Exploring the Open String Star Algebra – Applications to Tachyon Condensation

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Abstract

This is a set of informal introductory lectures on open string field theory, with special focus on the structure of the open string star algebra. The topics discussed are (1) Algebraic structure of OSFT, (2) CFT definition of star, (3) Neumann coefficients, (3) Conservation laws, (5) Surface states (6) Star algebra spectroscopy, (7) Star as Moyal product, and (8) Star algebra projectors.
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1 Algebraic structure of OSFT

In open string field theory, a kinetic operator $Q$, a multiplication rule $\ast$, and an inner product $\langle \cdot , \cdot \rangle$, are the key structures that allow the definition of a string action. All these structures are relevant to a vector space $\mathcal{H}$ where the string field lives. The kinetic operator $Q$ takes $\mathcal{H}$ to $\mathcal{H}$, the star product takes two elements of $\mathcal{H}$ and gives us a third, the product. Finally, the inner product $\langle \cdot , \cdot \rangle$ takes two elements of $\mathcal{H}$ and gives us a (complex) number. In this language the string action takes the form

$$S(\Phi) = -\frac{1}{g_o^2} \left[ \frac{1}{2} \langle \Phi , Q \Phi \rangle + \frac{1}{3} \langle \Phi , \Phi \ast \Phi \rangle \right], \quad (1)$$

where $g_o$ is the open string coupling constant. The consistency of such an action requires several conditions. The kinetic operator $Q$ satisfies the following identities

$$Q^2 A = 0,$$

$$Q(A \ast B) = (QA) \ast B + (-1)^A A \ast (QB) \quad (2)$$

$$\langle QA , B \rangle = -(-)^A \langle A , QB \rangle ,$$

The first one is the nilpotency condition. The second states that $Q$ is an odd derivation of the star product. The third shows that $Q$ is self-adjoint. There are also identities involving the inner product and the star operation

$$\langle A , B \rangle = (-)^{AB} \langle B , A \rangle$$

$$\langle A , B \ast C \rangle = \langle A \ast B , C \rangle \quad (3)$$

$$A \ast (B \ast C) = (A \ast B) \ast C .$$

In the sign factors, the exponents $A,B, \cdots$ denote the Grassmanality of the state, and should be read as $(-)^A \equiv (-)^{\epsilon(A)}$ where $\epsilon(A) = 0 \pmod{2}$ when $A$ is Grassmann even, and $\epsilon(A) = 1 \pmod{2}$ when $A$ Grassmann odd. The first property above is a symmetry condition, the second indicates that the inner product has a cyclicity property analogous to the similar property of the trace operation.

The last equation in (3) is the statement that the star product is associative. This is a fundamental property. In the case of string field theory the condition of associativity is so strict that there is just one possible way to define the product – the way devised by Witten.
Algebraically, some further information is necessary about the star product. We declare star to be an even product of degree zero. This, in plain English means that both the Grassmannality and the ghost number of the star product of two string fields is obtained from those of the string fields without any additional offset:

$$
\epsilon(A \star B) = \epsilon(A) + \epsilon(B)
$$

$$
gh(A \star B) = gh(A) + gh(B).
$$

(4)

In this language $Q$ is an odd operator of degree one:

$$
\epsilon(QA) = \epsilon(A) + 1
$$

$$
gh(QA) = gh(A) + 1.
$$

(5)

In the conventions we shall work the SL(2,R) vacuum $|0\rangle$ is assigned ghost number zero.

The above setup allows us to deduce some basic properties of the string field. In particular, its ghost number, and its Grassmannality. The Grassmannality of $\Phi$ can be deduced from the condition that the kinetic term of the string action must be non-vanishing. Using the above properties we have

$$
\langle \Phi, Q\Phi \rangle = (-1)^{\Phi(1+\Phi)} \langle Q\Phi, \Phi \rangle = \langle Q\Phi, \Phi \rangle = -(-1)^{\Phi} \langle \Phi, Q\Phi \rangle.
$$

(6)

It is clear that the string field $\Phi$ must be Grassmann odd. At this point we must use some CFT knowledge to decide on the Grassmannality of the SL(2,R) vacuum and on the ghost number of the string field. For bosonic strings we have that zero momentum tachyon states are of the form $tc_1 |0\rangle$, where $c_1$ is a ghost field oscillator. Since this oscillator is Grassmann odd, and the string field is also Grassmann odd, we must declare the SL(2,R) vacuum to be Grassmann even. Thus

$$
|0\rangle \text{ is a Grassmann even state of ghost number zero.}
$$

(7)

Since the $c_1$ oscillator carries ghost number one, we also deduce that the open string field must have ghost number one.

$$
|\Phi\rangle \text{ is a Grassmann odd state of ghost number one.}
$$

(8)

Equations (2), (3), (4), (5) and (8) guarantee that the string field action is invariant under the gauge transformations:

$$
\delta \Phi = Q\Lambda + \Phi \star \Lambda - \Lambda \star \Phi,
$$

(9)
for any Grassmann-even ghost-number zero state $\Lambda$. Moreover, variation of the action gives the field equation

$$Q\Phi + \Phi \ast \Phi = 0.$$  \hfill (10)

*Exercise* Verify that the string action in (1) is gauge invariant under the transformations (9).

It is convenient to use the above structures to define a multilinear object that given three string fields it yields a number:

$$\langle A, B, C \rangle \equiv \langle A, B \ast C \rangle$$  \hfill (11)

The middle equation in (3) implies the *cyclicity* of the multilinear form. A small calculation immediately gives:

$$\langle A, B, C \rangle = (-)^{A(B+C)} \langle B, C, A \rangle$$  \hfill (12)

A basic consistency check of the signs above is that the cubic term $\langle \Phi, \Phi, \Phi \rangle$ in the action (2) is strictly cyclic for odd $\Phi$, and therefore does not vanish.

### 1.1 Additional Structure: Twist Operator and Identity String Field

Open string theory has additional algebraic structure that sometimes plays a crucial role. One such structure arises from the twist operation, which reverses the parametrization of a string. From the algebraic viewpoint this is summarized by the existence of an operator $\Omega$ satisfying the following properties:

$$\Omega(QA) = Q(\Omega A)$$

$$\langle \Omega A, \Omega B \rangle = \langle A, B \rangle$$  \hfill (13)

$$\Omega(A \ast B) = (-)^{AB+1} \Omega(B) \ast \Omega(A).$$

The first property means that the BRST operator has zero twist, or does not change the twist property of the states it acts on. The second property states that the bilinear form is twist invariant. The third property is crucial. Up to signs twisting the star product of string fields amounts to multiplying the twisted states in *opposite order*. This change of order is a simple consequence of the basic multiplication rule where the second half of the first string must
be glued to the first have of the second one. The sign factor is also important. For the string field $\Phi$, which is Grassmann odd, it gives

$$\Omega(\Phi \ast \Phi) = + (\Omega \Phi) \ast (\Omega \Phi)$$

(14)

with the plus sign. This result, together with the first two equations in (13) immediately implies that the string field action in (2) is twist invariant

$$S(\Omega \Phi) = S(\Phi).$$

(15)

This invariance under twist transformations allows one to construct new string theories by truncating the spectrum to the subset of states that are twist even. Moreover, in solving the string field equations it will be possible to find consistent solutions by restricting oneself to the twist even subspace of the string field.

**Exercise.** Letting $\Omega_A$ denote the $\Omega$ eigenvalue of $A$, show that

$$\langle A, B, C \rangle = \Omega_A \Omega_B \Omega_C (-1)^{AB+BC+CA+1} \langle C, B, A \rangle.$$  

(16)

**Exercise.** Let $\Omega A_\pm = \pm A$ and $\epsilon(A_\pm) = 1$. Show that

$$\langle A_+, A_+, A_- \rangle = 0.$$  

(17)

**Exercise.** We will show later that the star product of the vacuum with itself is the vacuum plus Virasoro descendents:

$$|0\rangle \ast |0\rangle = |0\rangle + \cdots$$

(18)

Show that this implies that the vacuum is twist odd:

$$\Omega |0\rangle = -|0\rangle.$$  

(19)

The algebra of star has an identity element, usually written as $\mathcal{I}$. By definition, this state satisfies

$$\mathcal{I} \ast A = A \ast \mathcal{I} = A,$$

(20)

for all reasonable states $A$. Some properties of $\mathcal{I}$ are immediately deduced from the above definition

$$\mathcal{I}$$

is Grassmann even, ghost number zero, twist odd string field
The twist odd property follows from the twist property of products

$$\Omega(I \ast A) = (-1)^{A+1}(\Omega A) \ast (\Omega I) = -(\Omega A) \ast (\Omega I).$$  \hspace{1cm} (22)

Since the left hand side is also just $\Omega A$ it must follow that

$$\Omega I = -I.$$  \hspace{1cm} (23)

This is consistent with the fact that the SL(2,R) vacuum is also twist odd. Indeed the identity string field is just the vacuum plus Virasoro descendents of the vacuum, as we shall see later.

Finally, let us remark that any derivation $D$ of the star algebra should annihilate the identity:

$$D(I \ast A) = (DI) \ast A + I \ast DA = (DI) \ast A + DA.$$  \hspace{1cm} (24)

Since the left hand side also equals $DA$, one concludes that $(DI) \ast A = 0$ for all $A$, and thus the expectation that $DI = 0$. We will see that sometimes subtle effects prevent this from being the case.

### 1.2 Kinetic term computations

Let us illustrate some of the structure by a sample computation of the string action truncated to the zero momentum tachyon string field $|T\rangle = tc_1|0\rangle$. Here we evaluate the kinetic term $\langle T, QT\rangle$. To this end we need to use the normalization condition

$$\langle 0|c_{-1}c_0c_1|0\rangle = 1,$$  \hspace{1cm} (25)

which is appropriate if we have compactified all coordinates (including time). Otherwise the above inner product would need delta functions associated to momentum.

The tachyon field is said to be of level zero. The level $\ell$ of a state is related to the $L_0$ eigenvalue as

$$\ell = L_0 + 1.$$  \hspace{1cm} (26)

*Exercise:* Given $c(z) = \sum_n \frac{c_n}{z^n}$ show that

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$  \hspace{1cm} (27)

Now that we must compute a quantity precisely we should make clear the CFT definition of the inner product
Definition: \( \langle A, B \rangle = \langle bpz(A)|B \rangle \). Here \( bpz : \mathcal{H} \rightarrow \mathcal{H}^* \) is BPZ conjugation, which we review next.

Given a primary field \( \phi(z) \) of dimension \( d \), it has a mode expansion

\[
\phi(z) = \sum_n \phi_n z^{n+d} \quad \rightarrow \quad \phi_n = \oint \frac{dz}{2\pi i} z^{n+d+1} \phi(z).
\]  

(28)

We define

\[
bpz(\phi_n) \equiv \oint \frac{dt}{2\pi i} t^{n+d-1} \phi(t), \quad \text{with} \quad t = \frac{1}{z}.
\]  

(29)

Note that this simply defines the BPZ conjugation of the oscillator with the same formula as the oscillator itself (28) but referred to a coordinate at \( z = \infty \). This integral is evaluated by using the transformation law

\[
\phi(t)(dt)^d = \phi(z)(dz)^d.
\]  

(30)

We therefore get

\[
bpz(\phi_n) \equiv - \oint \frac{dz}{2\pi i} \frac{1}{z^n} \left( \frac{1}{z} \right)^{n+d-1} \phi(z)(z^2)^d.
\]  

(31)

The minus sign in front arises from a reversal of contour of integration (a contour circling \( t = 0 \) clockwise circles \( z = 0 \) counterclockwise). Moreover the transformation law was used to reexpress \( \phi(t) \) in terms of the field \( \phi(z) \) whose mode expansion is given. Simplifying the integral one finds

\[
bpz(\phi_n) = (-1)^{n+d} \phi_{-n}
\]  

(32)

This equation defines BPZ conjugation when we supplement it with the rule

\[
bpz(\phi_n_1, \cdots \phi_n_p | 0) = \langle 0 | bpz(\phi_n_1) \cdots bpz(\phi_n_p) \rangle.
\]  

(33)

This formula is correct as stated also when the oscillators are anticommuting. The only condition for its validity is that the various modes with mode numbers of the same sign must commute (or anticommute). Otherwise BPZ conjugation produces a sequence of oscillators in reverse order.

A nontrivial example of the above rules arises when trying to do BPZ conjugation of the modes \( L_n \) of the stress tensor. Even though the stress tensor \( T(z) \) is not a primary field, it transforms as a primary under \( SL(2,C) \) transformations and therefore it does transform as a dimension two primary under the inversion needed in the definition of BPZ. Thus we have

\[
bpz(L_n) = (-1)^n L_{-n}.
\]  

(34)
applied to a string of oscillators we must write
\[
bpz(L_{m_1} \cdots L_{m_p}|0) = \langle 0|bpz(L_{m_p}) \cdots bpz(L_{m_1}) \rangle. \tag{35}
\]

Since the dimension of the ghost field \( c(z) \) is minus one, we have \( bpz(c_1) = (-1)^{1+1}c_{-1} = c_{-1} \) and therefore \( bpz(c_1|0) = \langle 0|c_{-1}. \) With this we have
\[
\langle T, QT \rangle = t^2 \langle 0|c_{-1}c_0c_1|0 \rangle. \tag{36}
\]

Because of the form of the inner product only the term \( c_0L_0 \) in \( Q \) can contribute and we have
\[
\langle T, QT \rangle = t^2 \langle 0|c_{-1}c_0L_0c_1|0 \rangle = -t^2 \langle 0|c_{-1}c_0c_1|0 \rangle = -t^2. \tag{37}
\]

This completes the computation of the quadratic term in the tachyon potential. The negative sign obtained is the expected one, showing the instability of the \( t = 0 \) field configuration.

2 CFT definition of the open string star

The most direct way to define star in CFT language is through the relation given in (11)
\[
\langle A, B, C \rangle \equiv \langle A, B * C \rangle. \tag{38}
\]

We will define \( \langle A, B, C \rangle \), and since the inner product has already been defined and is non-degenerate, the above relation defines for us the star product.

Consider three states \( A, B, \) and \( C \) and their associated vertex operators \( O_A, O_B, \) and \( O_C. \) We define
\[
\langle A, B, C \rangle \equiv \left( f_1^D \circ O_A(0), f_2^D \circ O_B(0), f_3^D \circ O_C(0) \right)_D. \tag{39}
\]

Here the right-hand side denotes the CFT correlator of the conformal transforms of the vertex operators \( O_A, O_B, \) and \( O_C. \) The conformal transforms are specified by the functions \( f_i \) as we explain now. Let there be three canonical coordinates \( \xi_i, \) with \( i = 1, 2, 3. \) The three functions \( f_i(\xi_i) \) define maps from the upper half disks \( \mathbb{H}(\xi_i) \geq 0, |\xi_i| \leq 1 \) into a disk \( D, \) with the points \( \xi_i = 0 \) being taken into points in the boundary of the disk. The meaning of the conformal map of operators is that: \( f_i \circ O_A(0) \) is the operator \( O_A(\xi_i = 0) \) expressed in terms of local operators at \( f_i(\xi_i = 0). \) The disk \( D \) may have
the form of a unit disk, or can be the (conformally equivalent) upper half plane, or any other arbitrary form. Of course, the unit disk and the upper half plane are specially convenient for explicit computations.

For the SFT in hand, the worldsheets of the three strings are represented as the unit half-disks \(||\xi_i| \leq 1, \Im \xi_i \geq 0\), \(i = 1, 2, 3\), in three copies of the complex plane. The boundaries \(|\xi_i| = 1\) in the respective upper half-disks are the strings. Thus the point \(\xi_i = i\) is the string midpoint. The interaction defining the vertex is build by gluing the three half-disks to form a single disk. This is done the half-string identifications:

\[
\begin{align*}
\xi_1 \xi_2 &= -1, & \text{for } |\xi_1| = 1, \Re (\xi_1) \leq 0 \\
\xi_2 \xi_3 &= -1, & \text{for } |\xi_2| = 1, \Re (\xi_2) \leq 0 \\
\xi_3 \xi_1 &= -1, & \text{for } |\xi_3| = 1, \Re (\xi_3) \leq 0.
\end{align*}
\]

Note that the common interaction point \(P\), is indeed \(\xi_i = i\) (for \(i = 1, 2, 3\)), namely the mid-point of each open string \(|\xi_i| = 1, \Im \xi_i \geq 0\). The left half of the first string is glued with the right half of the second string, and the same is repeated cyclically. This construction defines a specific ‘three-punctured disk’, a genus zero Riemann surface with a boundary, three marked points (punctures) on this boundary, and a choice of local coordinates \(\xi_i\) around each puncture.

The calculation of the functions \(f_i^D(\xi)\) require a choice of disk \(D\). We begin with the case when the disk \(D\) is simply chosen to be the interior of the unit disk \(|w| < 1\). In this case the functions \(f_i^{Dw} \equiv f_i\) must map each half-disk to a 120° wedge of this unit disk. To construct the explicit maps that send \(\xi_i\) to the \(w\) plane, one notices that the \(\text{SL}(2,\mathbb{C})\) transformation

\[
h(z) = \frac{1 + i\xi}{1 - i\xi},
\]

maps the unit upper-half disk \(|\xi| \leq 1, \Im \xi \geq 0\) to the ‘right’ half-disk \(|h| \leq 1, \Re h \geq 0\), with \(z = 0\) going to \(h(0) = 1\). Thus the functions

\[
\begin{align*}
f_1(\xi_1) &= e^{2\pi i/3} \left(\frac{1 + i\xi_1}{1 - i\xi_1}\right)^2, \\
f_2(\xi_2) &= \left(\frac{1 + i\xi_2}{1 - i\xi_2}\right)^2, \\
f(\xi_3) &= e^{-2\pi i/3} \left(\frac{1 + i\xi_3}{1 - i\xi_3}\right)^2,
\end{align*}
\]

(42)
will send the three half-disks to three wedges in the $w$ plane, with punctures at $e^{\frac{2\pi}{3} i}$, 1, and $e^{-\frac{2\pi}{3} i}$ respectively. This specification of the functions $f_i(\xi)$ gives the definition of the cubic vertex. In this representation cyclicity (i.e., $\langle \Phi_1, \Phi_2, \Phi_3 \rangle = \langle \Phi_2, \Phi_3, \Phi_1 \rangle$) is manifest by construction. By SL(2,C) invariance, there are many other possible representations that give exactly the same off-shell amplitudes.

A useful choice is to map the interacting $w$ disk symmetrically to the upper half $z$-plane $H$. This is the convention that we shall mostly be using. We can therefore define the functions $f_i^H$ by composing the earlier maps $f_i$ (that send the half-disks to the $w$ unit disk) with the map $h^{-1}(w) = -i \frac{w-1}{w+1}$ taking this unit disk to the upper--half--plane, with the three punctures on the real axis,

\[
f_1^H(\xi_1) \equiv h^{-1} \circ f_1(\xi_1) = S(f_1^H(\xi_1)) = \sqrt{3} + \frac{8}{3} \xi_1 + \frac{16}{9} \sqrt{3} \xi_1^2 + \frac{248}{81} \xi_1^3 + O(\xi_1^4).
\]

\[
f_2^H(\xi_2) \equiv h^{-1} \circ f_2(\xi_2) = S(f_2^H(\xi_2)) = -\tan \left( \frac{2}{3} \arctan(\xi_2) \right) = \frac{2}{3} \xi_2 - \frac{10}{81} \xi_2^3 + O(\xi_2^5).
\]

\[
f_3^H(\xi_3) \equiv h^{-1} \circ f_3(\xi_3) = S(f_3^H(\xi_3)) = -\sqrt{3} + \frac{8}{3} \xi_3 - \frac{16}{9} \sqrt{3} \xi_3^2 + \frac{248}{81} \xi_3^3 + O(\xi_3^4).
\] (43)

The three punctures are at $f_1^H(0) = +\sqrt{3}, f_2^H(0) = 0, f_3^H(0) = -\sqrt{3}$, and the SL(2,R) map $S(z) = \frac{z-\sqrt{3}}{1+\sqrt{3}z}$ cycles them (thus $S \circ S \circ S(z) = z$).

This completes the definition of the string field theory action. When the disk $D$ is presented as a unit disk the functions $f_i$ in (39) are the functions given in equation (42). When the disk $D$ is presented as the upper half plane $H$ the relevant functions in (39) are the functions $f_i^H$ given in (43) above.

**Exercise:** Verify explicitly by a by-hand calculation that the first two terms in the expansion of $f_1^H$ and $f_3^H$, as well as the first term in $f_2^H$ are correct.

Let us now return to the computation of the tachyon action. For our string field $|T\rangle = t e_1 |0\rangle$ the interaction term $\langle T, T, T \rangle$ will be given by

\[
\langle T, T, T \rangle = t^3 \langle e_1, e_1, e_1 \rangle.
\] (44)
Since the vertex operator associated to $c_1|0\rangle$ is $c(z)$, using (39) we write:

$$\langle T, T, T \rangle = t^3 \langle f^H_1 \circ c(0), f^H_2 \circ c(0), f^H_3 \circ c(0) \rangle_H. \quad (45)$$

Since the field $c(z)$ is a primary of dimension minus one, we have

$$\frac{c(z)}{dz} = \frac{c(\xi)}{d\xi} \rightarrow c(\xi) = \frac{c(z)}{d\xi}. \quad (46)$$

Therefore

$$f \circ c(0) \equiv c(\xi = 0) = \frac{c(f(0))}{f'(0)}. \quad (47)$$

Using equations (43) to read the values of $f^H_1(0)$ and $\frac{df^H_1}{d\xi}(0)$ we therefore get, for example,

$$f^H_1 \circ c(0) = \frac{c(f^H_1(0))}{f^H_1(0)} = \frac{c(\sqrt{3})}{\frac{2}{3}}. \quad (48)$$

The other two insertions are dealt with similarly, and we find

$$\langle T, T, T \rangle = t^3 \left\langle \frac{c(\sqrt{3})}{\frac{2}{3}}, \frac{c(0)}{\frac{2}{3}}, \frac{c(-\sqrt{3})}{\frac{2}{3}} \right\rangle_H = 3^3 \frac{\sqrt{3}}{2^6} = \frac{81\sqrt{3}}{64}t^3K^3, \quad \langle c_1, c_1, c_1 \rangle = \frac{81\sqrt{3}}{64} = K^3. \quad (50)$$

This completes the calculation of an interaction term.

### 3 Neumann coefficients: the CFT approach to arbitrary vertices

When doing explicit computations in OSFT we need to consider interactions of fields other than the tachyon. The explicit computation of the previous section becomes a lot more involved for massive fields, and it is useful to find an automated procedure to deal with such calculations. One such procedure is the method of conservation laws. Another is the use of explicit Fock representations of the string vertex. This will be our subject of interest.
here. We will provide a self-contained derivation of the Neumann coefficients defining the three string vertex both in the matter and in the ghost sector. In fact our construction will be very general, and applies to three string interactions other than the one used in OSFT.

In the Fock space representation of the vertex, we must find a state \( \langle V_3 \rangle \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \) such that for any Fock space states \( A, B \) and \( C \) one finds that

\[
\langle A, B, C \rangle \equiv \langle V_3 | A \rangle_{(1)} | B \rangle_{(2)} | C \rangle_{(3)} .
\]

(51)

Since we provided in (39) a definition of the left hand side of the above equation, the vertex \( \langle V_3 \rangle \) is implicitly defined. Our procedure will be general in that the functions \( f_r(\xi) \) mapping the canonical half-disks to the upper half plane will be kept arbitrary. There is a natural ansatz for the vertex:

\[
\langle V_3 \rangle = \mathcal{N}((\langle 0 | c_{-1} c_0 \rangle)^{(3)})(\langle 0 | c_{-1} c_0 \rangle)^{(2)}(\langle 0 | c_{-1} c_0 \rangle)^{(1)}
\exp \left( -\frac{1}{2} \sum_{r,s} \sum_{n,m \geq 1} \alpha_{m}^{(r)} \alpha_{n}^{(s)} N_{mn}^{rs} \right) \exp \left( \sum_{r,s} \sum_{m \geq 0 \atop n \geq 1} b_{m}^{(r)} N_{mn}^{rs} \right) .
\]

(52)

Here \( \mathcal{N} \) is a normalization factor, which will be determined shortly. In fact, its determination is essentially the tachyon computation of the previous section. Moreover, note that the nontrivial oscillator dependence in the matter sector is in the form of an exponential of a quadratic form. This is a general result following from the free field property of the matter CFT. Having just a quadratic form is possible also for the ghost sector, but it requires a careful choice of vacua. This is because there is a sum rule regarding ghost number— if the vacua are not chosen conveniently, extra linear ghost factors are necessary in the vertex. Since the vertex state \( \langle V_3 \rangle \) is a bra we use out-vacua, in particular the vacuum \( \langle 0 | c_{-1} c_0 \rangle \). This is quite convenient because the ghost number conservation law is satisfied when each of the states \( A, B \) and \( C \) in (51) is of ghost number one. Indeed in each of the three state spaces we must have a total ghost number of three— two are supplied by the out-vacuum, and one by the in-state. This clearly allows the nontrivial ghost dependence of the vertex to be just a pure exponential with zero ghost number. A final point concerns the sum restrictions over the ghost oscillators. These simply arise because only oscillators that do not kill the vacuum \( \langle 0 | c_{-1} c_0 \rangle \) should appear in the exponential. Thus for the antiquantum oscillators \( b_m \) we find \( m \geq 0 \) and for the quantum oscillators \( c_n \) we find \( n \geq 1 \).
The normalization factor $N$ can be determined by finding the overlap of the vertex with three zero momentum tachyons $c_1|0\rangle$. In this case we have

$$\langle c_1, c_1, c_1 \rangle = \langle V_3|c_1\rangle_{(1)}\langle (c_1)_{(2)}|c_1\rangle_{(3)} = N,$$  

(53)

since all oscillators in the exponentials kill the zero momentum tachyon. In (50) we found the value of this constant for the case of the OSFT vertex. The calculation in the general case is not any more complicated and it is a good exercise!

**Exercise:** Show that for arbitrary functions $f_i(\xi)$, with $i = 1, 2, 3$, mapping to the UHP, the normalization constant of the three string vertex (52) is given by:

$$N = \frac{\langle f_1(0) - f_2(0)(f_1(0) - f_3(0))(f_2(0) - f_3(0)) \rangle}{\langle f_1(0)f_2(0)f_3(0) \rangle}.$$  

(54)

Our goal now is to find explicit expressions for the Neumann coefficients $N_{m_1}^{r_1 s_1}$ and $X_{m_1}^{r_1 s_1}$ in terms of the functions $f_i$ defining the vertex. We will begin our derivation by considering the case of the matter sector.

In the matter sector, the following conventions are necessary

$$i\partial X(z) = \sum \frac{\alpha_n}{n+1}, \quad \alpha_n = \oint \frac{dz}{2\pi i} z^n i\partial X,$$  

(55)

$$\langle i\partial X(z) i\partial X(w) \rangle = \frac{1}{(z-w)^3}, \quad [\alpha_n, \alpha_m] = n\delta_{m+n,0}.$$  

(56)

Our strategy for the derivation of the matter Neumann coefficients is to compute a particular expression in two different ways. The expression under consideration is

$$M = \langle V_3| i\partial X^{(r)}(z) i\partial X^{(s)}(w) c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)} \rangle.$$  

(57)

For our first computation we use the mode expansion (55) of the conformal fields and then simply evaluate using the oscillator form of the vertex

$$M = \sum_{m,n} \frac{1}{z^{m+1}w^{n+1}} \langle V_3| \alpha_m^{(r)} \alpha_n^{(s)} c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)} \rangle$$

$$= -N \sum_{m,n} z^{m-1}w^{n-1} mn N_{m_1}^{r_1 s_1}.$$  

(58)

\[1\] The derivation below needs small modifications when $r = s$, but the final result still holds in this case.
In the second evaluation we first rewrite $M$ as

$$M = \langle V_3 | i \partial X^{(r)}(z) i \partial X^{(s)}(w) c^{(1)}(0) c^{(2)}(0) c^{(3)}(0) |0\rangle_{(1)} |0\rangle_{(2)} |0\rangle_{(3)} \rangle, \quad (59)$$

and reinterpret as a correlator, in the spirit of (39). The three vertex operators being inserted are clearly $i \partial X^{(r)}(z) c(0)$, $i \partial X^{(s)}(w) c(0)$ and $c(0)$. We thus have

$$M = \left< f_r \circ \left( i \partial X(z) c(0) \right) \ f_s \circ \left( i \partial X(w) c(0) \right) \ f_t \circ c(0) \right>, \quad (60)$$

where $t \neq r, s$. This correlator is not hard to evaluate. The ghost part of it gives again the factor $N$. The matter part, using $i \partial X(z) = i \partial X(f(z)) \frac{df}{dz}$, and (56) finally gives

$$M = \mathcal{N} f_r^t(z) f_s^t(w) \left< i \partial X(f_r(z)) i \partial X(f_s(w)) \right>$$

$$= \mathcal{N} \frac{f_r^t(z) f_s^t(w)}{(f_r(z) - f_s(w))^2}. \quad (61)$$

Equating the results (58) and (61) of the two evaluations of $M$ we obtain:

$$\sum_{m,n} z^{m-1} w^{n-1} m n N_{mn}^{rs} = - \frac{f_r^t(z) f_s^t(w)}{(f_r(z) - f_s(w))^2}. \quad (62)$$

It is now simple to pick up the coefficients $N_{mn}^{rs}$ by contour integration over small circles surrounding $z = 0$ and $w = 0$. One finally finds

$$N_{mn}^{rs} = - \frac{1}{m n} \oint_{0} dz \frac{1}{2 \pi i} \oint_{0} dw \frac{1}{2 \pi i} \frac{f_r^t(z) f_s^t(w)}{(f_r(z) - f_s(w))^2}. \quad (63)$$

This is the desired expression for the Neumann coefficients of the matter sector. They can be used for any arbitrary vertex. The above contour integrals are straightforward to compute and they can be easily done by a computer in a series expansion. In terms of residues the expression above is equivalent to

$$N_{mn}^{rs} = - \frac{1}{m n} \text{Res}_{z=0} \text{Res}_{w=0} \left[ \frac{1}{z^{m}} \frac{1}{w^{n}} \frac{f_r^t(z) f_s^t(w)}{(f_r(z) - f_s(w))^2} \right]. \quad (64)$$

Exercise: Show that the contour integrals in (63) can be evaluated in any order. Do this both for the case when $r \neq s$ and for the case when $r = s$. 

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We now turn to the calculation of the ghost Neumann coefficients \( X_{mn}^{rs} \). For this we need mode expansions and two point functions for the ghost CFT. These are

\[
c(z) = \sum_n \frac{c_n}{z^{n+1}}, \quad b(z) = \sum_n \frac{b_n}{z^{n+2}}, \quad \langle c(z) b(w) \rangle = \frac{1}{z-w}. \tag{65}
\]

The strategy is once more based on the computation of a certain expression in two different ways. Indeed, we consider the overlap

\[
G = \langle V_3 \mid b^{(s)}(z) c^{(r)}(w) \mid c_1^{(1)} \lvert 0 \rangle (1) c_1^{(2)} \lvert 0 \rangle (2) c_1^{(3)} \lvert 0 \rangle (3) \rangle,
\]

and first evaluate it by using the mode expansion of the antighost and ghost fields, and then the explicit expression for the vertex in (52). In this way we find

\[
G = \sum_{m,n} z^{-n+2} w^{-m-1} \langle V_3 \mid b^{(s)}(z) c^{(r)}_{-m} c_1^{(1)} \lvert 0 \rangle (1) c_1^{(2)} \lvert 0 \rangle (2) c_1^{(3)} \lvert 0 \rangle (3) \rangle
= \mathcal{N} \sum_{m,n} z^{-n+2} w^{-m-1} X_{mn}^{rs}.
\tag{67}
\]

In the second computation \( G \) is interpreted as a correlator and we have

\[
G = \langle f_s \circ b(z) f_r \circ c(w) \mid f_1 \circ c(0) f_2 \circ c(0) f_3 \circ c(0) \rangle
= \frac{(f_s^{(1)}(z))^2}{f_1^{(1)}(w) f_2^{(1)}(0) f_3^{(1)}(0)} \langle b(f_s(z)) c(f_r(w)) c(f_1(0)) c(f_2(0)) c(f_3(0)) \rangle,
\tag{68}
\]

where we used the standard conformal maps of the relevant operators all of which are primary. The final correlator is just a correlator in the upper half plane and all arguments of the fields refer to the coordinates in the upper half plane. The correlator can be calculated by using OPE’s, but it is simpler to see its singularity structure and derive its normalization from a special configuration. Note for example that there must be zeroes when any pair of \( c \) fields approach each other. In particular this will include a factor \((f_1(0) - f_2(0))(f_1(0) - f_3(0))(f_2(0) - f_3(0))\) as in \( \mathcal{N} \) (see (54). We will also have poles when the antighost approaches any ghost. These considerations imply that

\[
G = \mathcal{N} \frac{(f_s^{(1)}(z))^2}{f_1^{(1)}(w)} \frac{1}{f_s(z) - f_r(w)} \prod_{i=1}^3 (f_r(w) - f_i(0)) \prod_{j=1}^3 (f_s(z) - f_j(0)). \tag{69}
\]
Having computed $G$ in two different ways, equating the results obtained in (67) and (69), and picking up the coefficients via contour integration, we find

$$X^{rs}_{mn} = \int \frac{dz}{2\pi i} \frac{1}{z^{n-1}} \int \frac{dw}{2\pi i} \frac{1}{w^{m+2}} \frac{(f_r'(z))^2}{f_r'(w)} \frac{1}{f_r(z) - f_r(0)} \prod_{j=1}^{3} (f_j(w) - f_j(0)),$$

This is the general result for the ghost Neumann coefficients. Again, for any vertex they are easily calculated by power series expansions and picking up residues. For particular vertices one can simplify somewhat the above expressions and find interesting relations. In fact, a fair amount of work can be done for the OSFT vertex in simplifying the above results. One can show that the matrices $N$ and $X$ are related, and while no closed form expressions are known for the coefficients, they can be generated quite efficiently from the simpler expressions.

Having explicit expressions where specific Neumann coefficients can be calculated exactly with finite number of operations, the problem of computing $\langle A, B, C \rangle$ for three Fock space states is solved also with finite number of operations.

4 Conservation Laws for Virasoro operators

We explain here a procedure that is the systematic off-shell implementation of the conventional Virasoro Ward Identities that allow the computation of correlators of Virasoro descendents in terms of those of Virasoro primaries. We view the vertex $\langle V_3 \rangle$ as an object where a negatively moded Virasoro operator in one state space can be converted into linear combinations of positively moded Virasoro operators in all state spaces, with readily calculable coefficients that capture the geometry of the interaction. Such relations allow recursive computation of all correlations involving the Virasoro operators. They are obtained, for the Virasoro case, by studying contour integrals of the type $\int T(z)\nu(z)dz$ where $\nu(z)$ is a globally defined vector field on the punctured surface defining the interaction vertex. The identities arise by contour deformation and by referring the objects inside the integrals to the coordinates chosen at the punctures. Particular cases of the conservation laws of the operator formalism have been used since very early times in string theory. For string interactions based on contact type interactions (as in light cone theories and classical closed string field theory) such relations have gone under the name of ‘overlap conditions’.
The identities are of the general form:

$$\langle V_3 \rangle L_{-k}^{(2)} = \langle V_3 \rangle \left( A^k \cdot c + \sum_{n \geq 0} a_n^k L_n^{(1)} + \sum_{n \geq 0} b_n^k L_n^{(2)} + \sum_{n \geq 0} c_n^k L_n^{(3)} \right), \quad (71)$$

where $A^k$, $a_n^k$, $b_n^k$ and $c_n^k$ are coefficients that will be determined below and depend on the geometry of the vertex. (By cyclicity, the same identity holds after letting (1) $\rightarrow$ (2), (2) $\rightarrow$ (3), (3) $\rightarrow$ (1).) The point of this identity is that the negatively moded Virasoro generator $L_{-k}^{(2)}$ acting on state space 2 is traded for a sum of positively moded generators acting on all the three state spaces, plus a central term. Since all the states in the background–independent subspace are of the form

$$\mathcal{H}_{\text{univ}}^{(1)} = \text{Span}\{L_{-j_1}^{m} \ldots L_{-j_p}^{m} L_{-i_1}^{g_h} \ldots L_{-i_p}^{g_h} c_1 | 0 \rightarrow j_i \geq 2, l_i \geq 1\}. \quad (72)$$

we see that by the conservation laws for matter and ghost Virasoro generators, and the commutation relations of the Virasoro algebra, we obtain a recursive procedure that allows one to express the coupling of any three states in the universal subspace in terms of the coupling $\langle c_1, c_1, c_1 \rangle$ of three tachyons.

### 4.1 Setting up Virasoro conservations

It is convenient to use the standard ‘doubling trick’ for open strings. We trade the holomorphic and antiholomorphic components of the stress tensor, defined in the upper–half $z$ plane, for a single holomorphic field $T(z)$ defined in the whole complex plane. With this convention, the cubic vertex is regarded as a 3–punctured sphere. We examine a stress–tensor with general central term,

$$T(z')T(z) \sim \frac{c/2}{(z' - z)^4} + \frac{2T(z)}{(z' - z)^2} + \frac{\partial T(z)}{z' - z} + \cdots \quad (73)$$

Under holomorphic change of variables,

$$\bar{T}(w) = \left( \frac{dz}{dw} \right)^2 T(z) + \frac{c}{12} S(z, w), \quad S(z, w) = \frac{d_z d_w}{d_w^2} - \frac{3}{2} \left( \frac{d_z}{d_w} \right)^2. \quad (74)$$

The Schwartzian derivative $S(z, w)$ vanishes if $z$ and $w$ are related by an SL(2,C) transformation. Under composition of conformal maps, $z \rightarrow \rho(z)$, $\rho \rightarrow w(\rho)$, one finds $S(w, z) = \left( \frac{dw}{dz} \right)^2 S(w, \rho) + S(\rho, z)$. 

Consider our representation of the cubic vertex in the full complex plane with punctures at $+\sqrt{3}$, 0 and $-\sqrt{3}$. We shall label the coordinate in the global plane as $z$, and the local coordinates around the punctures as $\xi_i$, 1 = 1,2,3. Let $v(z)$ be a holomorphic vector field $v(z)$, thus transforming as $\hat{v}(w) = \left( \frac{dz}{dw} \right)^{-1} v(z)$. We require $v(z)$ to be holomorphic everywhere in the $z$ plane, except at the punctures where it may have poles. Since in our convention the punctures are all located at finite points on the real axis, we need to impose regularity at infinity. Performing the change of variables $w = -1/z$, $\hat{v}(w) = z^{-2}v(z)$. Hence for $v(z)$ to be regular at infinity, $\lim_{z\to\infty} z^{-2}v(z)$ must be constant (or zero).

The purpose of considering the vector $v(z)$ is that the product $v(z)T(z)dz$ transforms as a 1–form (except for a correction due to the central term),

$$T(z)v(z)\,dz = \left( T(w) - \frac{c}{12} \, S(z, w) \right) \hat{v}(w)\,dw,$$  \hspace{1cm} (75)

and can be naturally integrated along 1–cycles. Moreover this 1–form is conserved, thanks to the holomorphy of $T(z)$ and $v(z)$, and integration contours in the $z$ plane can be continuously deformed as long as we do not cross a puncture. Consider a contour $\mathcal{C}$ which encircles the three punctures at $-\sqrt{3}$, 0 and $\sqrt{3}$ in the $z$ plane. For arbitrary vertex operators $\Phi_i$, the correlator

$$\langle \oint_{\mathcal{C}} v(z)T(z)dz \, f_1 \circ \Phi_1(0) \circ f_2 \circ \Phi_2(0) \circ f_3 \circ \Phi_3(0) \rangle$$ \hspace{1cm} (76)

vanishes identically, by shrinking the contour $\mathcal{C}$ to zero size around the point at infinity (which is a regular point). In this argument it is important that under the inversion $w = -1/z$, the Schwartzian derivative vanishes and thus there is no contribution from the central term in (75). Since the correlator (76) is zero for arbitrary $\Phi_i$, we can write

$$\langle V_3 | \oint_{\mathcal{C}} v(z)T(z)dz = 0 \rangle.$$ \hspace{1cm} (77)

Deforming the contour $\mathcal{C}$ into the sum of three contours $\mathcal{C}_i$ around the three punctures, and referring the 1–form to the local coordinates, we obtain the basic relation

$$\langle V_3 | \sum_{i=1}^3 \oint_{\mathcal{C}_i} d\xi_i \, v^{(i)}(\xi_i) \left( T(\xi_i) - \frac{c}{12} \, S(f_i(\xi), \xi_i) \right) \rangle = 0.$$ \hspace{1cm} (78)
The maps $f_i$, since they differ by SL(2,R) transformations, have the same Schwartzian derivative. An evaluation in power series gives\(^2\)

$$S(f_i, \xi_i) = -\frac{10}{9} + \frac{20}{9} \xi_i^2 - \frac{10}{3} \xi_i^3 \cdots, \quad i = 1, 2, 3. \quad (79)$$

Since this expression is regular at each puncture ($z_i = 0$), the central term can contribute to the conservation law (78) only for vector fields $v^{(i)}$ that have poles at the punctures.

We are looking for conservation laws of the form (71). Recalling that

$$L^{(i)}_{-k} = \frac{1}{2\pi i} \oint d\xi_i \xi_i^{-k+1} T^{(i)}(\xi_i), \quad (80)$$

we need a vector field which behaves as $v^{(2)} \sim \xi_2^{-k+1} + O(z_2)$ around puncture 2, and has a zero in the other two punctures, $v^{(1)} \sim O(\xi_1)$, $v^{(3)} \sim O(\xi_3)$. A vector field of this type has (for $k > 1$) a pole at the second puncture, and is regular around the other two punctures. Contributions from the central term will then only appear in the second state space.

### 4.2 The first few conservation laws

Consider using the globally defined vector field

$$v_1(z) = -\frac{2}{9} (z^2 - 3). \quad (81)$$

As discussed before, this has zero at punctures 1 and 3 and is regular at infinity. Using the transformation law $v^{(i)}_1(\xi_i) = v_1(z(\xi_i))/(dz/d\xi_i)$ we derive the Taylor expansion of the vector field referred to the each of the local coordinates

$$v^{(1)}_1(\xi_1) = -\frac{4}{3\sqrt{3}} \xi_1 + \frac{8}{27} \xi_1^2 - \frac{40}{81\sqrt{3}} \xi_1^3 + \frac{40}{729} \xi_1^4 + \frac{104}{729\sqrt{3}} \xi_1^5 + O(\xi_1^6)$$

$$v^{(2)}_1(\xi_2) = 1 + \frac{11}{27} \xi_2^2 - \frac{80}{729} \xi_2^4 + \frac{1136}{19683} \xi_2^6 + O(\xi_2^8)$$

$$v^{(3)}_1(\xi_3) = \frac{4}{3\sqrt{3}} \xi_3 + \frac{8}{27} \xi_3^2 + \frac{40}{81\sqrt{3}} \xi_3^3 + \frac{40}{729} \xi_3^4 - \frac{104}{729\sqrt{3}} \xi_3^5 + O(\xi_3^6) \quad (82)$$

\(^2\)For this and most explicit computations it is useful to use a symbolic manipulator such as Maple or Mathematica.
In this case the $v^{(i)}$ are regular around each puncture, so we get no contribution from the central term. Using (78) and noting that integration amounts to the replacement $v^{(i)}_n z_i^n \rightarrow v^{(i)}_n L^{(i)}_{n-1}$, we can immediately write the conservation law

$$
0 = \langle V_3 \rangle \left( -\frac{4}{3 \sqrt{3}} L_0 + \frac{8}{27} L_1 - \frac{40}{81 \sqrt{3}} L_2 + \frac{40}{729} L_3 + \frac{104}{729 \sqrt{3}} L_4 \cdots \right)_{(1)}
$$

$$
+ \langle V_3 \rangle \left( L_{-1} - \frac{11}{27} L_1 - \frac{80}{729} L_3 + \frac{1136}{19683} L_5 + \cdots \right)_{(2)}
$$

$$
+ \langle V_3 \rangle \left( \frac{4}{3 \sqrt{3}} L_0 + \frac{8}{81 \sqrt{3}} L_1 + \frac{40}{81 \sqrt{3}} L_2 + \frac{104}{729 \sqrt{3}} L_4 \cdots \right)_{(3)}.
$$

(83)

Thanks to the cyclicity of the string vertex, analogous identities hold by cycling the punctures, (1) $\rightarrow$ (2), (2) $\rightarrow$ (3), (3) $\rightarrow$ (1). Using the vector field

$$
v_2(z) = -\frac{4}{27} \frac{z^2 - 3}{z},
$$

we obtain

$$
0 = \langle V_3 \rangle \left( -\frac{8}{27} L_0 + \frac{80}{81 \sqrt{3}} L_1 - \frac{112}{243} L_2 + \frac{304}{729 \sqrt{3}} L_3 - \frac{400}{19683} L_4 \cdots \right)_{(1)}
$$

$$
+ \langle V_3 \rangle \left( L_{-1} + \frac{5}{54} c + \frac{16}{27} L_0 - \frac{19}{243} L_2 + \frac{800}{19683} L_4 + \cdots \right)_{(2)}
$$

$$
+ \langle V_3 \rangle \left( -\frac{8}{27} L_0 - \frac{80}{81 \sqrt{3}} L_1 - \frac{112}{243} L_2 - \frac{304}{729 \sqrt{3}} L_3 - \frac{400}{19683} L_4 \cdots \right)_{(3)}.
$$

(85)

Since $v_2(z)$ has a pole at puncture 2 we got a contribution from the central term. In general we can get conservation laws for $L^{(2)}_{-k}$ with $v_k(z) \sim (z^2 - 3) z^{-k+1}$. For $k > 2$, using this vector in (78) one obtains an identity that besides $L^{(2)}_{-k}$ involves other negatively moded Virasoro generators $L^{(2)}_{-k+2}$, $L^{(2)}_{-k+4}$, ... . It is straightforward to remove these terms by subtracting the conservation laws for smaller $k$.

We conclude this section with a discussion of the so-called ‘reparametrization invariances’ of the cubic vertex. It is well known that for $c = 0$ (total Virasoro generators) the combination

$$
K_n = L_n - (-1)^n L_{-n},
$$

(86)
is conserved on the vertex, that is
\[ \langle V_3 \rangle \left( K_n^{(1)} + K_n^{(2)} + K_n^{(3)} \right) = 0 . \tag{87} \]
These relations are special cases of the Virasoro conservation laws and can be obtained by adding the 3 cyclic versions of the \( L_{-n} \) conservation. A direct and more elegant derivation is as follows. Consider the vector field defined by \( v_n^{(i)}(\xi_i) = \xi_i^{n+1} - (-1)^n \xi_i^{-n+1} \) around each of the punctures. This vector field is globally defined since the expressions on each puncture are consistent with the gluing relations (40) of the string vertex. We have then
\[ \langle V_3 \rangle \sum_{i=1}^{3} \oint T^{(i)}(\xi_i) \left( \xi_i^{n+1} - (-1)^n \xi_i^{-n+1} \right) = 0 , \tag{88} \]
which immediately gives (87). Since the vector field \( v_n^{(i)} \) was directly given in terms of the local coordinates \( z_i \) and shown to be globally defined the reader may wonder how the central term violations of these identities would arise. Indeed, while the contour integrals can be canceled pairwise at the boundary of the local disks without extra contributions (the transition functions are projective), there is a subtlety at the interaction points (the points \( z_i = \pm i \) on the local coordinates and \( z = 0, \infty \) on the global disk). To deal with this properly one can cancel the contour integrals pairwise, but not all the way to the interaction points. This leaves three tiny contour integrals \( \sum_i \oint T(\xi_i)v(\xi_i)d\xi_i \) that add up to a contour surrounding each interaction point. To evaluate this one has to pass again to the coordinate \( z \) vanishing at the interaction point. A simple computation shows that the Schwarzian \( S(\xi, z) \) has a second order pole at \( z = 0 \). In addition, the vector \( v \) has a first order zero at \( z = 0 \). Thus a central charge contribution (in fact, of the right value) arises. One can show (see papers by Schnabl) that for even modes (87) becomes
\[ \langle V_3 \rangle \left( K_{2n}^{(1)} + K_{2n}^{(2)} + K_{2n}^{(3)} \right) = -3k_{2n} \langle V_3 \rangle . \tag{89} \]
where
\[ k_{2n} = \frac{5c}{54} \cdot n(-1)^n \tag{90} \]
The identities with \( n \) odd carry no anomaly.

5 Surface States, Identity and Sliver

In this section we use the technology developed before to get insight into the nature of the identity string field. We also learn how to compute ex-
plicitly in the level expansion a few star products, including that of two zero-momentum tachyons. We isolate a family of string fields associated to once-punctured disks. Such surface states, called wedge states because the local coordinate half-disk defines a wedge of the unit disk, form a subalgebra of the star algebra. Finally, we show using conservation laws that $\mathcal{H}_{\text{univ}}$ defines a subalgebra of the star algebra.

5.1 Presentations for surface states

For the purposes of the arguments to follow we will review the various coordinate systems used to describe surface states. A surface state $\langle \Sigma \rangle$ for the present purposes arises from a Riemann surface $\Sigma$ with the topology of a disk, with a marked point $P$, the puncture, lying on the boundary of the disk, and a local coordinate around it.

The $\xi$ coordinate. This is the local coordinate. The local coordinate, technically speaking is a map from the canonical half-disk $|\xi| \leq 1$, $\Im(\xi) \geq 0$ into the Riemann surface $\Sigma$, where the boundary $\Im(\xi) = 0, |\xi| < 1$ is mapped to the boundary of $\Sigma$ and $\xi = 0$ is mapped to the puncture $P$. The open string is the $|\xi| = 1$ arc in the half-disk. The point $\xi = i$ is the string midpoint. The surface $\Sigma$ minus the image of the canonical $\xi$ half-disk will be called $\mathcal{R}$. Using any global coordinate $u$ on the disk representing $\Sigma$, and writing

$$u = s(\xi), \quad \text{with} \quad s(0) = u(P), \quad (91)$$

the surface state $\langle \Sigma \rangle$ is then defined through the relation:

$$\langle \Sigma | \phi \rangle = \langle s \circ \phi(0) \rangle_{\Sigma}, \quad (92)$$

for any state $|\phi\rangle$. Here $\phi(x)$ is the vertex operator corresponding to the state $|\phi\rangle$ and $\langle . \rangle_{\Sigma}$ denotes the correlation function on the disk $\Sigma$. There is nothing special about a specific choice of global coordinate $u$, and the state $\langle \Sigma \rangle$ built with the above prescription does not change under a conformal map taking $u$ to some other coordinate and $\Sigma$ into a different looking (but conformally equivalent) disk. Nevertheless there are particularly convenient choices which we now discuss in detail.

The $z$-presentation. In this presentation the Riemann surface $\Sigma$ is mapped to the full upper half $z$-plane, with the puncture lying at $z = 0$. The image of the canonical $\xi$ half-disk is some region around $z = 0$. Thus if $z = f(\xi)$, we have

$$\langle \Sigma | \phi \rangle = \langle f \circ \phi(0) \rangle_{UHP}, \quad (93)$$
The \( \hat{z} \)-presentation. In this presentation the Riemann surface \( \Sigma \) is mapped such that the image of the canonical \( \xi \) half-disk is the full strip \( |\Re(\hat{z})| \leq \pi/4, \Im(\hat{z}) \geq 0 \), with \( \xi = i \) mapping to \( \hat{z} = i\infty \), and the open string mapping to the vertical half lines at \( \Re(\hat{z}) = \pm \frac{\pi}{4} \). This is implemented by the map
\[
\hat{z} = \tan^{-1} \xi .
\] (94)
The rest \( R \) of \( \Sigma \) will take some definite shape that will typically fail to coincide with the full upper-half \( \hat{z} \) plane. This shape actually carries the information about the surface \( \Sigma \).

The \( \hat{w} \)-presentation. In this presentation the Riemann surface \( \Sigma \) is mapped such that the image of the canonical \( \xi \) half-disk is the canonical half disk \( |\hat{w}| \leq 1, \Re(\hat{w}) \geq 0 \), with \( \xi = 0 \) mapping to \( \hat{w} = 1 \). This is implemented by the map
\[
\hat{w} = \frac{1 + i\xi}{1 - i\xi} \equiv h(\xi) .
\] (95)
The rest \( R \) of the surface will take some definite shape \( \hat{R} \) in this presentation. This shape actually carries the information about the surface \( \Sigma \). We also note that the \( \hat{w} \)-presentation and the \( \hat{z} \) presentation are related as
\[
\hat{w} = \exp(2i\hat{z}) .
\] (96)

The \( \xi \)-presentation. In this presentation the Riemann surface \( \Sigma \) is mapped into the \( \xi \)-plane by extending to the whole surface \( \Sigma \) the map that takes the neighborhood of the puncture \( P \in \Sigma \) into the \( \xi \)-half-disk. This extended map, of course, may require branch cuts. In this presentation the surface is the canonical \( \xi \) half-disk plus some region in the \( \xi \)-plane whose shape carries the information of the state. We call \( \Sigma_\xi \) the surface in this presentation. In this case the equation defining the state takes a particularly simple form since no conformal map is necessary
\[
\langle \Sigma | \phi \rangle = \langle \phi(0) \rangle_{\Sigma_\xi} .
\] (97)
The \( \xi \) presentation can be obtained from the \( \hat{w} \) presentation by the action of \( h^{-1} \).

5.2 The identity string field

A formal object which has often been considered in discussions of open string field theory is the identity element of the \( * \) algebra, a string field \( |\Sigma| \) which
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is formally expected to obey

\[ |I\rangle * |\Phi\rangle = |\Phi\rangle * |I\rangle = |\Phi\rangle \]  

for any string field \(|\Phi\rangle\). We now give a background independent characterization to \(|I\rangle\).

In CFT language, such functional is represented by a state \(\langle I| \in \mathcal{H}^*\) satisfying \(\langle I|\Phi\rangle \equiv \langle \Phi|\) for all \(|\Phi\rangle \in \mathcal{H}\), where the one point function is computed on a specific 1-punctured disk, one where the local coordinates are such that the left and right halves of the worldsheet boundary are glued together. \(|I\rangle\) is a surface state, since just as \(|V_3\rangle\) it encodes the correlators on a particular Riemann surface.

Representing as usual the worldsheet as the upper half-disk \(\{|\xi| \leq 1, \Re \xi \geq 0\}\), the function

\[ w = F^{360^\circ}(\xi) = \left(\frac{1 + i\xi}{1 - i\xi}\right)^2, \]  

sends this upper–half unit disk to the full unit disk in the \(w\) plane, mapping the two halves of the string \(\{|\xi| = 1, \Re \xi > 0\}\) and \(\{|\xi| = 1, \Re \xi < 0\}\) to the same interval \(\{\Re w = 0, -1 \leq \Re w \leq 0\}\). It is convenient to map back the disk to the upper half-plane. Our final choice of local coordinate is then

\[ z = f^{360^\circ}(\xi) = h^{-1}(F^{360^\circ}(\xi)) = \frac{2\xi}{1 - \xi^2}, \]  

where \(h\) was defined in (41). The puncture is at \(z = 0\), and the image of the unit upper half disk is the full upper half \(z\)-plane.

An explicit representation for the identity is now easy to write. If we can find the operator \(U_{f^{360^\circ}}\) that implements the conformal transformation \(f^{360^\circ}(z)\) in the CFT state space, we have

\[ \langle I| = \langle 0| U_{f^{360^\circ}} . \]  

Such operator must be written as the exponential of a linear combination of the total (matter + ghost) Virasoro generators [34]

\[ U_f = e^{v_0 L_0} e^{\sum_{n \geq 1} v_n L_n} . \]  

This makes manifest the background independence of \(|I\rangle\). Since \(f(0) = 0\) only positively moded Virasoro generators enter \(U_f\), and we have chosen to
separate out the global scaling component $e^{\nu_0 L_0}$. The coefficients $v_n$ can be
determined recursively from the Taylor expansion of $f$, by requiring

$$e^{\nu_0} = f'(0)$$

$$\exp \left( \sum_{n \geq 1} v_n z^{n+1} \partial_z \right) z = [f'(0)]^{-1} f(z) = z + a_2 z^2 + a_3 z^3 + \ldots.$$  

For example, for the first coefficients one finds

$$v_1 = a_2, \quad v_2 = -\frac{a_2^2}{2} + a_3, \quad v_3 = \frac{3}{2} a_2^2 - \frac{5}{2} a_2 a_3 + a_4.$$  

Taking $f = f^{\text{3Dss}}$ we obtain

$$U_{f^{\text{3Dss}}} = 2L_0 \exp \left( L_2 - \frac{1}{2} L_4 + \frac{1}{2} L_6 - \frac{7}{12} L_8 + \frac{2}{3} L_{10} + \cdots \right).$$  

By SL(2,R) invariance of the vacuum, $\langle 0 | L_0 = 0$, so that $\langle I \rangle$ does not in fact depend on the overall scaling factor $v_0$.

To obtain the ket $I$ simply recall that BPZ conjugation sends $L_n$ to $(-1)^n L_{-n}$,  

$$I = \exp \left( L_{-2} - \frac{1}{2} L_{-4} + \frac{1}{2} L_{-6} - \frac{7}{12} L_{-8} + \frac{2}{3} L_{-10} + \cdots \right) | 0 \rangle.$$  

This expression is well-defined in the level truncation scheme. Only a finite number of terms in the sum $\sum_n v_n L_{-n}$ are needed to write the expression of $I$ truncated at some given level. A beautiful expression for the identity string field was found by Ellwood et al. It reads

$$| I \rangle = \left( \prod_{n=2}^{\infty} \exp \left\{ -\frac{2}{2^n} L_{-2n} \right\} \right) e^{L_{-2}} | 0 \rangle$$

$$= \cdots \exp \left( -\frac{2}{2^3} L_{-2^3} \right) \exp \left( -\frac{2}{2^2} L_{-2^2} \right) \exp (L_{-2}) | 0 \rangle$$  

For $c = 0$ Virasoro generators we also have

$$\langle I | K_n = 0.$$  

\[\text{Incidentally, SL}(2,\mathbb{R})\text{ invariance also guarantees that the surface state is independent of the particular SL}(2,\mathbb{R})\text{ frame used to write the map } f. \text{ If } f = R f, \text{ where } R \text{ is an SL}(2,\mathbb{R})\text{ transformation, then } \langle 0 | U_f = \langle 0 | U_R U_f = \langle 0 | U_f. \text{ The composition law } U_{g f} = U_g U_f \text{ holds in a CFT with vanishing central charge, as is the case here.}\]
This is proved as follows. We consider the vector field $v(z) = z^{n+1} - (-)^n z^{-n+1}$ as before and confirm that: (i) it satisfies $v(-1/z)z^2 = v(z)$ as required by the identification $z \rightarrow -1/z$, (ii) the vector field has no poles anywhere else in the $\bar{z}$ plane. Checking (ii) requires verifying this is the case both for $\bar{z} = i$ (in fact, $v(\bar{z} = i)$ is zero)) and for $\bar{z} = \infty$ (in fact for $\bar{z} \rightarrow \infty$, $v(\bar{z}) \sim \bar{z}^2$, which is regular at $\bar{z} \rightarrow \infty$). On the other hand for general CFT we have

$$\langle I|K_{2n} = \frac{c}{2} n(-1)^n \langle I \rangle. \quad (109)$$

### 5.3 A subalgebra of wedge states

The observation that the identity is the surface state associated with the map $F_{n}^{360^\circ}$ naturally leads us to consider a generalization to ‘wedge-like’ states of arbitrary angle. As we shall explain, this family of states, arising from once punctured disks, has the interesting property of being a subalgebra of the $*$-algebra.

We begin by considering the map

$$w = F_{n}^{360^\circ}(z) = \left(\frac{1 + iz}{1 - iz}\right)^2,$$  \quad (110)

which sends the unit upper half disk in the $z$ plane to a wedge of angle $\frac{360^\circ}{n}$ in the $w$ plane. Such maps are used in the higher point interactions of open superstring field theory. As usual, we map back to the upper half plane:

$$\bar{z} = f_{n}^{360^\circ}(z) = h^{-1}(F_{n}^{360^\circ}(z)) = \tan\left(\frac{2}{n} \arctan(z)\right). \quad (111)$$

Let us then define the family of surface states

$$\langle \frac{360^\circ}{n} | \equiv \langle 0| U_{f_{n}^{360^\circ}}. \quad (112)$$

With the same method used to calculate the identity we can write an explicit expression for these wedge states:

$$\left| \frac{360^\circ}{n} \right> = \exp\left(\frac{n^2 - 4}{3n^2} L_{-2} + \frac{n^4 - 16}{30n^4} L_{-1} - \frac{(n^2 - 4)(176 + 128n^2 + 11n^4)}{1890n^6} L_{-6} + \ldots\right)\langle 0 \rangle \quad (113)$$
For \( n = 1 \) we recover the identity: \( |360^\circ\rangle = |I\rangle \) (see (106). For \( n = 2 \) we get the vacuum: \( |180^\circ\rangle = |0\rangle \). For \( n \to \infty \) we find a smooth limit

\[
|\Xi\rangle = \left| \frac{360^\circ}{\infty} \right\rangle = \exp \left( -\frac{1}{3} L_{-2} + \frac{1}{30} L_{-4} - \frac{11}{1890} L_{-6} + \frac{34}{467775} L_{-8} + \cdots \right)|0\rangle.
\]

(114)

The existence of the \( n \to \infty \) limit can be understood from the expression for the conformal map,

\[
f_{\frac{360^\circ}{n}}(z) \xrightarrow{n \to \infty} \frac{2}{n} \arctan(z).
\]

(115)

The map has a well-defined limit up to a vanishing scaling factor, which is immaterial in the definition of the surface state. This state \( |\Xi\rangle \) is the sliver.

The wedge surface states can be defined for continuous \( n \), as can be seen from (113). Let \( \gamma = \frac{360^\circ}{n} \) denote the wedge angle. In the \( \tilde{w} \) presentation the same state corresponds to a sector of angle \( 180^\circ(n - 1) \). Under star multiplication angles of sectors simply add. Thus the product of a wedge with \( n \) and one with \( m \) is one with \( n + m - 1 \).

\[
\left| \frac{360^\circ}{r_1} \right\rangle \ast \left| \frac{360^\circ}{r_2} \right\rangle = \left| \frac{360^\circ}{r_1 + r_2 - 1} \right\rangle.
\]

(116)

It is clear from this equation that

\[
|\Xi\rangle \ast |\Xi\rangle = |\Xi\rangle
\]

(117)

This is the projector property of the sliver state.

The wedge state \( \left| \frac{360^\circ}{3} \right\rangle = |120^\circ\rangle \) has an interesting interpretation. Consider the \( \ast \)-product of two vacuum states. We immediately have

\[
|0\rangle \ast |0\rangle = |180^\circ\rangle \ast |180^\circ\rangle = |120^\circ\rangle,
\]

(118)

where we made use of (116). This answer is easily understood if we recall that the SL(2,R) vacuum deletes punctures. It then follows from the disk presentation of the vertex that the resulting surface is a once punctured disk with a \( 120^\circ \) wedge. Using the \( n = 3 \) case of (113) we get:

\[
|0\rangle \ast |0\rangle = \exp \left( -\frac{5}{27} L_{-2} + \frac{13}{486} L_{-4} - \frac{317}{39366} L_{-6} + \frac{715}{236196} L_{-8} + \cdots \right)|0\rangle.
\]

(119)
Since ghost number is additive under star multiplication the subspace $\mathcal{H}_{\text{univ}}^{(0)}$ of ghost number zero states in $\mathcal{H}_{\text{univ}}$ is itself a subalgebra. There is an even smaller universal subalgebra at ghost number zero. Consider

$$\mathcal{H}_{\text{univ}}^{(0)}(L) \equiv \text{Span}\{L_{-j_1}^{\text{tot}} \cdots L_{-j_p}^{\text{tot}} | 0 \}, \quad j_i \geq 2.$$  \hfill (120)

Here $L_{-j}^{\text{tot}}$ denotes the combined matter and ghost ($c = 0$) Virasoro operators. Indeed since $|0\rangle \ast |0\rangle \in \mathcal{H}_{\text{univ}}^{(0)}(L)$ (see (119)) it follows from the conservation laws that any product of descendents of the vacuum will be a descendent of $|0\rangle \ast |0\rangle \in \mathcal{H}_{\text{univ}}^{(0)}(L)$ and thus a descendent of the vacuum. This confirms $\mathcal{H}_{\text{univ}}^{(0)}(L)$ is a subalgebra. It would be interesting to investigate it concretely.

6 Star Algebra Spectroscopy

Here we investigate the question whether the Neumann coefficient matrices can be diagonalized. Since these are infinite matrices, with complicated expressions the problem is quite nontrivial. One could have expected discrete eigenvalues that accumulate at some points, but the result is quite surprising. The Neumann matrices have continuous spectrum. It seems likely that a proper understanding of why this is the case may illuminate the structure of the Star algebra. The diagonalization of these matrices has already had several applications, some of which will be discussed here.

We will consider here the Neumann matrices relevant to the multiplication of zero momentum states. When multiplying zero momentum states we can use

$$|A \ast^m B\rangle_3 = \frac{1}{2} |A\rangle_2 \langle B| V_3\rangle,$$ \hfill (121)

where the three string vertex $|V_3\rangle$ is given by

$$|V_3\rangle = \exp(-E) |0, p\rangle_{123},$$ \hfill (122)

with

$$E = \frac{1}{2} \sum_{\mu, \nu} \eta_{\mu \nu} a^{(m)}_{\mu} a^{(n)}_{\nu}. \hfill (123)$$

Here $a^{(m)}_{\mu}$, $a^{(n)}_{\nu}$ are non-zero mode matter oscillators acting on the $r$-th string state normalized so that

$$[a^{(m)}_{\mu}, a^{(n)}_{\nu}] = \eta^{\mu \nu} \delta_{mn} \delta^{rs}, \quad m, n \geq 1.$$ \hfill (124)

\footnote{In our notation $i\sqrt{2} \partial X^\mu (z) = \sqrt{2} p^\mu + \sum_{n \neq 0} \sqrt{\eta} a^*_n z^{-n - 1} = \sum_n a_n^* z^{-n - 1}$, and $\partial X^\mu (z) \partial X^\nu (w) = - \eta^{\mu \nu} / 2 (z - w)^2$.}
We define by $V^{rs}$ the matrices $V^{rs}_{mn}$ with $m,n \geq 1$, and by $C_{mn}$ the twist matrix $(-1)^m \delta_{mn}$. We also define:

$$M^{rs} = CV^{rs}.$$  

(125)

Cyclic symmetry relates these matrices so that there are only three independent matrices $M^{11}$, $M^{12}$ and $M^{21}$. These matrices commute with each other and are real symmetric. Furthermore, we have the relations:

$$M^{12} + M^{21} = 1 - M^{11}, \quad M^{12}M^{21} = M^{11}(M^{11} - 1). \quad (126)$$

which allow us to determine the eigenvalues of $M^{12}$ and $M^{21}$ in terms of those of $M^{11}$. Our main goal will be the determination of the eigenvectors and eigenvalues of the matrix $M^{11}$. For convenience of notation, from now on we shall denote the matrices $M^{11}$ and $V^{11}$ by $M$ and $V$ respectively.

The first few elements of the matrix $M$ are

$$M = \begin{pmatrix}
-\frac{5}{27} & 0 & \frac{32}{243\sqrt{3}} & 0 & -\frac{416\sqrt{5}}{19683} & \cdots \\
0 & -\frac{13}{243} & 0 & \frac{512\sqrt{2}}{19683} & 0 & \cdots \\
\frac{32}{243\sqrt{3}} & 0 & -\frac{893}{19683} & 0 & \frac{1504\sqrt{5}}{59049\sqrt{3}} & \cdots \\
0 & \frac{512\sqrt{2}}{19683} & 0 & -\frac{5125}{177147} & 0 & \cdots \\
-\frac{416\sqrt{5}}{19683} & 0 & \frac{1504\sqrt{5}}{59049\sqrt{3}} & 0 & -\frac{41165}{1594323} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix} \quad (127)$$

The CFT method furnishes an integral expression for the elements of $M^{11}$. For this purpose we note the general formula [34]

$$M_{mn} = (-1)^{m+1} \sqrt{mn} \int_0^{2\pi} \frac{dw}{2\pi i} \int_0^{2\pi} \frac{dz}{2\pi i} \frac{1}{z^m w^n} \frac{f'(z) f'(w)}{(f(z) - f(w))^2}, \quad (128)$$

where, for the three string vertex,

$$f(z) = \left( \frac{1 + iz}{1 - iz} \right)^{2/3}. \quad (129)$$

Both $w$ and $z$ integration contours are circles around the origin, with the $w$ contour lying outside the $z$ contour, and both contours lying inside the unit circle.
6.1 \( K_1 \) and its Eigenvectors

Here we introduce a matrix \( K_1 \) representing the action of the star-algebra derivation \( K_1 = L_1 + L_{-1} \). We will be able to find explicit forms for the eigenvectors and eigenvalues of \( K_1 \). In particular we shall see that \( K_1 \) has a non-degenerate continuous spectrum. In the next section we shall show that \( K_1 \) and \( M \) commute, and thus the eigenvectors of \( K_1 \) are eigenvectors of \( M \). Further analysis will reveal the relation between the eigenvalues.

We begin our analysis by recalling that the operator

\[
K_1 = L_1 + L_{-1}
\]

is a derivation of the star algebra [30]. We use its action on positively moded oscillators \( a_n \) and

\[
\alpha_n \equiv \sqrt{n} a_n
\]

with \( n \geq 1 \) to define matrices \( K_1 \) and \( \tilde{K}_1 \)

\[
\begin{align*}
[K_1, v \cdot a] &\equiv (K_1 v) \cdot a - \sqrt{2}v_1 p \\
[K_1, w \cdot \alpha] &\equiv (\tilde{K}_1 w) \cdot \alpha - w_1 \alpha_0.
\end{align*}
\]

Here we have introduced the vector notation

\[
x \cdot a \equiv \sum_{n=1}^{\infty} x_n a_n, \quad y \cdot \alpha \equiv \sum_{n=1}^{\infty} y_n \alpha_n,
\]

and suppressed the Lorentz indices. Identifying \( v \cdot a \) to \( w \cdot \alpha \) and using eq.(131) we get,

\[
v_n = \sqrt{n} w_n.
\]

From (131), we have the relation

\[
(\tilde{K}_1)_{mn} = \sqrt{\frac{n}{m}} (K_1)_{mn}.
\]

Using the standard commutators:

\[
[L_m, \alpha^\mu_n] = -n \alpha^\mu_{m+n},
\]

we see that the matrix \( K_1 \) is symmetric, the diagonal elements are zero, and the only non-vanishing entries are one step away from the diagonal

\[
\begin{align*}
(\tilde{K}_1)_{nm} &= -(n - 1) \delta_{n-1,m} - (n + 1) \delta_{n+1,m}, \\
(K_1)_{nm} &= -\sqrt{n(n-1)} \delta_{n-1,m} - \sqrt{n(n+1)} \delta_{n+1,m}.
\end{align*}
\]
Since \( K_1 \) maps twist even to twist odd states, the associated matrices anti-commute with the matrix \( C \),

\[
\{K_1, C\} = \{\tilde{K}_1, C\} = 0.
\] (138)

Finally, since \( K_1 \) is invariant under hermitian conjugation, we have:

\[
[K_1, v \cdot a^\dagger] = -(K_1 v) \cdot a^\dagger + \sqrt{2} v_1 p.
\] (139)

It is also convenient to represent vectors of type \( w \) (\( \alpha_n \) basis) in terms of formal power series in a variable \( z \),

\[
f_w(z) \equiv \sum_{n=1}^{\infty} w_n z^n.
\] (140)

With this definition we note that

\[
f_{Cw} = f_w(-z).
\] (141)

Using eq.(136) (or directly from eq.(137) we see that the operator \( K_1 = L_1 + L_{-1} \) on the basis of functions of \( z \) is represented by the differential operator:

\[
\mathcal{K}_1 \equiv -(1 + z^2) \frac{d}{dz}.
\] (142)

More specifically, with the proviso that constant terms obtained after the action of the differential operator are to be dropped, we have

\[
f_{\tilde{K}_1 w}(z) = \mathcal{K}_1 f_w(z).
\] (143)

It immediately follows from this equation that

\[
\tilde{K}_1 w^{(\kappa)} = \kappa w^{(\kappa)} \quad \leftrightarrow \quad K_1 f_{w^{(\kappa)}} = \kappa f_{w^{(\kappa)}} + a,
\] (144)

where the constant \( a \) is used to account for the fact that the action of the differential operator must be supplemented by removing the constant term. Therefore the eigenvalue problem for the infinite matrix \( \tilde{K}_1 \) can be studied as the eigenvalue problem for the differential operator \( \mathcal{K}_1 \) on the space of formal power series. The differential equation above is readily integrated to find

\[
f_{w^{(\kappa)}}(z) = \frac{1}{\kappa} \exp(-\kappa \tan^{-1}(z)) + \frac{1}{\kappa} \equiv \sum_{n=1}^{\infty} w_n^{(\kappa)} z^n,
\] (145)
where the overall normalization has been chosen so that \( w_1^{(\kappa)} = 1 \). Expanding
the above in powers of \( z \) and using (140) one can read the coefficients \( w_n^{(\kappa)} \)
which, because of (144) provide an eigenvector of \( K_1 \). This shows that \( K_1 \)
has a \textit{non-degenerate continuous} spectrum. We can take \(-\infty < \kappa < \infty\),
and there is exactly one eigenvector for each value of \( \kappa \). Note also that
each eigenvector of \( K_1 \) provides an eigenvector \( w_n^{(\kappa)} \) of \( K_1 \) with the same
eigenvalue using the relation \( v_n^{(\kappa)} = \sqrt{n} w_n^{(\kappa)} \). This follows from eq. (134).

It follows from (141) and (145) that \( C w^{(\kappa)} = -w^{(-\kappa)} \) and therefore we
can form linear combinations of definite twist
\[
w_{\pm}^{(\kappa)} \equiv \frac{1}{2} \left( w^{(-\kappa)} \mp w^{(\kappa)} \right) ,
\]
that satisfy
\[
C w_{\pm}^{(\kappa)} = \pm w_{\pm}^{(\kappa)} .
\]

6.2 Wedge States, \( K_1 \) and \( M \)

A general wedge state \(|N\rangle \) can be expressed as
\[
|N\rangle = \exp \left( -\frac{1}{2} a^{\dagger} \cdot (CT_N) \cdot a \right) |0\rangle \equiv \exp \left( -E_N \right) |0\rangle ,
\]
where \[12\]
\[
T_N = \frac{T + (-T) N^{-1}}{1 - (-T) N} .
\]
The matrices \( T_N \) are related to the Neumann coefficients of the \( N \)-th com-
plete overlap string vertex, \( V_N = CT_N \). Important special cases are \( T_\infty = T \)
(the sliver)\(^5\) and \( T_3 = M \).

The eigenvectors of \( T_N \) are the same for all the matrices in the family,
and the eigenvalues are related according to the above formula. So we can
simply focus on \( T \) and/or \( M \). We shall first establish that the eigenvectors of
\( K_1 \) are eigenvectors of \( M \) and of \( M^{12} \) and \( M^{21} \). Then we find the eigenvalues
of \( M, M^{12} \), and \( M^{21} \) corresponding to a given eigenvector.

6.3 Eigenvectors of \( T \) and \( M \)

In this subsection we shall show that the eigenvectors of \( K_1 \) are eigenvectors
of \( M \) and of \( M^{12} \) and \( M^{21} \). We will do this in two stages. We first show this

\(^5\)This follows from (149) if the eigenvalues of \( T \) lie in the range \([-1, 0]\), as has been
found numerically. We shall return to this point later.
is true for $M$ and all wedge state matrices $T_N$. Then we turn to the case of the matrices $M^{12}$ and $M^{21}$.

We first note that the derivation $K_1$ annihilates all wedge states $|N \to N\rangle$. Indeed, $K_1$ annihilates the identity, which corresponds to $N = 1$, and the SL(2,R) vacuum, which corresponds to $N = 2$. Since all higher $N$ wedge states can be obtained by star multiplication of $N = 2$ states we have $K_1 |N\rangle = 0$. We now show that as a consequence of this, the matrices $T_N$ commute with $K_1$. Indeed, using eq. (148 we have

$$0 = K_1 |N\rangle = K_1 \exp \left(-E_N \right) |0\rangle = -[K_1, E_N] \exp \left(-E_N \right) |0\rangle,$$

(150)
since $[K_1, E_N]$ commutes with $E_N$. Using (139), and noting that the momentum operator kills any wedge state, we have that the above equation gives

$$0 = \left( \frac{1}{2} a^\dagger \cdot (K_1 C T_N + C T_N K_1) \cdot a^\dagger \right) |N\rangle = \left( \frac{1}{2} a^\dagger \cdot (C \{T_N, K_1\}) \cdot a^\dagger \right) |N\rangle.$$ 

(151)

Since the multiplicative factor acting on the wedge states above consists of creation operators only, the factor itself must vanish identically. This implies that

$$[T_N, K_1] = 0.$$

(152)

Since the spectrum of $K_1$ is non-degenerate, all eigenvectors of $K_1$ must be eigenvectors of $T_N$. Furthermore, since $T_N$ commutes with $C$, we see from eq.(147 that $w^{(\pm \kappa)}$ describe degenerate eigenvectors of $T_N$, and $w^{(\kappa)} \pm w^{(-\kappa)}$ are simultaneous eigenvectors of $T_N$ and $C$. We should note, however, that the relation $[T_N, K_1] = 0$ holds only for infinite-dimensional matrices $T_N$ and $K_1$ and is only approximate if we truncate $T_N$ and $K_1$ to finite dimensional matrices.

A similar argument can be used to show that $K_1$ commutes with the matrices $M^{12}$ and $M^{21}$. The derivation property of $K_1$ implies that

$$\left( K_1^{(1)} + K_1^{(2)} + K_1^{(3)} \right) |V_3\rangle = 0,$$

(153)

where the expression for the vertex was given in (122) and (123). It suffices for the present purposes to work at zero momentum, and the above equation implies that

$$\left( K_1^{(1)} + K_1^{(2)} + K_1^{(3)} \right) \exp \left(-\frac{1}{2} \sum_{r,s} a^{(r)\dagger} C M^{rs} a^{(s)\dagger} \right) |0\rangle = 0.$$

(154)
Since the $K$’s annihilate the vacuum we pick commutators that give

$$
\left\{ \frac{1}{2} \sum_{p,q} a^{(p)\dagger} C [K_1, M^{pq}] a^{(q)} \right\} \exp\left\{ \frac{-1}{2} \sum_{r,s} a^{(r)\dagger} C M^{rs} a^{(s)} \right\} |0\rangle = 0 . \quad (155)
$$

This condition implies that $[K_1, M^{pq}] = 0$, as claimed. Once again, the non-degeneracy of the $K_1$ spectrum implies that $K_1$ eigenvectors are eigenvectors of $M^{12}$ and $M^{21}$.

### 6.4 Relating the eigenvalues of $M$ and $K_1$ via the $B$ matrices

While the $K_1$ operator helps us determine the eigenvectors of $M$ and $T_N$, so far it has not given us information about the corresponding eigenvalues as the precise relation between $M$ and $K_1$ has not been found. We shall find this relation by introducing a new matrix $B$, much simpler than $M$, and that shares all the eigenvectors of $M$. The relation of $B$ to $T$ (or $M$) is calculable analytically and thus the relation between their eigenvalues is fixed. Furthermore the action of $B$ on the eigenvector $w^{(k)}_n$ will also be calculable analytically. This in turn, will determine the action of $M$ and $T$ on $w^{(k)}_n$.

We define $B$ as the leading expansion of $T_N$, when $N$ is very close to two:

$$
T_{2+\epsilon} = \epsilon B + O(\epsilon^2) . \quad (156)
$$

Expanding the formula (149) we find the relation of $B$ with the sliver matrix $T$,

$$
B = -\frac{T \ln(-T)}{1 - T^2} . \quad (157)
$$

Notice that as $T \to -1$, $B \to -1/2$, and as $T \to 0$, $B \to 0$. So the spectrum of $B$ is expected to lie on the interval $[-1/2, 0]$.

To obtain an expression for the matrix elements of $B$ we consider in more detail the wedge state $|2 + \epsilon\rangle$. We have, on the one hand,

$$
|2 + \epsilon\rangle = \exp(\epsilon V_-)|0\rangle = |0\rangle + \epsilon V_-|0\rangle + O(\epsilon^2) \quad (158)
$$

for an appropriate vector field

$$
V_- = \sum_{n=2}^{\infty} v_n L_{-n} . \quad (159)
$$
To find $V_-$, recall that wedge states $|N\rangle = \exp\left(\sum_{k=2}^{\infty} c_k^{(N)} L_{-k}\right) |0\rangle$ are defined by requiring[30]

$$\exp\left(\sum_{k=2}^{\infty} c_k^{(N)} z^{k+1} \partial_z\right) z = \frac{N}{2} \tan\left(\frac{2}{N} \arctan(z)\right)$$

(160)

From eqs.(158)-(160) we have $c_k^{(2+\epsilon)} = \epsilon v_k$. Expanding the right-hand side of (160) for $N = 2 + \epsilon$ we get

$$\sum_n v_n z^{n+1} = \frac{1}{2} (z - (1 + z^2) \tan^{-1} z) .$$

(161)

This gives

$$V_- = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n - 1)(2n + 1)} L_{-2n} .$$

(162)

On the other hand we also have from (148) and (156) that

$$|2 + \epsilon\rangle = \exp\left(\epsilon - \frac{1}{2} a^\dagger \cdot (CB) \cdot a^\dagger\right) |0\rangle = |0\rangle - \epsilon a^\dagger \cdot (CB) \cdot a^\dagger |0\rangle + O(\epsilon^2) .$$

(163)

Comparing the right-hand sides of (158) and (163), using eq.(162), and the equation

$$L_{-m} = \frac{1}{2} \sum_p \alpha_{m+p}^\mu \alpha_{-p}^\nu \eta_{\mu \nu} \text{ for } m > 0 ,$$

(164)

we finally find

$$B_{mn} \equiv -\frac{(-1)^{\frac{n+m}{2}} \sqrt{m n}}{(m + n)^2 - 1} \text{ for } n + m \text{ even}$$

$$B_{mn} \equiv 0 \text{ for } n + m \text{ odd} .$$

(165)

Note that the matrix $B$ is much simpler than the matrix $T$ or $M$.

Since $T_N$ commutes with $K_1$ for every $N$, so must $B$. Thus the eigenvectors $v_n^{(\kappa)} = \sqrt{n} w_n^{(\kappa)}$ must also be eigenvectors of $B$. Our goal now is to find an expression for the eigenvalues $\beta(\kappa)$ of $B$ associated to the eigenvectors $v^{(\kappa)}$. For this we consider the eigenvalue equation

$$\sum_{n \geq 1} B_{mn} v_n^{(\kappa)} = \beta(\kappa) v_m^{(\kappa)} .$$

(166)

---

6 This vector field has also been considered independently by Schnabl [33].
Since this relation holds for every \( m \) we have, in particular,
\[
\beta(\kappa) = \frac{1}{v_{\frac{1}{2}}(\kappa)} \sum_{n \geq 1} B_{1n} v_n^{(\kappa)} = \frac{1}{w^{(\kappa)}} \sum_{q=1}^{n} \frac{(-1)^q}{2q+1} w_{2q-1}^{(\kappa)}.
\]
(167)

If we define
\[
F'(z) = \sum_{q=1}^{n} \frac{(-1)^q}{2q+1} w_{2q-1}^{(\kappa)} z^{2q+1},
\]
(168)
then we may rewrite (167) as
\[
\beta(\kappa) = \frac{F(1)}{w_{\frac{1}{2}}^{(\kappa)}} = F(1),
\]
(169)

since, as seen from eq.(145), \( w_{\frac{1}{2}}^{(\kappa)} = 1 \). On the other hand, we have,
\[
\frac{dF'(z)}{dz} = \sum_{q=1}^{n} (-1)^q w_{2q-1}^{(\kappa)} z^{2q} = \frac{1}{2} iz \left( f_{w^{\kappa}}(iz) - f_{w^{\kappa}}(-iz) \right),
\]
(170)
where \( f_{w^{\kappa}}(z) \) has been defined in eq.(145). We can easily integrate this equation (with the boundary condition \( F(0) = 0 \)) to get
\[
\beta(\kappa) = F(1) = \frac{1}{\kappa} \int_0^1 dz \sin(\kappa \tanh^{-1}(z)) = \frac{1}{2} \frac{\kappa \pi / 2}{\sinh(\kappa \pi / 2)}. \]
(171)

This is the eigenvalue of the matrix \( B \) associated to the eigenvector \( v^{(\kappa)} \). Using eq.(157) we can determine the corresponding eigenvalue of \( T \) to be:
\[
\tau(\kappa) = -e^{-|\kappa| \pi / 2}.
\]
(172)

In deriving (172) we have noted that eq.(157) does not determine \( T \) uniquely for a given \( A \), and we need some additional input. This comes from the requirement that the eigenvalue of \( T \) lies between \(-1\) and \(0\), a fact found in numerical experiments.

Finally, using eq.(149), we can determine the eigenvalue \( \mu(\kappa) \) of \( M = T_3 \) to be
\[
\mu(\kappa) = \frac{\tau(\kappa) + (\tau(\kappa))^2}{1 + (\tau(\kappa))^3} = -\frac{1}{1 + 2 \cosh(\kappa \pi / 2)}. \]
(173)

This completes the determination of the eigenvectors and eigenvalues of the matrix \( M \). Furthermore, eq.(173) also provides us a simple expression for \( M \) in terms of the matrix \( K_1 \):
\[
M = -(1 + 2 \cosh(K_1 \pi / 2))^{-1}. \]
(174)
The set of eigenvalues are then
\[
\begin{align*}
\mu^{11}(\kappa) &= -\frac{1}{1 + 2\cosh(\pi \kappa/2)}, \\
\mu^{12}(\kappa) &= -(1 + \exp(\pi \kappa/2)) \mu(\kappa), \\
\mu^{21}(\kappa) &= -(1 + \exp(-\pi \kappa/2)) \mu(\kappa).
\end{align*}
\]

(175)

7 Star Product as Continuous Moyal Product

In a stimulating paper Bars [38] has described the open string star product in terms of the Moyal product. Bars proposed that each even position mode (except for the zero mode) together with a specific linear combination of the Fourier transforms of the odd position modes forms a Moyal pair. The various Moyal pairs are mutually commuting and they all have the same noncommutativity parameter \(\theta \neq 0\). The precise treatment of the center-of-mass or string midpoint was left for future work.

Here we review an investigation of the open string star product in terms of Moyal products using the diagonalization of the Neumann matrices defining the oscillator form of the three-open-string vertex, and the orthonormality and completeness of the eigenvectors as proven in [25]. These results imply that there is a basis of oscillators where the exponential in the vertex takes diagonal form, and thus furnishes a representation of the product in terms of mutually commuting algebras.

We will show that each commuting factor in this product is a Moyal product, and thus each algebra is a (one dimensional) Heisenberg algebra. Since the spectrum of the Neumann matrices is continuous, we get a continuous tensor product of Heisenberg algebras with a smoothly varying noncommutativity parameter. It should be emphasized that we work on the subspace of zero momentum functionals, that is, functionals that are independent of the center of mass coordinate of the string.

Our Moyal coordinates are constructed as follows. For each \(\kappa \in (-\infty, \infty)\) the Neumann matrices \(M, M^{12},\) and \(M^{21}\) have a common eigenvector \(v(\kappa)\) with eigenvalues \(\mu(\kappa), \mu^{12}(\kappa)\) and \(\mu^{21}(\kappa)\) respectively. The signal of noncommutativity is the fact that \(\mu^{12}(\kappa) \neq \mu^{21}(\kappa)\). Under the action of open string twist, the eigenvectors \(v(\kappa)\) and \(v(-\kappa)\) mix, and one can conveniently define twist odd and twist even eigenvectors for \(\kappa \in [0, \infty)\). The unpaired eigenvector at \(\kappa = 0\) is actually twist odd, and has featured in several studies.
of vacuum string field theory and sliver states [17, 32, 27, 4]. It will have 
an important role here as well. The definite twist combinations are degenerate 
eigenvectors of $M$, and while they not eigenvectors of $M^{12}$ and $M^{21}$, 
these matrices have simple action on them. We find that for each $\kappa > 0$, 
the twist even and the twist odd eigenvectors respectively define the Moyal 
coordinates $x(\kappa)$ and $y(\kappa)$, with noncommutativity parameter given as\textsuperscript{7} 
\[ \theta(\kappa) = 2 \tanh \left( \frac{\pi \kappa}{4} \right), \quad \kappa \geq 0. \] 
(176) 
Note that for $\kappa = 0$, where there is only one eigenvector, and thus only 
one coordinate, there is no scope for a Moyal product, and consistently the 
noncommutativity vanishes. The various coordinates for different values of 
$\kappa$ commute, so that we have 
\[ [x(\kappa), y(\kappa')] = i \theta(\kappa) \delta(\kappa - \kappa'). \] 
(177) 

The description of the star product in terms of the above Moyal products 
is as follows. Given a string field $\Psi(X(\sigma))$, we view it as a function $\Psi(\{x_{2n}\}, \{x_{2n-1}\})$ of the even and odd modes in the expansion of $X(\sigma)$. Then we Fourier transform $\Psi$ on the odd modes $x_{2n-1}$ into variables $p_{2n-1}$ (that correspond to the eigenvalues of $\hat{p}_{2n-1}$) obtaining a functional $\hat{\Psi}(\{x_{2n}\}, \{p_{2n-1}\})$. This functional is now written in terms of coordinates $(x(\kappa), y(\kappa))$ using the invertible relations 
\[ x(\kappa) = \sqrt{2} \sum_{n=1}^{\infty} v_{2n}(\kappa) \sqrt{2n} x_{2n}, \quad y(\kappa) = -\sqrt{2} \sum_{n=1}^{\infty} \frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}} p_{2n-1}, \] 
(178) 
that, as mentioned before, use the twist even and twist odd eigenvectors. The resulting field $\Psi^{M}(x(\kappa), y(\kappa))$ is star multiplied just using the Moyal product on the underlying coordinates.\textsuperscript{8} As we will see, this continuous Moyal product can be written as a functional Moyal product, using the language of path integrals over the coordinates $(x(\kappa), y(\kappa))$. For $\kappa = 0$ the surviving coordinate $y(\kappa = 0)$ is predicted to be commutative. As can be seen in (178) it corresponds to a specific linear combination of momentum modes. This combination precisely selects the momentum carried by half of 

\textsuperscript{7}With units inserted back there is an $\alpha'$ factor in the right-hand side of (176). So, as usual, $\theta$ has dimensions of $(\text{length})^2$.

\textsuperscript{8}Strictly speaking, the two products are related by multiplication by an overall (infinite) constant factor.
the string. As we explain in the text, for zero momentum functionals, the open string vertex treats the momentum carried by half of the string as a commuting coordinate.

Alternatively, we can invert (178) and use (177) to interpret the Witten star product as the Moyal product in the noncommutative space \((x_{2m}, p_{2m-1})\) with a nondiagonal form of noncommutative parameter

\[
[x_{2n}, p_{2m-1}]_* = i\Theta^{2n,2m-1} \quad n, m \geq 1.
\]  

(179)

where the matrix \(\Theta\) above coincides with one of the matrices relevant in the half-string formalism. In this way Witten’s prescription for identifying half strings may be interpreted as identifying the end points of dipoles in the noncommutative space (179). It is interesting that \(p_{2m-1}\) have the interpretation of coordinates and thus \(x_{2m-1}\) should be interpreted as momenta.

### 7.1 Oscillator vertex for the Moyal product

The three string vertex is ordinarily expressed as a quadratic form in oscillators acting on the vacuum. More precisely, we have oscillators from the three state spaces, those of the two input string fields and that of the output product, acting on the tensor product of vacua. In order to understand how to interpret such a vertex as a Moyal product we will calculate here an oscillator-form three-vertex for the Moyal multiplication in two dimensional space.

It is useful first to recall how ordinary commutative pointwise multiplication of functions can be encoded in a three-vertex written in oscillator form. This result was given in [32] (eqn. (3.15)):

\[
|V_3\rangle = \left(\frac{2}{3\sqrt{\pi}}\right)^{1/2} \exp\left[ \frac{1}{6}(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger + a_3^\dagger a_3^\dagger) - \frac{2}{3} (a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger + a_3^\dagger a_1^\dagger) \right] |0\rangle,
\]

(180)

where the subscripts one and two on the oscillators refer to the first function and second function to be multiplied respectively, while the subscript three is associated to the product. This result goes along with the definitions

\[
\hat{x} = \frac{i}{\sqrt{2}}(a - a^\dagger), \quad \hat{p} = \frac{1}{\sqrt{2}}(a + a^\dagger),
\]

(181)

and the construction of the position states

\[
\langle x | = \frac{1}{\pi^{1/4}} |0\rangle \exp\left(-\frac{1}{2}x^2 + \sqrt{2}ix a + \frac{1}{2}aa^\dagger\right).
\]

(182)
Given two functions \( f(x) = \langle x | f \rangle \) and \( g(x) = \langle x | g \rangle \), then \( (f \cdot g)(x) = \langle x | (f)(g) | V_3 \rangle \). We are after the generalization of (180) for the case of Moyal multiplication. It should be noted from the outset that for a Moyal product we need two coordinates — so the vertex will be a modification of the above involving two sets of oscillators. When the noncommutativity is set to zero, we must recover two copies of the above vertex.

We begin with the standard definition of the Moyal product in momentum space for \( R^{2d} \):

\[
(f \ast g)(k) = \int \frac{dk_1}{(2\pi)^{2d}} \frac{dk_2}{(2\pi)^{2d}} (2\pi)^{2d} \delta(k_1 + k_2 - k) e^{-\frac{i}{2} k_{\mu} \Theta^{\mu\nu} k_{\nu}} f(k_1) g(k_2).
\]

(183)

with Fourier transformation definitions

\[
h(k) = \int dx e^{ikx} h(x), \quad h(x) = \int \frac{dk}{(2\pi)^{2d}} e^{-ikx} h(k).
\]

(184)

We pass to coordinate space to find a kernel \( K(x_1, x_2, x_3) \) defined from

\[
(f \ast g)(x_3) \equiv \int dx_1 dx_2 K(x_1, x_2, x_3) f(x_1) g(x_2).
\]

(185)

Using the formula for momentum space we have:

\[
K(x_1, x_2, x_3) = \int \frac{dk_1}{(2\pi)^{2d}} \frac{dk_2}{(2\pi)^{2d}} \exp\left(-\frac{1}{2} k_1 \cdot i \Theta \cdot k_2 + k_1 \cdot i(x_1 - x_3) + k_2 \cdot i(x_2 - x_3)\right).
\]

(186)

Here \( \Theta \) is a \( 2d \times 2d \) matrix and we use \( \cdot \) to denote scalar product, or sum over \( 2d \) component indices. The integral can be done by completing squares and one finds the standard result

\[
K(x_1, x_2, x_3) = \frac{1}{\pi^{2d} \det \Theta} \exp\left(-2i (x_1 - x_3) \Theta^{-1} (x_2 - x_3)\right),
\]

(187)

or, in manifestly cyclic form

\[
K(x_1, x_2, x_3) = \frac{1}{\pi^{2d} \det \Theta} \exp\left(-2i [x_1 \Theta^{-1} x_2 + x_2 \Theta^{-1} x_3 + x_3 \Theta^{-1} x_1]\right).
\]

(188)

Now restrict attention to two dimensional noncommutative space, i.e., set \( d = 1 \). Write a general cyclic ansatz for a three-vertex, by considering oscillators \( (a, b) \) associated to the coordinates \( x^\mu \), with \( \mu = 1, 2 \).

\[
(x_1, y_1 ; x_2, y_2 ; x_3, y_3) \leftrightarrow (a_1 \dagger, b_1 \dagger ; a_2 \dagger, b_2 \dagger ; a_3 \dagger, b_3 \dagger).
\]

(189)
The position eigenstates can be defined using (181) together with the analogous relations

\[ \hat{y} = \frac{i}{\sqrt{2}} (b - b^\dagger); \quad \hat{q} = \frac{1}{\sqrt{2}} (b + b^\dagger). \]  

Using 2-component notation \( \vec{x} = (x, y), \vec{a} = (a, b) \), we have

\[ \langle \vec{x} \rangle = \frac{1}{\sqrt{\pi}} |0\rangle \exp \left( -\frac{1}{2} \vec{x} \cdot \vec{x} + \sqrt{2} i \vec{a} \cdot \vec{x} + \frac{i}{2} \vec{a} \cdot \vec{a} \right), \]
\[ \langle \vec{p} \rangle = \frac{1}{\sqrt{\pi}} |0\rangle \exp \left( -\frac{1}{2} \vec{p} \cdot \vec{p} + \sqrt{2} \vec{a} \cdot \vec{p} - \frac{1}{2} \vec{a} \cdot \vec{a} \right). \]  

The three-vertex will be written as

\[ |V_3\rangle = N \exp \left( -\frac{1}{2} \vec{a}_1^\dagger V \vec{a}_1^\dagger \right) |0\rangle, \]  

where \( V \) is a \( 6 \times 6 \) matrix, and \( \vec{A} \) a six component vector

\[ V = \begin{pmatrix} u & v & v^T \\ v^T & u & v \\ v & v^T & u \end{pmatrix}, \quad \vec{A}^\dagger = (\vec{a}_1^\dagger, \vec{a}_2^\dagger, \vec{a}_3^\dagger). \]  

with \( u, v \) two by two matrices. We now require that

\[ K(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \left( |\vec{x}_1\rangle \otimes |\vec{x}_2\rangle \otimes |\vec{x}_3\rangle \right) |V_3\rangle = \frac{1}{\pi^2 \det \Theta} \exp \left( -\frac{1}{2} \vec{X} K \vec{X} \right), \]  

where use was made of (188), and we have defined

\[ K = (2i) \begin{pmatrix} 0 & \Theta^{-1} & -\Theta^{-1} \\ -\Theta^{-1} & 0 & \Theta^{-1} \\ \Theta^{-1} & -\Theta^{-1} & 0 \end{pmatrix}, \quad \vec{X} = (\vec{x}_1, \vec{x}_2, \vec{x}_3). \]  

In this notation, (191) allows one to write

\[ \langle X \rangle = |\vec{x}_1\rangle \otimes |\vec{x}_2\rangle \otimes |\vec{x}_3\rangle = \frac{1}{\pi \sqrt{\pi}} |0\rangle \exp \left( -\frac{1}{2} \vec{X} \cdot \vec{X} + \sqrt{2} i \vec{A} \cdot \vec{X} + \frac{1}{2} \vec{A} \cdot \vec{A} \right), \]  

Substituting back into the main condition (194) and doing the contraction we find:

\[ \frac{1}{\pi^2 \det \Theta} \exp \left( -\frac{1}{2} \vec{X} K \vec{X} \right) = \frac{N}{\pi \sqrt{\pi}} \frac{1}{\sqrt{\det(1 + V)}} \exp \left( -\frac{1}{2} \vec{X} \frac{1 - V}{1 + V} \vec{X} \right), \]
and conclude that

\[ V = \frac{1 - K}{1 + K}, \quad N = \frac{\sqrt{\det(1 + V)}}{\sqrt{\pi} \det \Theta}. \]  

(198)

We now take \( \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \), and find that for \( \theta \neq 0 \) the matrix \( (1 + K) \) appearing in \( V \) is invertible. The resulting expression for \( V \) in terms of the \( u, v \) matrices in (193) is

\[ u = -\frac{4 + \theta^2}{12 + \theta^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v = \frac{4}{12 + \theta^2} \begin{pmatrix} 2 & i\theta \\ -i\theta & 2 \end{pmatrix}. \]

(199)

Moreover

\[ N = \frac{8}{\sqrt{\pi}(12 + \theta^2)} = \frac{2}{3\sqrt{\pi}} \frac{1}{(1 + \frac{\theta^2}{12})}. \]

(200)

All in all, the form of the vertex is

\[
|V_3(\theta)\rangle = \frac{2}{3\sqrt{\pi}} \frac{1}{1 + \frac{\theta^2}{12}} \exp\left[-\frac{1}{2} \left(\frac{-4 + \theta^2}{12 + \theta^2}\right) (a_1^\dagger a_1^\dagger + b_1^\dagger b_1^\dagger + \text{cyclic}) - \left(\frac{8}{12 + \theta^2}\right) (a_1^\dagger a_2^\dagger + b_1^\dagger b_2^\dagger + \text{cyclic}) - \left(\frac{4i\theta}{12 + \theta^2}\right) (a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger + \text{cyclic})\right] |0\rangle. 
\]

(201)

This is the desired form of the vertex for canonical Moyal. The limit \( \theta \to 0 \) is smooth and gives:

\[
|V_3(\theta = 0)\rangle = \frac{2}{3\sqrt{\pi}} \exp\left[\frac{1}{6} (a_1^\dagger a_1^\dagger + b_1^\dagger b_1^\dagger + \text{cyclic}) \right] - \frac{2}{3} (a_1^\dagger a_2^\dagger + b_1^\dagger b_2^\dagger + \text{cyclic})\right] |0\rangle. 
\]

(202)

Happily, this agrees with the commutative result reviewed in (180).

### 7.2 Open string star in the continuous oscillator basis

We now examine the three string vertex and use the recent diagonalization of the Neumann matrices to rewrite the vertex in a basis of oscillators with a continuous mode label. We will then perform a redefinition of the oscillators in such a way to allow a comparison with the Moyal form of the vertex determined in the previous section.
The eigenvector \( v_n(\kappa) \) is given by the generating function:\(^9\)

\[
f_\kappa(z) = \sum_{n=1}^{\infty} \frac{v_n(\kappa)}{\sqrt{n}} z^n = \frac{1}{N(\kappa)\frac{1}{2} \kappa} (1 - e^{-\kappa \tan^{-1} z}),
\]

(203)

where \( N(\kappa) \) is given by \([27]\)

\[
N(\kappa) = \frac{2}{\kappa} \sinh \frac{\pi \kappa}{2}.
\]

(204)

Note also that

\[
\mu^{rs}(-\kappa) = \mu^{sr}(\kappa),
\]

(205)

where we have defined \( \mu^{r+1,s+1} = \mu^{rs} \) for superscripts mod 3. The twist matrix \( C \) has a well-defined action on the eigenvectors

\[
\sum_{n=1}^{\infty} C_{mn} v_n(\kappa) = -v_m(-\kappa), \quad C_{mn} = (-1)^m \delta_{mn}.
\]

(206)

It follows from this equation that the even and odd components of the eigenvectors satisfy the relations

\[
v_{2n+1}(-\kappa) = v_{2n+1}(\kappa), \quad v_{2n}(-\kappa) = -v_{2n}(\kappa).
\]

(207)

The eigenfunctions can be shown to be orthogonal and complete,

\[
\sum_{n=1}^{\infty} v_n(\kappa_1)v_n(\kappa_2) = \delta(\kappa_1 - \kappa_2),
\]

\[
\int_{-\infty}^{\infty} dk v_m(\kappa)v_n(\kappa) = \delta_{m,n}.
\]

(208)

Due to the relations (207), we can separate the even and odd modes and write the completeness and orthogonality relations (208)

\[
2 \sum_{n=1}^{\infty} v_{2n-1}(\kappa_1)v_{2n-1}(\kappa_2) = \delta(\kappa_1 - \kappa_2), \quad 2 \sum_{n=1}^{\infty} v_{2n}(\kappa_1)v_{2n}(\kappa_2) = \delta(\kappa_1 - \kappa_2),
\]

\[
2 \int_{-\infty}^{\infty} dk v_{2n}(\kappa)v_{2n}(\kappa) = \delta_{m,n}, \quad 2 \int_{-\infty}^{\infty} dk v_{2n-1}(\kappa)v_{2n-1}(\kappa) = \delta_{m,n}.
\]

(209)

\(^9\)The \( v_n \)'s here differ from those in \([20]\) by the inclusion of the normalization factor \( N(\kappa)^{-\frac{1}{2}} \).
These properties allow us to introduce new oscillators whose mode number is a continuous parameter. From (209), it is convenient to introduce new continuous oscillators $e_{\kappa}$ and $o_{\kappa}$ associated to the even and odd mode sums respectively:

$$
o_{\kappa}^\dagger = -\sqrt{2} i \sum_{n=1}^{\infty} v_{2n-1}(\kappa) a_{2n-1}^\dagger, \quad e_{\kappa}^\dagger = \sqrt{2} \sum_{n=1}^{\infty} v_{2n}(\kappa) a_{2n}^\dagger \tag{210}
$$

$$
a_{2n}^\dagger = \sqrt{2} \int_0^\infty d\kappa v_{2n}(\kappa) e_{\kappa}^\dagger, \quad a_{2n-1}^\dagger = \sqrt{2} i \int_0^\infty d\kappa v_{2n-1}(\kappa) o_{\kappa}^\dagger \tag{211}
$$

We have introduced a factor of $i$ in the first equation so that they have the same BPZ conjugation property

$$
b_{pz}(e_{\kappa}) = -o_{\kappa}^\dagger, \quad b_{pz}(o_{\kappa}) = -e_{\kappa}^\dagger. \tag{212}
$$

The new oscillators satisfy the commutation relations

$$[o_{\kappa}, o_{\kappa'}^\dagger] = [e_{\kappa}, e_{\kappa'}^\dagger] = 0, \quad [o_{\kappa}, e_{\kappa'}^\dagger] = \delta(\kappa - \kappa'). \tag{213}
$$

Note that for $\kappa = 0$, the $e_{\kappa=0}$ oscillator vanishes, and we only have $o_{\kappa=0}$. This is because the $\kappa = 0$ eigenvector is $C$-odd. Note also that the change of basis from the discrete oscillators into the continuous one (210) is a unitary transformation, as can be checked using (209).

Using equations (211) and the completeness relations (209), after some algebra the three-string vertex

$$|V_3\rangle = \exp\left[-\frac{1}{2} \sum_{r,s} \sum_{m,n} a_m^{\dagger} (CMrs)_{mn} a_n^s |0\rangle, \tag{214}
$$

can then be written in terms of $o_{\kappa}, e_{\kappa}$ basis as

$$|V_3\rangle = \exp \left[ \sum_{r,s} \int_0^\infty d\kappa \left\{ -\frac{1}{4} (\mu^{rs}(\kappa) + \mu^{sr}(\kappa)) \left( e_{\kappa}^{(r)} e_{\kappa}^{(s)\dagger} + o_{\kappa}^{(r)} o_{\kappa}^{(s)\dagger} \right) + \frac{i}{4} (\mu^{rs} + \mu^{sr}) \left( o_{\kappa}^{(r)} e_{\kappa}^{(s)\dagger} - e_{\kappa}^{(r)} o_{\kappa}^{(s)\dagger} \right) \right\} \right] |0\rangle \tag{215}
$$

where |0\rangle is now interpreted as a “continuous tensor product” of oscillator ground states for $e_{\kappa}, o_{\kappa}$. More explicitly, (215) can be written as

$$|V_3\rangle = \exp \left[ \int_0^\infty d\kappa \left\{ -\frac{1}{2} \mu(\kappa) \left( o_{\kappa}^{(1)} o_{\kappa}^{(1)\dagger} + e_{\kappa}^{(1)} e_{\kappa}^{(1)\dagger} \right) + \text{cyc} \right\} -\frac{1}{2} (\mu^{12}(\kappa) + \mu^{21}(\kappa)) \left( o_{\kappa}^{(1)} o_{\kappa}^{(2)\dagger} + e_{\kappa}^{(1)} e_{\kappa}^{(2)\dagger} \right) + \text{cyc} \right] |0\rangle \tag{216}
$$

$$-\frac{i}{2} (\mu^{12}(\kappa) - \mu^{21}(\kappa)) \left( e_{\kappa}^{(1)} o_{\kappa}^{(2)\dagger} - o_{\kappa}^{(1)} e_{\kappa}^{(2)\dagger} \right) + \text{cyc} \right] |0\rangle \tag{216}
$$
This is the desired form of the open string three-vertex. In the next section we identify the Moyal structures explicitly. The above matter vertex is restricted to the $p = 0$ sector, and is appropriately normalized.

7.3 Identification of Moyal structures

In this section we use the results from the two previous sections to show that the star product indeed corresponds to Moyal products for a continuous set of variables parameterized by $\kappa \in [0, \infty)$. We identify these variables and also give an interpretation for the commuting mode encountered at $\kappa = 0$.

We have now completed our preliminary work finding both the oscillator representation of the Moyal product in section 2, eqn. (201), and a presentation (216) of the three-string vertex as a continuous tensor product of three-vertices. Comparing the exponents in these vertices we see that an identification

$$\left(a^\dagger, b^\dagger\right) \leftrightarrow \left(e^\dagger_\kappa, o^\dagger_\kappa\right), \quad (217)$$

requires the conditions

$$\mu(\kappa) = \frac{-4 + \theta^2}{12 + \theta^2},$$

$$\mu^{12}(\kappa) + \mu^{21}(\kappa) = \frac{16}{12 + \theta^2},$$

$$\mu^{12}(\kappa) - \mu^{21}(\kappa) = \frac{8\theta}{12 + \theta^2}. \quad (218)$$

Note that the consistency condition $\mu + \mu^{12} + \mu^{21} = 1$ is satisfied. Making use of the explicit expressions (175) we find that the above equations are all satisfied by choosing

$$\theta(\kappa) = 2 \tanh \left(\frac{\pi \kappa}{4}\right). \quad (219)$$

This is the noncommutativity parameter associated to the Moyal algebras. Note in particular that for $\kappa = 0$ where the pair of oscillators collapses to just $o_{\kappa=0}$ we have a commutative product. The noncommutativity parameter grows as $\kappa$ grows but is bounded: $\theta(\kappa) \leq 2$.

Note that the Witten vertex differs from the Moyal vertex by a $c$-number prefactor, as can be seen by comparing (201) with (216). This means that, strictly speaking, the open string field theory product $\mathcal{W}$ and the Moyal product $\ast$

$$f \ast_W g = C' f \ast g, \quad (220)$$
Having identified the noncommutative parameter we must now determine what are the explicit forms of the coordinates that enter the Moyal product. It is natural to expect that those will be continuous coordinate modes $x(\kappa)$ associated to the usual position modes $x_n$ by relations similar to those in (210). As hinted at before, we will have to use even and odd modes, and a Fourier transformation of the odd modes will be necessary.

For this purpose we consider the position and momentum eigenstates
\[
\langle \hat{X}(\sigma) \rangle = \langle 0 | \exp \left( -\vec{x} \cdot E^{-2} \cdot \vec{x} + 2i \vec{a} \cdot E^{-1} \cdot \vec{a} + \frac{1}{2} \left( \vec{a} \cdot \vec{a} \right) \right) ,
\]
\[
\langle \hat{P}(\sigma) \rangle = \langle 0 | \exp \left( -\frac{1}{4} \vec{p} \cdot E^2 \cdot \vec{p} + \vec{a} \cdot E \cdot \vec{p} - \frac{1}{2} \left( \vec{a} \cdot \vec{a} \right) \right) ,
\]
where
\[
\hat{x} = \frac{i}{2} E \cdot (a - a^\dagger) , \quad \hat{p} = E^{-1} \cdot (a + a^\dagger) , \quad E_{mn} = \sqrt{\frac{2}{n}} \delta_{mn} .
\]

(This is working in the $\alpha' = 1/2$ convention.) More explicitly
\[
\hat{x}_n = \frac{i}{\sqrt{2n}} \cdot (a_n - a_n^\dagger) , \quad \hat{p}_n = \sqrt{\frac{n}{2}} (a_n + a_n^\dagger) , \quad n \geq 1 ,
\]
and the zero mode expressions are those in (181). The above go along with the expansions
\[
\hat{X}(\sigma) = \hat{x}_0 + \sqrt{2} \sum_{n=1}^{\infty} \hat{x}_n \cos n \sigma , \quad \pi \hat{P}(\sigma) = \hat{p}_0 + \sqrt{2} \sum_{n=1}^{\infty} \hat{p}_n \cos n \sigma .
\]

We must now define new coordinate and momentum operators associated to the oscillators $(e_\kappa, e_\kappa^\dagger)$ and $(o_\kappa, o_\kappa^\dagger)$ we have introduced. We let
\[
(\hat{x}_\kappa, \hat{q}_\kappa) \leftrightarrow (e_\kappa, e_\kappa^\dagger) , \quad (\hat{y}_\kappa, \hat{l}_\kappa) \leftrightarrow (o_\kappa, o_\kappa^\dagger) ,
\]
using the standard correspondences for zero-modes (181):
\[
\hat{x}_\kappa = \frac{i}{\sqrt{2}} (e_\kappa - e_\kappa^\dagger) , \quad \hat{q}_\kappa = \frac{1}{\sqrt{2}} (e_\kappa + e_\kappa^\dagger) ,
\]
\[
\hat{y}_\kappa = \frac{i}{\sqrt{2}} (o_\kappa - o_\kappa^\dagger) , \quad \hat{l}_\kappa = \frac{1}{\sqrt{2}} (o_\kappa + o_\kappa^\dagger) .
\]

The coordinate operators $\hat{x}_\kappa$ and $\hat{y}_\kappa$ are the ones for which the 3-string vertex has been put in Moyal form. We must therefore now express them in terms of
the original operators \((\hat{x}_n, \hat{p}_n)\). For this we use (210) to first pass to \((a_n, a_n^\dagger)\) oscillators and then (223) to pass to \((\hat{x}_n, \hat{p}_n)\) operators. One immediately finds

\[
\hat{x}_\kappa = \sqrt{2} \sum_{n=1}^{\infty} \nu_{2n}(\kappa) \sqrt{2n} \; \hat{x}_{2n},
\]

(227)

\[
\hat{y}_\kappa = -\sqrt{2} \sum_{n=1}^{\infty} \frac{\nu_{2n-1}(\kappa)}{\sqrt{2n-1}} \; \hat{p}_{2n-1},
\]

(228)

\[
\hat{q}_\kappa = \sqrt{2} \sum_{n=1}^{\infty} \frac{\nu_{2n}(\kappa)}{\sqrt{2n}} \; \hat{p}_{2n},
\]

(229)

\[
\hat{t}_\kappa = \sqrt{2} \sum_{n=1}^{\infty} \nu_{2n-1}(\kappa) \sqrt{2n-1} \; \hat{x}_{2n-1}.
\]

(230)

Here we see that the Moyal coordinates, the eigenvalues of \(\hat{x}_\kappa\) and \(\hat{y}_\kappa\), are respectively (i) linear combination of even conventional coordinates, and (ii) linear combinations of odd momenta. Thus the Moyal structure is canonical for the multiplication of string functionals where the odd coordinates \(x_{2n-1}\) are replaced by odd momenta \(p_{2n-1}\) via Fourier transformation. In other words given string functionals \(\Psi_i([x_{2n}], [x_{2n-1}])\) to be star multiplied, use of Moyal product requires the transformations

\[
\Psi_i([x_{2n}], [x_{2n-1}]) \rightarrow \Psi_i([x_{2n}], [p_{2n-1}]) \rightarrow \Psi_i^M(x(\kappa), y(\kappa)),
\]

(231)

where \(x(\kappa)\) and \(y(\kappa)\) denote the eigenvalues of \(\hat{x}_\kappa\) and \(\hat{y}_\kappa\) respectively. The first arrow above denotes Fourier transformation, and in the second arrow we just reexpress the coordinate and momenta in terms of continuous variables using (227) and (228). In this final form, with superscript \(M\) for Moyal, the star product is just canonical Moyal product with \(\theta(\kappa)\) for each \(\kappa \geq 0\), in a way that will be made more precise in the next subsection.

It is of interest to identify the nature of the commuting coordinate associated to \(\kappa = 0 \rightarrow \theta(\kappa) = 0\). Indeed for \(\kappa = 0\) the eigenvector \(\nu(\kappa = 0)\) only has odd components, and therefore, from (227) and (228) only \(\hat{y}_{\kappa=0}\) survives. This is explicitly

\[
\hat{y}_{\kappa=0} = -\sqrt{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \hat{p}_{2n-1} = -\sqrt{2}(\hat{p}_1 - \frac{1}{3}\hat{p}_3 + \frac{1}{5}\hat{p}_5 - \cdots).
\]

(232)

We can identify this right-hand side using (224). We note that the above linear combination is, with \(p_0 = 0\), just the momentum carried by half the
\[
y_{k=0} = -\sqrt{2} \int_0^{\pi/2} \pi \hat{P}(\sigma) d\sigma = -\sqrt{2} \pi \hat{P}_L.
\]

(233)

Our result therefore states that \( \hat{P}_L \) must behave as an ordinary commuting coordinate as far as the string field vertex is concerned. Indeed, this is the case for zero momentum functionals, as we explain now. Since the open string vertex is defined by gluing right half strings to left half strings it implements the following conditions when multiplying string (1) times string (2) to give string (3):

\[
\begin{align*}
\hat{P}_R^{(3)} &= -\hat{P}_L^{(1)}, & \hat{P}_R^{(1)} &= -\hat{P}_L^{(2)}, & \hat{P}_R^{(2)} &= -\hat{P}_L^{(2)}, \\
\end{align*}
\]

(234)

But for zero momentum string states \( \hat{P}_R^{(1)} = -\hat{P}_L^{(1)} \) and \( \hat{P}_R^{(2)} = -\hat{P}_L^{(2)} \), and as a consequence we have that the vertex requires

\[
\hat{P}_L^{(1)} = \hat{P}_L^{(2)} = \hat{P}_L^{(3)},
\]

(235)

which is the statement that \( \hat{P}_L \) eigenvalues behave as a commuting coordinate at the string vertex.

It is interesting to note that for zero momentum functionals \( \hat{P}_L \) is, up to a constant, the same as the operator \( \hat{P}_L - \hat{P}_R \). The connection of the \( \kappa = 0 \) eigenvector to \( \hat{P}_L - \hat{P}_R \) is clear from the observation of [32] that this eigenvector implies that the sliver functional is invariant under opposite rigid displacements of the two halves of the string. More precisely, the zero-momentum sliver is annihilated by the action of \( \hat{P}_L - \hat{P}_R \). It follows that the commuting coordinate \( \hat{y}_{k=0} \sim \hat{P}_L \) vanishes on the sliver. Of course, it will not vanish on general zero-momentum functionals.

8 Star Algebra Projectors

In star multiplying two surface states to give a third we glue the surfaces to form a new surface. For a projector the gluing of the two identical surfaces must give the same surface we start with. This is only possible because the surfaces need only be conformally equivalent. Projectors represent somewhat degenerate surfaces, and this type of degeneration is crucial in making the star product give a surface conformally equivalent to the original one.

The mathematical property guaranteeing this result is explained below.

Consider a set of surfaces parametrized by \( t \in [0,1] \). For each \( t \) different from one, the surface \( R(t) \) is a finite region of the complex plane with the
topology of a disk. As \( t \) goes to one the region varies smoothly throughout but develops a thin neck and at \( t = 1 \) it pinches, breaking into two pieces \( R_1 \) and \( R_2 \), both of which are finite disks. Let \( P \) denote the pinching point, common to \( R_1 \) and \( R_2 \), and assume there are no other pinching points. Because disks can always be mapped to disks, any \( R(t) \) with \( t < 1 \) can be mapped to \( R_1 \). The map in fact is not unique due to \( \text{SL}(2, \mathbb{R}) \) invariance of the disk. We now claim that:

\textbf{Claim:} There exists a family of conformal maps \( m(t) : R(t) \to R_1 \), for \( t \in [0, 1] \), continuous in \( t \), where \( m(1) \) is the identity map over \( R_1 \) and maps all of \( R_2 \) to \( P \).

The intuition here is that as far as one of the sides of the pinching surface is concerned, call it side one, all that is going on on the other side, side two, can be viewed as happening near the pinching point. The complete side two, lying on the other side of the neck, can be mapped to a vanishingly small region while the conformal map is accurately close to the identity on side one.

### 8.1 A universal eigenvector of \( V^f \) for all projectors

Give a projector, we can represent it as the exponentials of matter (and ghost) oscillators acting on the vacuum. One can show that the matrix \( V^{f}_{mn} \) associated with any projector has the property that it has an eigenvector of eigenvalue one, the eigenvector being the same as the \( \kappa = 0 \) eigenvector of \( K_{1} \)[20]. This generalizes the same property obeyed by the sliver Neumann matrix.

An intuitive explanation for this property can be given using the half-string interpretation. The eigenvector in question implies that the projector wave-functional is invariant under constant and opposite translation of the half-strings [32]. If we denote by \( P_L \) and \( P_R \) respectively the momentum carried by the left and right half-strings we have that the eigenvector condition is interpreted as the condition that

\[ (P_L - P_R)|\Sigma\rangle = 0, \quad (236) \]

where \( |\Sigma\rangle \) is the projector surface state. Being a surface state defined with Neumann boundary condition the total momentum \( P_L + P_R \) carried by the state vanishes. Thus the condition above is simply the statement that \( P_L \) and \( P_R \) annihilate \( |\Sigma\rangle \). But this must be so, since \( |\Sigma\rangle = |\Sigma_L\rangle \otimes |\Sigma_R\rangle \), where
$|\Sigma_L\rangle$ and $|\Sigma_R\rangle$ are themselves surface states defined with Neumann boundary condition.

### 8.2 The Butterfly State

The butterfly state, just as any surface state, is completely defined by a map from $\xi$ to the upper half $z$ plane ($z$-presentation). We thus write

$$z = \frac{\xi}{\sqrt{1 + \xi^2}} \equiv f_B(\xi), \quad (237)$$

and define the butterfly state $|B\rangle$ through the relation:

$$\langle B|\phi\rangle = \langle f_B \circ \phi(0)\rangle_{U HP}. \quad (238)$$

In the $z$-presentation the surface is the full upper half plane, and therefore in order to gain intuition about the type of state this is, we can plot the image of the canonical $\xi$ half-disk in the $z$-plane. The open string $|\xi| = 1, \Im(\xi) \geq 0$ is seen to map to the hyperbola $x^2 - y^2 = \frac{1}{2}$ (in the upper half plane, with $z = x + iy$). We note that $z(\xi = i) = \infty$ and thus, as expected for a projector, the open string midpoint coincides with the boundary of the disk.

Further insight into the nature of the state is obtained by examination of the disk in the $\tilde{z}$-presentation. To this end we use (94) to recognize that (237) can be rewritten as

$$z = \sin(\tan^{-1}(\xi)) = \sin \tilde{z} \quad (239)$$

This maps the image of the local coordinate in the $\tilde{z}$-presentation to the image of the local coordinate in the $z$-presentation. As explained before, the surface need not fill the upper-half $\tilde{z}$-plane. To figure out the extension of the surface in the $\tilde{z}$ presentation we simply invert the previous equation to write

$$\tilde{z} = \sin^{-1} z. \quad (240)$$

This transformation maps the full upper half $z$-plane into the region $|\Re(\tilde{z})| \leq \pi/2, \Im(\tilde{z}) \geq 0$. Note that the vertical lines $\Re(\tilde{z}) = \pm\pi/2$ are images of the boundary and not identification lines. Even though the surface occupies a portion of the $\tilde{z}$-plane the boundary reaches the point at infinity, and so does the midpoint (as expected). The above conformal map is perhaps most easily thought about in differential form, where it belongs to the class of Schwarz-Christoffel transformations. We have

$$d\tilde{z} = \frac{dz}{\sqrt{(1 - z)(1 + z)}} \quad (241)$$
The real line in the $z$-plane is mapped into a polygon in the $\hat{z}$ presentation, where the turning points are $z = \pm 1$ and the turning angles are both $\pi/2$.

Finally, we give the $\hat{w}$ presentation. Using (96) the region $|\Re(\hat{z})| \leq \pi/2, \Im(\hat{z}) \geq 0$ of the $\hat{z}$ presentation turns into the full disk with a pair of cuts zooming into the $\hat{w}$ origin from $\hat{w} = -1$. Indeed the boundary of the surface is the arc $e^{i\theta}$ with $0 < \theta < \pi$ together with the line going from $\hat{w} = -1$ to $\hat{w} = 0$, plus the backwards line from $\hat{w} = 0$ to $\hat{w} = -1$ plus the arc $e^{i\theta}$ with $-\pi < \theta < 0$. It is perhaps in this presentation that it is clearest that the string midpoint $\hat{w} = 0$ touches the boundary of the disk.

9 Open Problems

We end this informal review with a list of problems that would be important to solve in the near future:

- Finding the analytic solution representing the tachyon vacuum in Witten’s OSFT.
- Finding the analytic solution representing a marginal deformation in Witten’s OSFT.
- Finding the physical interpretation of spectroscopy results with zero modes, and with magnetic field.

- Solving the ghost number zero equation $(L_0 - 1)\Phi + \Phi \ast \Phi = 0$.

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References


