K-Theory and Arithmetic

Manfred Kolster*

McMaster University, Department of Mathematics, Hamilton, Ontario, Canada

(Dedicated to H. Bass on the occasion of his 70th birthday)

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*kolster@mcmail.cis.mcmaster.ca

Abstract

The main focus in these notes is on the relation between special values of zeta-functions of global fields and orders of cohomology groups and algebraic K-theory groups attached to rings of integers, a relation extending Dirichlet's analytic class number formula. The function field case serves as a motivation for the study of étale cohomology and its relation to Iwasawa theory. We describe the basic properties of étale cohomology groups, and study in particular Galois descent and co-descent for Galois extensions, the modifications – positive étale cohomology – needed to treat the case of the prime 2, and suitable "globalizations". We provide a brief introduction to Iwasawa theory, concentrating on the relation between the Main Conjecture and the orders of certain étale cohomology groups. The connection to algebraic K-groups is obtained via étale Chern characters, and we discuss the recent progress on the Quillen-Lichtenbaum Conjecture, which is implied by a conjecture of Bloch-Kato on the Galois symbols in Milnor K-theory. This also leads to a reformulation of the Lichtenbaum Conjectures in terms of motivic cohomology rather than K-theory. Finally, we describe the impact of these results on the study of the K-groups of $\mathbb{Z}$ and their relation to Vandiver's Conjecture.

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0 Introduction

Around 1736 Euler computed the following infinite series

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n^2} &= \frac{\pi^2}{6}\\ \sum_{n\geq 1}\frac{1}{n^4} &= \frac{\pi^4}{90}\\ \ldots & \quad \ldots\\ \sum_{n\geq 1}\frac{1}{n^{12}} &= \frac{691}{6825\cdot 93555}\cdot\pi^{12}\\ \ldots &= \ldots\end{aligned}$$

In fact, he calculated the sums for all even exponents in terms of Bernoulli numbers:

Theorem 0.1. [Euler]. *For* $m \geq 1$*:*

$$\sum_{n\geq 1}\frac{1}{n^{2m}} = (-1)^{m+1}\frac{B_{2m}}{(2m)!}2^{2m-1}\pi^{2m}.$$

The Bernoulli numbers B_n were defined by Jacob Bernoulli around 1713 as the coefficients of the power series of $\frac{x}{e^x-1}$ around 0:

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!}.$$

They are rational numbers, which can easily be calculated from the recursion formula

$$\frac{B_n}{n!} = -\sum_{k=0}^{n-1}\frac{B_k}{k!}\frac{1}{(n-k+1)!}.$$

This yields for example

$$B_0 = 1,\quad B_1 = -\frac{1}{2},\quad B_2 = \frac{1}{6},\quad B_{2m+1} = 0\quad \text{for } m \geq 1.$$

In modern terminology, Euler computed the values of the Riemann zeta-function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

at even positive integers $s = 2m$. The Riemann zeta-function, which is a priori only defined for $Re(s) > 1$ has a meromorphic continuation throughout the complex plane with a single simple pole at $s = 1$ (the harmonic series) and residue

$$\lim_{s \to 1} (s-1)\zeta(s) = 1.$$

Furthermore, it satisfies a functional equation relating the values at s and $1-s$:

$$\pi^{-\frac{s}{2}} \, \Gamma\left(\frac{s}{2}\right) \, \zeta(s) = \pi^{\frac{s-1}{2}} \, \Gamma\left(\frac{1-s}{2}\right) \, \zeta(1-s).$$

Here $\Gamma(s)$ denotes the Γ-function, which is a non-zero meromorphic function with simple poles at $s = -m$ for $m = 0, 1, \cdots$ and residues $(-1)^m/m!$. Let us consider the functional equation at $s = 2m$. Using the facts that $\Gamma(m) = (m-1)!$ and

$$\Gamma\left(\frac{1}{2} - m\right) = (-1)^m \frac{2^{2m} m!}{(2m)!} \pi^{1/2}$$

we obtain

Corollary 0.2. *For all $m \geq 1$:*

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}.$$

As examples we note that

$$\zeta(-1) = \frac{1}{12}, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-11) = \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}.$$

Since the Γ-function has a simple pole at $s = 0$ with residue 1, the functional equation shows that

$$\zeta(0) = -\frac{1}{2}.$$

Not too much is known about the values of the zeta-function at *odd* positive integers $1 + 2m$. In 1979 Apéry [1] showed that $\zeta(3)$ is irrational. A striking generalization is due to Rivoal, who proved that $\zeta(1 + 2m)$ is irrational for infinitely many m (cf. [78]) and at least once in the range $2 \leq m \leq 10$ ([79]). To interpret the values we use again the functional equation. The Γ-function has a simple pole at $-m$, which has to be compensated by a simple zero of the zeta-function at $s = -2m$. We denote by $\zeta^*(-2m)$ the first non-vanishing coefficient in a Taylor-expansion around $s = -2m$ and call this the *special value* of the zeta-function at $s = -2m$. We obtain

$$\zeta(1 + 2m) = (-1)^m \cdot \frac{2^{2m+1}}{(2m)!} \cdot \pi^{2m} \cdot \zeta^*(-2m),$$

in particular

$$\zeta(3) = -4\pi^2 \cdot \zeta^*(2).$$

What we have seen so far is that in order to study values of the zeta-function at positive integers it suffices to study special values of the zeta-function at non-positive integers.

In order to understand the special values even at $s = 0$ we have to generalize from $\mathbb{Q}$ to an algebraic number field F with r_1 real embeddings and r_2 pairs of conjugate complex embeddings. Let o_F denote the ring of integers in F, and for any non-zero ideal $I \subset o_F$ let $N(I) = |o_F/I|$ denote the number of elements in the finite quotient ring o_F/I. Dedekind generalized the Riemann zeta-function and defined the zeta-function of F as

$$\zeta_F(s) = \sum_{0 \neq I \subset o_F} \frac{1}{N(I)^s},$$

which is again convergent for $Re(s) > 1$, can be extended to a meromorphic function on $\mathbb{C}$ with a single simple pole at $s = 1$, and satisfies a functional equation relating $\zeta_F(s)$ and $\zeta_F(1 - s)$: Let

$$A = 2^{-r_2} \pi^{-(\frac{r_1 + 2r_2}{2})} \sqrt{|d(F)|},$$

where $d(F)$ denotes the discriminant of F. The functional equation then reads:

$$A^s \Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta_F(s) = A^{1-s} \Gamma(\frac{1-s}{2})^{r_1} \Gamma(1-s)^{r_2} \zeta_F(1-s).$$

If $s = n$ is a natural number ≥ 1, then the product of the Gamma-factors on the right-hand side of the functional equation yields a pole of total order r_2, if n is even, and of total order $r_1 + r_2$, if n is odd. For $s = n = 1$, the left-hand side of the functional equation has a simple pole, hence the order of vanishing of the zeta-function of F at $s = 0$ is equal to $r_1 + r_2 - 1$. For all $n \geq 2$ the left-hand side of the functional equation at $s = n$ is a positive real number, hence the order of vanishing of $\zeta_F(s)$ at $s = 1 - n$, $n \geq 2$, is equal to r_2 if n is even and equal to $r_1 + r_2$, if n is odd. If we denote by d_n the order of vanishing of the zeta-function at $s = 1 - n$, then we can summarize the information as

$$d_n = \begin{cases} r_1 + r_2 - 1 & \text{if } n = 1 \\ r_1 + r_2 & \text{if } n \geq 3 \text{ is odd} \\ r_2 & \text{if } n \geq 2 \text{ is even} \end{cases}$$

Dirichlet computed the residue of the zeta-function of F at $s = 1$. This result, the so-called Analytic Class Number Formula, computes the special value $\zeta_F^*(0)$ via the functional equation as follows:

Theorem 0.3. [Dirichlet]

$$\zeta_F^*(0) = -\frac{h_F}{w_F} \cdot R_F,$$

where h_F denotes the class number of F, w_F the number of roots of unity of F, and R_F is the Dirichlet regulator.

We note that the order of vanishing of $\zeta_F(s)$ at $s = 0$ is equal to the rank of the unit group of o_F. The Dirichlet regulator map provides a logarithmic embedding of the lattice $o_F^*/\mu(F)$ into the real vectorspace $\mathbb{R}^{r_1+r_2-1}$ and the Dirichlet regulator is defined as the covolume of the image lattice. As an example we consider $F = \mathbb{Q}(\sqrt{2})$: The fundamental unit is $1 + \sqrt{2}$, and the Dirichlet regulator is simply $log(1 + \sqrt{2})$.

Theorem 0.3 allows the following algebraic interpretation of the special value of the Riemann zeta-function at $s = 0$: $\zeta(0) = -\frac{1}{2}$ simply expresses the fact, that the regulator is trivial, the class number of $\mathbb{Z}$ is equal to 1 and the order of the group of roots of unity in $\mathbb{Z}$ is 2.

Let us consider now the special values at $1 - n$ for $n \geq 2$.

It was only after the invention of the algebraic K-theory functors from rings to abelian groups by Grothendieck (K_0), Bass (K_1), Milnor (K_2) and finally Quillen (K_n, $n \geq 1$) (cf. section 1) that an attempt could be made to interpret the special values at $1-n$ for $n \geq 2$ algebraically in the spirit of Dirichlet's class number formula. The following facts are known about the K-theory of o_F (cf. section 1): First of all

$$K_0(o_F) \cong \mathbb{Z} \oplus Cl\,(F), \quad K_1(o_F) \cong o_F^*,$$

so that Theorem 0.3 can be reformulated as

$$\zeta^*(0) = -\frac{|K_0(o_F)_{tors}|}{|K_1(o_F)_{tors}|} \cdot R_F.$$

Borel (cf. [13]) determined the group structure of Quillen's higher K-theory groups: For $n \geq 2$ the K-theory groups $K_{2n-2}(o_F)$ are finite and

$$K_{2n-1}(o_F) \cong \mathbb{Z}^{d_n} \oplus (\text{finite}).$$

In particular, for $n \geq 1$, the order of vanishing of ζ_F at $s = 1-n$ is equal to the rank of $K_{2n-1}(o_F)$. In fact, Borel computed the rank by defining higher regulator maps

$$\rho_n^B(F) : K_{2n-1}(o_F) \longrightarrow \mathbb{R}^{d_n}.$$

He showed that the kernel is finite and the image is a lattice of rank d_n. The covolume of this lattice is called the *Borel regulator* and denoted by $R_n^B(F)$. As a consequence Borel obtained

$$\zeta_F^*(1-n) = q_n \cdot R_n^B(F)$$

(cf. [14]) with a non-zero rational number q_n, hence a qualitative result in the direction of generalizing Dirichlet's class number formula. Borel's result confirms in the special case of number rings a general conjecture of Beilinson about the relation between special values of L-functions and regulators for smooth projective varieties over number fields (cf. [7]). Around 1971 Lichtenbaum (cf. [64]) gave a conjectural interpretation of the rational numbers q_n. He suggested that the correct generalization of Dirichlet's class number formula should read:

Lichtenbaum Conjecture. *For all $n \geq 2$:*

$$\zeta_F^*(1-n) = \pm \frac{|K_{2n-2}(o_F)|}{|K_{2n-1}(o_F)_{tors}|} \cdot R_n^B(F)$$

up to powers of 2.

A special case of this conjecture was formulated earlier by Birch and Tate (cf. [93]):

Birch-Tate Conjecture. *For a totally real number field*

$$\zeta_F(-1) = \pm \frac{|K_2(o_F)|}{w_2(F)}.$$

Here for any integer $n \geq 1$ we denote by $w_n(F)$ the largest integer m, such that $Gal\,(\bar{F}/F)$ acts trivially on the n-fold tensorproduct $\mu_m^{\otimes n}$. In terms of Galois cohomology this means

$$w_n(F) = |H^0(F, \mathbb{Q}/\mathbb{Z}(n))|.$$

The motivation for this conjecture came from Tate's proof (cf. [93]) of an analogous formula in the function field case, which we describe below. In the special case $F = \mathbb{Q}$ we know that

$$\zeta(-1) = -\frac{1}{12}.$$

On the other hand $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $w_2(\mathbb{Q}) = 24$, which verifies the Birch-Tate Conjecture for $\mathbb{Q}$. Since it is known (cf. [59]) that $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$, we see that the Lichtenbaum Conjecture (including the prime 2) reads

$$\zeta(-1) = -2 \cdot \frac{|K_2(\mathbb{Z})|}{|K_3(\mathbb{Z})|},$$

a result, which lead Lichtenbaum to exclude powers of 2 in his formulation.

We note that the sign of $\zeta_F^*(1-n)$, $n \geq 2$, is easy to determine from the functional equation. One obtains:

$$\mathrm{sign}\,(\zeta_F^*(1-n)) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 4 \\ (-1)^{r_1+r_2} & \text{if } n \equiv 2 \mod 4 \\ (-1)^{r_1} & \text{if } n \equiv 3 \mod 4 \\ (-1)^{r_2} & \text{if } n \equiv 0 \mod 4 \end{cases}$$

Let us have a closer look at the function field case, which – as always – provided a lot of inspiration for the number field situation:

Let X be a smooth projective connected curve over a finite field $\mathbb{F}_q$ of characteristic p. Let N_r denote the number of rational points of $\bar{X}$ over $\mathbb{F}_{q^r}$, where $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ denotes the corresponding curve over an algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. The zeta-function of X is defined by

$$Z_X(T) = exp\left(\sum_{r=1}^{\infty} N_r \cdot \frac{T^r}{r}\right)$$

considered as a formal power series in $\mathbb{Q}[[T]]$. Weil has shown that

$$Z_X(T) = \frac{P_1(T)}{(1-T)(1-qT)}$$

with

$$P_1(T) = \prod_{i=1}^{2g}(1-\alpha_i T),$$

where g is the genus of X and the α_i are the eigenvalues of the Frobenius acting on the Jacobian variety J of $\bar{X}$. They have absolute value $q^{\frac{1}{2}}$.

To see the relation with the Dedekind zeta-functions of number fields one defines

$$\zeta_X(s) = Z_X(q^{-s})$$

for $s \in \mathbb{C}$. Let F denote the function field of X. Thus F is a finite extension of $\mathbb{F}_q(T)$, and

$$\zeta_X(s) = \prod_{\mathfrak{p}} \frac{1}{(1-N(\mathfrak{p})^{-s})},$$

where the product extends over *all* primes in F.

To determine the values of the zeta-functions at $s = 1-n$ for $n \geq 2$ one uses étale cohomology. In fact, this cohomology theory was invented by Grothendieck et al. to prove part of the Weil Conjectures for smooth projective varieties over $\mathbb{F}_q$. Let us fix a prime $l \neq p$. For our purposes we can desribe the étale cohomology groups $H^*_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ as the Galois cohomology groups of the maximal unramified extension of F. Let us assume for simplicity that $\mu_l \subset \mathbb{F}_q$. Let $F_\infty = F(\mu_{l^\infty})$. Then F_∞/F is a $\mathbb{Z}_l$-extension, *i.e.*, $\Gamma = Gal\,(F_\infty/F)$ is isomorphic to $\mathbb{Z}_l$, generated by the Frobenius automorphism γ. Let X_∞ denote the Galois group of the maximal abelian unramified pro-l-extension of F_∞. This is a module over the

Iwasawa-algebra $\Lambda = \mathbb{Z}_l[[T]]$. It is easy to see from the Hochschild-Serre spectral sequence that

$$H^1_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))^{\Gamma} \cong Hom\,(X_\infty, \mathbb{Q}_l/\mathbb{Z}_l(n))^{\Gamma}.$$

Iwasawa-theory now shows that the Kummer dual $Hom\,(X_\infty, \mathbb{Q}_l/\mathbb{Z}_l(1))$ of X_∞ is canonically isomorphic to the l-primary part J_l of the Jacobian of F_∞, hence

$$Hom\,(X_\infty, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong J_l(n-1).$$

We obtain

$$H^1_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong J_l(n-1)^{\Gamma},$$

the order of which is equal to the order of the kernel of $1 - q^{n-1}\gamma$ acting on J_l, hence equal to the l-part of $P_1(q^{1-n})$. We have shown the following:

$$\zeta_X(1-n) \sim_l \frac{|H^1_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}{(1-q^{n-1})(1-q^n)}.$$

Here $\sim_l$ means that both sides have the same l-adic valuation. The denominator can also be interpreted in terms of étale cohomology:

$$|H^0_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))| = q^n - 1$$

and

$$|H^2_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))| = q^{n-1} - 1.$$

Moreover, since all the cohomology groups involved are finite, we have

$$H^i_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^{i+1}_{\text{ét}}(X, \mathbb{Z}_l(n)).$$

We obtain:

Theorem 0.4. *Let X be a smooth projective connected curve over a finite field $\mathbb{F}_q$ of characteristic p. For all $n \geq 2$ and all primes $l \neq p$ we have*

$$\begin{aligned}\zeta_X(1-n) \quad &\sim_l \quad \frac{|H^1_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}{|H^0_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))| \cdot |H^2_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|} \\ &\sim_l \quad \frac{|H^2_{\acute{e}t}(X, \mathbb{Z}_l(n))|}{|H^1_{\acute{e}t}(X, \mathbb{Z}_l(n))| \cdot |H^3_{\acute{e}t}(X, \mathbb{Z}_l(n))|}.\end{aligned}$$

Theorem 0.4 expresses the l-primary part of the zeta-function as an Euler characteristic in étale cohomology. It generalizes to arbitrary smooth projective connected schemes (cf. [5]).

Since we considered so far projective curves, which is the right framework in the function field case, we have not obtained a complete analogy with the number field case due to the presence of infinite primes. Let o_F denote the integral closure of $\mathbb{F}_q[T]$ in the function field F. The maximal ideals in o_F are the *finite* primes in F, and the associated zeta-function is defined via

$$\zeta_F(s) = \prod_{\mathfrak{p}\, fin.} \frac{1}{(1 - N(\mathfrak{p})^{-s})}.$$

Therefore the two zeta-functions differ by the product of the Euler factors at the infinite primes of F:

$$\zeta_F(s) = \zeta_X(s) \cdot \prod_{v|\infty} (1 - N(v)^{-s}).$$

To obtain now an analogous description for the zeta-function $\zeta_F(s)$ we use the exact localization sequence in étale cohomology (cf. section 2):

$$\begin{aligned} 0 \quad \longrightarrow \quad & H^1_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow H^1_{\text{ét}}(o_F, \mathbb{Q}_l/\mathbb{Z}_l(n)) \\ \longrightarrow \quad & \bigoplus_{v|\infty} H^0(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n-1)) \longrightarrow H^2_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow 0 \end{aligned}$$

together with the facts that

$$H^2_{\text{ét}}(o_F, \mathbb{Q}_l/\mathbb{Z}_l(n)) = 0$$

and

$$|H^0(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n-1))| = N(v)^{n-1} - 1.$$

The cohomological analog of Lichtenbaum's Conjecture in the function field case is therefore:

Theorem 0.5. *Let F be a global field of characteristic $p > 0$. For all $n \geq 2$ and all primes $l \neq p$ we have*

$$\zeta_F(1-n) \sim_l \frac{|H^2_{\acute{e}t}(o_F, \mathbb{Z}_l(n))|}{w_n(F)}.$$

To obtain a global formula let us define

$$H^2(o_F, \mathbb{Z}(n)) = \prod_{l \neq char F} H^2_{\text{ét}}(o_F, \mathbb{Z}_l(n)),$$

and let $h_n(F)$ denote the order of this finite group. Then

Corollary 0.6. *Let F be a global field of characteristic $p > 0$. For all $n \geq 2$ we have*

$$\zeta_F(1-n) = \pm \frac{|H^2(o_F, \mathbb{Z}(n))|}{w_n(F)} = \pm \frac{h_n(F)}{w_n(F)}.$$

For a number field F with ring of integers o_F and a prime l we use the étale cohomology groups, that can be viewed as the Galois cohomology groups of the maximal algebraic extension of F, which is unramified outside primes above l and ∞, in other words, we are looking at the étale cohomology groups of $spec\, o_F[1/l]$ for all l. We denote these cohomology groups simply by $H^i_{\text{ét}}(o'_F, \mathbb{Z}_l(n))$, where $o'_F = o_F[\frac{1}{l}]$. The relation to the algebraic K-theory groups of the ring of integers in a global field F is given via étale Chern characters for $l \neq char\,(F)$, $i = 1, 2$,

$$K_{2n-i}(o_F) \otimes \mathbb{Z}_l \longrightarrow H^i_{\text{ét}}(o'_F, \mathbb{Z}_l(n))$$

defined by Soulé and Dwyer-Friedlander (cf. [85],[23], section 2), which are surjective for all $l \neq 2$ and for $l = 2$ if F is a function field or if $\sqrt{-1} \in F$. They are conjectured to be isomorphisms in these cases (*Quillen-Lichtenbaum Conjecture*). It follows from Tate's computation of the K_2-group of a global field (cf. [94]) that the Quillen-Lichtenbaum Conjecture is true for $n = i = 2$, *i.e.*,

$$K_2(o_F) \otimes \mathbb{Z}_l \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}_l(n))$$

for all $l \neq char\,(F)$. In general, the Quillen-Lichtenbaum Conjecture is a consequence of a Conjecture of Bloch and Kato (cf. section 2), which seems to have been proven by Rost and Voevodsky. Since - by a result of Geisser-Levine [31] - $K_m(F)(p) = 0$ for all fields F of characteristic p, and all $m \geq 2$, we can reformulate Corollary 0.6. under the assumption of the Quillen-Lichtenbaum Conjecture as follows:

Theorem 0.7. *Let F be a global field of characteristic $p > 0$. If the Quillen-Lichtenbaum Conjecture is true for all $l \neq p$, then we have for all $n \geq 2$:*

$$\zeta_F(1-n) = \pm\frac{|K_{2n-2}(o_F)|}{w_n(F)} = \pm\frac{|K_{2n-2}(o_F)|}{|K_{2n-1}(o_F)|}.$$

The special case $n = 2$: $\zeta_F(-1) = \pm\frac{|K_2(o_F)|}{w_2(F)}$ is due to Tate and led to the formulation of the Birch-Tate Conjecture for number fields.

Let us return now to the situation of a number field F. Iwasawa (cf. [44]) extensively studied the theory of $\mathbb{Z}_p$-extensions for any prime p, in particular the cyclotomic $\mathbb{Z}_p$-extension of a *totally real* number field (cf. section 3). The analog of the Jacobian is a certain Iwasawa-module, whose characteristic polynomial is related to p-adic L-functions via the so-called Main Conjecture, which has been proven by Wiles (cf. [101]) for odd primes p and for $p = 2$ in the case $F = \mathbb{Q}$. As a consequence we obtain the following analog of 0.6:

Theorem 0.8. [Wiles] *Let F be a totally real number field with ring of integers o_F. For all* even *integers $n \geq 2$:*

$$\zeta_F(1-n) = \pm\frac{h_n(F)}{w_n(F)}$$

up to powers of 2. If F is abelian over $\mathbb{Q}$, then the formula also gives the correct powers of 2.

Again, for $n = 2$ this can be reformulated in terms of K-groups and verifies the Birch-Tate Conjecture:

Corollary 0.9. *Let F be a totally real abelian number field. Then*

$$\zeta_F(-1) = \pm\frac{|K_2(o_F)|}{w_2(F)}.$$

For arbitrary totally real number fields the same result holds up to powers of 2.

The deviation between $K_3(F) = K_3(o_F)$ and $H^0(F, \mathbb{Q}/\mathbb{Z}(n))$ is known (cf. [67]): There is an exact sequence

$$0 \to K_3^M(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1(F)} \to K_3(F) \to H^0(F, \mathbb{Q}/\mathbb{Z}(n)) \to 0.$$

Here $K_3^M(F)$ denotes Milnor's K_3-group (cf. section 1). Corollary 0.9 can therefore be reformulated as:

Corollary 0.10. *Let F be a totally real abelian number field of degree r_1. Then*

$$\zeta_F(-1) = \pm 2^{r_1} \cdot \frac{|K_2(o_F)|}{|K_3(o_F)|}.$$

For arbitrary totally real number fields the same result holds up to powers of 2.

For arbitrary $n \geq 2$ the deviation in the 2-torsion between K-theory and étale cohomology is known from Voevodsky's proof of the Milnor Conjecture (cf. [97],[49],[81]). If we assume that the Quillen-Lichtenbaum Conjecture holds for odd primes p, then we can reformulate Theorem 0.8 both in terms of K-groups and in terms of motivic cohomology groups $H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n))$ (cf. [30]):

Theorem 0.11. *Let F be a totally real abelian number field of degree r_1 and let $n \geq 2$ be even. If the Quillen-Lichtenbaum Conjecture holds for odd primes p, then*

$$\zeta_F(1-n) = \pm\, 2^{r_1} \cdot \frac{|K_{2n-2}(o_F)|}{|K_{2n-1}(o_F)|} = \pm \frac{|H^2_{\mathcal{M}}(o_F, \mathbb{Z}(n))|}{|H^1_{\mathcal{M}}(o_F, \mathbb{Z}(n))_{\text{tors}}|}$$

for all even $n \geq 2$.

So far we have dealt with the situation that the zeta-function does not vanish at $1-n$, hence no regulators are involved. There is an analogous relative situation, which does not involve regulators, and which is essential for the general case: Let L/F be a CM-extension of number fields, and let χ denote the non-trivial Artin character of $Gal\,(L/F)$. Then

$$\zeta_L(s) = \zeta_F(s) \cdot L_F(\chi, s),$$

where $L_F(\chi, s)$ denotes the Artin L-function attached to χ. For *odd* integers $n \geq 1$ the orders of vanishing of $\zeta_L(s)$ and $\zeta_F(s)$ at $s = 1-n$ are the same, hence $L_F(\chi, 1-n)$ is a non-zero rational number. For $n = 1$ this number is classically determined by the relative class number formula (cf. [99], Chapter 4):

$$L_F(\chi, 0) = \frac{2^{r_1}}{Q} \cdot \frac{h^-}{w(L)},$$

where $r_1 = [F : \mathbb{Q}]$, $h^- = \frac{h_L}{h_F}$ is the relative class number, and Q is the so-called Q-index:

$$Q = [o_L^* : o_F^* \cdot \mu(L)],$$

which is equal to 1 or 2. For larger odd values of n the analog is:

Theorem 0.12. **[53]** *Let L/F be an abelian CM-extension of number fields and $n \geq 3$ odd. Then*

$$L_F(\chi, 1-n) = \pm \frac{2^{r_1+1}}{Q_n} \cdot \frac{h_n^-}{w_n(L)}.$$

For general CM-extensions the same result is true up to powers of 2.

Here

$$h_n^- = \frac{h_n(L)}{h_n(F)}$$

and Q_n is the following generalized Q-index, which again is equal to either 1 or 2:

$$Q_n = [H^1_{\text{ét}}(L, \mathbb{Z}_2(n)) : H^1_{\text{ét}}(F, \mathbb{Z}_2(n)) \cdot H^0(L, \mathbb{Q}_2/\mathbb{Z}_2(n))].$$

In the following main result towards the general Lichtenbaum Conjectures both Theorem 0.7 and Theorem 0.12 are needed, although only up to 2-torsion.

Theorem 0.13. **[54]** *Let F be an abelian number field. Then for all $n \geq 2$:*

$$\zeta_F^*(1-n) = \pm \frac{h_n(F)}{w_n(F)} \cdot R_n(F)$$

up to powers of 2.

The formulation uses the Beilinson regulator $R_n(F)$ and not the Borel regulator. The difference between the two regulators is now known due to a recent result of Burgos-Gil [16], who showed that the Borel regulator map is twice the Beilinson regulator map, hence

$$R_n^B(F) = 2^{d_n} R_n(F),$$

where - as before - d_n is the rank of $K_{2n-1}(F)$.

It should be pointed out that the formulation in [54] is not quite correct, it includes some Euler factors, which were removed in [57]. Part of the proof was also based on an incorrect result by Villemot. The necessary adjustments of the proof can be found in an appendix to [8]. An outline of the proof is given in section 4. For a different proof cf. [42].

If we assume that the Quillen-Lichtenbaum Conjecture holds for odd primes p, then Theorem 0.13 shows that Lichtenbaum's original conjecture is true for abelian fields. On the other hand the 2-primary information in Theorem 0.12 indicates that one is forced to replace K-groups by motivic cohomology groups to obtain a formulation including the exact power of 2. Let $\bar{H}^1_{\mathcal{M}}(F,\mathbb{Z}(n))$ denote $H^1_{\mathcal{M}}(F,\mathbb{Z}(n))$ modulo torsion. This is a free $\mathbb{Z}$-module of rank d_n. Let $R_n^{\mathcal{M}}(F)$ denote the covolume of the image of $\bar{H}^1_{\mathcal{M}}(F,\mathbb{Z}(n))$ under the Beilinson regulator map. This is the motivic regulator and the correct motivic formulation of the Lichtenbaum Conjecture including 2-primary information should be:

Motivic formulation of the Lichtenbaum Conjecture. *For any number field F and any integer $n \geq 2$:*

$$\zeta_F^*(1-n) = \pm \frac{|H^2_{\mathcal{M}}(F,\mathbb{Z}(n))|}{|H^1_{\mathcal{M}}(F,\mathbb{Z}(n))_{\text{tors}}|} \cdot R_n^{\mathcal{M}}(F).$$

If L/F is a CM-extension of number fields and if $n \geq 3$ is odd, then one can show that

$$[R_n^{\mathcal{M}}(L) : R_n^{\mathcal{M}}(F)] = \frac{2^{r_1}}{Q_n},$$

hence the motivic formulation of the Lichtenbaum Conjecture is consistent with Theorem 0.12. A recent result of Ion Rada, a student of mine, shows that for complex abelian fields and *odd* integers $n \geq 3$ the cohomological formulation in Theorem 0.13 also gives the correct powers of 2, up to a certain Iwasawa μ-invariant, which should be trivial.

Let us now return to Euler's computations of the values of the Riemann zeta-function at odd negative integers $1-n$, $n \geq 2$ even. Corollary 0.2 and Theorem 0.8 yield

$$\zeta(1-n) = -\frac{B_n}{n} = \pm \frac{h_n(\mathbb{Q})}{w_n(\mathbb{Q})}.$$

As we pointed out the numbers $w_n(\mathbb{Q})$ are easy to calculate: $w_n(\mathbb{Q})$ is the maximal number m, such that *Gal* $(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ has exponent m. Therefore Euler's result essentially computes $h_n(\mathbb{Q})$ in terms of Bernoulli numbers, and therefore - assuming the Quillen-Lichtenbaum Conjecture for odd p - the orders of the K-theory groups $K_{2n-2}(\mathbb{Z})$ for *even* $n \geq 2$ (cf. section 5).

Examples.

1. For $n = 4$ *we have*

$$\zeta(-3) = -\frac{1}{120} = -\frac{h_4(\mathbb{Q})}{w_4(\mathbb{Q})}.$$

We have

$$w_4(\mathbb{Q}) = 2^4 \cdot 3 \cdot 5 = 240,$$

so that

$$h_4(\mathbb{Q}) = 2,$$

which predicts

$$K_6(\mathbb{Z}) = 0.$$

2. For $n = 12$ *we have*

$$\zeta(-11) = \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13} = \frac{h_{12}(\mathbb{Q})}{w_{12}(\mathbb{Q})}.$$

We have

$$w_{12}(\mathbb{Q}) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13,$$

so that

$$h_{12}(\mathbb{Q}) = 2 \cdot 691,$$

which predicts

$$K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691\mathbb{Z}.$$

The structure of the cohomology groups $H^2(\mathbb{Z}, \mathbb{Z}(n))$ and of the K-groups $K_{2n-2}(\mathbb{Z})$ is related to Vandiver's Conjecture. It can be shown (cf. [58] and section 5) that Vandiver's Conjecture for the prime p is equivalent to the vanishing of $H^2_{\text{ét}}(\mathbb{Z}, \mathbb{Z}_p(n))$ for all *odd* $n \geq 3$. If we assume Vandiver's Conjecture for all p, then the motivic cohomology groups $H^2_{\mathcal{M}}(\mathbb{Z}, \mathbb{Z}(n))$ are cyclic for all *even* $n \geq 2$. Using the Quillen-Lichtenbaum Conjecture for odd primes this translates into: $K_{4m}(\mathbb{Z})$ is trivial for all $m \geq 1$ if and only if Vandiver's Conjecture holds for all primes, and in this case the K-groups $K_{4m-2}(\mathbb{Z})$ are cyclic for all $m \geq 1$. The result of Rognes-Soulé (cf. [80]) that $K_4(\mathbb{Z})$ is trivial can be viewed as supporting evidence for Vandiver's Conjecture.

Let us assume that Vandiver's Conjecture holds for all primes. Then for $n \geq 3$ odd we have $h_n(\mathbb{Q}) = 1$. Since $w_n(\mathbb{Q}) = 2$ for n odd, Theorem 0.13 implies that

$$\zeta^*(1-n) = \pm R_n(\mathbb{Q})$$

up to powers of 2. The K-groups $K_{2n-1}(\mathbb{Q})$ have rank 1 for $n \geq 3$ odd, and the Beilinson regulator $R_n(\mathbb{Q})$ should be simply given by the n-th polylogarithm evaluated on a generator of $K_{2n-1}(\mathbb{Q})$. In particular, for $n = 3$ we obtain

$$\zeta(3) = -\pi^2 R_3(\mathbb{Q})$$

up to powers of 2, and Goncharov (cf. [35]) has shown that in fact $R_3(\mathbb{Q})$ is given by a trilogarithm.

1 K-theory of rings of integers in number fields

For an arbitrary ring R with 1 the *Grothendieck group* $K_0(R)$ is the free abelian group generated by isomorphism classes $[P]$ of finitely generated projective left R-modules P modulo the subgroup generated by

$$[P] + [Q] - [P \oplus Q].$$

Two isomorphism classes $[P]$ and $[Q]$ are equal in $K_0(R)$ if and only if P and Q are stably isomorphic, *i.e.*, $P \oplus R^n \cong Q \oplus R^n$ for some n.

Examples.

1. If R is a local ring or a principal ideal domain, then $K_0(R) \cong \mathbb{Z}$.

2. If R is the endomorphism ring of an infinite-dimensional vectorspace, then $K_0(R)$ is trivial.

3. If R is a Dedekind domain, then there is a natural isomorphism

$$K_0(R) \cong \mathbb{Z} \oplus Cl(R),$$

where $Cl(R)$ is the class-group of R (cf. e.g. [71]).

The abelian group $K_1(R)$ is defined as the abelianization of the infinite general linear group $GL(R) = \cup_{n\geq 1} GL_n(R)$:

$$K_1(R) = GL(R)/[GL(R), GL(R)].$$

Let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by all elementary matrices

$$E_{ij}(a) = I_n + ae_{ij}, \quad 1 \leq i \neq j \leq n, \quad a \in R,$$

where e_{ij} are the standard matrix units. Let $E(R) = \cup_{n\geq 1} E_n(R)$. The generators $E_{ij}(a)$ satisfy the following "trivial" relations:

$$\begin{array}{rcll} E_{ij}(a)E_{ij}(b) & = & E_{ij}(a+b), & \\ [E_{ij}(a), E_{jk}(b)] & = & E_{ik}(ab) & \text{if } i \neq k, \\ [E_{ij}(a), E_{lk}(b)] & = & 1 & \text{if } i \neq k,\ j \neq l. \end{array}$$

The second relation shows that $E_n(R) = [E_n(R), E_n(R)]$ is a perfect group for $n \geq 3$, in particular $E(R)$ is perfect. It is easy to see ("Whitehead's Lemma") that

$$E(R) = [GL(R), GL(R)].$$

Examples.

1. If R is a local ring, then $K_1(R) \cong R^*/[R^*, R^*]$ (Dieudonné determinant).

2. Assume that R is commutative. Then the determinant induces a canonical splitting

$$K_1(R) \cong R^* \oplus SK_1(R),$$

where $SK_1(R) = SL(R)/E(R)$. Moreover, by a result of Suslin [89] the group $E_n(R)$ is normal in $GL_n(R)$, and therefore one obtains unstable K_1-groups

$$K_{1,n} = GL_n(R)/E_n(R)$$

for $n \geq 3$. These groups stabilize depending on the Krull dimension of R, in particular for a Dedekind domain one has $SK_1(R) = SL_n(R)/E_n(R)$ for $n \geq 3$.

3. If R is the ring of integers in a number field, then $SK_1(R) = 0$, hence

$$K_1(R) \cong R^*.$$

This result is deep and follows from the solution of the congruence subgroup problem in [3].

To define Milnor's group $K_2(R)$ of a ring R with 1 we note that the relations between elementary matrices listed above hold independent of the ring R, hence are universal relations in $E(R)$. Let $St(R)$ denote the free abelian group with generators $x_{ij}(a)$, $i \neq j$, $a \in R$, modulo the subgroup generated by the following universal relations:

$$\begin{array}{rcll} x_{ij}(a)x_{ij}(b) & = & x_{ij}(a+b), & \\ [x_{ij}(a), x_{jk}(b)] & = & x_{ik}(ab) & \text{if } i \neq k, \\ [x_{ij}(a), x_{lk}(b)] & = & 1 & \text{if } i \neq k,\ j \neq l. \end{array}$$

$St(R)$ is the so-called *Steinberg group*. Sending $x_{ij}(a)$ to $E_{ij}(a)$ defines a surjective map $St(R) \to E(R)$. Since $E(R)$ is perfect, it has a universal central extension, and it can be shown that the Steinberg group $St(R)$ is the universal central extension of $E(R)$ (cf. [71]). Milnor defines $K_2(R)$ as the kernel of the natural surjection $St(R) \to E(R)$. In particular, $K_2(R)$ is the center of $St(R)$, hence abelian. Furthermore, the theory of universal central extensions implies that

$$K_2(R) \cong H_2(E(R), \mathbb{Z}).$$

(cf. [71])

For a field F the group $K_2(F)$ has been computed by Matsumoto as the universal symbol group:

$$K_2(F) = F^* \otimes F^* / \langle u \otimes 1 - u; u \neq 1 \rangle,$$

in other words $K_2(F)$ is defined by generators $\{u, v\}$, $u, v \in F^*$ and relations

$$\begin{aligned} \{uv, w\} &= \{u, w\}\{v, w\} \\ \{u, vw\} &= \{u, v\}\{u, w\} \\ \{u, 1-u\} &= 1. \end{aligned}$$

Immediate consequences are

$$\begin{aligned} \{u, v\}^{-1} &= \{v, u\} \\ \{u, -u\} &= 1. \end{aligned}$$

Examples. *1. For a finite field $\mathbb{F}_q$ the group $K_2(\mathbb{F}_q)$ is trivial. ([71])*

2. $K_2(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} \oplus$ (uniquely divisible). ([90])

3. $K_2(\mathbb{C})$ is uniquely divisible. ([90])

4. $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, a result due to Silvester (cf. [71]).

Motivated by Matsumoto's theorem Milnor introduced higher K-theory groups $K_n^M(F)$ of a field F (cf. [70]) - now called *Milnor K-groups* - as the quotient of the n-fold tensor product $F^* \otimes F^* \otimes \cdots \otimes F^*$ by the subgroup generated by all $u_1 \otimes \cdots \otimes u_n$, such that $u_i + u_j = 0$ for some $i \neq j$. Let m be an integer prime to the characteristic of F. Then the cup-product induces a homomorphism

$$g_{n,m} : K_n^M(F)/m \longrightarrow H^n(F, \mu_m^{\otimes n}),$$

called the *Galois symbol*. Based on the relation to quadratic forms Milnor conjectured that for $m = 2$ the Galois symbols $g_{n,2}$ should be isomorphisms for all n. This Conjecture was extended by Bloch-Kato to all values of m. For global fields these conjectures follow from Tate's fundamental description of the K_2 of a global field F in terms of Galois cohomology in [93], and from the computation of the higher Milnor K-groups:

$$K_n^M(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$$

for $n \geq 3$ by Bass-Tate in [4].

Merkurjev-Suslin proved the Bloch-Kato Conjecture for $n = 2$ and arbitrary fields F, *i.e.*, they showed that the homomorphism

$$g_{2,m} : K_2(F)/m \to H^2(F, \mu_m^{\otimes 2})$$

is an isomorphism for all m, thus extending Tate's result to arbitrary fields (cf. [66]).

Voevodsky ([97]) proved the Milnor Conjecture for arbitrary fields and provided a general approach to the Bloch-Kato Conjecture. It seems that Rost and Voevodsky recently completed the proof of the Bloch-Kato Conjecture. The striking consequences of these results to the K-theory of rings of integers will be discussed in the next section.

Let us turn now to Quillen's definition of higher algebraic K-groups: Recall that

$$K_1(R) \cong H_1(GL(R), \mathbb{Z})$$

and

$$K_2(R) \cong H_2(E(R), \mathbb{Z}),$$

so that in the classical situation the K-groups are closely related to the integral homology of $GL(R)$. Quillen's idea was to look for a topological space, whose integral homology as a space is closely related to the integral homology of $GL(R)$, and to define the higher K-groups as homotopy groups of that space.

As a first step one looks at the classifying space $BGL(R)$ of $GL(R)$. Up to homotopy equivalence this space is characterized by the properties that it is connected and its homotopy groups are:

$$\pi_1(BGL(R)) \cong GL(R)$$

$$\pi_i(BGL(R)) = 0 \text{ for } i \geq 2.$$

Furthermore:

$$H_n(BGL(R), \mathbb{Z}) \cong H_n(GL(R), \mathbb{Z})$$

for all $n \geq 0$. Quillen's +–construction now adds 2-cells and 3-cells to $BGL(R)$ in such a way that the new space $BGL(R)^+$ has the same integral homology as $BGL(R)$ and the inclusion $BGL(R) \to BGL(R)^+$ induces the quotient map

$$\pi_1(BGL(R)) \cong GL(R) \to \pi_1(BGL(R)^+ \cong GL(R)/E(R).$$

For any ring R and $n \geq 1$ the higher K-theory groups $K_n(R)$ are now defined as

$$K_n(R) = \pi_n(BGL(R)^+).$$

We obtain a homomorphism

$$\begin{aligned} K_n(R) &= \pi_n(BGL(R)^+) \to H_n(BGL(R)^+, \mathbb{Z}) \\ &= H_n(BGL(R), \mathbb{Z}) = H_n(GL(R), \mathbb{Z}) \end{aligned}$$

called the *Hurewicz homomorphism*, which gives the expected relation to the integral homology of $GL(R)$.

Of course, for $n = 1$, we obtain as before

$$K_1(R) = H_1(GL(R), \mathbb{Z}) = GL(R)^{ab} = GL(R)/E(R).$$

Let $n = 2$. Since $BE(R)^+$ is the universal covering space of $BGL(R)^+$ and $\pi_1(BE(R)^+)$ is trivial, Hurewicz' theorem implies that

$$K_2(R) = \pi_2(BGL(R)^+) = \pi_2(BE(R)^+) = H_2(BE(R)^+) = H_2(E(R), \mathbb{Z}),$$

hence Quillen's definition of $K_2(R)$ agrees with Milnor's.

The Kronecker product of matrices induces a homomorphism

$$GL(R) \times GL(R) \to GL(R),$$

which induces on $BGL(R)^+$ the structure of a commutative H-group (cf. [65]). Roughly speaking this means that $BGL(R)^+$ satisfies the axioms of a commutative group up to homotopy equivalence. As a consequence one obtains natural products

$$K_i(R) \otimes K_j(R) \to K_{i+j}(R)$$

for all $i, j \geq 0$. These products satisfy

$$xy = (-1)^{ij} yx$$

for $x \in K_i(R)$ and $y \in K_j(R)$ ([65]). As an example, for $i = j = 1$ and F a field, the product $F^* \times F^* \to K_2(F)$ is given by

$$(u, v) \mapsto \{u, v\}^{-1}.$$

Another consequence of the fact that $BGL^+(R)$ is an H-space is that the Hurewicz map is injective modulo torsion, *i.e.*, that

$$K_n(R) \otimes \mathbb{Q} \longrightarrow H_n(GL(R), \mathbb{Q}))$$

is injective. The image can be identified with the so-called *primitive part* of the rational homology, a result, which has been used by Borel (cf. [13]) to define regulator maps on higher algebraic K-theory groups.

Let us state now some of Quillen's main results:

Theorem 1.1. (Quillen [74]) *Let $F = \mathbb{F}_q$ be a finite field with q elements. Then for all $n \geq 1$:*

$$K_{2n}(\mathbb{F}_q) = 0$$

$$K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}.$$

We note that this can be reformulated in terms of Galois cohomology as

$$K_{2n}(\mathbb{F}_q) \cong H^2(\mathbb{F}_q, \mathbb{Z}(n)) = 0$$

and

$$K_{2n-1}(\mathbb{F}_q) \cong H^1(\mathbb{F}_q, \mathbb{Z}(n)) \cong H^0(\mathbb{F}_q, \mathbb{Q}/\mathbb{Z}(n)).$$

As is well-known the zeta-function of the finite field $\mathbb{F}_q$ equals

$$\zeta_{\mathbb{F}_q}(s) = \frac{1}{1 - q^{-s}},$$

hence for $n > 0$ we obtain

$$\zeta_{\mathbb{F}_q}(-n) = -|K_{2n-1}(\mathbb{F}_q)|^{-1} = -\frac{|K_{2n}(\mathbb{F}_q)|}{|K_{2n-1}(\mathbb{F}_q)|} = -\frac{|H^2(\mathbb{F}_q, \mathbb{Z}(n))|}{|H^1(\mathbb{F}_q, \mathbb{Z}(n))|},$$

one of the results which inspired the Lichtenbaum Conjecture.

Quillen also calculated the higher K-groups of the algebraic closure $\bar{\mathbb{F}}_p$:

Theorem 1.2. (Quillen [74]) *For $n \geq 1$:*

$$K_{2n}(\bar{\mathbb{F}}_p) = 0$$

and

$$K_{2n-1}(\bar{\mathbb{F}}_p) \cong \oplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l.$$

The fact that the K-theory of $\bar{\mathbb{F}}_p$ has no p-torsion is true more generally:

Theorem 1.3. (Geisser-Levine [31]) *Let F be a field of positive characteristic p. Then the K-groups of F have no p-torsion.*

For rings of integers in global fields Quillen showed

Theorem 1.4. (Quillen [76]) *Let o_F be the ring of integers in a global field. For all $n \geq 0$ the K-theory groups $K_n(o_F)$ are finitely generated.*

The ranks of these finitely generated groups were determined by Borel for number fields:

Theorem 1.5. (Borel [13]) *For $n \geq 1$ the groups $K_{2n}(o_F)$ are finite and*

$$rk_{\mathbb{Z}}\left(K_{2n-1}(o_F)\right) = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd} > 1 \\ r_2 & \text{if } n \text{ is even.} \end{cases}$$

The finiteness of $K_2(o_F)$ had been previously obtained by Garland ([29]).

The proofs of both these results use *Geometry of numbers*, as in the classical case of Dirichlet's Unit Theorem and the proof of the finiteness of the class number.

Borel's results gave the first indication that the even K-groups of a ring of integers behave like higher-dimensional analogs of the class groups, and the odd K-groups like higher-dimensional analogs of the unit groups.

The powerful topological definition of higher algebraic K-groups has many interesting consequences, one of them is the long exact localization sequence, which comes from the exact sequence of a fibration. For Dedekind rings R with finite residue fields $k_{\mathfrak{p}}$ and quotient field F Theorem 1.1 implies that the long exact localization sequence gives 5-term exact sequences of the form

$$0 \to K_{2n}(R) \to K_{2n}(F) \to \oplus_{\mathfrak{p}} K_{2n-1}(k_{\mathfrak{p}}) \to K_{2n-1}(R) \to K_{2n-1}(F) \to 0$$

for $n \geq 1$. Soulé (cf. [86]) showed that for the ring of integers o_F in a global field the result gets even better:

$$K_{2n-1}(o_F) \cong K_{2n-1}(F)$$

for all $n \geq 2$, and there are short exact sequences

$$0 \to K_{2n}(o_F) \to K_{2n}(F) \to \oplus_{\mathfrak{p}} K_{2n-1}(k_{\mathfrak{p}}) \to 0$$

for all $n \geq 1$. The first result gives an indication of the fact that higher K-theory of rings of integers is in a sense "easier" than the classical K_0 and K_1. For $n = 1$ the last sequence reads

$$0 \to K_2(o_F) \to K_2(F) \xrightarrow{\lambda} \oplus_{\mathfrak{p}} K_1(k_{\mathfrak{p}}) \to 0.$$

The map $\lambda : K_2(F) \to \oplus_{\mathfrak{p}} K_1(k_{\mathfrak{p}})$ is explicitly given by the so-called *tame symbol*: For each prime ideal $\mathfrak{p}$ consider the map

$$\lambda_{\mathfrak{p}} : K_2(F) \to K_1(k_{\mathfrak{p}}) = k_{\mathfrak{p}}^*$$

defined by

$$\lambda_{\mathfrak{p}}(\{u, v\}) = (-1)^{v_{\mathfrak{p}}(u)v_{\mathfrak{p}}(v)} \cdot \frac{u^{v_{\mathfrak{p}}(v)}}{v^{v_{\mathfrak{p}}(u)}} \bmod \mathfrak{p},$$

where $v_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic valuation, and take $\lambda = \oplus \lambda_{\mathfrak{p}}$. Therefore $K_2(o_F) = ker\, \lambda$, and hence $K_2(o_F)$ is also referred to as the *tame kernel*.

2 Chern characters and étale cohomology

The theory of Chern classes and Chern characters is to a large extent purely formal and is e.g. described in [33] and in Schneider's survey article in [7]. It extends the classical definition of Grothendieck's of Chern classes for vectorbundles over quasi-projective varieties. The Chern classes are essentially defined on the homology groups $H_n(GL(R),\mathbb{Z})$ and take values in any "reasonable" cohomology theory. The Chern classes on higher K-theory groups are then simply obtained by composition with the Hurewicz map.

Let us first consider Chern characters with values in étale cohomology groups. For our purposes we can use a description of étale cohomology in terms of Galois cohomology, valid for global fields F and rings of integers o_F. Fix a prime $p \neq char\,(F)$. Let $\Omega_F^{(p)}$ denote the maximal algebraic extension of F, which is unramified outside primes above p and infinite primes, and let $G_F^{(p)} = Gal(\Omega_F^{(p)}/F)$. The étale cohomology groups $H^*_{\text{ét}}(spec\, o_F[\frac{1}{p}], \mu_{p^m}^{\otimes n})$ of the scheme $spec\, o_F[\frac{1}{p}]$ with values in the étale sheaf $\mu_{p^m}^{\otimes n}$ can be identified with the Galois cohomology groups $H^*(G_F^{(p)}, \mu_{p^m}^{\otimes n})$. Here the Galois group $G_F^{(p)}$ acts diagonally on the n-fold tensor product $\mu_{p^m}^{\otimes n}$. To simplify notations we will write $H^*_{\text{ét}}(o'_F, \mu_{p^m}^{\otimes n})$ or $H^*_{\text{ét}}(o'_F, \mathbb{Z}/p^m(n))$, where o'_F indicates that we are working over $o_F[\frac{1}{p}]$ and not over o_F. We will also denote by

$$H^*_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) = \varprojlim H^*_{\text{ét}}(o'_F, \mu_{p^m}^{\otimes n})$$

the p-adic étale cohomology groups and set

$$H^*_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \varinjlim H^*_{\text{ét}}(o'_F, \mu_{p^m}^{\otimes n}).$$

We note the following relationship: For each $n \in \mathbb{Z}$ the exact sequence

$$0 \to \mathbb{Z}_p(n) \to \mathbb{Q}_p(n) \to \mathbb{Q}_p(n)/\mathbb{Z}_p(n) \to 0$$

gives rise to a long exact sequence in étale cohomology and the kernels and cokernels of the boundary maps

$$\delta_i : H^{i-1}_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) \ \ (i \geq 1)$$

can be described as follows (cf. [94]): The kernel of δ_i is the maximal divisible subgroup of $H^{i-1}_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and the image of δ_i is the torsion subgroup of $H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$. In particular this implies that the torsion subgroup of $H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ is isomorphic to $H^0_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$:

$$H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))_{\text{tors}} \cong H^0_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$$

The following is known about the finitely generated p-adic étale cohomology groups for rings of integers in global fields:

Proposition 2.1. *1.* $H^0_{ét}(o'_F, \mathbb{Z}_p(n)) = 0$ *for* $n \neq 0$ *([85]).*

2. For $k \geq 3$*:* $H^k_{ét}(o'_F, \mathbb{Z}_p(n)) = 0$ *if* p *is odd ([85]).*

3. For $k \geq 3$*:* $H^k_{ét}(o'_F, \mathbb{Z}_2(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r_1(F)} & \text{if } k+n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$

The relation between the étale cohomology groups of $spec\, o_F[\frac{1}{p}]$ and $spec\, F$ – the latter being isomorphic to the continuous Galois cohomology groups of F – is provided by Soulé's exact localisation sequence (cf. [85]), a consequence of the Leray spectral sequence in étale cohomology:

Proposition 2.2. *There is a long exact sequence*

$$0 \to H^1_{ét}(o'_F, \mu_{p^m}^{\otimes n}) \to H^1_{ét}(F, \mu_{p^m}^{\otimes n}) \to \oplus_v H^0_{ét}(k_v, \mu_{p^m}^{\otimes n-1}) \to H^2_{ét}(o'_F, \mu_{p^m}^{\otimes n}) \to \cdots$$

with v *running through the finite places of* F *not dividing* p*.*

Since $H^0_{\text{ét}}(k_v, \mathbb{Z}_p(n-1)) = 0$ for $n \neq 1$, passing to the projective limit in the first part of the localization sequence gives the following interesting consequence:

Corollary 2.3. *For all* $n \neq 1$ *there is an isomorphism*

$$H^1_{ét}(o'_F, \mathbb{Z}_p(n)) \cong H^1_{ét}(F, \mathbb{Z}_p(n)).$$

For $n = 1$ the étale cohomology groups $H^i_{\text{ét}}(o'_F, \mu_{p^m})$ have a classical interpretation. Let $\mathbb{G}_m$ denote the multiplicative group scheme. The étale cohomology groups $H^i_{\text{ét}}(o'_F, \mathbb{G}_m)$ are known (cf. [68], Chapter III):

$$\begin{aligned} H^0_{\text{ét}}(o'_F, \mathbb{G}_m) &\cong o_F[\frac{1}{p}]^* \\ H^1_{\text{ét}}(o'_F, \mathbb{G}_m) &\cong Cl(o_F[\frac{1}{p}]) \\ H^2_{\text{ét}}(o'_F, \mathbb{G}_m) &\cong Br(o_F[\frac{1}{p}]). \end{aligned}$$

Here $Br(o_F[\frac{1}{p}])$ denotes the subgroup of the Brauer group $Br(F)$ of F consisting of all classes of central simple F-algebras, which are split outside of primes above p. As an abelian group $Br(o_F[\frac{1}{p}])$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{s_p-1}$, where s_p is the number of primes in F above p.

Multiplication by p^m yields a short exact sequence

$$0 \to \mu^{p^m} \to \mathbb{G}_m \xrightarrow{p^m} \mathbb{G}_m \to 0,$$

hence short exact sequences for $i = 1, 2$:

$$0 \to H^{i-1}_{\text{ét}}(o'_F, \mathbb{G}_m)/p^m \to H^i_{\text{ét}}(o'_F, \mu_{p^m}) \to {}_{p^m}H^i_{\text{ét}}(o'_F, \mathbb{G}_m) \to 0.$$

Here we denote for any abelian group A by ${}_{p^m}A$ the subgroup of A of all elements of exponent p^m. Let us now simply write U'_F for $o_F[\frac{1}{p}]^*$, $Cl(F)'$ for $Cl(o_F[\frac{1}{p}])$ and $Br(F)'$ for $Br(o_F[\frac{1}{p}])$. Then we can rewrite these short exact sequences as follows:

$$0 \to U'_F/U'^{p^m}_F \to H^1_{\text{ét}}(o'_F, \mu_{p^m}) \to {}_{p^m}Cl(F)' \to 0$$

and

$$0 \to Cl(F)'/Cl(F)'^{p^m} \to H^2_{\text{ét}}(o'_F, \mu_{p^m}) \to {}_{p^m}Br(F)' \to 0.$$

Passing to the inverse limit yields

$$H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p \otimes U'_F$$

and

$$H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(1)) \cong Cl(F)'(p) \oplus \mathbb{Z}_p^{s_p-1}.$$

The localization sequence provides a description of $H^1_{\text{ét}}(o'_F, \mu_{p^m})$ as a subgroup of $H^1_{\text{ét}}(F, \mu_{p^m}) \cong F^*/F^{*p^m}$ (Kummer isomorphism). Let Δ_F denote the subgroup of F^* of all x, such that the $\mathfrak{p}$-adic valuation of x is divisible by p^m for all non-p-adic $\mathfrak{p}$. Then Δ_F contains F^{*p^m} and

$$H^1_{\text{ét}}(o'_F, \mu_{p^m}) \cong \Delta_F/F^{*p^m}.$$

We note that in general the exact sequence

$$0 \to \mathbb{Z}/p^m\mathbb{Z}(n) \to \mathbb{Z}_p(n) \xrightarrow{p^m} \mathbb{Z}_p(n) \to 0$$

gives rise to the so-called *Bockstein sequences* in étale cohomology: For $i \geq 1$ we have short exact sequences

$$0 \to H^{i-1}_{\text{ét}}(o'_F, \mathbb{Z}_p(n))/p^m \to H^i_{\text{ét}}(o'_F, \mathbb{Z}/p^m\mathbb{Z}(n)) \to {}_{p^m}H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) \to 0.$$

For $n = 1$ and $i = 1, 2$ these yield exactly the sequences discussed above.

The relation between algebraic K-theory and p-adic étale cohomology groups in degrees 1 and 2 is provided by étale Chern characters defined by Soulé in [85]: He constructed homomorphisms

$$ch_{i,n}^{(p)} : K_{2n-i}(o_F) \otimes \mathbb{Z}_p \to H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$$

for $i = 1, 2$, $n \geq 2$, and p odd, and proved surjectivity. A different approach, which worked also for the prime 2 and used étale K-theory, was given by Dwyer-Friedlander ([23]). They extended the surjectivity result to the prime 2 provided $\sqrt{-1}$ is contained in F. The Quillen-Lichtenbaum Conjecture predicts the following:

Quillen-Lichtenbaum Conjecture 2.4. *For a global field F the étale Chern characters*

$$ch_{i,n}^{(p)} : K_{2n-i}(o_F) \otimes \mathbb{Z}_p \to H^i_{\acute{e}t}(o'_F, \mathbb{Z}_p(n))$$

are isomorphisms for $n \geq 2$, $i = 1, 2$, unless $p = 2$ and F is a number field with a real embedding.

For $2n - i = 2$ the results of Tate and Soulé imply that the Quillen-Lichtenbaum Conjecture holds for all primes p including $p = 2$, *i.e.*, there are isomorphisms

$$K_2(o_F)(p) \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(2))$$

for all p. It is worth mentioning that many results about the structure of $K_2(o_F)$ for number rings can be obtained via these isomorphisms from the corresponding results about étale cohomology groups. We will give examples below.

For $2n - i = 3$ results of Merkurjev-Suslin [67] and Levine [61] show that the Quillen-Lichtenbaum Conjecture is true for $K_3(o_F) = K_3(F)$. Moreover they also show that the kernel of the surjection $K_3(F) \otimes \mathbb{Z}_2 \to H^1_{\text{ét}}(F, \mathbb{Z}_2(2))$ is isomorphic to Milnor's $K_3^M(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$, *i.e.*, we obtain a short exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{r_1(F)} \to K_3(F) \otimes \mathbb{Z}_2 \to H^1_{\text{ét}}(F, \mathbb{Z}_2(2)) \to 0.$$

For any field F the quotient $K_3(F)/K_3^M(F)$ is called the *undecomposable* K_3 of F and denoted by $K_3(F)_{ind}$.

In general, the Quillen-Lichtenbaum Conjecture is a consequence of the Bloch-Kato Conjecture - we will sketch a proof of this below. On the other hand the Bloch-Kato Conjecture seems to have been proven by Rost and Voevodsky, and therefore the same would also be true for the Quillen-Lichtenbaum Conjecture.

It is worth mentioning that Hesselholt and Madsen ([40]) proved the Quillen-Lichtenbaum Conjecture for *local* fields.

For $2n - i = 1$, hence $n = i = 1$, we also have a Chern character

$$ch_{1,1}^{(p)} : K_1(o_F[1/p]) \otimes \mathbb{Z}_p \to H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(1)).$$

We have seen above that

$$H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(1)) \cong o_F[1/p]^* \otimes \mathbb{Z}_p,$$

and the Chern character $ch_{1,1}^{(p)}$ can be identified with the determinant.

The relation between K-groups and dyadic étale cohomology groups in the exceptional case that F has a real embedding is given below in Theorem 2.8.

From the results of Soulé and Dwyer-Friedlander together with Theorem 1.5 we obtain the structure of the étale cohomology groups $H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ for $i = 1, 2$ and $n \geq 2$. It should be pointed out that at present there is no "direct" approach, *i.e.*, one which does not use Borel's structure theorem for higher K-groups.

Corollary 2.5. *Let $n \geq 2$. Then $H^2_{ét}(o'_F, \mathbb{Z}_p(n))$ is finite and trivial for almost all primes p, and*

$$rk_{\mathbb{Z}}\, H^1_{ét}(o'_F, \mathbb{Z}_p(n)) = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd } > 1 \\ r_2 & \text{if } n \text{ is even.} \end{cases}$$

We note the following two consequences:

Since $H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ is finite for $n \geq 2$, we obtain $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p(n)) = 0$, hence the boundary map

$$\delta_3 : H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^3_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$$

is an isomorphism, and therefore for $n \geq 2$ we obtain $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$ unless $p = 2$, n odd, and F has a real embedding. We note that the vanishing of $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(0))$ is equivalent to Leopoldt's Conjecture. The vanishing of $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ for negative n is a Conjecture of P. Schneider (cf. [82]). Of course $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong Br(F)'(p) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{s_p-1}$ does not vanish in general.

For fixed $n \geq 2$ the étale cohomology group $H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ is torsion precisely when F is totally real and n is odd. If this is the case then the boundary map δ_2 gives an isomorphism

$$H^1_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n)).$$

Assume now that F contains a primitive p-th root of unity. Then

$$H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(n)) \cong \mathbb{Z}/p\mathbb{Z}(n-1) \otimes H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(1)),$$

hence

$$rk_p\,(H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(n))) = rk_p\,(H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(1))).$$

As before, let s_p denote the number of p-adic primes in F. By Dirichlet's unit theorem the quotient U'_F/U'^p_F is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{r_1+r_2+s_p}$, in particular U'_F/U'^p_F is a pure subgroup of $H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(n))$, hence the sequence

$$0 \to U'_F/U'^p_F \to H^1_{\text{ét}}(o'_F, \mu_p) \to {}_pCl(F)' \to 0$$

splits, and we obtain

$$rk_p\,(H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(1))) = r_1 + r_2 + s_p + rk_p\,(Cl(F)').$$

On the other hand the Bockstein sequence

$$0 \to H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))/p \to H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(n)) \to {}_pH^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) \to 0$$

splits as well, since

$$H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))/p \cong \begin{cases} (\mathbb{Z}/p\mathbb{Z})^{r_1+r_2+1} & \text{if } n \text{ is odd} \\ (\mathbb{Z}/p\mathbb{Z})^{r_2+1} & \text{if } n \text{ is even} \end{cases}$$

is also a pure subgroup of $H^1_{\text{ét}}(o'_F, \mathbb{Z}/p\mathbb{Z}(n))$. We therefore obtain the following calculation of the p-rank of $H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$:

Proposition 2.6. *If F contains a primitive p-th root of unity, then for all $n \geq 2$*

$$rk_p\left(H^2_{ét}(o'_F, \mathbb{Z}_p(n)\right) = \begin{cases} s_p - 1 + rk_p\left(Cl(F)'\right) & \text{if } n \text{ is odd} \\ r_1 + s_p - 1 + rk_p\left(Cl(F)'\right) & \text{if } n \text{ is even} \end{cases}$$

In particular, still assuming $\mu_p \subset F$, we see that $H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ vanishes if and only if F contains only one p-adic prime, the class group $Cl(F)'$ has no p-torsion, and F is totally complex in case n is even. For $n = 2$ we recover Tate's p-rank formulas for $K_2(o_F)$ (cf. [94]).

We now turn our attention to the relation between K-groups and motivic cohomology. There have been several definitions of suitable motivic cohomology groups for smooth projective varieties X over a field F. We use Bloch's higher Chow groups to define motivic cohomology groups for a field F:

$$H^i_{\mathcal{M}}(F, \mathbb{Z}(n)) := CH^n(F, \mathbb{Z}(2n - i)).$$

These cohomology groups are isomorphic to the ones defined by Levine (cf. [62]) and by Voevodsky (cf. [91]).

In a similar way one defines motivic cohomology groups with finite coefficients. The definition also extends to Dedekind rings (cf. [30]).

Pushin (cf. [73]) has constructed Chern classes and characters from K-theory to motivic cohomology, in particular we have for a global field F Chern characters

$$ch^{\mathcal{M}}_{i,n} : K_{2n-i}(F) \to H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$$

for $n \geq 2$ and $i = 1, 2$, which induce the étale Chern characters after tensoring by $\mathbb{Z}_p$.

Let us now give the essential ingredients in the well-known proof that the Quillen-Lichtenbaum Conjecture follows from the Bloch-Kato Conjecture:

Theorem 2.7. *Let p be an odd prime, and assume that the Bloch-Kato Conjecture holds for powers of p, i.e., the Galois symbol*

$$g_{n,p^\nu} : K^M_n(F)/p^\nu \to H^n(F, \mu_{p^\nu}^{\otimes n})$$

is an isomorphims for all $\nu \geq 1$ and all fields of characteristic $\neq p$. Then the Quillen-Lichtenbaum Conjecture holds for the prime p.

Proof. (Sketch) Let F be a field of characteristic $\neq p$. On of the main consequences of the Bloch-Kato Conjecture is the following comparison between motivic cohomology and étale cohomology: There is an isomorphism

$$H^i_{\mathcal{M}}(F, \mathbb{Z}/p^\nu(n)) \cong \begin{cases} H^i_{\text{ét}}(F, \mathbb{Z}/p^\nu(n)) & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This was proved in [92] for fields with resolution of sigularities and in [32] in general. An easy implication is that $H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$ is uniquely p-divisible for $i < 0$ and also for $i = 0$, if F contains only finitely many p-power roots of unity. If F is a global field, then we also obtain that $H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$ is p-divisible for $i \geq 3$ and uniquely p-divisible for $i > 3$ unless $p = 2$ and F has a real embedding. The link between motivic cohomology and K-theory is provided by the following third quadrant spectral sequence, due to Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky ([12],[81]):

$$E_2^{pq} = H^{p-q}_{\mathcal{M}}(F, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(F).$$

The groups E_2^{0q} on the q-axis of the spectral sequence are isomorphic to Milnor's K-theory groups:

$$H^i_{\mathcal{M}}(F, \mathbb{Z}(i) \cong K^M_i(F)$$

by a result of Suslin-Voevodsky [92]. The study of the action of the Adams operations on the spectral sequence by Gillet-Soulé [34] yields that all differentials in the spectral sequence are torsion, and that it degenerates modulo groups of finite exponent (cf. also [49]). Since for a global field F the odd K-groups are finitely generated and the even K-groups are torsion we obtain isomorphisms

$$K_{2n-2}(F) \cong H^2_{\mathcal{M}}(F, \mathbb{Z}(n))$$
$$K_{2n-1}(F) \cong H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$$

for all $n \geq 2$ up to 2-torsion. We note that in the function field case neither the K-groups nor the motivic cohomology groups have $char(F)$-torsion (cf. [31]).

To relate these results to the K-groups of the ring of integers, one can use the localization sequence in motivic cohomology due to Geisser [30], which relates motivic cohomology groups of a Dedekind ring to the motivic cohomology of its field of fractions, and is very similar to Soulé's localization sequence in étale cohomology.

This results in isomorphisms

$$H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$$

for *all* primes p, $n \geq 2$ and $i = 1, 2$ and isomorphisms - up to 2-torsion -

$$K_{2n-2}(o_F) \cong H^2_{\mathcal{M}}(o_F, \mathbb{Z}(n))$$

$$K_{2n-1}(o_F) \cong H^1_{\mathcal{M}}(o_F, \mathbb{Z}(n)).$$

□

The situation for the prime 2 is similar, but much more complicated, since the presence of real places prevents the spectral sequence from collapsing. Nevertheless the details have been worked out by Kahn and Rognes-Weibel and the consequences of Voevodsky's proof of the Milnor Conjecture for the 2-primary Chern character are summarized in the following unconditional result:

Theorem 2.8. [Rognes-Weibel [81], Kahn [49]] *For* $i = 1, 2$ *and* $2n-i \geq 2$ *the* 2*-primary Chern character*

$$ch^{(2)}_{i,n} : K_{2n-i}(o_F) \otimes \mathbb{Z}_2 \to H^i_{ét}(o'_F, \mathbb{Z}_2(n))$$

is

$$\begin{cases} \textit{an isomorphim} & \textit{if } 2n-i \equiv 0,1,2,7 \bmod 8, \\ \textit{surjective with kernel} \cong (\mathbb{Z}/2\mathbb{Z})^{r_1} & \textit{if } 2n-i \equiv 3 \bmod 8, \\ \textit{injective with cokernel} \cong (\mathbb{Z}/2\mathbb{Z})^{r_1} & \textit{if } 2n-i \equiv 6 \bmod 8. \end{cases}$$

In the case that $n \equiv 3 \bmod 4$*, there is an exact sequence*

$$0 \to K_{2n-1}(o_F) \otimes \mathbb{Z}_2 \to H^1_{ét}(o'_F, \mathbb{Z}_2(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1}$$

$$\to K_{2n-2}(o_F) \otimes \mathbb{Z}_2 \to H^2_{ét}(o'_F, \mathbb{Z}_2(n)) \to 0.$$

We see that the relation between higher algebraic K-groups of o_F (tensored by $\mathbb{Z}_2$) and dyadic étale cohomology is quite complicated for certain indices. On the other hand the motivic cohomology groups of o_F behave much better: For all $n \geq 2$ and $i = 1, 2$:

$$H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n)) \otimes \mathbb{Z}_2 \cong H^i_{\text{ét}}(o'_F, \mathbb{Z}_2(n)).$$

Moreover, if we assume the Quillen - Lichtenbaum Conjecture for odd primes p, then the motivic Chern characters

$$ch_{i,n}^{\mathcal{M}} : K_{2n-i}(F) \to H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$$

are isomorphisms up to 2-torsion and the 2-primary information is the same as in Theorem 2.8.

Even if we do not assume the Quillen-Lichtenbaum Conjecture for odd primes p we can still find global "models" $H^i(o_F, \mathbb{Z}(n))$, $i = 1, 2$, for the étale cohomology groups (cf. [18]). For $i = 2$ this is easy. We simply define

$$H^2(o_F, \mathbb{Z}(n)) = \prod_p H^2_{\text{ét}}(o_F[\frac{1}{p}], \mathbb{Z}_p(n)).$$

This is a finite group, and the Chern characters for each prime p yield a homomorphism

$$K_{2n-2}(o_F) \to H^2(o_F, \mathbb{Z}(n)).$$

For $i = 1$ the construction is more involved. Let

$$H^1_\infty(o_F, \mathbb{Z}(n)) := \prod_p H^1_{\text{ét}}(o_F[\frac{1}{p}], \mathbb{Z}_p(n)).$$

This is a finitely generated module over $\hat{\mathbb{Z}}$. The étale Chern characters $ch_{1,n}^{(p)}$ yield a homomorphism

$$ch_{1,n} : K_{2n-1}(o_F) \otimes \hat{\mathbb{Z}} \to H^1_\infty(o_F, \mathbb{Z}(n))$$

with finite cokernel T of 2-power order, hence we obtain a short exact sequence of finitely generated $\hat{\mathbb{Z}}$-modules

$$0 \to ch_{1,n}(K_{2n-1}(o_F)) \otimes \hat{\mathbb{Z}} \to H^1_\infty(o_F, \mathbb{Z}(n)) \to T \to 0.$$

We now use the fact that for any finitely generated $\mathbb{Z}$-modules M, N the natural map

$$\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} Ext^i_{\mathbb{Z}}(M, N) \mapsto Ext^i_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M, \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} N)$$

is an isomorphism. For $i > 0$ these groups are finite and hence we obtain in particular an isomorphism

$$Ext^1_{\mathbb{Z}}(M, N) \cong Ext^1_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M, \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} N).$$

If we apply this to $M = ch_{1,n}(K_{2n-1}(o_F))$ and $N = T$, then we see that there exists a finitely generated $\mathbb{Z}$-module $H^1(o_F, \mathbb{Z}(n))$, which fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ch_{1,n}(K_{2n-1}(o_F)) & \longrightarrow & H^1(o_F, \mathbb{Z}(n)) & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ch_{1,n}(K_{2n-1}(o_F)) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} & \longrightarrow & H^1_{\infty}(o_F, \mathbb{Z}(n)) & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

In fact, $H^1(o_F, \mathbb{Z}(n))$ is uniquely determined (cf. [18]). Since the left vertical arrow is injective, we see that $H^1(o_F, \mathbb{Z}(n))$ injects into $H^1_{\infty}(o_F, \mathbb{Z}(n))$ and moreover for each prime p we have

$$H^1(o_F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^1_{\text{ét}}(o_F[\frac{1}{p}], \mathbb{Z}_p(n)).$$

We see that for $i = 1, 2$ the Chern characters $ch_{i,n}$ induce exact sequences

$$0 \to ker\,(ch_{i,n}) \to K_{2n-i}(o_F) \to H^i(o_F, \mathbb{Z}(n)) \to coker\,(ch_{i,n}) \to 0.$$

Of course, if the Quillen-Lichtenbaum Conjecture holds, then

$$H^i(o_F, \mathbb{Z}(n)) \cong H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n)),$$

$ch_{i,n} = ch^{\mathcal{M}}_{i,n}$ is Pushin's Chern character and the kernel and cokernel are finite 2-groups determined by the results in Theorem 2.8.

We mention that for $n = 2$ we have the following special cases due to Tate, Merkurjev-Suslin and Levine:

$$K_2(o_F) \cong H^2(o_F, \mathbb{Z}(2))$$

and

$$K_3(F)_{ind} \cong H^1(o_F, \mathbb{Z}(2)).$$

Assume now that E/F is a finite Galois extension of number fields with Galois group G. Let p be a prime and let S denote a finite set of primes of F containing all primes above p as well as all primes which ramify in F. Let o_F^S denote the ring of S-integers of F and let o_E^S denote the integral closure of o_F^S in E. The extension $spec\, o_E^S \to spec\, o_F^S$ is then étale, and we want to study Galois descent and Galois codescent properties for the étale cohomology groups $H^i_{\text{ét}}(o_E^S, \mathbb{Z}_p(n))$ for $n \geq 2$ and $i = 1, 2$. We recall that for $i = 1$ the cohomology groups $H^1_{\text{ét}}(o_E^S, \mathbb{Z}_p(n))$ are independent of the set S and isomorphic to $H^1_{\text{ét}}(E, \mathbb{Z}_p(n))$.

We note that in the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G, H^j_{\text{ét}}(o_E^S, \mathbb{Z}_p(n)) \Rightarrow E^{i+j} = H^{i+j}_{\text{ét}}(o_F^S, \mathbb{Z}_p(n))$$

all E_2-terms with $j = 0$ vanish, since $H^0_{\text{ét}}(o_E^S, \mathbb{Z}_p(n)) = 0$. For p odd the terms with $j \geq 3$ vanish as well. This easily implies the following result:

Proposition 2.9. *Let E/F be a Galois extension of number fields with Galois group G. Then*

$$H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))^G \cong H^1_{\acute{e}t}(F, \mathbb{Z}_p(n)).$$

Furthermore, if p is an odd prime, and S a finite set of primes in F containing all primes above p and all finite ramified primes, then the following sequence is exact:

$$0 \to H^1(G, H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))) \to H^2_{\acute{e}t}(o_F^S, \mathbb{Z}_p(n)) \to H^2_{\acute{e}t}(o_E^S, \mathbb{Z}_p(n))^G \to$$

$$H^2(G, H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))) \to 0.$$

In a similar fashion we can use the Tate spectral sequence for p odd (cf. [84],[50]), which in cohomological formulation reads

$$E_2^{ij} = H_{-i}(G, H^j_{\text{ét}}(o_E^S, \mathbb{Z}_p(n)) \Rightarrow E^{i+j} = H^{i+j}_{\text{ét}}(o_F^S, \mathbb{Z}_p(n))$$

for $i \leq 0$ and $j \geq 0$ to obtain:

Proposition 2.10. *For p odd the corestriction induces an isomorphism*

$$H^2_{\acute{e}t}(o_E^S, \mathbb{Z}_p(n))_G \cong H^2_{\acute{e}t}(o_F^S, \mathbb{Z}_p(n)).$$

From Propositions 2.9 and 2.10 we obtain the following interesting consequence for $i = 1, 2$. The general case can be deduced from this (cf. [47]).

Corollary 2.11. *For p odd and all $i \in \mathbb{Z}$ there are canonical isomorphisms*

$$\hat{H}^{i-2}(G, H^2_{\acute{e}t}(E, \mathbb{Z}_p(n))) \cong \hat{H}^i(G, H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))).$$

The situation for the prime 2 is more complicated, since the higher étale cohomology groups do not necessarily vanish, and the Galois descent and codescent properties of $H^2_{\text{ét}}(E, \mathbb{Z}_p(n))$ are more complicated. One way to deal with these difficulties is to modify the dyadic étale cohomology groups. This was done in [18] based on [47] with the definition of *positive* étale cohomology groups $H^i_+(o_K^S, \mathbb{Z}_2(n))$ for an arbitrary number field K and an arbitrary finite set of primes S in K containing the dyadic primes. In our context the two main features of $H^i_+(o_K^S, \mathbb{Z}_2(n))$ are:

1. $H^i_+(o_K^S, \mathbb{Z}_2(n)) = 0$ for $i \neq 1, 2$ (We use here the indexing of [47], not of [18]).

2. There is a long exact sequence

$$0 \to \oplus_{v|\infty} H^0(G_v, \mathbb{Z}_2(n)) \to H^1_+(o_K^S, \mathbb{Z}_2(n)) \to H^1_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to$$
$$\oplus_{v|\infty} H^1(G_v, \mathbb{Z}_2(n)) \to H^2_+(o_K^S, \mathbb{Z}_2(n)) \to H^2_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to$$
$$\oplus_{v|\infty} H^2(G_v, \mathbb{Z}_2(n)) \to 0,$$

where G_v denotes the decomposition group of the infinite prime v.

The cohomology groups $H^i(G_v, \mathbb{Z}_2(n))$ are easy to calculate:

$$H^0(G_v, \mathbb{Z}_2(n)) = \begin{cases} \mathbb{Z}_2(n) & \text{if } n \text{ is even or } v \text{ is complex} \\ 0 & \text{otherwise} \end{cases}$$

and for $i \geq 1$

$$H^i(G_v, \mathbb{Z}_2(n)) = \begin{cases} 0 & \text{if } v \text{ is complex or } i+ \text{ is odd} \\ \mathbb{Z}/2Z & \text{otherwise.} \end{cases}$$

Therefore we get a more precise relation between positive étale cohomology and étale cohomology depending on the parity of n: For n odd we have an exact sequence

$$0 \to (\mathbb{Z}_2(n))^{2r_2(K)} \to H^1_+(o_K^S, \mathbb{Z}_2(n)) \to H^1_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to (\mathbb{Z}_2(n))^{r_1(K)} \to$$
$$H^2_+(o_K^S, \mathbb{Z}_2(n)) \to H^2_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to 0,$$

whereas for n even we obtain two short exact sequences

$$0 \to (\mathbb{Z}_2(n))^{[K:\mathbb{Q}]} \to H^1_+(o_K^S, \mathbb{Z}_2(n)) \to H^1_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to 0$$

and

$$0 \to H^2_+(o_K^S, \mathbb{Z}_2(n)) \to H^2_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to (\mathbb{Z}_2(n))^{r_1(K)} \to 0.$$

We remark that for odd n the map $H^1_{\text{ét}}(o_K^S, \mathbb{Z}_2(n)) \to (\mathbb{Z}_2(n))^{r_1(K)}$ factors through

$$H^1_{\text{ét}}(o_K^S, \mathbb{Z}_2(n))/2 \hookrightarrow H^1(o_K^S, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow K^*/K^{*2},$$

and hence can be viewed as a signature map.

Because the positive étale cohomology groups vanish in degrees $\neq 1, 2$ the analogs of Propositions 2.9, 2.10 and Corollary 2.11 hold for these groups. Using the relation to étale cohomology this allows to describe Galois descent and codescent for $H^2_{\text{ét}}(o_E^S, \mathbb{Z}_2(n))$ for the Galois extension E/F (cf. [53]). Let r_∞ denote the number of real primes in F, which ramify in E and let - for n odd - 2^{s_∞} denote the order of the cokernel of the signature map $H^1_{\text{ét}}(E, \mathbb{Z}_2(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty}$.

Proposition 2.12. *For n even there are exact sequences*

$$0 \to H^2_{ét}(o_E^S, \mathbb{Z}_2(n))_G \to H^2_{ét}(o_F^S, \mathbb{Z}_2(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty} \to 0$$

and

$$0 \to H^1(G, H^1_{ét}(E, \mathbb{Z}_2(n))) \to H^2_{ét}(o_F^S, \mathbb{Z}_2(n)) \to H^2_{ét}(o_E^S, \mathbb{Z}_2(n))^G \to$$
$$H^2(G, H^1_{ét}(E, \mathbb{Z}_2(n))) \to 0.$$

For n odd there are exact sequences

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{s_\infty} \to H^2_{ét}(o_E^S, \mathbb{Z}_2(n))_G \to H^2_{ét}(o_F^S, \mathbb{Z}_2(n)) \to 0$$

and

$$0 \to H^1(G, H^1_{ét}(E, \mathbb{Z}_2(n))) \to H^2_{ét}(o_F^S, \mathbb{Z}_2(n)) \to H^2_{ét}(o_E^S, \mathbb{Z}_2(n))^G \to$$
$$H^2(G, H^1_{ét}(E, \mathbb{Z}_2(n))) \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty - s_\infty} \to 0.$$

All the p-adic results can be lifted to the global situation (cf. [18] for details) and give the following:

Theorem 2.13. *Let E/F be a Galois extension of number fields with Galois group G, and let S denote a set of primes in F containing all ramified primes.*

i) For each $n \geq 2$ there are isomorphisms

$$H^1(F, \mathbb{Z}(n)) \cong H^1(E, \mathbb{Z}(n))^G.$$

ii) For each even $n \geq 2$ there are exact sequences

$$0 \to H^2(o_E, \mathbb{Z}(n))_G \to H^2(F, \mathbb{Z}(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty} \to 0$$

and

$$0 \to H^1(G, H^1(E, \mathbb{Z}(n))) \to H^2(o_F^S, \mathbb{Z}(n)) \to H^2(o_E^S, \mathbb{Z}(n))^G \to$$
$$H^2(G, H^1(E, \mathbb{Z}(n))) \to 0.$$

iii) For each odd $n \geq 3$ there are exact sequences

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{s_\infty} \to H^2(o_E^S, \mathbb{Z}(n))_G \to H^2(o_F^S, \mathbb{Z}(n)) \to 0$$

and

$$0 \to H^1(G, H^1(E, \mathbb{Z}(n))) \to H^2(o_F^S, \mathbb{Z}(n)) \to H^2(o_E^S, \mathbb{Z}(n))^G \to$$
$$H^2(G, H^1(E, \mathbb{Z}(n))) \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty - s_\infty} \to 0.$$

We remark that for $n = 2$ these results describe in fact Galois descent and codescent for $K_2(o_E^S)$, results which are due to Kahn [47] and Bak-Rehmann [2], respectively. In general these results describe Galois descent and codescent for motivic cohomology provided the Quillen-Lichtenbaum Conjecture holds for odd primes.

3 Iwasawa theory

Now that we have more or less complete information about the relation between algebraic K-theory and étale cohomology for our number rings o_F, we want to discuss briefly the relation between étale cohomology and Iwasawa-theory, which will allow us to actually interpret orders of étale cohomology groups in some cases.

Let F be a number field and p a prime number. A Galois extension F_∞/F is called a $\mathbb{Z}_p$-*extension*, if $\Gamma := Gal(F_\infty/F) \cong \mathbb{Z}_p$. Since the closed subgroups of $\mathbb{Z}_p$ are of the form 0 or $p^n\mathbb{Z}_p$, we have for each $n \geq 0$ a unique subfield F_n of degree p^n over F and $Gal(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}_p$. Hence we obtain a filtration

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\infty,$$

such that $[F_n : F] = p^n$ and $F_\infty = \cup_{n\geq 0} F_n$.

A typical example of a $\mathbb{Z}_p$-extension F_∞/F is the so-called *cyclotomic* $\mathbb{Z}_p$-*extension*, which is constructed as follows: Let $E_\infty = F(\mu_{p^\infty})$. Then $Gal(E_\infty/F) \cong \mathbb{Z}_p \times \Delta$, where Δ is finite. Now we take $F_\infty = E_\infty^\Delta$. The number of independent $\mathbb{Z}_p$-extensions of a number field F is always $\geq 1 + r_2(F)$ and $= 1 + r_2(F)$ if and only if Leopoldt's Conjecture holds for the field F and the prime p. If, in particular, F is a totally real abelian number field, then the cyclotomic $\mathbb{Z}_p$-extension is the only $\mathbb{Z}_p$-extension of F.

Within a $\mathbb{Z}_p$-extension F_∞/F ramification is very restricted, in fact the extension is p-ramified, *i.e.*, unramified outside primes above p. In particular this implies that Γ is a quotient of $G_F^{(p)}$, and therefore we obtain spectral sequences relating the étale cohomology groups of $o_F[\frac{1}{p}]$ and of $o_\infty[\frac{1}{p}]$, where o_∞ denotes the integral closure of o_F in F_∞.

Let γ denote a topological generator of Γ, and let $\Gamma_n = Gal(F_n/F)$. Passing to the inverse limit we obtain the *Iwasawa-algebra* $\mathbb{Z}_p[[\Gamma]] := \varprojlim \mathbb{Z}_p[\Gamma_n]$. The group rings $\mathbb{Z}_p[\Gamma_n]$ are quite complicated, but the Iwasawa-algebra has a rather simple structure, it is isomorphic to the power series ring $\Lambda := \mathbb{Z}_p[[T]]$, the isomorphism being induced by $\gamma \mapsto 1+T$. Λ is a two-dimensional Noetherian local Krull domain, and the structure of finitely generated Λ-modules is known up to pseudo-isomorphism. If M and N are finitely generated Λ-modules, then we write $M \sim N$ if there exists a pseudo-isomorphism $f : M \to N$, *i.e.*, a module homomorphism with finite kernel and cokernel. The structure theorem for finitely generated Λ-modules now says that for every finitely generated Λ-module M there is a pseudo-isomorphism

$$M \sim \Lambda^r \oplus \bigoplus_{i=1}^{m} \Lambda/\mathfrak{p}_i^{n_i}.$$

Here $\mathfrak{p}_i$ are certain height 1 prime ideals of Λ, hence either equal to (p) or to $(F(T))$, where $F(T)$ is an irreducible Weierstrass polynomial, *i.e.*, an irreducible polynomial of the form

$$F(T) = T^n + b_{n-1}T^{n-1} + \cdots + b_0$$

with $p|b_i$ for all i. The prime ideals $\mathfrak{p}_i$ and the integers $r \geq 0, m \geq 0$ and $n_i \geq 1$ are uniquely determined by M. The ideal $\prod_{i=1}^{m} \mathfrak{p}_i^{n_i}$ is the *characteristic ideal* of M. Any generator $G(T)$ of the characteristic ideal has the form

$$G(T) = p^{\mu} \cdot F(T) \cdot U(T),$$

where $F(T)$ is a Weierstrass polynomial and $U(T)$ is a unit in Λ. The polynomial

$$f(T) := p^{\mu} \cdot F(T)$$

is the *characteristic polynomial* of M. The exponent μ is the μ-invariant of M and $\lambda := deg f(T)$ is called the λ-invariant of M.

The following results are extremely useful: Assume that M is a finitely generated Λ-torsion module with characteristic polynomial $f(T)$. We denote by M^{Γ} the invariants of M under Γ and by $M_{\Gamma} = M/(\gamma - 1)M$ the coinvariants of M. The following statements are equivalent (cf. [19]):

(a) M^{Γ} is finite
(b) M_{Γ} is finite
(c) $f(0) \neq 0$.

If any of these conditions are satisfied, then

$$\frac{|M^{\Gamma}|}{|M_{\Gamma}|} = |f(0)|_p = p^{-v_p(f(0))},$$

where v_p denotes the p-adic valuation.

Let us assume now that F_{∞}/F is the cyclotomic $\mathbb{Z}_p$-extension. Let $E_{\infty} = F(\mu_{p^{\infty}})$ and let $G_{\infty} = Gal(E_{\infty}/F) \cong \Gamma \times \Delta$, where $\Delta \cong Gal(F(\zeta_{2p})/F$.

Since E_∞ contains all p-power roots of unity, the Galois group G_∞ acts on μ_{p^∞} and this action gives rise to the *cyclotomic character*

$$\rho : G_\infty \to \mathbb{Z}_p^*$$

defined by

$$\zeta^\sigma = \zeta^{\rho(\sigma)}$$

for all $\sigma \in G_\infty$ and all $\zeta \in \mu_{p^\infty}$. We denote by κ the restriction of ρ to Γ and by ω the restriction of ρ to Δ. ω is the *Teichmüller character.*

Let M be a $\mathbb{Z}_p$-module with a G_∞-action, denoted by $m \mapsto m^\sigma$. For $n \in \mathbb{Z}$ the n-th Tate twist $M(n)$ of M is defined as the $\mathbb{Z}_p$-module M with the new G_∞-action

$$m \mapsto \rho(\sigma)^n \cdot m^\sigma.$$

In particular, $\mathbb{Z}_p(1) \cong \varprojlim \mu_{p^n} =: \mathcal{T}$, which is the so-called *Tate-module*, and $\mathbb{Q}_p/\mathbb{Z}_p(1) \cong \mu_{p^\infty}$. In general: $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$. If M and N are two $\mathbb{Z}_p$-modules with a G_∞-action, then we turn $Hom_{\mathbb{Z}_p}(M, N)$ into a G_∞-module in the following way: For $f \in Hom_{\mathbb{Z}_p}(M, N)$ and $\sigma \in G_\infty$ we define f^σ via

$$f^\sigma(m) = (f(m^{\sigma^{-1}}))^\sigma.$$

It is easy to see that with this definition of the G_∞-action on Hom-groups we obtain canonical isomorphisms for all $n \in \mathbb{Z}$:

$$Hom_{\mathbb{Z}_p}(M(n), \mathbb{Q}_p/\mathbb{Z}_p) \cong Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(-n)) \cong Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)(-n).$$

Assume now that M is a Λ-torsion module with characteristic polynomial $f(T)$. Then for all multiples n of the order of Δ the characteristic polynomial of $M(n)$ is given by

$$f(\kappa(\gamma)^{-n}(1+T) - 1).$$

The most interesting Λ-modules arise as Galois groups of certain abelian pro-p extensions of F_∞, where F_∞/F is an arbitrary $\mathbb{Z}_p$-extension of a number field F. Assume then that K_∞ is an abelian pro-p extension of F_∞, let $X = Gal(K_\infty/F_\infty)$, and assume that K_∞/F is again a Galois extension (although not necessarily abelian). Let $G = Gal(K_\infty/F)$. We obtain an extension of $\mathbb{Z}_p$-modules

$$0 \to X \to G \to \Gamma \to 0.$$

Since X is abelian, Γ acts on X by inner automorphisms, and this action turns X into a compact Λ-module. As examples we can take for K_∞ the maximal abelian unramified pro-p extension of F_∞, usually denoted by L_∞, or the maximal subextension of L_∞, in which all p-adic primes of F_∞ split completely, usually denoted by L'_∞. The corresponding Galois groups $X_\infty := Gal(L_\infty/F_\infty)$ and $X'_\infty := Gal(L'_\infty/F_\infty)$ are examples of finitely generated Λ - torsion modules.

The main example in the current framework is the following: Let M_∞ denote the maximal abelian pro-p-extension of F_∞, which is unramified outside primes above p and infinite primes, and let $\mathcal{X} = Gal\,(M_\infty/F_\infty)$. This is the so-called *standard Iwasawa module.* It is a finitely generated Λ-module. Let us again specialize to the case of the cyclotomic $\mathbb{Z}_p$-extension. Iwasawa has shown that in this case $\mathcal{X}$ has no non-trivial finite Λ-submodules and that the Λ-rank of $\mathcal{X}$ is equal to r_2. In particular, $\mathcal{X}$ is a Λ-torsion module if and only if F is totally real, which we will assume from now on. As before, we let $E = F(\zeta_{2p})$ and $E_\infty = F(\mu_{p^\infty})$. To simplify this exposition we will assume that F is equal to the maximal real subfield E^+ of E. For $p = 2$, this is automatically satisfied. For p odd the general argument uses eigenspaces of powers of the Teichmueller character of the standard Iwasawa-module over E^+.

Consider now an *even* positive integer $n \geq 2$. Let us also fix a topological generator γ of Γ. Recall that $\Omega_F^{(p)}$ denotes the maximal algebraic extension of F, which is unramified outside primes above p and infinity, and that $G_F^{(p)} = Gal(\Omega_F^{(p)}/F)$. Since F_∞/F is unramified outside primes above p, we have $F_\infty \subset \Omega_F^{(p)}$ and we denote by $G_{F_\infty}^{(p)}$ the Galois group of $\Omega_F^{(p)}/F_\infty$. The cohomology groups $H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ are finite (cf. Corollary 2.2), and hence

$$H^1_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n)).$$

We now consider the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(\Gamma, H^q(G_{F_\infty}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \Longrightarrow H^{p+q}(G_F^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

Since the cohomological p-dimension of Γ is equal to 1 all terms E_2^{pq} with $p > 0$ vanish, hence in particular we obtain an isomorphism

$$H^1(G_F^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^1(G_{F_\infty}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma.$$

Since n is even and $[E_\infty : F_\infty] = 2$, the module $\mathbb{Q}_p/\mathbb{Z}_p(n)$ is a trivial $G_{F_\infty}^{(p)}$-module and we obtain

$$H^1(G_{F_\infty}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n)) = Hom\,(G_{F_\infty}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n)) = Hom\,(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)),$$

since $\mathcal{X}$ is the pro-p-part of the abelianization of $G_{F_\infty}^{(p)}$. If we denote by # the Pontrjagin - dual and use the fact that

$$Hom\,(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong Hom\,(\mathcal{X}(-n), \mathbb{Q}_p/\mathbb{Z}_p),$$

then we have shown that

$$H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) \cong H^1_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) = H^1(G_F^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong (\mathcal{X}(-n)_\Gamma)^\#.$$

Since $H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ is finite, so is $\mathcal{X}(-n)_\Gamma$, hence also $\mathcal{X}(-n)^\Gamma$. But $\mathcal{X}$ has no non-trivial finite submodules, and therefore $\mathcal{X}(-n)^\Gamma = 0$. Let $f(T)$ denote the characteristic polynomial of the Iwasawa module $\mathcal{X}$. As we pointed out above $f(\kappa(\gamma)^n(1+T)-1)$ is then the characteristic polynomial of $\mathcal{X}(-n)$, and the order of the finite group $\mathcal{X}(-n)_\Gamma$ is obtained up to a p-adic unit by evaluating the characteristic polynomial at $T = 0$.

We obtain:

$$|H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n))| \sim_p f(\kappa(\gamma)^n - 1).$$

Here again $\sim_p$ indicates that both sides have the same p-adic valuation.

A similar, but much easier calculation for the Iwasawa module $\mathbb{Z}_p$ over F_∞, whose characteristic polynomial equals T, shows that

$$|H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))| = |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n)| \sim_p \kappa(\gamma)^n - 1.$$

A consequence of the *Main Conjecture* in Iwasawa theory is the following:

$$\zeta_F(1-n) \sim_p \zeta_{F,p}(1-n) \sim_p \frac{f(\kappa(\gamma)^n - 1)}{\kappa(\gamma)^n - 1}.$$

For odd primes p the Main Conjecture was proved by Wiles for arbitrary totally real number fields F (cf. [101]) and for the prime 2 he proved the Conjecture over $\mathbb{Q}$.

To summarize the discussion, recall that

$$H^2(o_F, \mathbb{Z}(n)) \cong \prod_p H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n)).$$

Let us denote the order of $H^2(o_F, \mathbb{Z}(n))$ simply by $h_n(F)$ in analogy with the class number, and let us denote the order of $H^0(F, \mathbb{Q}/\mathbb{Z}(n))$ simply by $w_n(F)$. Then we obtain the following special case of the Lichtenbaum Conjectures (for details at the prime 2 cf. [52]):

Theorem 3.1. *Let F be a totally real abelian number field and $n \geq 2$ an even integer. Then*

$$\zeta_F(1-n) = \pm \frac{h_n(F)}{w_n(F)}.$$

The same result holds up to 2-torsion for an arbitrary totally real number field.

In a similar manner one can prove a "higher" relative class number formula: Let L/F be a CM-extension of number fields and let $n \geq 3$ be an *odd* integer. Let χ denote the non-trivial character of $Gal\,(L/F)$.

We recall (cf. [99], Chapter 4) that the classical relative class number formula reads

$$L_F(\chi, 0) = \frac{2^{r_1}}{Q} \cdot \frac{h^-}{w(L)},$$

where $Q = [o_L^* : o_F^* \cdot \mu_L]$ is the Q-index, which is equal to 1 or 2.

The "higher" relative class number h_n^- is simply defined as

$$h_n^- = \frac{h_n(L)}{h_n(F)}.$$

We note that the order of the zeroes of the zeta-functions of F and L at $s = 1 - n$ are the same, and therefore the L-function $L_F(\chi, s)$ does not vanish at $s = 1 - n$. The precise result is the following

Theorem 3.2. **[53]** *Let L/F be a CM-extension of number fields and $n \geq 3$ an odd integer. If L is abelian, then*

$$L_F(\chi, 1-n) = \pm \frac{2^{r_1+1}}{Q_n} \cdot \frac{h_n^-}{w_n(L)},$$

where

$$Q_n = [H^1_{\acute{e}t}(L, \mathbb{Z}_2(n)) : H^1_{\acute{e}t}(F, \mathbb{Z}_2(n)) \cdot H^0(L, \mathbb{Q}_2/\mathbb{Z}_2(n))]$$

is a generalized Q-index, which is again equal to 1 *or* 2.

For an arbitrary CM-extension the same result holds up to 2*-torsion.*

As in the previous result the obstruction to obtaining the result in full generality lies again in the Main Conjecture for the prime 2.

Let us consider some examples, where this result proves to be useful: Let $L = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let us take $n = 3$. Since $K_4(\mathbb{Z}) = 0$, the relative class number formula can be used to compute $K_4(o_L)$ under the Quillen-Lichtenbaum Conjecture for odd primes p:

For $d = 1, 2, 3$ we obtain $K_4(o_L) = 0$, and for $d = 5$ we obtain $K_4(o_L) \cong \mathbb{Z}/15$.

4 The Lichtenbaum Conjecture

The Beilinson regulator map is simply obtained by composing the various embeddings of F into $\mathbb{C}$ with a Chern character

$$ch_n : K_{2n-1}(\mathbb{C}) \to H^1_{\mathcal{D}}(spec\,(\mathbb{C}), \mathbb{R}(n)) \overset{\text{can.}}{\cong} \mathbb{R}(n-1)$$

into Deligne-cohomology. Here $\mathbb{R}(n-1) = (2\pi i)^{n-1}\mathbb{R}$. We refer to the excellent article of Neukirch in [7] for details. We obtain

$$\rho_n : K_{2n-1}(F) \to K_{2n-1}(F) \otimes \mathbb{Q} \to (\mathbb{R}(n-1)^{Hom\,(F,\mathbb{C})})^+,$$

where complex conjugation acts on the set of embeddings and on the coefficients $\mathbb{R}(n-1)$.

The Beilinson regulator map ρ_n is twice the Borel regulator map by a recent result of Burgos Gil [15], and therefore by Borel's results the kernel of ρ_n is torsion. The image of ρ_n therefore is a full lattice in the real vectorspace $(\mathbb{R}(n-1)^{Hom\,(F,\mathbb{C})})^+$ of dimension d_n, and we denote by $R_n(F)$ the covolume of this lattice.

In a similar manner we can define a motivic regulator map

$$\rho_n^{\mathcal{M}} : H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \to H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong K_{2n-1}(F) \otimes \mathbb{Q} \to (\mathbb{R}(n-1)^{Hom\,(F,\mathbb{C})})^+$$

and we denote by $R_n^{\mathcal{M}}(F)$ the corresponding covolume.

The special cases, which we considered in the previous section, indicate that the extended form of the Lichtenbaum Conjectures – including powers of 2 – should read:

$$\zeta_F^*(1-n) = \pm \frac{h_n(F)}{w_n(F)} \cdot R_n^{\mathcal{M}},$$

hence that the correct version should be in terms of motivic cohomology rather than K-theory.

In [54],[55] essentially the following result was obtained:

Theorem 4.1. *Assume that F is an abelian number field. Then up to powers of 2 the formula*

$$\zeta_F^*(1-n) = \pm \frac{h_n(F)}{w_n(F)} \cdot R_n$$

is true.

In fact, in [54] a slightly different result was obtained including some Euler factors. This, however, is incorrect, and the Euler factors are removed in [57]. An updated proof, which also removes connections to a wrong result of Villemot's, can be found in [8].

The idea behind the proof is quite classical: Let $F = \mathbb{Q}(\zeta_N)$, $N > 2$, be a non-trivial cyclotomic field, and let $G = Gal\,(F/\mathbb{Q})$. The Beilinson regulator map has been calculated by Beilinson on a certain sublattice of $\bar{K}_{2n-1}(F) = K_{2n-1}(F)/tors.$, which is analogous to the group of cyclotomic units, and the index of this sublattice is related to $h_n(F)$. Let us make this a little bit more precise: Let

$$L_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

denote the n-th polylogarithm function, which is defined on $|z| < 1$ and extended analytically to $\mathbb{C} \setminus [1, \infty)$. Beilinson has shown (cf. Neukirch's article in [7] and [43]) that there is a map of G-sets

$$\epsilon_n : \mu_N \setminus \{1\} \to \bar{K}_{2n-1}(F) \otimes \mathbb{Z}[1/2],$$

such that for $\zeta \in \mu_N \setminus \{1\}$ the composite $\rho_n \circ \epsilon_n$ is given by

$$\zeta \mapsto N^{n-1}(n-1)!(\dots, L_n(\alpha\zeta), \dots)$$

with α running through the embeddings of F into $\mathbb{C}$.

Let $B_n(F)$ denote the lattice in $\bar{K}_{2n-1}(F) \otimes \mathbb{Z}[1/2]$ generated by the image of ϵ_n and let $B_n^{prim}(F)$ denote the lattice generated by the image of *primitive* N-th roots of unity.

The covolume of $\rho_n(B_n^{prim}(F))$ is now calculated as

$$\text{covolume}\left(\rho_n(B_n^{prim}(F))\right) = \prod_{\substack{\tilde{\chi} \\ \tilde{\chi}(-1)=(-1)^{n-1}}} \frac{N^{n-1} \cdot (n-1)!}{(2\pi i)^{n-1}}\, l_n(n, \tilde{\chi}),$$

where $\tilde{\chi}$ runs through the characters mod N with given parity and

$$l_n(n, \tilde{\chi}) = \sum_{a \bmod N} \tilde{\chi}(a) L_n(e^{2\pi i a/N}).$$

Let χ denote the *primitive* character of conductor f_χ belonging to $\tilde{\chi}$, and let

$$l_n(n, \chi) = \sum_{a \bmod f_\chi} \chi(a) L_n(e^{2\pi i a/f_\chi})$$

denote the corresponding Gauss sum. The relation between $l_n(n,\chi)$ and $l_n(n,\tilde{\chi})$ is the following:

$$\prod_{p|N}(1-\chi(p)p^{n-1})\cdot f_\chi^{n-1}\cdot l_n(n,\chi)=N^{n-1}l_n(n,\tilde{\chi}).$$

Now, for a primitive character χ with parity $\chi(-1)=(-1)^{n-1}$ the Gauss sum $l_n(n,\chi)$ is related to the value of the derivative $L'(1-n,\chi)$ of the Dirichlet L-function $L(s,\chi)$ at $s=1-n$ via

$$L'(1-n,\chi)=\frac{1}{2}\cdot\frac{1}{(2\pi i)^{n-1}}\cdot(n-1)!\cdot f_\chi^{n-1}\cdot l_n(n,\chi)$$

(cf. [38]).

Summarizing this discussion we obtain:

$$\text{covolume}\left(\rho_n(B_n^{prim}(F))\right)=$$

$$\frac{1}{2^{r_1}}\prod_{\substack{\chi\\ \chi(-1)=(-1)^{n-1}}}\left(\prod_{p|N}(1-\chi(p)p^{n-1}))\,L'(1-n,\chi)\right).$$

It is shown in [57] that the Euler factor

$$\prod_{\substack{\chi\\ \chi(-1)=(-1)^{n-1}}}\left(\prod_{p|N}(1-\chi(p)p^{n-1})\right)$$

is precisely the index of $B_n^{prim}(F)$ in $B_n(F)$. On the other hand the zeta-function $\zeta_F(s)$ splits naturally into a product of L-functions:

$$\zeta_F(s)=\prod_\chi L(s,\chi),$$

and we obtain for the special value at $s=1-n$:

$$\zeta_F(1-n)^*=\prod_{\substack{\chi\\ \chi(-1)=(-1)^{n-1}}}L'(1-n,\chi)\cdot\prod_{\substack{\chi\\ \chi(-1)=(-1)^{n}}}L(1-n,\chi).$$

In the previous section we calculated in particular the values of

$$\prod_{\substack{\chi\\ \chi(-1)=(-1)^{n}}}L(1-n,\chi)=\begin{cases}\zeta_{F^+}(1-n) & \text{if } n \text{ is even}\\ \dfrac{\zeta_F(1-n)^*}{\zeta_{F^+}(1-n)^*} & \text{if } n \text{ is odd.}\end{cases}$$

The proof of theorem 4.1 (in the special case $F = \mathbb{Q}(\zeta_N)$) is now reduced to showing that

$$[\bar{K}_{2n-1}(F) : B_n(F)] \sim_p \begin{cases} h_n(F^+) & \text{if } n \text{ is odd} \\ h_n(F)^- & \text{if } n \text{ is even} \end{cases}$$

for all odd primes p. This is done using the étale Chern characters again to map to étale cohomology. The group $B_n(F)$ maps to the group of *cyclotomic elements* of Soulé and Deligne in $H^1_{\text{ét}}(F, \mathbb{Z}_p(n))$ and the index is calculated using Poitou-Tate duality and Iwasawa-theory.

5 K-theory and some classical conjectures in number theory

Let p denote an *odd* prime number. For a number field F with ring of integers o_F and an arbitrary integer N the étale cohomology group $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N))$ is p-divisible, since the boundary map

$$\delta_3 : H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N)) \to H^3_{\text{ét}}(o'_F, \mathbb{Z}_p(N)) = 0$$

is trivial, p being odd. As we have seen in section 2, for $N \geq 2$ the groups $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N))$ are trivial as a consequence of Borel's result on the finiteness of $H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(N))$. For $N = 1$ the corank of $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is equal to $s_p - 1$, where s_p denotes the number of p-adic primes of F. For $N = 0$ the vanishing of $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(0))$ is equivalent to Leopoldt's Conjecture for the prime p and the field F. More generally, Schneider has conjectured that the groups $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N))$ vanish for all $N \neq 1$. It is easy to see that the following two conditions are equivalent:

1. $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N)) = 0$
2. $rk_{p^n}\ (H^2_{\text{ét}}(o'_F, \mathbb{Z}/p^n\mathbb{Z}(N)) = 0$ for some $n \geq 1$.

Let $d_n = [F(\zeta_{p^n}) : F]$. Then for all integers k:

$$H^2_{\text{ét}}(o'_F, \mathbb{Z}/p^n\mathbb{Z}(N)) \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}/p^n\mathbb{Z}(N + kd_n)),$$

and the second condition above is easily seen to be equivalent to the following periodicity statement:

3. There exists $n \geq 1$ such that

$$H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(i)) \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(N))$$

for all $i \equiv N \mod d_n$.

Let us assume that F is totally real. Then for odd $N < 0$ Schneider's Conjecture is true as a consequence of a "Spiegelungssatz" and the vanishing of $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(1-N))$ (cf. [82]). Hence let us assume that N is even. Since d_n is even as well, we can use the results of section 3 to relate the vanishing of $H^2_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N))$ to a periodicity statement for values of the zeta-function $\zeta_F(s)$:

Theorem 5.1. [51]

Let F be totally real and $N \in \mathbb{Z}$ even.

a) If $N \neq 0$, then the following 2 conditions are equivalent:

1. $H^2_{ét}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(N)) = 0$.
2. *There exists $n \geq 1$, such that*

$$\zeta_F(1-i) \sim_p \zeta_F(1-j)$$

for all $i, j \geq 2$, $i \equiv j \equiv N \mod d_n$.

b) For $N = 0$ the following 2 conditions are equivalent:

1. *Leopoldt's Conjecture holds for p and the field F.*
2. $\zeta_F(1-d_n)$ *is* not *p-integral for some $n \geq 1$.*

Proof. We have seen in section 3 that for i even, $i \geq 2$:

$$\zeta_F(1-i) \sim_p \frac{|H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(i))|}{w_i(F)}.$$

The result follows from the equivalence of conditions 1. and 3. above, and the fact that for n large:

$$w_i(F) \sim_p w_j(F) \quad \text{if } i \equiv j \equiv N \neq O \mod d_n$$

and

$$w_{d_n}(F) \sim_p p^{d_n}.$$

□

We note that the conditions in part a) hold, whenever $N \geq 2$, and therefore the second condition can be viewed as a generalization of the Kummer congruences for Bernoulli numbers.

Let us go back to the simplest possible situation $F = \mathbb{Q}$, and let us recall a few facts about the K-groups of $\mathbb{Z}$:

$$\begin{aligned} K_0(\mathbb{Z}) &= \mathbb{Z} \\ K_1(\mathbb{Z}) &= \mathbb{Z}/2\mathbb{Z} \\ K_2(\mathbb{Z}) &= \mathbb{Z}/2\mathbb{Z} \\ K_3(\mathbb{Z}) &= \mathbb{Z}/48 \quad [59] \\ K_4(\mathbb{Z}) &= 0 \quad [79] \\ K_5(\mathbb{Z}) &= \mathbb{Z} \quad [24] \\ K_6(\mathbb{Z}) &= 3-\text{torsion} \quad [24] \end{aligned}$$

Furthermore, $K_m(\mathbb{Z})$ is finite for all $m > 0$, $m \not\equiv 1 \bmod 4$, whereas for $m \equiv 1 \bmod 4$ we obtain $K_m(\mathbb{Z}) = \mathbb{Z} \oplus$ (finite).

Theorem 2.8 simplifies and yields the following information about the kernels and cokernels of the 2-primary Chern characters $ch^{(2)}_{i,n}$ for $i = 1, 2$ and $2n - i \geq 2$:

$$ch^{(2)}_{i,n} : K_{2n-i}(\mathbb{Z}) \otimes \mathbb{Z}_2 \to H^i_{\text{ét}}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(n))$$

$$\begin{cases} \text{is an isomorphim} & \text{if } 2n-i \equiv 0,1,2,4,7 \bmod 8, \\ \text{surjective with kernel } \mathbb{Z}/2\mathbb{Z} & \text{if } 2n-i \equiv 3 \bmod 8, \\ \text{injective with cokernel } \mathbb{Z}/2\mathbb{Z} & \text{if } 2n-i \equiv 5,6 \bmod 8. \end{cases}$$

For $2n - 1 \equiv 3 \mod 8$ the extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2 \to H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(n)) \to 0$$

is nontrivial (cf. [81]). The cohomology groups $H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(n))$, are easily calculated using the results from section 2:

$$H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(n)) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even} \end{cases}$$

Therefore we obtain the following results about the 2-torsion in $K_{2n-2}(\mathbb{Z})$:

$$K_{2n-2}(\mathbb{Z})(2) \cong \begin{cases} 0 & \text{if } n \equiv 0,1,3 \mod 4 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 2 \mod 4 \end{cases}$$

In a similar way we obtain

$$H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}_2(n)) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd} \\ H^0(\mathbb{Q}, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \text{if } n \text{ is even.} \end{cases}$$

If we assume that the Quillen-Lichtenbaum Conjecture holds for odd primes, then we obtain for odd n:

$$K_{2n-1}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 3 \mod 4 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 1 \mod 4, \end{cases}$$

whereas for even n the group $K_{2n-1}(\mathbb{Z})$ is finite. In this case it is isomorphic to $H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(n))$ if $n \equiv 0 \mod 4$ and cyclic of order $2 \cdot |H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(n))|$ if $n \equiv 2 \mod 4$.

For even $n \geq 2$, we can compute the orders $h_n(\mathbb{Q})$ of the groups $H^2(\mathbb{Z}, \mathbb{Z}(n))$ and – assuming the Quillen-Lichtenbaum Conjecture for odd primes – of the K-groups $K_{2n-2}(\mathbb{Z})$ via the results of section 3:

$$\zeta(1-n) = \pm\frac{h_n(\mathbb{Q})}{w_n(\mathbb{Q})} = \pm 2 \cdot \frac{|K_{2n-2}(\mathbb{Z})|}{|K_{2n-1}(\mathbb{Z})|}.$$

On the other hand we know from Euler's result that

$$\zeta(1-n) = -\frac{B_n}{n},$$

where the B_n are the Bernoulli numbers. Let us write

$$\left|-\frac{B_n}{n}\right| = \frac{N_n}{D_n}$$

with N_n and D_n relatively prime. Then it is well-known that

$$D_n = \prod_{\substack{p \\ p-1|n}} p^{n_p+1},$$

where $n_p = ord_p(n)$. It is easy to compare this with the order $w_n(\mathbb{Q})$ of $H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(n))$:

$$D_n = \frac{w_n(\mathbb{Q})}{2},$$

and therefore by our Lichtenbaum formula:

$$N_n = \frac{h_n(\mathbb{Q})}{2}$$

as well. Now we note that for $n \equiv 0 \mod 4$ we have - still assuming the Quillen-Lichten-baum Conjecture -:

$$|K_{2n-1}(\mathbb{Z})| = w_n(\mathbb{Q}) = 2 \cdot D_n$$

and

$$|K_{2n-2}(\mathbb{Z})| = \frac{1}{2} \cdot h_n(\mathbb{Q}) = N_n,$$

whereas in the case that $n \equiv 2 \bmod 4$ the situation is different:

$$|K_{2n-1}(\mathbb{Z})| = 2w_n(\mathbb{Q}) = 4 \cdot D_n$$

and

$$|K_{2n-2}(\mathbb{Z})| = h_n(\mathbb{Q}) = 2 \cdot N_n.$$

This would imply for example that $K_6(\mathbb{Z}), K_{14}(\mathbb{Z})$ are trivial, $K_{10}(\mathbb{Z})$, $K_{18}(\mathbb{Z})$ are cyclic of order 2, and $K_{22}(\mathbb{Z})$ is cyclic of order 691. However, all these calculations merely give the order of the finite K-groups and cohomology groups, but not their structure.

The structure is related to Vandiver's Conjecture, an observation due to Kurihara [58]: Let us fix an odd prime p, and let $E = \mathbb{Q}(\zeta_p)$. Let A_E denote the p-part of the class group of E, and let Δ denote the Galois group of $E/\mathbb{Q}$. Then

$$A_E \cong \oplus_{i=0}^{p-1} A_E^{[i]}$$

is the decomposition of A_E into eigenspaces with respect to the powers ω^i of the Teichmüller character. Clearly $A_E^{[0]} = 0$. Since the class group of o_E is equal to the class group of $o_E[\frac{1}{p}]$, the results of section 3 show that

$$(A_E/p)^{[i]} \cong H^2_{\text{ét}}(o'_E, \mu_p)^{[i]} \cong H^2_{\text{ét}}(o'_E, \mu_p^{p-i})^{\Delta} \cong H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{p}], \mu_p^{\otimes p-i}).$$

Therefore

Proposition 5.2. *For even i, $2 \le i \le p-3$ the eigenspace $A_E^{[i]}$ is trivial if and only if $H^2_{ét}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(p-i)) = 0$, hence if and only if $p \nmid h_{p-i}(\mathbb{Q})$.*

Assuming the Quillen-Lichtenbaum Conjecture the latter condition is equivalent to $K_{2(p-1-i)}(\mathbb{Z})(p) = 0$, hence

Corollary 5.3. [Kurihara] [58] *Assume that the Quillen-Lichtenbaum Conjecture holds for p. Then Vandiver's Conjecture is equivalent to the vanishing of $K_{4*}(\mathbb{Z})(p)$.*

As a consequence – using the fact that $K_4(\mathbb{Z}) = 0$ – we obtain the following

Corollary 5.4.
$$A_E^{[p-3]} = 0.$$

The vanishing of the K-theory of $\mathbb{Z}$ in dimensions a multiple of 4 also implies that the K-groups of $\mathbb{Z}$ in dimensions $\equiv 2 \bmod 4$ are cyclic, and hence the structure of the K-theory of $\mathbb{Z}$ would be completely known.

Let us mention another application of the technique of expressing eigenspaces of class groups in terms of étale cohomology groups over $\mathbb{Z}$:

Assume that i is odd, $3 \leq i \leq p-2$, and let $j = p - i$. Then as we observed above:

$$p \Big| |A_E^{[i]}| \iff p \Big| |H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{p}], \mu_p^{\otimes j})| \iff p \Big| |H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}(j))|.$$

But
$$H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}(j)) \sim_p \frac{B_j}{j} \sim_p B_j = B_{p-i},$$

hence we obtain the famous theorem of Herbrandt-Ribet:

Theorem 5.5. *For odd i, $3 \leq i \leq p-2$:*

$$p \Big| |A_E^{[i]}| \iff p | B_{p-i}.$$

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References

[1] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Astérisque **61** (1979), 11–13.

[2] A. Bak and U. Rehmann, *K_2-analogs of Hasse's norm theorems*, Comm. Math. Helv. **59** (1984), 1–11.

[3] H. Bass, J. Milnor and J.-P. Serre, *Solution of the congruence subgroup problem for* $SL_n\,(n \geq 3)$ *and* $Sp_{2n}\,(n \geq 2)$, Publ. Math. I.H.E.S. **33** (1967), 59–137.

[4] H. Bass and J. Tate, *The Milnor ring of a global field (with an appendix on euclidean quadratic imaginary fields, by J. Tate)*, Lecture Notes in Math. vol. 342, Springer Verlag, 1973, 349–446.

[5] P. Bayer and J. Neukirch, *On Values of Zeta Functions and l-adic Euler Characteristics*, Invent. Math. **50** (1978), 35–64.

[6] A. Beilinson, *Higher regulators and values of L-functions*, Journal Soviet Math. **30** (1985), 2036-2070.

[7] M. Rapoport, N. Schappacher and P. Schneider (eds.), *Beilinson's conjectures on special values of L-functions*, Perspectives in Mathematics vol. 4, Academic Press, Boston, 1988.

[8] D. Benois and T. Nguyen Quang Do, *Les nombres de Tamagawa locaux et la conjecture de Bloch et Kato pour les motifs* $\mathbb{Q}(m)$ *sur un corps abelien*, Ann. Sci. École Norm. Sup. (4) **35** (2000), 641–672.

[9] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1086), 267–304.

[10] S. Bloch and K. Kato, *p-adic étale cohomology*, Publ. Math. I.H.E.S. **63** (1986), 107–152.

[11] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives* The Grothendieck Fest-schrift, vol. 1, Progr. Math 86, Birkhäuser, Boston, 1990, pp. 333–400.

[12] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, Preprint (1994).

[13] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. **7** (1977), 613–636.

[14] A. Borel, *Cohomologie de SL_n et valeurs de fonction zêta*, Ann. Sci. Scuola Norm. Sup. Pisa **4** (1974), 235–272.

[15] J.I. Burgos Gil, *The regulators of Beilinson and Borel*, CRM Monograph Series, vol. 15, AMS, Providence, RI, 2002.

[16] D. Burns and C. Greither, *On the equivariant Tamagawa number conjecture for Tate motives*, preprint (2000).

[17] J.W.S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, 1967.

[18] T. Chinburg, M. Kolster, G. Pappas and V. Snaith, *Galois structure of K-groups of rings of integers*, K-Theory **14** (1998), 319–369.

[19] J. Coates, *p-adic L-functions and Iwasawa theory*, Algebraic Number Theory (A. Fröhlich, ed.), Academic Press, London, 1977, pp. 269–353.

[20] J. Coates, *On K_2 and some classical conjectures in algebraic number theory*, Ann. of Math. **95** (1972), 99–116.

[21] J. Coates and S. Lichtenbaum, *On l-adic zeta functions*, Ann. of Math. **98** (1973), 498–550.

[22] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois Groups over $\mathbb{Q}$ (Y. Ihara, k. Ribet, J.-P. Serre, eds.), Math. Sci. Research Inst. Publ. vol. 16, Springer, Berlin, 1989, pp. 79–298.

[23] W. Dwyer and E. Friedlander, *Algebraic and étale K-theory*, Trans. AMS **292** (1985), 247–280.

[24] P. Elbaz-Vincent, H. Gangl and C. Soulé, *Quelques calculs de la cohomologie de $GL_N(\mathbb{Z})$ et la K- théorie de $\mathbb{Z}$*, preprint (2002).

[25] L.J. Federer, *Regulators, Iwasawa modules and the Main conjecture for* $p = 2$, Number theory related to Fermat's Last Theorem (N. Koblitz, ed.), Birkhäuser Verlag, 1982, pp. 289–296.

[26] J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonction L*, Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proceedings of Symposia in Pure Math. vol. 55, Part I, AMS, Providence, 1994, pp. 599–706.

[27] E. Friedlander and A.A. Suslin, *The spectral sequence relating K-theory to motivic cohomology*, preprint (2000).

[28] E. Friedlander and V. Voevodsky, *Bivariant cycle cohomology*, Chapter 4 in [98]

[29] H. Garland, *A finiteness theorem for K_2 of a number field*, Ann. of Math. **94** (1971), 534–548.

[30] T. Geisser, *Motivic cohomology over Dedekind domains*, preprint (2001).

[31] T. Geisser and M. Levine, *The K-Theory of Fields of Characteristic p*, Invent. Math. **139** (2000), 459–493.

[32] T. Geisser and M. Levine, *The Bloch-Kato Conjecture and a Theorem of Suslin-Voevodsky*, J. Reine Angew. Math. **530** (2001), 55–103.

[33] H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Adv. Math. **40** (1981), 203–289.

[34] H. Gillet and Ch. Soulé, *Filtrations on Higher Algebraic K-theory*, Proc. of Symp. in Pure Math. **67** (1999).

[35] A. Goncharov, *The Classical Trilogarithm, Algebraic K-Theory of Fields, and Dedekind Zeta-Functions*, Bull. AMS **24, 1** (1991), 155–162.

[36] A. Goncharov, *Geometry of Configurations, Polylogarithms, and Motivic Cohomology*, Adv. Math. **114, No. 2** (1995), 197–318.

[37] C. Greither, *Class groups of Abelian fields, and the Main Conjecture*, Ann. Inst. Fourier, Grenoble **42** (1992), 449–499.

[38] B.H. Gross, *On the values of Artin L-functions*, unpublished (1980).

[39] B. Harris and G. Segal, *K_i groups of rings of algebraic integers*, Annals of Math. **101** (1975), 20–33.

[40] L. Hesselholt and I. Madsen, *The K-theory of local fields*, Ann. of Math. (to appear).

[41] A. Huber and G. Kings, *Degeneration of l-adic Eisenstein classes and of the elliptic polylog*, Invent. Math. **135**, (1999), 545–594.

[42] A. Huber and G. Kings, *Bloch-Kato conjecture and main conjecture of Iwasawa theory for Dirichlet characters*, preprint (2000).

[43] A. Huber and J. Wildeshaus, *Classical Motivic Polylogarithm according to Beilinson and Deligne*, Doc. Math. **3** (1998), 27–133.

[44] K. Iwasawa, *On $\mathbb{Z}_l$-extensions of algebraic number fields*, Ann. of Math. **98** (1973), 246–326.

[45] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.

[46] B. Kahn, *Some conjectures on the algebraic K-theory of fields I: K-theory with coefficients and étale cohomology*, Algebraic K-theory: Connections with Geometry and Topology (J. F. Jardine and V. P. Snaith, eds.), NATO ASI Series C, vol. 279 Kluwer, 1989, pp. 117–176.

[47] B. Kahn, *Descente galoisienne et K_2 des corps de nombres K-Theory*, **7**, (1993), 55–100.

[48] B. Kahn, *On the Lichtenbaum-Quillen conjecture*, Algebraic K-Theory and Algebraic Topo-logy (P Goerss, J.F. Jardine, eds.), Nato Proceedings Lake Louise, vol, 407, Kluwer, Dordrecht, 1993, pp. 147–166.

[49] B. Kahn, *The Quillen–Lichtenbaum Conjecture at the prime* 2, preprint (1997).

[50] B. Kahn, *Deux Théorèmes de comparison en cohomology étale; Applications*, Duke Math. J. **69** (1993), 137–165.

[51] M. Kolster, *Remarks on étale K-theory and Leopoldt's conjecture*, Séminaire de Théorie des Nombres, Paris, 1991-1992 (S. David, ed.), Progress in Math., vol. 116, Birkhäuser, Boston, 1993, pp. 37–62.

[52] M. Kolster, *Cohomological version of the Lichtenbaum conjecture at the prime* 2, Appendix to [81].

[53] M. Kolster, *Higher relative class number formulae*, Math. Annalen **323** (2002), 667–692.

[54] M. Kolster, T. Nguyen Quang Do and V. Fleckinger, *Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures*, Duke Math. J. **84** (1996), 679–717.

[55] M. Kolster, T. Nguyen Quang Do and V. Fleckinger, *Correction to "Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures"*, Duke Math. J. **90** (1997), 641–643.

[56] M. Kolster and T. Nguyen Quang Do, *Syntomic regulators and special values of p-adic L-functions*, Invent. Math. **133** (1998), 417–447.

[57] M. Kolster and T. Nguyen Quang Do, *Cyclotomic lattices with distribution relations*, in preparation

[58] M. Kurihara, *Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$*, Compositio Math. **45 No.1** (1992), 223–236.

[59] R. Lee and R.H. Szczarba, *The group $K_3(\mathbb{Z})$ is cyclic of order* 48, **104** Ann. of Math. (1976), 31–60.

[60] R. Lee and R.H. Szczarba, *On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$*, with an Addendum by C. Soulé, Duke Math. J. **45 No. 1** (1978), 101–132.

[61] M. Levine, *The indecomposable K_3 of a field*, Ann. Sci. École Norm. Sup. **22** (1989), 255–344.

[62] M. Levine, *Mixed Motives*, Mathematical Surveys and Monographs, vol. 57 American Mathematical Society, 1998.

[63] S. Lichtenbaum, *On the values of zeta and L-functions, I*, Annals of Math. **96** (1972), 338–360.

[64] S. Lichtenbaum, *Values of zeta functions, étale cohomology, and algebraic K-theory*, Lecture Notes in Math., vol, 342, Springer Verlag, 1973, pp. 489–501.

[65] J.-L. Loday, *K-théorie algébrique et représentations de groupes*, Ann. Sci. École Norm. Sup. **9** (1976), 309–377.

[66] A.S. Merkurjev and A.A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Math. USSR **21** (1983), 307–340.

[67] A.S. Merkurjev and A.A. Suslin, *The K_3 of a field*, Math. USSR **36** (1990), 541–565.

[68] J.S. Milne, *Étale Cohomology*, Princeton University Press, 1980.

[69] J.S. Milne, *Arithmetic Duality Theorems*, Academic Press, 1986.

[70] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1970), 318–344.

[71] J. Milnor, *Introduction to Algebraic K-Theory*, Annals of Mathematical Studies, vol. 72, Princeton University Press, New Jersey, 1971.

[72] T. Nguyen Quang Do, *Une étude cohomologique de la partie 2-primaire de $K_2(\mathcal{O})$*, K-Theory **3** (1990), 523–542.

[73] O. Pushin, *Higher Chern classes and Steenrod operations in motivic cohomology*, preprint (2001).

[74] D. Quillen, *On the cohomology and K-theory of the general linear group over a finite field*, Annals of Math. **96** (1972), 552–586.

[75] D. Quillen, *Higher algebraic K-theory, I*, Lecture Notes in Math., vol. 341, Springer Verlag, 1973, pp. 85–147.

[76] D. Quillen, *Finite generation of the groups K_i of rings of algebraic integers*, Lecture Notes in Math., vol. 341, Springer Verlag, 1973, pp. 179–198.

[77] D. Quillen, *Higher algebraic K-theory*, Proc. Intern. Congress Math., Vancouver, 1974, vol. I, Canad. Math. Soc., 1975, pp. 171–176.

[78] T. Rivoal, *La fonction Zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C.R. Acad. Sci. Paris, **331** (2000), 267–270.

[79] T. Rivoal, *Irrationalité d'au moins un des neuf nombres $\zeta(5), \zeta(7), \cdots \zeta(21)$*, preprint (2001).

[80] J. Rognes, *Approximating $K_*(\mathbb{Z})$ through Degree Five*, K-Theory, vol. 7, 1993. pp. 175–200.

[81] J. Rognes and C. Weibel, *Two-primary algebraic K-theory of rings of integers in number fields*, Journal of the AMS **13** (2000), 1–54.

[82] P. Schneider, *Über gewisse Galoiscohomologiegruppen*, Math. Z. **168** (1979), 181–205.

[83] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer Verlag, 1979.

[84] J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics, vol. 5, Springer-Verlag, 1964.

[85] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295.

[86] C. Soulé, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), 317–345.

[87] C. Soulé, *On higher p-adic regulators*, Algebraic K-theory, LNM, vol. 854, Springer, Berlin, 1981, pp. 372–401.

[88] C. Soulé, *Eléments cyclotomiques en K-théorie*, Astérisque **147/148** (1987), 225–257.

[89] A.A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izv. **11**, (1977), 221–238.

[90] A.A. Suslin, *On the K-theory of local fields*, J. Pure Appl. Algebra **34** (1984), 301–318.

[91] A.A. Suslin, *Higher Chow groups and étale cohomology*, Cycles, Transfers, and Motivic Homology Theories (V. Voevodsky, A.A. Suslin, E.M. Friedlander, eds.), Annals of Math. Studies, vol. 143, Princeton University Press, 2000, pp. 239–254.

[92] A. A. Suslin and V. Voevodsky, *The Bloch–Kato conjecture and motivic cohomology with finite coefficients*, preprint (1995); revised version (1999).

[93] J. Tate, *Symbols in Arithmetic*, Lecture at the International Congress in Nice (1970).

[94] J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. **36** (1976), 257–274.

[95] J. Tate, *Duality theorems in Galois cohomology over number fields*, Proc. Intern. Congress Math., Stockholm, 1962, Inst. Mittag–Leffler, 1963, pp. 234–241.

[96] V. Voevodsky, *Triangulated categories of motives over a field*, preprint (1995).

[97] V. Voevodsky, *The Milnor Conjecture*, preprint (1996).

[98] V. Voevodsky, A. A. Suslin and E. Friedlander, *Cycles, Transfers, and Motivic Homology Theories*, Annals of Math. Studies, vol. 143, Princeton University Press, New JA. A. Suslinersey, 2000.

[99] L.C. Washington, *Introduction to Cyclotomic Fields*, Graduate Texts in Mathematics, vol. 83, Springer Verlag, 1982.

[100] C. Weibel, *The 2-torsion in the K-theory of the integers*, C.R. Acad. Sci. Paris **324** (1997), 615–620.

[101] A. Wiles, *The Iwasawa Conjecture for totally real fields*, Annals of Math. **131** (1990), 493–540.

[102] D. Zagier, *Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields*, Proccedings of the Texel Conference on Arithmetical Algebraic Geometry, 1990 (G. Van der Geer, F. Oort, J. Steenbrink, eds.), Progress in Math., vol. 89, Birkhäuser, Boston, 1991, pp. 391–430.