomorphisms of sheaves

$$\pi_n(E)_m \cong \pi_n(\omega^{\mathbb{G}_m}(E(m)[m]))$$

By the preceding lemma the canonical morphisms

$$\omega^{\mathbb{G}_m}(E(m)[m]) \to \underline{Hom}(\mathbb{G}_m, \omega^{\mathbb{G}_m}(E(m+1)[m+1]))$$

are isomorphisms. By Lemma 4.3.11 one gets canonical isomorphisms

$$\pi_n(E)_m \cong \pi_n(\omega^{\mathbb{G}_m}(E(m)[m])) \cong \pi_n(\underline{Hom}(\mathbb{G}_m, \omega^{\mathbb{G}_m}(E(m+1)[m+1])))$$
$$\cong (\pi_n(\omega^{\mathbb{G}_m}(E(m+1)[m+1])))_1 \cong (\pi_n(E)_{m+1})_{-1}$$

Thus for a fixed integer $n \in \mathbb{Z}$, the collection of $\pi_n(E)_m$, $m \in \mathbb{Z}$, forms a homotopy module, which is denoted

$$\pi_n(E)_*$$

and called the n-homotopy module of E.

Conversely, let M_* be a homotopy module. We construct a $S^1 \wedge \mathbb{G}_m$-spectrum denoted HM_* as follows. Its n-th term is the simplicial sheaf $K(M_n, n)\Delta^{op}Shv_\bullet(Shv(\mathcal{V}_{Nis}))$. The structure morphism is the obvious composition

$$K(M_n, n) \wedge S^1 \wedge \mathbf{G}_m \to K(M_n, n+1) \wedge \mathbf{G}_m \to K(M_{n+1}, n+1)$$

In the following we identify $\mathcal{SH}^{\mathbb{P}^1}(k)$ and $\mathcal{SH}^{S^1\wedge\mathbb{G}_m}(k)$, and thus will consider HM_* as a $\mathbb{P}^1$-spectrum as well.

Using all our previous results, now, it is not difficult to prove the:

Theorem 5.2.6 *The functor from the category of homotopy modules to the stable homotopy category of $\mathbb{P}^1$-spectra*

$$M_* \mapsto HM_*$$

is fully faithful and induces an equivalence between the category of homotopy modules and the heart of the homotopy t-structure. Its inverse is induced by the functor

$$E \mapsto \pi_0(E)_*$$

We observe that as a consequence of the symmetric monoidal structure on $\mathcal{SH}(k)$ we get a symmetric monoidal structure on the category of homotopy modules[41] by setting

$$M_* \otimes N_* := \pi_0((HM_*) \wedge (HN_*))$$

Of course the unit in that symmetric monoidal structure is $\pi_0(S^0)_*$ and we observe also that $\pi_0(\mathbb{G}_m)_*$ is automatically an invertible object (with inverse $\pi_0(\mathbb{G}_m{}^{-1})_*$).

5.3 Examples

Rost's cycles modules [33]. Let M_* be a Rost's cycle module on k [33]. We denote by $\underline{M}_n$ the presheaf on $Sm(k) : X \mapsto A^0(X, M_n)$. It is a strictly homotopy invariant sheaf in the Nisnevich topology. There is also an isomorphism $(M_n)_{-1} \cong M_{n-1}$ which follows from the results in [33]. Thus such a cycle module define a homotopy module M_* and a $\mathbb{P}^1$-spectrum HM_*.

[41] which we will identify with the heart of the homotopy t-structure

Example 5.3.1 The fundamental example of Rost's cycle module is given by *unramified Milnor K-theory*

$$\underline{K}^M_*$$

Recall from [18] that the Milnor K-theory of a field F is the graded algebra

$$\underline{K}^M_*(F)$$

obtained as the quotient of the tensor algebra (over $\mathbb{Z}$)

$$Tens_{\mathbb{Z}}(F^\times)$$

on the $\mathbb{Z}$-module $F^\times$ (multiplicative group of F) by the two sided ideal generated by the Steinberg relations

$$u \otimes (1-u)$$

for each $u \in F^\times - \{1\}$.

The sheaf[42] $\underline{K}^M_n$ has value on an irreducible $X \in Sm(k)$ the group of *unramified elements* in $K^M_n(k(X))$ (where $k(X)$ means the function field of X)

$$\underline{K}^M_n(X) := Ker\big(\ K^M_n(k(X)) \overset{\oplus_{x\in X^{(1)}}\partial_x}{\rightarrow} \oplus_{x\in X^{(1)}} K^M_{n-1}(\kappa(x))\ \big)$$

i.e. the kernel of the sum of all the residue morphisms [18]

$$\partial_x : K^M_n(k(X) \to K^M_{n-1}(\kappa(x))$$

indexed by the set $X^{(1)}$ of points of codimension one in X.

Motivic cohomology spectrum A reformulation of one of the basic definitions and computations in [35] is the:

Theorem 5.3.2 *For any $n < 0$ one has*

$$\pi_n(\mathbb{H})_* = 0$$

and the canonical morphism

$$\pi_0(\mathbb{H})_* \to \underline{K}^M_*$$

is an isomorphism.

[42] indeed strictly $\mathbb{A}^1$-invariant sheaf

Indeed the value of the sheaf $\pi_n(\mathbb{H})_m$ on $Spec(k)$ is the group

$$[S^n, \mathbb{H}(m)[m]] \cong H^{m-n}(Spec(k); \mathbb{Z}(m))$$

In particular it vanishes for $n < 0$ and any m by [35] which proves the first part. The second part is a reformulation of the fact that the isomorphism $H^1(X; \mathbb{Z}(1)) = \mathcal{O}(X)^\times$ induces (by cup-products) an isomorphism

$$K_n^M(k) \cong H^n(Spec(k); \mathbb{Z}(n))$$

In the same spirit one can prove the following basic statement

Theorem 5.3.3 *The canonical morphism*

$$\pi_0(M\mathbf{G}\ell)_* \to \underline{K}_*^M$$

is an isomorphism.

The proof of that result will be given in [24], see also the discussion in the next section below. We observe as a consequence the

Corollary 5.3.4 *The canonical map*

$$k^\times = Hom_{\mathcal{H}_\bullet(k)}(Spec(k)_+, \mathbb{G}_m) \to [S^0, M\mathbb{G}l \wedge \mathbb{G}_m] = M\mathbb{G}l^{1,1}(Spec(k))$$

induces by cup-product, for any integer $n \geq 0$, an isomorphism

$$K_n^M(k) \cong M\mathbb{G}l^{n,n}(Spec(k)) = [S^0, M\mathbb{G}l(n)[n]] = [S^0, M\mathbb{G}l \wedge (\mathbf{G}_m)^{\wedge n}]$$

We observe that by the connectivity Theorem 4.2.10, the groups

$$M\mathbb{G}l^{n,n}(Spec(k))$$

vanish for $n < 0$ so in fact the corollary holds for any $n \in \mathbb{Z}$.

6 $\pi_0(S^0)$ and Milnor-Witt K-theory of fields

In this section, we will simply denote by $\mathbb{G}_m$ the $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1}(\mathbb{G}_m)$ and $\mathbb{P}^1$ the $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1)$.

So far we haven't said anything on the question we addressed in the introduction to understand the groups

$$[S^m, {\mathbb{G}_m}^{\wedge n}] = \pi_m(S^0)_n(Spec(k))$$

An obvious corollary of Theorem 4.2.10 is the fact that

$$[S^m, {\mathbb{G}_m}^{\wedge n}] = 0 \text{ for any } m < 0$$

Our last part will now address the problem of computing

$$[S^0, {\mathbb{G}_m}^{\wedge n}]$$

for all $n \in \mathbb{Z}$.

6.1 The element ϵ

Let $\epsilon \in [S^0, S^0]$ be the morphism induced by the pointed morphism

$$\mathbb{G}_m \to \mathbb{G}_m, u \mapsto u^{-1}$$

Lemma 6.1.1 *1) The pointed morphism*

$$f : \mathbb{P}^1 \to \mathbb{P}^1, [x, y] \mapsto [y, x]$$

corresponds in $[S^0, S^0]$ to $-\epsilon$.

2) The permutation morphism

$$\mathbb{G}_m \wedge \mathbb{G}_m \cong \mathbb{G}_m \wedge \mathbb{G}_m \in [\mathbb{G}_m \wedge \mathbb{G}_m, \mathbb{G}_m \wedge \mathbb{G}_m] = [S^0, S^0]$$

corresponds to ϵ.

The first point is an easy consequence of that fact that f restricts to the morphism $\mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^{-1}$, and that it permutes the two canonical affine line in $\mathbb{P}^1$.

To prove the second point, taking into account that $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$ it is sufficient to prove that the permutation

$$\mathbb{P}^1 \wedge \mathbb{P}^1 \cong \mathbb{P}^1 \wedge \mathbb{P}^1$$

corresponds to $-\epsilon$. Consider the canonical $\mathbb{A}^1$-weak equivalences $\mathbb{P}^1 \to \mathbb{A}^1/\mathbb{A}^1 - \{0\}$ and the isomorphism of pointed sheaves 3.3.2

$$\mathbb{A}^1/\mathbb{A}^1 - \{0\} \wedge \mathbb{A}^1/\mathbb{A}^1 - \{0\} \cong \mathbb{A}^2/\mathbb{A}^2 - \{0\}$$

The permutation is given by the action of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and because the last one has determinant $+1$ our permutation is equal in $[S^0, S^0]$ to the morphism defined by the first matrix (any matrix of $SL_2(k)$ is a product of elementary matrices and thus is $\mathbb{A}^1$-homotopic to the identity in $\mathcal{H}_\bullet(k)$). Thus the permutation of $\mathbb{P}^1 \wedge \mathbb{P}^1$ corresponds in fact to the morphism $\mathbb{P}^1 \to \mathbb{P}^1, [x, y] \mapsto [-x, y]$. But the same trick (using the action of $Gl_2(k)$ on $\mathbb{P}^1$) shows that the latter is $\mathbb{A}^1$-homotopic to $\mathbb{P}^1 \to \mathbb{P}^1, [x, y] \mapsto [y, x]$, which by 1) is $-\epsilon$.$\square$

We observe that the groups $[S^0, {\mathbb{G}_m}^{\wedge n}]$, $n \in \mathbb{Z}$, form a graded associative algebra. The product is just induced by the smash-product[43].

Corollary 6.1.2 *The $\mathbb{Z}$-graded algebra*

$$[S^0, {\mathbb{G}_m}^{\wedge *}]$$

is ϵ-graded commutative in the sense that if α has degree n and β has degree m then

$$\alpha.\beta = \epsilon^{nm} \beta.\alpha$$

This of course follows immediately from 2) of Lemma 6.1.1.

[43]It can be shown to be also induced by the composition in $\mathcal{SH}(k)$

6.2 The Hopf map

The Hopf map[44] is the canonical morphism of k-schemes

$$Hopf : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1, (x, y) \mapsto [x, y]$$

We observe it is pointed if we point $\mathbb{A}^2 - \{0\}$ by $(1, 1)$ and $\mathbb{P}^1$ by $[1, 1]$.

Lemma 6.2.1 *The cone of the Hopf map is canonically isomorphic in the pointed $\mathbb{A}^1$-homotopy category to the 2-dimensional projective space $\mathbb{P}^2$.*

Indeed let $E(\lambda) = \mathbb{A}^2 - \{0\} \times_{\mathbb{G}_m} \mathbb{A}^1$ denote the total space of the canonical line bundle on $\mathbb{P}^1$. Then the Hopf map factors as the open immersion $\mathbb{A}^2 - \{0\} \to E(\lambda)$ followed by the projection $E(\lambda) \to \mathbb{P}^1$ which is an $\mathbb{A}^1$-weak equivalence. Thus the cone of the Hopf map is isomorphic in $\mathcal{H}_\bullet(k)$ to the quotient $E(\lambda)/\mathbb{A}^2 - \{0\}$. But $E(\lambda)$ is isomorphic to the open complement of the closed $[1, 0, 0] \in \mathbb{P}^2$. Moreover the complement of the projective line at ∞, $\mathbb{P}^1 \subset \mathbb{P}^2$ is $\mathbb{A}^2$ and the intersection of the two open subset $E(\lambda)$ and $\mathbb{A}^2$ in $\mathbb{P}^2$ is $\mathbb{A}^2 - \{0\}$. Thus we get a (cartesian and) cocartesian square

$$\begin{array}{ccc} \mathbb{A}^2 - \{0\} & \to & \mathbb{A}^2 \\ \downarrow & & \downarrow \\ E(\lambda) & \to & \mathbb{P}^2 \end{array}$$

which gives the required isomorphism

$$cone(Hopf) \cong E(\lambda)/\mathbb{A}^2 - \{0\} = \mathbb{P}^2/\mathbb{A}^2 \cong \mathbb{P}^2 \quad \square$$

The (co)cartesian square, corresponding to the open covering of $\mathbb{A}^2 - \{0\}$ by $\mathbb{G}_m \times \mathbb{A}^1$ and $\mathbb{A}^1 \times \mathbb{G}_m$:

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \to & \mathbb{G}_m \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \times \mathbb{G}_m & \to & \mathbb{A}^2 - \{0\} \end{array}$$

defines an isomorphism in $\mathcal{H}_\bullet(k)$

$$\mathbb{A}^2 - \{0\} \cong (\mathbb{G}_m)^{\wedge 2} \wedge S^1$$

Recall also from 3.1.10 that $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(k)$. Thus the Hopf map can be considered as a morphism

$$Hopf : (\mathbb{G}_m)^{\wedge 2} \wedge S^1 \to \mathbb{G}_m \wedge S^1$$

[44] or Hopf fibration

We still denote by $Hopf \in [\mathbb{G}_m, S^0]$ the morphism in $\mathcal{SH}(k)$ induced by the Hopf map after applying the suspension spectrum and after simplification by $\mathbb{G}_m \wedge S^1$, which is invertible.

Recall also the following well-known trick in algebraic topology:

Lemma 6.2.2 *Let $\mathcal{X}$ and $\mathcal{Y}$ be pointed simplicial sheaves. Then there is a canonical $\mathcal{SH}(k)$-isomorphism*[45]

$$\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X}) \vee \Sigma^\infty_{\mathbb{P}^1}(\mathcal{Y}) \vee \Sigma^\infty_{\mathbb{P}^1}(\mathcal{X} \wedge \mathcal{Y}) \cong \Sigma^\infty_{\mathbb{P}^1}(\mathcal{X} \times \mathcal{Y})$$

The proof as usual consists, because any pointed suspension being a co-group object in $\mathcal{H}_{s,\bullet}(\mathcal{V}_{Nis})$, in splitting the cofiber sequence

$$\mathcal{X} \vee \mathcal{Y} \to \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \wedge \mathcal{Y}$$

after applying the suspension spectrum functor $\Sigma^\infty_{\mathbb{P}^1}$ by describing a left inverse $\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X}\times\mathcal{Y}) \to \Sigma^\infty_{\mathbb{P}^1}(\mathcal{X})\vee\Sigma^\infty_{\mathbb{P}^1}(\mathcal{Y})$ to the natural inclusion. For this we observe using the additive structure that it suffices to take the "sum" of the two obvious morphisms induced by the two projections $\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X} \times \mathcal{Y}) \to \Sigma^\infty_{\mathbb{P}^1}(\mathcal{X})$ and $\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X} \times \mathcal{Y}) \to \Sigma^\infty_{\mathbb{P}^1}(\mathcal{Y})$.$\square$

We apply the previous splitting with $\mathcal{X} = \mathcal{Y} = \mathbb{G}_m$, so that

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{G}_m \vee \mathbb{G}_m \vee (\mathbb{G}_m \wedge \mathbb{G}_m)$$

As a consequence, the product morphism of the multiplicative group

$$\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$$

induces a morphism

$$\mathbb{G}_m \vee \mathbb{G}_m \vee (\mathbb{G}_m \wedge \mathbb{G}_m) \to \mathbb{G}_m$$

and we denote by

$$\eta : \mathbb{G}_m \wedge \mathbb{G}_m \to \mathbb{G}_m$$

the morphism induced on the factor $\mathbb{G}_m \wedge \mathbb{G}_m$.

Thus $\eta \in [\mathbb{G}_m \wedge \mathbb{G}_m, \mathbb{G}_m] = [\mathbb{G}_m, S^0]$ and the Hopf map can be interpreted as elements in

$$[\mathbb{G}_m, S^0]$$

[45]In fact this isomorphism hold $in\mathcal{H}_\bullet(k)$ after one suspension

Lemma 6.2.3 *In* $[\mathbb{G}_m, S^0]$ *one has the equations*

$$Hopf = \eta.\epsilon = \eta$$

The first equality follows rather formally from the fact that in $\mathbb{P}^1$ one has for $(x,y) \in {\mathbb{G}_m}^2$:

$$[x,y] = [1, x.y^{-1}]$$

and the second follows from 2) of Lemma 6.1.1 and the commutativity of the product on the multiplicative group.

η and orientable homotopy modules Let's still denote by

$$\eta : \pi_0(\mathbb{G}_m)_* \to \pi_0(S^0)_*$$

the induced morphism by η on $\pi_0()_*$.

We have the following fundamental

Lemma 6.2.4 *In the abelian category* $\pi_*(k)$ *of homotopy modules, there is an exact sequence*

$$\pi_0(\mathbb{G}_m)_* \xrightarrow{\eta} \pi_0(S^0)_* \to \underline{K}_*^M \to 0$$

This result easily implies Theorem 5.3.3 because the map

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^2)(-1)[-2] = cone(\eta) \to M\mathbb{G}l$$

is obtained by adding "0-connected cells" and thus induces an isomorphism on $\pi_0()_*$.

Definition 6.2.5 *A homotopy module* M_* *will be say to be* orientable *if the morphism of homotopy modules*

$$Id_{M_*} \otimes \eta : M_* \otimes \pi_0(\mathbb{G}_m)_* \to M_*$$

is zero.

In other words, using Lemma 6.2.4, the category of orientable homotopy modules is exactly the category of modules over the monoid $\underline{K}_*^M \in \pi_*(k)$.

The fact that $\pi_0(M\mathbb{G}l)_* = \underline{K}_*^M$ also means that these orientable homotopy modules have a canonical structure of $M\mathbb{G}l$-module.

Conjecture 3 *The functor above functor $M_* \mapsto HM_*$ induces an equivalence between the category of Rost's cycle modules and the category of oriented homotopy modules.*

Remark 6.2.6 Define a *motivic homotopy module* over k to be a pair

$$(M_*, \mu_*)$$

consisting of a $\mathbb{Z}$-graded homotopy invariant sheaf with transfers [38] M_* together with, for each $n \in \mathbb{Z}$, an isomorphism of abelian sheaves :

$$M_n \cong (M_{n+1})_{-1},$$

By the results of Voevodsky in [37], any homotopy invariant sheaf with transfers is strictly homotopy invariant and thus any motivic homotopy module M_* defines a homotopy module. We also conjecture that this functor should induce an equivalence between the category of motivic homotopy modules with the full subcategory of orientable homotopy modules. We observe that F. Déglise has given in [10] an equivalence between the category of motivic homotopy modules and that of Rost's cycle modules.

6.3 Milnor-Witt K-theory and $\oplus_{n\in\mathbb{Z}}[S^0, {\mathbb{G}_m}^{\wedge n}]$

This section is based on a collaboration with Mike Hopkins.

We start by pointing out some obvious elements in the graded group

$$[S^0, {\mathbb{G}_m}^{\wedge *}]$$

First to any $u \in k^\times$ is associated a pointed morphism $Spec(k)_+ \to \mathbb{G}_m$ (in $\mathcal{H}_\bullet(k)$ say) which induces by taking the suspension spectra a morphism

$$[u] : S^0 \to \mathbb{G}_m$$

thus defining the element $[u] \in [S^0, \mathbb{G}_m]$.

We also have defined in the previous section the Hopf element

$$\eta \in [\mathbb{G}_m, S^0] = [S^0, {\mathbb{G}_m}^{-1}]$$

Definition 6.3.1 *Let F be a field. We denote by $T_*^{MW}(F)$ the free graded associative algebra with a generator $[u]$ of degree $+1$ for each $u \in F^\times$ and with one generator η of degree -1.*

We denote by $K_*^{MW}(F)$ the quotient of $T_*^{MW}(F)$ by the relations of the following type :

1 *For each pair* $(a,b) \in (F^\times)^2$ *:*

$$[ab] = [a] + [b] + \eta.[a].[b]$$

2 *(Steinberg relation) For each* $a \in F^\times - \{1\}$ *:*

$$[a].[1-a] = 0$$

3 *For each* $u \in F^\times$

$$[u].\eta = \eta.[u]$$

4

$$\eta^2.[-1] + 2\eta = 0$$

$K_*^{MW}(F)$ *is called the* Milnor-Witt K-theory *of* F*.*

Remark 6.3.2 Clearly, the associative graded algebra $K_*^{MW}(F)/\eta$ is the Milnor K-theory $K_*^M(F)$ of F.

One of our main reason to introduce the Milnor-Witt K-theory of a field F is the following Theorem[46]

Theorem 6.3.3 *The canonical graded ring homomorphism*

$$T_*^{MW}(k) \to \oplus_{n\in\mathbb{Z}}[S^0, {\mathbb{G}_m}^{\wedge n}]$$

$$[u] \mapsto [u] \in [S^0, \mathbb{G}_m]$$

$$\eta \mapsto \eta \in [S^0, {\mathbb{G}_m}^{-1}]$$

satisfies the 4 *relations of Definition 6.3.1 and thus induces a graded ring homomorphism*

$$K_*^{MW}(k) \to \oplus_{n\in\mathbb{Z}}[S^0, {\mathbb{G}_m}^{\wedge n}]$$

For any $u \in F^\times$, set:

$$< u > := \eta[u] + 1 \in K_0^{MW}(F)$$

[46]Obtained in collaboration with Mike Hopkins

Lemma 6.3.4 For any $u \in k^\times$, the morphism $< u >: S^0 \to S^0$ defined by the previous formula corresponds to the pointed morphism (of pointed smooth k-schemes)

$$f_u : \mathbb{P}^1 \to \mathbb{P}^1, [x, y] \mapsto [ux, y]$$

Indeed, this morphism is clearly the unreduced suspension the morphism $\mathbb{G}_m \to \mathbb{G}_m, x \mapsto u.x$ which can be written as the composition of the product $Id \times u : \mathbb{G}_m \times \mathrm{Spec}\,(k) \to \mathbb{G}_m \times \mathbb{G}_m$ and of the product of $\mathbb{G}_m$, which implies the result.

Remark 6.3.5 One can now "compute" $\epsilon \in [S^0, S^0]$ as follows. By Lemma 6.1.1 and its proof we know that $-\epsilon$ is the morphism $f_{-1} : \mathbb{P}^1 \to \mathbb{P}^1, [x, y] \mapsto [-x, y]$. This one now is $< -1 >$ by the previous Lemma. Thus

$$\epsilon = - < -1 >$$

We observe then that relation **4** can be also rewritten as

$$\eta.\epsilon = \eta$$

which is known to be true by 6.2.3.

Sketched proof of Theorem 6.3.3. The first relation simply follows from the definition of η as a factor of the suspension of the product of $\mathbb{G}_m$. The relation **2** follows from the following fundamental result:

Theorem 6.3.6 *[29] The canonical composed pointed morphism*

$$\mathbb{G}_m - \{1\} \stackrel{(u,1-u)}{\to} \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$$

becomes trivial after applying the suspension spectrum.

The relation **3** follows from an easy computation and relation **4** was established in the previous Remark.$\square$

The following lemma is not difficult:

Lemma 6.3.7 *1)* $< \forall (u, v) \in (F^\times)^2,\ uv >=< u > . < v >\in K_0^{MW}(F)$.

2) $\forall u \in F^\times,\ < u^2 >= 1$.

3) $\forall u \in F^\times - \{1\},\ < u > + < 1 - u >= 1+ < u(1 - u) >$.

Recall that the Grothendieck-Witt ring of F is the Grothendieck group $GW(F)$ of the monoid of isomorphism classes of quadratic forms over F. For any $u \in F^\times$, we denote by $< u > \in GW(F)$ the (class of the) quadratic form of rank one $u.X^2$. It is known[47] [34] that the relation from the previous lemma between the $< u >$'s gives a presentation of the ring $GW(F)$. Thus as a corollary we get a canonical ring homomorphism

$$\Phi : GW(F) \to K_0^{MW}(F), < u > \mapsto < u >$$

In fact it is not difficult to prove:

Lemma 6.3.8 *This ring homomorphism*

$$\Phi : GW(F) \to K_0^{MW}(F), < u > \mapsto < u >$$

is an isomorphism.

In degree 0, the Theorem above thus defines a ring homomorphism[48]

$$GW(k) \to [S^0, S^0]$$

This was first obtained in collaboration with J. Lannes using a Lefchetz formula for the morphism $f_u : \mathbb{P}^1 \to \mathbb{P}^1, [x,y] \mapsto [ux, y]$ (observe that the fixed points of f_u are precisely 0 and ∞). Since then, there has been several different constructions of this homomorphism by Jean Lannes, Markus Rost, and finally the one above by Hopkins and the author.

Now, let $h := 1 + < -1 > \in GW(F) = K_0^{MW}(F)$ denote the hyperbolic plane. Then relation **4** is also equivalent to

$$\eta.h = 0$$

Recall [34] that the *Witt ring* of F is the quotient of $GW(F)$ by h. As a consequence, we see that the multiplication by η

$$GW(F) = K_0^{MW}(F) \to K_{-1}^{MW}(F)$$

kills the hyperbolic plane $h \in GW(F)$ and induces homomorphisms

$$W(F) \to K_{-1}^{MW}(F) \to K_{-2}^{MW}(F) \to \cdots \to K_{-n}^{MW}(F) \to \dots$$

One easily checks:

Lemma 6.3.9 *In the previous diagram, the homomorphisms are all isomorphisms.*

[47] here we assume $char(F) \neq 2$
[48] at least if $char(k) \neq 2$

6.4 The basic theorem

Theorem 6.4.1 *For any (perfect) field k of char $\neq 2$ the canonical homomorphism*

$$K_*^{MW}(k) \to \oplus_{n\in\mathbb{Z}}[S^0, (\mathbb{G}_m)^n]_{\mathcal{SH}(k)}$$

is an isomorphism.

Remark 6.4.2 In degree 0 the previous statement says that the homomorphism

$$GW(k) \cong [S^0, S^0]$$

is an isomorphism. The fact that such a theorem would hold was rather clear from our work in [20, 23], where, for any field k of characteristic 0 we showed, using the both the proof of the Milnor conjecture on Galois mod. 2 cohomology as well as the computation by Voevodsky of the motivic Steenrod algebra, that the Adams spectral sequence for the sphere spectrum based on mod 2 motivic cohomology converges to the graded ring associated to the filtration of the Grothendieck-Witt ring of quadratic forms over k by powers of the ideal of even dimensional forms[49].

Remark 6.4.3 In negative degrees , the Theorem gives isomorphisms

$$W(k) \to [{\mathbb{G}_m}^{\wedge n}, S^0] = [S^0, {\mathbb{G}_m}^{\wedge -n}]$$

for any $n > 0$.

Remark 6.4.4 We observe that statement 6.4.1 implies Lemma 6.2.4 by Remark 6.3.2.

The proof uses recent results by Hornbostel [12], Panin [27] and Arason and Elman [2].

Hornbostel proved in [12] that there is a $\mathbb{P}^1$-spectrum $\mathbb{K}O$ (resp. $\mathbb{K}W$) which represents hermitian algebraic K-theory (resp. Balmer's Witt groups [4]).

By the results in Section 5.2, this implies that the associated sheaf to $U \mapsto W(U)$ (the Witt group of U) is a strictly $\mathbb{A}^1$-invariant homotopy sheaf

[49]In fact we showed there using these ideas that Voevodsky's results imply that the spectral sequence degenerate in the critical area. This allowed us to give a new proof of the Milnor conjecture on the graded ring of the Witt ring of k

that we denote by $\underline{W}$ (unramified Witt groups). In fact this was also proven directly recently by Panin [27].

Now recall the following standard definitions for a Field F. Let $I(F)$ denote the kernel of the mod rank homomorphism

$$GW(k) \to \mathbb{Z}$$

For any $u \in F^{\times}$ we denote $<< u >>:= 1- < u > \in I(F)$. Obviously the symbols $<< u >>$ generate $I(F)$ as an abelian group. Through the map $GW(F) \to W(F)$ the symbol $<< u >>$ is mapped to the (class of) the Pfister form $1+ < -u >$. Clearly the diagram of commutative rings :

$$\begin{array}{ccc} GW(F) & \to & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(F) & \to & \mathbb{Z}/2 \end{array}$$

is cartesian. Thus the quotient map $GW(F) \to W(F)$ identifies $I(k)$ with the kernel of the mod 2 rank homomorphism $W(k) \to \mathbb{Z}/2$.

For any integer $n \in \mathbb{Z}$ we set $I^n(F) := (I(F))^n$ for $n \geq 0$ and $I^n(F) := W(F)$ for $n \leq 0$.

Let's denote by $J^1(k)$ the fiber product of the diagram :

$$\begin{array}{ccc} & & F^{\times} \\ & & \downarrow \\ I(F) & \to & I(F)/I(F)^2 \end{array}$$

It is in a canonical way a module over $GW(F)$. For any $u \in F^{\times}$ we denote by $\{u\} \in J^1(F)$ the element corresponding to the opposite of the Pfister form $< 1 > + < -u > \in I(k)$ and $u \in F^{\times}$.

Recall from [5] that we set for $n \geq 0$:

$$J^n(F) := I(F)^n \times_{I(F)^n/I(F)^{(n+1)}} K_n^M(F)$$

and that we set $J^n(F) = W(F)$ for $n < 0$. The $J^n(F)$, $n \in \mathbb{Z}$, form altogether a $\mathbb{Z}$-graded associative ring that we denote $J^*(F)$.

Theorem 6.4.5 *The correspondence $u \mapsto \{u\} \in J^1(F)$ and $\eta \mapsto 1 \in W(F) = J^{-1}(F)$ induces a graded ring homomorphism*

$$K_*^{MW}(k) \to J^*(k)$$

which is an isomorphism[50].

[50] for $char(F) \neq 2$

The proof of this theorem uses some results by Arason and Elman [2], which relies on the Milnor conjecture. On the way we can reformulate Arason and Elman result as follows:

Definition 6.4.6 *Let us define the Witt K-theory of the field F as the quotient*

$$K_*^W(F) := K_*^{MW}(F)/h$$

Theorem 6.4.7 *The canonical graded ring homomorphism*

$$K_*^W(F) \to I^*(F)$$

is an isomorphism.

All these results allows us to define a homotopy module

$$\underline{K}_*^{MW}$$

with value $K_*^{MW}(F)$ on fields and with a canonical map

$$\pi_0(S^0)_* \to \underline{K}_*^{MW}$$

Theorem 6.3.3 gives a section of that morphism on each field. We prove then that these sections define altogether a section of monoids

$$\underline{K}_*^{MW} \to \pi_0(S^0)_*$$

which has to be an isomorphism because $\pi_0(S^0)_*$ is the unit object. □

Acknowledgments. It is a pleasure to thank very much Mike Hopkins and Vladimir Voevodsky for several discussions on and around the subject of these notes.

References

[1] J. F. Adams, Stable homotopy and generalised homology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974.

[2] J. Arason and R. Elman, Powers of the fundamental ideal in the Witt ring, Journal of Algebra **239**, 150-160 (2001).

[3] M. Artin, A. Grothendieck, J.-L. Verdier, Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos. (French) Séminaire de Gomtrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972.

[4] P. Balmer, Triangular Witt groups. II. From usual to derived. Math. Z. 236 (2001), no. 2, 351–382.

[5] J. Barge et F. Morel, Cohomologie des groupes linéaires, K-théorie de Milnor et groupes de Witt, C.R. Acad. Sci. Paris, t. 328, Série I, p. 191-196, 1999.

[6] A.A. Beilinson; J. Bernstein; P. Deligne. Faisceaux pervers. (French) [Perverse sheaves] Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astrisque, 100, Soc. Math. France, Paris, 1982.

[7] A. K. Bousfield and E. M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, pp. 80–130, Lecture Notes in Math., 658, Springer, Berlin, 1978.

[8] Brown and Gersten, Algebraic K-theory as generalized sheaf cohomology. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 266–292. Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973.

[9] A. Grothendieck, La théorie des classes de Chern. (French) Bull. Soc. Math. France 86 1958 137–154.

[10] F. Déglise, Modules homotopiques, Thèse, in preparation.

[11] P. Gabriel et M. Zisman, Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35 Springer-Verlag New York, Inc., New York 1967.

[12] J. Hornbostel, A^1-representability of hermitian K-theory and Witt groups, preprint, http://www.math.uiuc.edu/K-theory/0578/.

[13] L. Illusie, Complexe cotangent et déformations. I & II. (French) Lecture Notes in Mathematics, Vol. 239 & 283. Springer-Verlag, Berlin-New York, 1971, 1972.

[14] J.F. Jardine, Simplicial presheaves. J. Pure Appl. Algebra 47 (1987), no. 1, 35–87.

[15] J.F. Jardine, Motivic symmetric spectra. Doc. Math. 5 (2000), 445–553 (electronic).

[16] J. P. May, Simplicial objects in algebraic topology. Van Nostrand Mathematical Studies, No. 11 D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London 1967.

[17] J. Milne, Étale cohomology. Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980.

[18] J. Milnor, Algebraic K-theory and quadratic forms. Invent. Math. 9 1969/1970 318–344.

[19] F. Morel, *Théorie homotopique des schémas*, Astérisque 256 (1999).

[20] F. Morel, Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques, C.R. Acad. Sci. Paris, t. 328, Série I, p. 963-968, 1999.

[21] F. Morel, Voevodsky's proof of Milnor's conjecture Bull. Amer. Math. Soc. 35 (1998), pp. 123-143.

[22] F. Morel, Stable $\mathbb{A}^1$-homotopy theory I, preprint, http:// www.math.jussieu.fr/~morel/listepublications.html

[23] F. Morel, Suites spectrales d'Adams et conjectures de Milnor, En préparation.

[24] F. Morel, Stable $\mathbb{A}^1$-homotopy theory II, in preparation.

[25] F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, Publications Mathématiques de l'I.H.É.S. no **90**.

[26] Y. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), 241–342, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, Kluwer Acad. Publ., Dordrecht, 1989.

[27] I. Panin, private communication.

[28] P. Hu, S-module in the category of schemes, preprint, http://www.math.uiuc.edu/K-theory/0396/

[29] P. Hu and I. Kriz, The Steinberg relation in $\mathbb{A}^1$-stable homotopy. Internat. Math. Res. Notices 2001, no. 17, 907–912.

[30] D. Quillen, *Homotopical algebra.* Lecture Notes in Mathematics, No. **43** Springer-Verlag, Berlin-New York 1967.

[31] D. Quillen, Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[32] Michel Raynaud, *Anneaux locaux henséliens*, (French) Lecture Notes in Mathematics, Vol. 169 Springer-Verlag, Berlin-New York 1970.

[33] M. Rost, Chow groups with coefficients. Doc. Math. 1 (1996), No. 16, 319–393 (electronic).

[34] W. Scharlau, *Quadratic and Hermitian forms.* Grundlehren der Mathematischen Wissenschaften, **270**. Springer-Verlag, Berlin, 1985.

[35] A. Suslin and V. Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[36] R. Thomason and T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories. The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., 88, Birkhuser Boston, Boston, MA, 1990.

[37] V. Voevodsky, Cohomological theory of presheaves with transfers, in *Cycles, transfers, and motivic homology theories*, V. Voevodsky; A. Suslin; E.M. Friedlander. Annals of Mathematics Studies, 143. Princeton University Press, Princeton, NJ, 2000.

[38] V. Voevodsky, Triangulated categories of mixed motives over a field, in *Cycles, transfers, and motivic homology theories*, V. Voevodsky; A. Suslin; E.M. Friedlander. Annals of Mathematics Studies, 143. Princeton University Press, Princeton, NJ, 2000.

[39] V. Voevodsky. The $\mathbb{A}^1$-homotopy theory. Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 579–604 (electronic).

[40] V. Voevodsky, The Milnor conjecture, preprint, http://www.math.uiuc.edu/K-theory/0170/