

CFT & FRW

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1 Introduction

The holographic principle is based on the idea that for a given volume V the state of maximal entropy is given by the largest black hole that fits inside V . 't Hooft and Susskind [1] argued on this basis that the microscopic entropy S associated with the volume V should be less than the Bekenstein-Hawking entropy

$$S \leq \frac{A}{4G} \quad (1)$$

of a black hole with horizon area A equal to the surface area of the boundary of V . Here the dependence on Newton's constant G is made explicit, but as usual \hbar and c are set to one.

To shed further light on the holographic principle and the entropy bounds derived from it, we consider a standard cosmology of a closed radiation dominated Friedmann-Robertson-Walker (FRW) universe with general space-time dimension

$$D = n + 1.$$

The metric takes the form

$$ds^2 = -dt^2 + R^2(t)d\Omega_n^2 \quad (2)$$

where $R(t)$ represents the radius of the universe at a given time t and $d\Omega_n^2$ is a short hand notation for the metric on the unit n -sphere S^n . Hence, the spatial section of a $(n+1)$ d closed FRW universe is an n -sphere with a finite volume

$$V = \text{Vol}(S^n)R^n.$$

The holographic bound is in its naive form (1) not really applicable to a closed universe, since space has no boundary. Furthermore, the argumentation leading to (1) assumes that it's possible to form a black hole that fills the entire volume. This is not true in a cosmological setting, because the expansion rate H of the universe as well as the given value of the total energy E restrict the maximal size of black hole. As will be discussed in these notes, this will lead to a modified version of the holographic bound.

The radiation in the universe is described by a conformal field theory (CFT) with a very large central charge c . In a finite volume the energy E has a Casimir contribution proportional to c . Due to this Casimir effect, the entropy S is no longer a purely extensive function of E and V . The entropy of a $(1+1)$ d CFT is given by the well-known Cardy formula [2]

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}, \quad (3)$$

where L_0 represents the product ER of the energy and radius, and the shift of $\frac{c}{24}$ is caused by the Casimir effect. In these notes we show that, after making the appropriate identifications for L_0 and c , the same Cardy formula is also valid for CFTs in other dimensions. This is rather surprising, since the standard derivation of the Cardy formula based on modular invariance only appears to work for $n = 1$. By defining the central charge c in terms of the Casimir energy, we are able to argue that the Cardy formula is universally valid. Specifically, we will show that with the appropriate identifications, the entropy S for a $n+1$ dimensional CFT with an AdS-dual is exactly given by (3).

There appears to be a deep and fundamental connection between the holographic principle, the entropy formulas for the CFT, and the FRW equations for a radiation dominated universe. In $n+1$ dimensions the FRW equations are given by

$$H^2 = \frac{16\pi G}{n(n-1)} \frac{E}{V} - \frac{1}{R^2} \quad (4)$$

$$\dot{H} = -\frac{8\pi G}{n-1} \left(\frac{E}{V} + p \right) + \frac{1}{R^2} \quad (5)$$

where $H = \dot{R}/R$ is the Hubble parameter, and the dot denotes as usual differentiation with respect to the time t . The FRW equations are usually written in terms of the energy density $\rho = E/V$, but for the present study it is more convenient to work with the total energy E and entropy S instead of their respective densities ρ and $s = S/V$. Note that the cosmological constant has been put to zero.

Entropy and energy momentum conservation together with the equation of state $p = E/nV$ imply that E/V and p decrease in the usual way like $R^{-(n+1)}$. Hence, the cosmological evolution follows the standard scenario for a closed radiation dominated FRW universe. After the initial Big Bang, the universe expands until it reaches a maximum radius, the universe subsequently re-collapses and ends with a Big Crunch. No surprises happen in this respect.

The fun starts when one compares the holographic entropy bound with the entropy formulas for the CFT. We will show that when the bound is saturated the FRW equations and entropy formulas of the CFT merge together into one set of equation. One easily checks on the back of an envelope that via the substitutions

$$\begin{aligned}
2\pi L_0 &\Rightarrow \frac{2\pi}{n}ER \\
2\pi \frac{c}{12} &\Rightarrow (n-1)\frac{V}{4GR} \\
S &\Rightarrow (n-1)\frac{HV}{4G}
\end{aligned}
\tag{6}$$

the Cardy formula (3) *exactly* turns into the $n + 1$ dimensional Friedmann equation (4). This observation appears as a natural consequence of the holographic principle. In sections 2 and 3 we introduce three cosmological bounds each corresponding to one of the equations in (6) The Cardy formula is presented and derived in section 4. In section 5 we introduce a new cosmological bound, and show that the FRW equations and the entropy formulas are exactly matched when the bound is saturated. In section 6 we present a graphical picture of the entropy bounds and their time evolution.

2 Cosmological entropy bounds

This section is devoted to the description of three cosmological entropy bounds: the Bekenstein bound, the holographic Bekenstein-Hawking bound, and the Hubble bound. The relation with the holographic bound proposed by Fischler-Susskind and Bousso (FSB) will also be clarified.

2.1 The Bekenstein bound

Bekenstein [4] was the first to propose a bound on the entropy of a macroscopic system. He argued that for a system with limited self-gravity, the total entropy S is less or equal than a multiple of the product of the energy and the linear size of the system. In the present context, namely that of a closed radiation dominated FRW universe with radius R , the appropriately normalized Bekenstein bound is

$$S \leq S_B \tag{7}$$

where the Bekenstein entropy S_B is defined by

$$S_B \equiv \frac{2\pi}{n}ER. \tag{8}$$

The bound is most powerful for relatively low energy density or small volumes. This is due to the fact that S_B is super-extensive: under $V \rightarrow \lambda V$ and $E \rightarrow \lambda E$ it scales like $S_B \rightarrow \lambda^{1+1/n} S_B$.

For a radiation dominated universe the Bekenstein entropy is constant throughout the entire evolution, since $E \sim R^{-1}$. Therefore, once the Bekenstein bound is satisfied at one instance, it will remain satisfied at all times as long as the entropy S does not change. The Bekenstein entropy is the most natural generalization of the Virasoro operator $2\pi L_0$ to arbitrary dimensions, as is apparent from (6). Indeed, it is useful to think about S_B not really as an entropy but rather as the energy measured with respect to an appropriately chosen conformal time coordinate.

2.2 The Bekenstein-Hawking bound

The Bekenstein-bound is supposed to hold for systems with limited self-gravity, which means that the gravitational self-energy of the system is small compared to the total energy E . In the current situation this implies, concretely, that the Hubble radius H^{-1} is larger than the radius R of the universe. So the Bekenstein bound is only appropriate in the parameter range $HR \leq 1$. In a strongly self-gravitating universe, that is for $HR \geq 1$, the possibility of black hole formation has to be taken into account, and the entropy bound must be modified accordingly. Here the general philosophy of the holographic principle becomes important.

It follows directly from the Friedmann equation (4) that

$$HR \lesssim 1 \quad \Leftrightarrow \quad S_B \lesssim (n-1) \frac{V}{4GR} \quad (9)$$

Therefore, to decide whether a system is strongly or weakly gravitating one should compare the Bekenstein entropy S_B with the quantity

$$S_{BH} \equiv (n-1) \frac{V}{4GR}. \quad (10)$$

When $S_B \leq S_{BH}$ the system is weakly gravitating, while for $S_B \geq S_{BH}$ the self-gravity is strong. We will identify S_{BH} with the holographic Bekenstein-Hawking entropy of a black hole with the size of the universe. S_{BH} indeed grows like an area instead of the volume, and for a closed universe it is the closest one can come to the usual expression $A/4G$.

As will become clear in these notes, the role of S_{BH} is not to serve as a bound on the total entropy, but rather on a sub-extensive component of

the entropy that is association with the Casimir energy of the CFT. The relation (6) suggests that the Bekenstein-Hawking entropy is closely related to the central charge c . Indeed, it is well-known from $(1+1)d$ CFT that the central charge characterizes the number of degrees of freedom may be even better than the entropy. This fact will be further explained in sections 5 and 6, when we describe a new cosmological bound on the Casimir energy and its associated entropy.

2.3 The Hubble entropy bound

The Bekenstein entropy S_B is equal to the holographic Bekenstein-Hawking entropy S_{BH} precisely when $HR = 1$. For $HR > 1$ one has $S_B > S_{BH}$ and the Bekenstein bound has to be replaced by a holographic bound. A naive application of the holographic principle would imply that the total entropy S should be bounded by S_{BH} . This turns out to be incorrect, however, since a purely holographic bound assumes the existence of arbitrarily large black holes, and is irreconcilable with a finite homogeneous entropy density.

Following earlier work by Fischler and Susskind [5], it was argued by Easter and Lowe [6], Veneziano [7], Bak and Rey [8], Kaloper and Linde [9], that the maximal entropy inside the universe is produced by black holes of the size of the Hubble horizon, see also [10]. Following the usual holographic arguments one then finds that the total entropy should be less or equal than the Bekenstein-Hawking entropy of a Hubble size black hole times the number N_H of Hubble regions in the universe. The entropy of a Hubble size black hole is roughly $HV_H/4G$, where V_H is the volume of a single Hubble region. Combined with the fact that $N_H = V/V_H$ one obtains an upper bound on the total entropy S given by a multiple of $HV/4G$. The presented arguments of [6, 8, 9, 7] are not sufficient to determine the precise pre-factor, but in the following subsection we will fix the normalization of the bound by using a local version of the Fischler-Susskind-Bousso formulation of the holographic principle. The appropriately normalized entropy bound takes the form

$$S \leq S_H \quad \text{for} \quad HR \geq 1 \quad (11)$$

with

$$S_H \equiv (n-1) \frac{HV}{4G}. \quad (12)$$

The Hubble bound is only valid for $HR \geq 1$. In fact, it is easily seen that for $HR \leq 1$ the bound will at some point be violated. For example, when the universe reaches its maximum radius and starts to re-collapse the Hubble

constant H vanishes, while the entropy is still non-zero.¹ This should not really come as a surprise, since the Hubble bound was based on the idea that the maximum size of a black hole is equal to the Hubble radius. Clearly, when the radius R of the universe is smaller than the Hubble radius H^{-1} one should reconsider the validity of the bound. In this situation, the self-gravity of the universe is less important, and the appropriate entropy bound is

$$S \leq S_B \quad \text{for} \quad HR \leq 1 \quad (13)$$

2.4 The Hubble bound and the FSB prescription

Fischler, Susskind, and subsequently Bousso [12], have proposed an ingenious version of the holographic bound that restricts the entropy flow through contracting light sheets. The FSB-bound works well in many situations, but, so far, no microscopic derivation has been given. Wald and collaborators [13] have shown that the FSB bound follows from local inequalities on the entropy density and the stress energy. The analysis of [13] suggests the existence a local version of the FSB entropy bound, one that does not involve global information about the causal structure of the universe, see also [11]. The idea of to formulate the holographic principle via entropy flow through light sheets also occurred in the work of Jacobson [14], who used it to derive an intriguing relation between the Einstein equations and the first law of thermodynamics. In this subsection, a local FSB bound will be presented that leads to a precisely normalized upper limit on the entropy in terms of the Hubble constant.

According to the original FSB proposal, the entropy flow S through a contracting light sheet is less or equal to $A/4G$, where A is the area of the surface from which the light sheet originates. The following infinitesimal version of this FSB prescription will lead to the Hubble bound. For every $n-1$ dimensional surface at time $t + dt$ with area $A + dA$ one demands that

$$dS \leq \frac{dA}{4G}, \quad (14)$$

where dS denotes the entropy flow through the infinitesimal light sheets originating at the surface at $t + dt$ and extending back to time t , and dA represents the increase in area between t and $t + dt$. For a surface that is kept fixed in co-moving coordinates the area A changes as a result of the

¹To avoid this problem a different covariant version of the Hubble bound was proposed in [11].

Hubble expansion by an amount

$$dA = (n-1)H A dt, \quad (15)$$

where the factor $n-1$ simply follows from the fact that $A \sim R^{n-1}$. Now pick one of the two past light-sheets that originate at the surface: the inward or the outward going. The entropy flow through this light-sheet between t and $t+dt$ is given by the entropy density $s = S/V$ times the infinitesimal volume $A dt$ swept out by the light-sheet. Hence,

$$dS = \frac{S}{V} A dt. \quad (16)$$

By inserting this result together with (15) into the infinitesimal FSB bound (14) one finds that the factor $A dt$ cancels on both sides and one is left exactly with the Hubble bound $S \leq S_H$ with the Hubble entropy S_H given in (12). We stress that the relation with the FSB bound was merely used to fix the normalization of the Hubble bound, and should not be seen as a derivation.

3 Time-evolution of the entropy bounds

Let us now return to the three cosmological entropy bounds discussed in section 2. The Friedmann equation (4) can be re-written as an identity that relates the Bekenstein-, the Hubble-, and the Bekenstein-Hawking entropy. One easily verifies that the expressions given in (8), (10), and (12) satisfy the quadratic relation

$$S_H^2 + (S_B - S_{BH})^2 = S_B^2. \quad (17)$$

It is deliberately written in a Pythagorean form, since it suggests a useful graphical picture of the three entropy bounds. By representing each entropy by a line with length equal to its value one finds that due to the quadratic Friedmann relation (17) all three fit nicely together in one diagram, see figure 1. The circular form of the diagram reflects the fact that S_B is constant during the cosmological evolution. Only S_H and S_{BH} depend on time.

Let us introduce a conformal time coordinate via

$$R d\eta = (n-1) dt \quad (18)$$

and let us compute the η -dependence of S_{BH} and S_H . For S_{BH} this easily follows from: $\dot{S}_{BH} = (n-1)H S_{BH} = (n-1)R^{-1}S_H$. For S_H the calculation

is a bit more tedious, but with the help of the FRW equations, the result can eventually be put in the form

$$\begin{aligned}\frac{dS_H}{d\eta} &= S_B - S_{BH}, \\ \frac{dS_{BH}}{d\eta} &= -S_H.\end{aligned}\tag{19}$$

These equations show that the conformal time coordinate η can be identified with the angle η , as already indicated in figure 1. As time evolves the Hubble entropy S_H rotates into the combination $S_B - S_{BH}$ and visa versa. Equation (19) can be integrated to

$$\begin{aligned}S_H &= S_B \sin \eta \\ S_{BH} &= S_B(1 - \cos \eta)\end{aligned}\tag{20}$$

The conformal time coordinate η plays the role of the time on a cosmological clock that only goes around once: at $\eta = 0$ time starts with a Big Bang and at $\eta = 2\pi$ it ends with a Big Crunch. Note that η is related to the parameter HR via

$$HR = \cot \frac{\eta}{2}\tag{21}$$

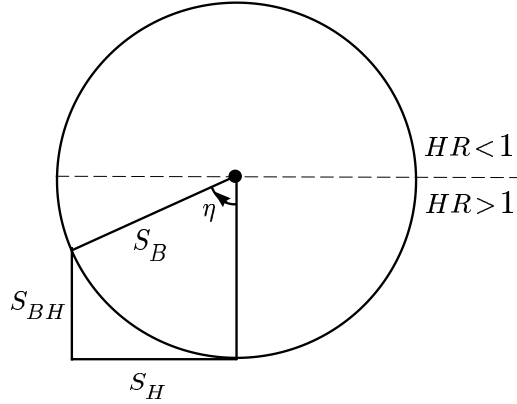


Figure 1: A graphical representation of the Bekenstein entropy S_B , the Hubble entropy S_H and the Bekenstein-Hawking entropy S_{BH} . The angle η corresponds to the conformal time coordinate. The value of each entropy is represented by an actual distance: S_B is constant, while S_H and S_{BH} change with time.

So far we have not yet included the CFT into our discussion. We will see that the entropy of the CFT will ‘fill’ part of the diagram, and in this way give rise to a special moment in time when the entropy bounds are saturated.

4 Casimir energy and the Cardy formula

We now turn to the discussion of the entropy of the CFT that lives inside the FRW universe. We begin with a study of the finite temperature Casimir energy with the aim to exhibit its relation with the entropy of the CFT. Subsequently a universal Cardy formula will be derived that expresses the entropy in terms of the energy and the Casimir energy, and is valid for all values of the spatial dimension n .

4.1 The Euler relation and Casimir energy

In standard textbooks on cosmology [15, 16] it is usually assumed that the total entropy S and energy E are extensive quantities. This fact is used for example to relate the entropy density s to the energy density ρ and pressure p , via $Ts = \rho + p$. For a thermodynamic system in finite volume V the energy $E(S, V)$, regarded as a function of entropy and volume, is called extensive when it satisfies $E(\lambda S, \lambda V) = \lambda E(S, V)$. Differentiating with respect to λ and putting $\lambda = 1$ leads to the Euler relation²

$$E = V \left(\frac{\partial E}{\partial V} \right)_S + S \left(\frac{\partial E}{\partial S} \right)_V \quad (22)$$

The first law of thermodynamics $dE = TdS - pdV$ can now be used to re-express the derivatives via the thermodynamic relations

$$\left(\frac{\partial E}{\partial V} \right)_S = -p, \quad \left(\frac{\partial E}{\partial S} \right)_V = T. \quad (23)$$

The resulting equation $TS = E + pV$ is equivalent to the previously mentioned relation for the entropy density s .

For a CFT with a large central charge the entropy and energy are not purely extensive. In a finite volume the energy E of a CFT contains a non-extensive Casimir contribution proportional to c . This is well known in (1+1) dimensions where it gives rise to the familiar shift of $c/24$ in the L_0 Virasoro operator. The Casimir energy is the result of finite size effects in the quantum fluctuations of the CFT, and disappears when the volume becomes infinitely large. It therefore leads to sub-extensive contributions to the total energy E . Usually the Casimir effect is discussed at zero temperature [17],

²We assume here that there are no other thermodynamic functions like a chemical or electric potential. For a system with a 1st law like $TdS = dE + pdV + \mu dN + \Phi dQ$ the Euler relation reads $TS = E + pV + \mu N + \Phi Q$.

but a similar effect occurs at finite temperature. The value of the Casimir energy will in that case generically depend on the temperature T .

We will now define the Casimir energy as the violation of the Euler identity (22)

$$E_C \equiv n(E + pV - TS) \quad (24)$$

Here we inserted for convenience a factor equal to the spatial dimension n . From the previous discussion it is clear that E_C parameterizes the sub-extensive part of the total energy. The Casimir energy will just as the total energy be a function of the entropy S and the volume V . Under $S \rightarrow \lambda S$ and $V \rightarrow \lambda V$ it scales with a power of λ that is smaller than one. On general grounds one expects that the first subleading correction to the extensive part of the energy scales like

$$E_C(\lambda S, \lambda V) = \lambda^{1-2/n} E_C(S, V) \quad (25)$$

One possible way to see this is to write the energy as an integral over a local density expressed in the metric and its derivatives. Derivatives scale like $\lambda^{-1/n}$ and because derivatives come generally in pairs, the first subleading terms indeed has two additional factors of $\lambda^{-1/n}$. The total energy $E(S, V)$ may be written as a sum of two terms

$$E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V) \quad (26)$$

where the first term E_E denotes the purely extensive part of the energy E and E_C represents the Casimir energy. Again the factor $1/2$ has been put in for later convenience. By repeating the steps that lead to the Euler relation one easily verifies the defining equation (24) for the Casimir energy E_C .

4.2 Universality of the Cardy formula and the Bekenstein bound

Conformal invariance implies that the product ER is independent of the volume V , and is only a function of the entropy S . This holds for both terms E_E and E_C in (26). Combined with the known (sub-)extensive behavior of E_E and E_C this leads to the following general expressions

$$E_E = \frac{a}{4\pi R} S^{1+1/n} \quad E_C = \frac{b}{2\pi R} S^{1-1/n}$$

where a and b are a priori arbitrary positive coefficients, independent of R and S . The factors of 4π and 2π are put in for convenience. With these expressions, one now easily checks that the entropy S can be written as

$$S = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_C(2E - E_C)}. \quad (27)$$

If we ignore for a moment the normalization, this is exactly the Cardy formula: insert $ER = L_0$ and $E_C R = c/12$, and one recovers (3). It is obviously an interesting question to compute the coefficients a and b for various known conformal invariant field theories. This should be particularly straightforward for free field theories, such as $d = 4$ Maxwell theory and the self-dual tensor theory in $d = 6$. This question is left for future study.

Given the energy E the expression (27) has a maximum value. For all values of E , E_C and R one has the inequality

$$S \leq \frac{2\pi}{\sqrt{ab}} ER$$

This looks exactly like the Bekenstein bound, except that the pre-factor is in general different from the factor $2\pi/n$ used in the previous section. In fact, in the following subsection we will show that for CFTs with an AdS-dual description, the value of the product ab is exactly equal to n^2 , so the upper limit is indeed exactly given by the Bekenstein entropy. Although we have no proof of this fact, we believe that the Bekenstein bound is universal. This implies that the product ab for all CFTs in $n+1$ dimensions is larger or equal than n^2 . Only then it is guaranteed that the upper limit on the entropy is less or equal than S_B .

The upper limit is reached when the Casimir energy E_C is equal to the total energy E . Formally, when E_C becomes larger than E the entropy S will again decrease. Although in principle this is possible, we believe that in actual examples the Casimir energy E_C is bounded by the total energy E . So, from now on we assume that

$$E_C \leq E \quad (28)$$

In the next subsection we provide further evidence for this inequality.

From now on we will assume that we are dealing with a CFT for which $ab = n^2$. In the next section I will show that this includes all CFTs that have an AdS-dual description.

4.3 The Cardy formula derived from AdS/CFT

Soon after Maldacena's AdS/CFT-correspondence [18] was properly understood [19, 20] it was convincingly argued by Witten [21] that the entropy, energy and temperature of CFT at high temperatures can be identified with the entropy, mass, and Hawking temperature of the AdS black hole previously considered by Hawking and Page [22]. Using this duality relation the following expressions can be derived for the energy and entropy³ for a $D = n + 1$ dimensional CFT on $R \times S^n$:

$$\begin{aligned} S &= \frac{c}{12} \frac{V}{L^n} \\ E &= \frac{c}{12} \frac{n}{4\pi L} \left(1 + \frac{L^2}{R^2}\right) \frac{V}{L^n} \end{aligned} \quad (29)$$

The temperature again follows from the first law of thermodynamics. One finds

$$T = \frac{1}{4\pi L} \left((n+1) + (n-1) \frac{L^2}{R^2} \right). \quad (30)$$

The length scale L of the thermal CFT arises in the AdS/CFT correspondence as the curvature radius of the AdS black hole geometry. The expression for the energy clearly exhibits a non-extensive contribution, while also the temperature T contains a corresponding non-intensive term. Inserting the equations (29,30) into (24) yields the following result for the Casimir energy

$$E_C = \frac{c}{12} \frac{n}{2\pi R} \frac{V}{L^{n-1}R}. \quad (31)$$

Now let us come to the Cardy formula. The entropy S , energy E and Casimir energy E_C are expressed in c , L and R . Eliminating c and L leads to a unique expression for S in terms of E , E_C and R . One easily checks that it takes the form of the Cardy formula

$$S = \frac{2\pi R}{n} \sqrt{E_C (2E - E_C)} \quad (32)$$

³These expressions differ somewhat from the presented formulas in [21] due to the fact that (i) the $D + 1$ dimensional Newton constant has been eliminated using its relation with the central charge, (ii) the coordinates have been re-scaled so that the CFT lives on a sphere with radius equal to the black hole horizon. We will not discuss the AdS perspective in these notes, since the essential physics can be understood without introducing an extra dimension. The discussion of the CFT/FRW cosmology from an AdS perspective will be described elsewhere [3].

In the derivation of these formulas it was assumed that $R \gg L$. One may worry therefore that these formulas are not applicable in the early universe. Fortunately this is not a problem because during an adiabatic expansion both L and R scale in the same way so that R/L is fixed. Hence the formulas are valid provided the (fixed) ratio of the thermal wave-length and the radius R is much smaller than one. Effectively this means, as far as the CFT is concerned, we are in a high temperature regime. We note further that with in this parameter range, the Casimir energy E_C is indeed smaller than the total energy E .

Henceforth, we will assume that the CFT that describes the radiation in the FRW universe will have an entropy given by (32) with the specific normalization of $2\pi/n$. Note that if we take $n = 1$ and make the previously mentioned identifications $ER = L_0$ and $E_C R = c/12$ that this equation exactly coincides with the usual Cardy formula. We will therefore in the following refer to (32) simply as the Cardy formula. To check the precise coefficient of the Cardy formula for a CFT we have made use of the AdS/CFT correspondence. The rest of our discussions in the preceding and in the following sections do not depend on this correspondence.

5 A new cosmological bound

In this section a new cosmological bound will be presented, which is equivalent to the Hubble bound in the strongly gravitating phase, but which unlike the Hubble bound remains valid in the phase of weak self-gravity. When the bound is saturated the FRW equations and the CFT formulas for the entropy and Casimir energy completely coincide.

5.1 A cosmological bound on the Casimir energy

Let us begin by presenting another criterion for distinguishing between a weakly or strongly self-gravitating universe. When the universe goes from the strongly to the weakly self-gravitating phase, or vice-versa, the Bekenstein entropy S_B and the Bekenstein-Hawking entropy S_{BH} are equal in value. Given the radius R , we now define the ‘Bekenstein-Hawking’ energy E_{BH} as the value of the energy E for which S_B and S_{BH} are exactly equal. This leads to the condition

$$\frac{2\pi}{n} E_{BH} R \equiv (n-1) \frac{V}{4GR}. \quad (33)$$

One may interpret E_{BH} as the energy required to form a black hole with the size of the entire universe. Now, one easily verifies that

$$\begin{aligned} E &\leq E_{BH} & \text{for} & \quad HR \leq 1 \\ E &\geq E_{BH} & \text{for} & \quad HR \geq 1. \end{aligned} \tag{34}$$

Hence, the universe is weakly self-gravitating when the total energy E is less than E_{BH} and strongly gravitating for $E > E_{BH}$.

We are now ready to present a proposal for a new cosmological bound. It is not formulated as a bound on the entropy S , but as a restriction on the Casimir energy E_C . The physical content of the bound is the Casimir energy E_C by itself can not be sufficient to form a universe-size black hole. Concretely, this implies that the Casimir energy E_C is less or equal to the Bekenstein-Hawking energy E_{BH} . Hence, we postulate

$$E_C \leq E_{BH} \tag{35}$$

To put the bound in a more conventional notation one may insert the definition (24) of the Casimir energy together with the defining relation (33) of the Bekenstein-Hawking energy. We leave this to the reader.

The virtues of the new cosmological bound are: (i) it is universally valid and does not break down for a weakly gravitating universe, (ii) in a strongly gravitating universe it is equivalent to the Hubble bound, (iii) it is purely holographic and can be formulated in terms of the Bekenstein-Hawking entropy S_{BH} of a universe-size black hole, (iv) when the bound is saturated the laws of general relativity and quantum field theory converge in a miraculous way, giving a strong indication that they have a common origin in a more fundamental unified theory.

The first point on the list is easily checked because E_C decays like R^{-1} while E_{BH} goes like R^{-n} . Only when the universe re-collapses and returns to the strongly gravitating phase the bound may again become saturated. To be able to proof the other points on the list of advertised virtues, we have to take a closer look to the FRW equations and the CFT formulas for the entropy an entropy.

5.2 A cosmological Cardy formula

To show the equivalence of the new bound with the Hubble bound let us write the Friedmann equation as an expression for the Hubble entropy S_H in terms of the energy E , the radius R and the Bekenstein-Hawking energy

E_{BH} . Here, the latter is used to remove the explicit dependence on Newton's constant G . The resulting expression is unique and takes the form

$$S_H = \frac{2\pi}{n} R \sqrt{E_{BH} (2E - E_{BH})} \quad (36)$$

This is exactly the Cardy formula (32), except that the role of the Casimir energy E_C in CFT formula is now replaced by the Bekenstein-Hawking energy E_{BH} . Somehow, miraculously, the Friedmann equation knows about the Cardy formula for the entropy of a CFT!

With the help of (36) is now a straightforward matter to prove that when $HR \geq 1$ the new bound $E_C \leq E_{BH}$ is equivalent to the Hubble bound $S \leq S_H$. First, let us remind that for $HR \geq 1$ the energy E satisfies $E \geq E_{BH}$. Furthermore, we always assume that the Casimir energy E_C is smaller than the total energy E . The entropy S is a monotonically increasing function of E_C as long as $E_C \leq E$. Therefore in the range

$$E_C \leq E_{BH} \leq E \quad (37)$$

the maximum entropy is reached when $E_C = E_{BH}$. In that case the Cardy formula (32) for S exactly turns into the cosmological Cardy formula (36) for S_H . Therefore, we conclude that S_H is indeed the maximum entropy that can be reached when $HR \geq 1$. Note that in the weakly self-gravitating phase, when $E \leq E_{BH}$, the maximum is reached earlier, namely for $E_C = E$. The maximum entropy is in that case given by the bekenstein entropy S_B . The bifurcation of the new bound in two entropy bounds is a direct consequence of the fact that the Hubble bound is written as the square-root of a quadratic expression.

5.3 A limiting temperature

So far we have focussed on the entropy and energy of the CFT and on the first of the two FRW equations, usually referred to as the Friedmann equation. We will now show that also the second FRW equation has a counterpart in the CFT, and will lead to a constraint on the temperature T . Specifically, we will find that the bound on E_C implies that the temperature T in the early universe is bounded from below by

$$T_H \equiv -\frac{\dot{H}}{2\pi H} \quad (38)$$

The minus sign is necessary to get a positive result, since in a radiation dominated universe the expansion always slows down. Further, we assume

that we are in the strongly self-gravitating phase with $HR \geq 1$, so that there is no danger of dividing by zero.

The second FRW equation in (5) can now be written as a relation between E_{BH} , S_H and T_H that takes the familiar form

$$E_{BH} = n(E + pV - T_H S_H) \quad (39)$$

This equation has exactly the same form as the defining relation $E_C = n(E + pV - TS)$ for the Casimir energy. In the strongly gravitating phase we have just argued that the bound $E_C \leq E_{BH}$ is equivalent to the Hubble bound $S \leq S_H$. It follows immediately that the temperature T in this phase is bounded from below by T_H . One has

$$T \geq T_H \quad \text{for } HR \geq 1 \quad (40)$$

When the cosmological bound is saturated all inequalities turn into equalities. The Cardy formula and the defining Euler relation for the Casimir energy in that case exactly match the Friedmann equation for the Hubble constant and the FRW equation for its time derivative.

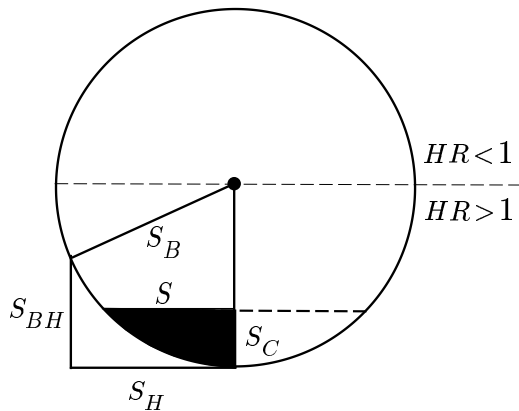


Figure 2: The entropy S and Casimir entropy S_C fill part of the cosmological entropy diagram. The diagram shows: (i) the Bekenstein bound $S \leq S_B$ is valid at all times (ii) the Hubble bound $S \leq S_H$ restricts the allowed range of η in the range $HR > 1$, but is violated for $HR < 1$, (iii) the new bound $S_C \leq S_{BH}$ is equivalent to the Hubble bound for $HR > 1$, and remains valid for $HR < 1$.

6 The entropy bounds revisited

We now return to the cosmological entropy bounds introduced in sections 2 and 3. In particular, we are interested in the way that the entropy of the CFT may be incorporated in the entropy diagram described in section 3. For this purpose it will be useful to introduce a non-extensive component of the entropy that is associated with the Casimir energy.

The cosmological bound $E_C \leq E_{BH}$ can also be formulated as an entropy bound, not on the total entropy, but on a non-extensive part of the entropy that is associated with the Casimir energy. In analogy with the definition of the Bekenstein entropy (8) one can introduce a 'Casimir' entropy defined by

$$S_C \equiv \frac{2\pi}{n} E_C R. \quad (41)$$

For $d = (1+1)$ the Casimir entropy is directly related to the central charge c . One has $S_C = 2\pi c/12$. In fact, it is more appropriate to interpret the Casimir entropy S_C as a generalization of the central charge to $n+1$ dimensions than what is usually called the central charge c . Indeed, if one introduces a dimensionless 'Virasoro operator' $\tilde{L}_0 \equiv \frac{1}{2\pi} S_B$ and a new central charge $\tilde{c} \equiv \frac{1}{2\pi} S_C$, the $n+1$ dimensional entropy formula (32) is exactly identical to (3).

The Casimir entropy S_C is sub-extensive because under $V \rightarrow \lambda V$ and $E \rightarrow \lambda E$ it goes like $S_C \rightarrow \lambda^{1-1/n} S_C$. In fact, it scales like an area! This is a clear indication that the Casimir entropy has something to do with holography. The total entropy S contains extensive as well as sub-extensive contributions. One can show that for $E_C \leq E$ the entropy S satisfies the following inequalities

$$S_C \leq S \leq S_B \quad (42)$$

where both equal signs can only hold simultaneously. The precise relation between S and its super- and sub-extensive counterparts S_B and S_C is determined by the Cardy formula, which can be expressed as

$$S^2 + (S_B - S_C)^2 = S_B^2. \quad (43)$$

This identity has exactly the same form as the relation (17) between the cosmological entropy bounds, except that in (17) the role of the entropy and Casimir entropy are taken over by the Hubble entropy S_H and Bekenstein-Hawking entropy S_{BH} . This fact will be used to incorporate the entropy S and the Casimir entropy S_C in the entropy diagram introduced in section 3.

The cosmological bound on the Casimir energy presented in the section 4 can be formulated as an upper limit on the Casimir entropy S_C . From the definitions of S_C and E_{BH} it follows directly that the bound $E_C \leq E_{BH}$ is equivalent to

$$S_C \leq S_{BH} \quad (44)$$

where we made use of the relation (33) to re-write E_{BH} again in terms of the Bekenstein-Hawking entropy S_{BH} . Thus the bound puts a holographic upper limit on the d.o.f. of the CFT as measured by the Casimir entropy S_C .

In figure 2 we have graphically depicted the quadratic relation between the total entropy S and the Casimir entropy S_C in the same diagram we used to related the cosmological entropy bounds. From this diagram it easy to determine the relation between the new bound and the Hubble bound. One clearly sees that when $HR > 1$ that the two bounds are in fact equivalent. When the new bound is saturated, which means $S_C = S_{BH}$, then the Hubble bound is also saturated, *ie.* $S = S_H$. The converse is not true: there are two moments in the region $HR < 1$ when the $S = S_H$, but $S_C \neq S_{BH}$. In our opinion, this is an indication that the bound on the Casimir energy has a good chance of being a truly fundamental bound.

7 Summary and conclusion

We have used the holographic principle to study the bounds on the entropy in a radiation dominated universe. The radiation has been described by a continuum CFT in the bulk. Surprisingly the CFT appears to know about the holographic entropy bounds, and equally surprising the FRW-equations know about the entropy formulas for the CFT. Our main results are summarized in the following two tables. Table 1. contains an overview of the bounds that hold in the early universe on the temperature, entropy and Casimir energy. In table 2. the Cardy formula for the CFT and the Euler relation for the Casimir energy are matched with the Friedmann equations written in terms of the quantities listed in table 1.

| <i>CFT-bound</i> | <i>FRW-definition</i> |
|-------------------|-----------------------------------|
| $T \geq T_H$ | $T_H \equiv -\dot{H}/2\pi H$ |
| $S \leq S_H$ | $S_H \equiv (n-1)HV/4G$ |
| $E_C \leq E_{BH}$ | $E_{BH} \equiv n(n-1)V/8\pi GR^2$ |

Table 1: Summary of cosmological bounds

| <i>CFT-formula</i> | <i>FRW-equation</i> |
|---------------------------------------------|-----------------------------------------------------|
| $S = \frac{2\pi R}{n} \sqrt{E_C(2E - E_C)}$ | $S_H = \frac{2\pi R}{n} \sqrt{E_{BH}(2E - E_{BH})}$ |
| $E_C \equiv n(E + pV - TS)$ | $E_{BH} = n(E + pV - T_H S_H)$ |

Table 2: Matching of the CFT-formulas with the FRW-equations

The presented relation between the FRW equations and the entropy formulas precisely holds at this transition point, when the holographic bound is saturated or threatens to be violated. The miraculous merging of the CFT and FRW equations strongly indicates that both sets of these equations arise from a single underlying fundamental theory.

The discovered relation between the entropy, Casimir energy and temperature of the CFT and their cosmological counterparts has a very natural explanation from a RS-type brane-world scenario [23] along the lines of [24]. The radiation dominated FRW equations can be obtained by studying a brane with fixed tension in the background of a AdS-black hole. In this description the radius of the universe is identified with the distance of the brane to the center of the black hole. At the Big Bang the brane originates from the past singularity. At some finite radius determined by the energy of the black hole, the brane crosses the horizon. It keeps moving away from the black hole, until it reaches a maximum distance, and then it falls back into

the AdS-black hole. The special moment when the brane crosses the horizon precisely corresponds to the moment when the cosmological entropy bounds are saturated. This world-brane perspective on the cosmological bounds for a radiation dominated universe are described in detail in [3].

We have restricted our attention to matter described by a CFT in order to make our discussion as concrete and coherent as possible. Many of the used concepts, however, such as the entropy bounds, the notion of a non-extensive entropy, the matching of the FRW equations, and possibly even the Cardy formula are quite independent of the equation of state of the matter. One point at which the conformal invariance was used is in the diagrammatic representation of the bounds. The diagram is only circular when the energy E goes like R^{-1} . But it is possible that a similar non-circular diagram exists for other kinds of matter. It would be interesting to study other examples in more detail.

Finally, the cosmological constant has been put to zero, since only in that case all of the formulas work so nicely. It is possible to modify the formalism to incorporate a cosmological constant, but the analysis becomes less transparent. In particular, one finds that the Hubble entropy bound needs to be modified by replacing H with the square root of $H^2 - \Lambda/n$. At this moment we have no complete understanding of the case $\Lambda \neq 0$, and postpone its discussion to future work.

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