Lectures on D-Branes on Calabi-Yau Manifolds

Michael R. Douglas[*]

Department of Physics and Astronomy, Rutgers University, Piscataway, USA
and
I.H.E.S., Le Bois-Marie, Bures-sur-Yvette, France

Lectures given at the
Spring School on Superstrings and Related Matters
Trieste, 2 – 10 April 2001

LNS027005

[*]mrd@physics.rutgers.edu

Abstract

This is a write-up of lectures given at the 2001 Spring School at the Abdus Salam ICTP.

Contents

1 Introduction

Dirichlet branes play many important roles in the modern discussion of superstring compactification and duality. They provide a very general way to embed gauge theories into string theory, which has led to remarkable physical conjectures such as M(atrix) theory and AdS/CFT. They have also led to remarkably detailed connections between physics and mathematics, such as a rederivation of the ADHM construction of instantons.

In these lectures, we will give some introduction to the problem of finding BPS D-branes in weakly coupled type II string compactification on Calabi-Yau manifolds. This problem is prototypical for the case of $\mathcal{N}=1$ supersymmetry in four dimensions, and as such has received a lot of study, especially in the special case of non-compact Calabi-Yaus such as resolved orbifolds. Almost all of the ideas apply to compact Calabi-Yaus as well, and although these examples are not as well understood at present, no fundamental barrier has been found to progress in this direction, and this could lead to a much more complete understanding of $\mathcal{N}=1$ compactification than we now have. This is the general direction our lectures will head in (though we won't get very far).

Rather than start from geometry of Calabi-Yau, we start from general principles which apply to all D-brane problems, then specialize to the case of $\mathcal{N}=1$ supersymmetry on the world-volume, and finally to special features of the Calabi-Yau case.

As with most discussions of string compactification, unless one restricts attention to very special regions in moduli space where the underlying conformal field theory is exactly solvable, which we do not want to do, making a proper discussion involves a good deal of mathematics which is unfamiliar to most physicists. In these lectures, we will discuss a bit of the theory of holomorphic bundles – since the gauge fields on B-type BPS branes live in such bundles, this is obviously relevant – and that of coherent sheaves, a generalization which describes singular limits of bundles.

We will not start with this however, but rather with the mathematics which underlies quiver theories, namely homological algebra and category theory. This is even less familiar, but turns out to be very well motivated in these problems. Let us start out by giving the basic dictionary, to explain why this is.

A category is defined mathematically as a set of *objects*, and for each pair of objects A and B a set of *morphisms* from A to B. There is a multiplication law: given a morphism from A to B and one from B to C, the product is a morphism from A to C.

This data satisfies certain axioms, which are exactly the general properties we want for each open string boundary condition to be an object, and each morphism to be an open string. The multiplication law is just the operator product expansion between open string vertex operators.

There is much more structure in the physical problem than the open string spectrum of course. One needs to distinguish "matter" and "gauge" vertex operators; one needs to interpret the superpotential and D-flatness conditions; and so on. It turns out that essentially all of the structure of the low energy world-volume theory has known mathematical counterparts. We will explain some of this as we go along.

What saves us from having to do this completely abstractly is that all of this structure is already visible in quiver gauge theories. Indeed, the mathematical notion of "quiver" was defined as a particularly simple source of algebras and categories, long before any physical applications emerged. Physically, these theories contain the minimal structure required to discuss the following problem: one takes a basic finite set of "generating objects" and all morphisms between these, and tries to form all the BPS branes as bound states.

This project leads to many further questions. How many branes do we need, and how do we find such a set? Suppose we start with a brane defined using a different construction: can we find some canonical way to decompose this brane into these fundamental constituents? Are there natural symmetries of the spectrum which map between different generating sets? As we discuss, paths in Calabi-Yau moduli space are associated with monodromies which should produce such symmetries. Of course we also need to discuss marginal stability and the variation of the spectrum in this context. Finally, is this a useful way to consider the problem? Can we compute the spectrum, moduli spaces and so forth this way?

Although we will not have time to go into it deeply, there is a a powerful underlying concept which simplifies all of this further discussion, the derived category. In physical terms, this is a structure derived from the original category of boundary conditions, which describes arbitrary bound states of branes and antibranes, and keeps track of everything which does not depend on the precise identification of which are branes and which are antibranes.

The derived category enters at many points in the discussion. For example, it turns out that for string size Calabi-Yau, not all branes are bundles or coherent sheaves; some are more general bound states of branes and antibranes which correspond to "non-classical" objects in the derived category. As a second example, the relationship between quiver theories describing different sets of generating branes can be understood in terms of "Fourier-Mukai transformations," which can be simply formulated as acting on the derived category, and which reduce for concrete quiver theories to "Seiberg dualities" between these theories. Although we will not be able to get into details of this (many of which are presently under investigation), the concepts we discuss should serve as a good introduction to these directions.

2 Topological considerations

D-branes carry Ramond-Ramond charge and this is the most obvious topological classification we can make. If we consider a brane B wrapping an arbitrary cycle Σ in the internal space M, and which looks like a D-particle in $3+1$ dimensions, it will carry electric and magnetic charges under the $3+1$ $U(1)^r$ gauge group of the bulk theory. Although the considerations we discuss now are more general, let us assume M is a CY_3. Then, if we start with IIa theory, we will have $r = b_{1,1} + 1$ coming from odd rank potentials, while if we start with IIb it will have $r = b_{2,1} + 1$.

A simpler topological invariant which can be derived from these charges is the "intersection form," whose simplest physical definition is as the integer appearing in the Dirac-Schwinger-Zwanziger charge quantization condition

$$\langle B_1, B_2 \rangle = e_1 \cdot m_2 - e_2 \cdot m_1 \tag{1}$$

in an appropriate basis. In the case of 3-branes, this is entirely geometric and counts the signed intersection number of the two cycles Σ_1 and Σ_2. Poincaré duality then tells us that there exists a basis for $H_3(M, \mathbb{Z})$ which makes this form unimodular (determinant 1). More generally it would pair $H_p(M, \mathbb{Z})$ with $H_{n-p}(M, \mathbb{Z})$ to produce a unimodular form; this is sometimes referred to as a "perfect pairing."

For $2p$-branes, computing the intersection form in this definition requires using the general formula for RR charges of branes carrying gauge fields,

$$\int C \wedge \mathrm{Tr} e^F \wedge \sqrt{\hat{A}(M)}.$$

The DSZ term then becomes

$$\langle B_1, B_2 \rangle = \int \mathrm{Tr} e_1^F \ \mathrm{Tr} e^{-F_2} \hat{A}(M). \tag{2}$$

This is exactly the index of the Dirac operator $\not{D}$ for bifundamental fermions coupled to the gauge field $-A_1 \otimes 1 + 1 \otimes A_2$. This is no coincidence but can be derived from stringy considerations. The basic point is that $\langle B_1, B_2 \rangle$ can be computed from a string annulus diagram with all boundary conditions taken to be Ramond, leading in the closed string channel to the part of the RR closed string exchange proportional to the Levi-Civita symbol ϵ, and in the open string channel to the index

$$\langle B_1, B_2 \rangle = \mathrm{Tr}_{B_1,B_2}(-1)^F$$

which of course is equal to (2).

Thus, the intersection form also counts the massless fermion content of the combined world-volume theory of the B_1 and B_2 branes. Let the number of fermions with charges $(-1, +1)$ under the $U(1)$ gauge groups of the two branes and four dimensional left and right chiralities be n_L and n_R, then $n_L - n_R = \langle B_1, B_2 \rangle$. We could also write all the fermions as left chirality of course by complex conjugating the right chirality.

This quantity is also the natural definition of intersection form in K theory. We will not talk specifically about K theory very much as it will be subsumed in the derived category framework we will develop later. The simplest argument for the relevance of K theory to topological classification of branes however is short and well worth keeping in mind. It is simply that a brane B should be identified topologically with anything one can get by adding another brane X, its antibrane $\bar{X}$, and performing any continuous variations on this configuration. This can be expressed mathematically by a simple construction: let a class in the K theory of "branes" be a pair of branes (E, F) subject to the relation $(E, F) \cong (E \oplus X, F \oplus X)$ for all X. This uses very little structure of the branes and indeed the objects under discussion could be almost anything to make this definition; we just need to know how to take direct sums such as $E \oplus X$ and decide when two direct sums produce the same object. Now we know many branes in the large volume limit, namely those which wrap the entire space M and carry arbitrary vector bundles on M, and it is plausible that these already carry all the topological charges, leading to the classification by K theory of vector bundles.

At this writing, it has not really been proven that this is the right classification. The issue is the torsion part, classes $[X]$ which satisfy $n[X] \cong 0$ for some finite n. The rest of the K theory agrees with cohomology and thus the RR charge considerations we started with, but one might imagine that the true answer in some string theory example might not distinguish these torsion classes, or might make more distinctions, presumably associated with singularities which would not be governed by the large volume Yang-Mills equations. Nevertheless the classification by K theory of vector bundles works in all examples studied to date and seems to fit well with our general understanding of string theory.

To get a simple example with torsion, consider a Calabi-Yau M with $\pi_1(M) = G$ some finite group. One might well expect that a string wound around a nontrivial element of $\pi_1(M)$ would be topologically stable, and use this to construct new topological classes of D1-branes. It can be shown that $H^2(M, \mathbb{Z}) \cong H_1(M, \mathbb{Z}) = G/[G, G]$ the abelianization of G and that this appears in $K^0(M)$, so if G is abelian this works. (The story if G is not abelian is less clear).

Any such M can be obtained as a quotient of a simply connected $\tilde{M}$ by a free action of a symmetry group G, so this particular type of torsion can probably be understood by close study of the theory of branes on $\tilde{M}$. We refer to [6, 7] for examples of this. It is not known whether other types of torsion which cannot be understood this way exist in $K(M)$ for M Calabi-Yau; of course other K groups appropriate to type I theories, orientifolding, H fields and so on will generally have other types of torsion.

We move on however and assume that M is a simply connected CY_3, in which case one can prove that $K^*(M) \cong H^*(M, \mathbb{Z})$, and all of the topological information is summarized in the intersection form. All of our considerations would still hold in the presence of torsion, we would just have further conserved quantum numbers which we would not be making explicit.

2.1 Noncompact manifolds

We need to generalize the previous discussion to handle noncompact manifolds such as the local orbifolds we will discuss below. Although all of the same definitions of intersection form can be used, Poincaré duality takes a different form: it relates the homology $H_p(M, \mathbb{Z})$ to the homology with compact support $H_{n-p}(M, \mathbb{Z})$, and provides a perfect pairing between these.

We will use this below, but we still would like a way to decide whether we have a complete basis for the charge lattice. We will be most interested in BPS branes of finite energy which wrap cycles of compact support, so we want a pairing purely in this sector.

We will do this below by using the pairing provided by the index (2) on the compact manifold of interest. This is also an intersection form in a mathematical sense but it should be distinguished from (1) as it is not symmetric or antisymmetric. Physically, we are dropping certain degrees of freedom (fermions partner to normal deformations of the brane) in this definition. However it still serves our purpose as if this pairing is unimodular, one has a complete basis for the K theory of the compact space. It will also turn out that this index will provide much more information about the fermion spectrum than we would get if we restricted attention to (1).

3 Constraints on brane world-volume theories

A good way to study the dynamics of a collection of branes is to derive their effective world-volume theory, which includes only modes which are visible at the energy scales of interest. For analyzing the vacuum structure this means only modes which can become massless somewhere on the moduli space.

We restrict attention in these lectures to weakly coupled type II theory, i.e. we define our effective Lagrangians using only sphere and disk world-sheets, and treat the resulting world-volume theories as classical, solving equations of motion to find vacua.

For D-branes, we can think of this world-volume theory as derived by starting with a complete open string field theory, and then keeping only the potentially massless modes. We could of course define it using world-sheet conformal field theory instead, and we would only be keeping boundary couplings which can become relevant or marginal.

There are some obvious constraints which are already visible at this level. First, the world-volume theory has only a trivial dependence on the number of dimensions in which the brane extends in the flat Minkowski dimensions. If we derive it for branes filling these dimensions, the lower dimensional cases can be defined by trivial Kaluza-Klein reduction, taking fields to be constant in the dimensions we reduce. Components of the vector potential will become coordinates in these dimensions.

The generating branes will each have gauge group at least $U(1)$ and it is natural to restrict attention to those with gauge group $U(1)$, because a world-volume theory with a larger unbroken gauge symmetry (in weakly coupled type II theory it must be $U(N)$) does not really describe a single brane. If we consider D-particles, such a theory will have moduli which enable us to split the brane into N constituents at different positions in Minkowski space, and it is better to consider these as generating branes instead. Thus we define a *simple* configuration of a gauge theory to be one which breaks the gauge symmetry to be $U(1)$ and take the B_i to be simple. Furthermore, we only need to classify the simple branes, as the others are just direct sums of these configurations.

A theory with N_i copies of the elementary brane B_i will then have gauge group

$$G = \prod_i U(N_i),$$

by the usual Chan-Paton arguments. We also know that the matter fields will all transform in the adjoint or bifundamentals of the gauge groups, and the action (computed from the disk world-sheet) can be written as a single trace.

The matter content is constrained by the index theorem arguments discussed above. Let n_{ij} be the number of massless fermions with charge $(\bar{N}_i, N_j)$; we then have

$$n_{ij} - n_{ji} = \langle B_i, B_j \rangle.$$

3.1 Constraints from $\mathcal{N} = 1$ supersymmetry

To go any farther we need some statement about the spectrum of scalar fields. We will now make the important simplifying assumption that the combined world-volume theory has $\mathcal{N} = 1$ supersymmetry in $d = 4$ (so, four supercharges). This is the supersymmetry preserved by a BPS brane in CY_3 compactification and thus this assumption would seem very natural in the problem of classifying BPS branes. On further reflection, however, it is not at all obvious, because typically the generating branes B_i will each preserve a different $\mathcal{N} = 1$ subalgebra contained in the bulk $\mathcal{N} = 2$ supersymmetry, and thus the combination will break all supersymmetry. This does not invalidate the assumption because we can still claim that this configuration, which in particular has zero vev for any open string modes associated to pairs of branes, has spontaneously broken an underlying $\mathcal{N} = 1$ supersymmetry visible in the ground state. This is what we will implicitly be claiming and will verify in examples, but eventually we will find that this picture cannot always be taken literally, and will have to generalize this assumption. Nevertheless it is close enough to the truth to justify devoting a good deal of attention to the particularities of the $\mathcal{N} = 1$ case.

In an $\mathcal{N} = 1$ supersymmetric Lagrangian, the massless fermion content determines the massless field content. Massless fields can be in either chiral or vector multiplets. A bifundamental must be a chiral multiplet, while an adjoint could be either, but we know that each set of N_i generating branes will have exactly the gauge bosons of $U(N_i)$ and no more, so knowing just the integers n_{ij}, the number of massless fermions of each chirality, completely determines the field content. (If one explicitly derives the world-volume theory from world-sheet considerations, there is no difficulty in identifying the fermions which are the gauginos: their vertex operator is just the spectral flow operator, as we discuss later.) The index however is not enough information.

This data can be conveniently summarized in a "quiver diagram," in which we denote each gauge group with a node, and each bifundamental chiral multiplet with an arrow. Thus we obtain an oriented graph with n_{ij} links between nodes i and j, each representing a chiral matter field $X^a_{i,j}$.

This graph summarizes an infinite set of supersymmetric gauge theories, distinguished by the choice of a rank N_i for each gauge group. Let $V_i \cong \mathbb{C}^{N_i}$ carry the fundamental representation of the group $U(N_i)$, then $X^a_{i,j}$ is in $V_i^* \otimes V_j$.

This is a lot of information already and the remaining data we need to know to find supersymmetric vacua is the superpotential W, the D-flatness conditions, and some qualitative information about the Kähler potential, say that it is nonsingular on the moduli space.

The superpotential is a gauge invariant function of chiral superfields which can be written as a single trace. It can be written as a sum of monomials, each of which could be denoted by a closed loop in the quiver. A supersymmetric vacuum must satisfy the F-flatness conditions

$$0 = F_a = \frac{\partial W}{\partial X^a}, \tag{3}$$

a matrix equation for each chiral superfield X^a.

The D-flatness conditions are largely determined by the spectrum and gauge representations, but there is one further input: each $U(N_i)$ factor in the gauge group (so each node) can come with a single real parameter, the Fayet-Iliopoulos term ζ_i. A supersymmetric vacuum must then satisfy the D-flatness conditions,

$$0 = D_i = \sum_{a=1}^{n_{ij}} (X^a)^\dagger X^a - \sum_{a=1}^{n_{ji}} X^a (X^a)^\dagger - \zeta_i \mathbf{1}. \tag{4}$$

Supersymmetric vacua are gauge equivalence classes of solutions of these two sets of equations, and we will be interested in the general problem of finding such vacua which break the gauge symmetry to $U(1)$.

There is a variation on the D-flatness condition which is relevant for our D-brane problems coming from the fact that D-brane world-volume theories have an additional inhomogeneous $\mathcal{N} = 1$ supersymmetry, the shift $\delta\chi = \epsilon'$ of the decoupled gaugino in the diagonal $U(1)$ factor of the gauge group. We need to allow for vacua which break the linearly realized supersymmetry but preserve some combination of the two as well.

Supersymmetry breaking by D terms shows up in an inhomogeneous transformation law for the gaugino. Adding to this the overall inhomogeneous supersymmetry, we have

$$\delta\chi^a = D^a\epsilon + \epsilon',$$

and we see that these more general supersymmetric vacua can be found by the prescription of solving the D-flatness conditions with an overall constant shift $\zeta_a \to \zeta_a + \xi$ of all of the FI terms.

3.2 Finding supersymmetric vacua

An effective way to think about this problem is to first classify solutions of F-flatness modulo complex gauge equivalence, and then check which of these solutions can solve D-flatness as well. The complexified gauge group is

$$G_{\mathbb{C}} = \prod_i GL(N_i, \mathbb{C}),$$

and it acts on a bifundamental as

$$X_{i,j} \to g_i^{-1} X_{i,j} g_j. \tag{5}$$

For general (nonunitary) g, this is a symmetry of the holomorphic part of the theory (the F-flatness conditions in particular) but not of the D-flatness conditions. Thus, any solution of F-flatness in fact comes with an entire $G_{\mathbb{C}}$-orbit of solutions, and in this first stage of the problem it is not natural to distinguish the points on a given orbit. One can then try to find a g_i in (5) which solves (4).

A major advantage of this two-step procedure is that the second step is very well understood mathematically, and indeed we will be able to quote a general theorem which tells us precisely when solutions of D-flatness do and do not exist. As motivation, let us review two very familiar examples.

First, consider $U(1)$ theory with n chiral multiplets z^i of charge $+1$. The D-flatness condition is

$$\sum_i |z^i|^2 = \zeta. \tag{6}$$

Clearly there are three cases: for $\zeta > 0$ there are solutions whose moduli space is $\mathbb{CP}^{n-1}$; for $\zeta = 0$ there is a unique solution $z = 0$, while for $\zeta < 0$ there are no solutions. This exhibits the fact that the moduli space of solutions of D-flatness will depend on the specific values of the FI parameters and can even disappear. It also illustrates the fact that not every $G_{\mathbb{C}}$-orbit will contain a solution of D-flatness. In the first case, the orbit $z^i = 0$ cannot solve (6) while all the others can; in the second case the situation is reversed, while in the final case of course none will.

Second, consider $U(N)$ theory with a single adjoint chiral superfield X. In this case one cannot usefully introduce an FI term, so the D-flatness condition is

$$[X^\dagger, X] = 0.$$

A matrix satisfying this equation is referred to as "normal" and it can be diagonalized; the moduli space is the space of sets of N eigenvalues x_i (the ordering does not matter thanks to a remaining S_N discrete subgroup of the gauge symmetry), so is $\mathbb{C}^N/S_N$.

Let us again compare with the $G_{\mathbb{C}}$ orbits. This includes the normal matrices but also matrices which cannot be diagonalized, such as

$$\begin{pmatrix} x_1 & 1 \\ 0 & x_2 \end{pmatrix}$$

to give the simplest example. Such matrices cannot solve the D-flatness conditions and thus our previous result for the moduli space was correct, but we still can ask: what distinguishes these $G_{\mathbb{C}}$ orbits from the ones which can solve D-flatness?

There is an answer to this question which is fairly well known by physicists but only applies to the case of zero FI terms. It is that the moduli space of solutions of D-flatness is parameterized by a complete set of independent holomorphic gauge invariant polynomials formed from the original fields. In the matrix example, these could be taken to be $\mathrm{Tr}X^k$ for $1 \le k \le N$. Clearly these suffice to distinguish different sets of eigenvalues, but they are not a good system of coordinates to describe all matrices up to gauge equivalence, as any non-diagonalizable matrix will have the same invariants as some diagonalizable matrix. On the other hand, the non-diagonalizable matrices never solve the D-flatness conditions, so these are a good system of coordinates to describe these solutions.

This connection between invariants and solutions of D-flatness is very general and provides a very satisfactory description of the moduli space, but only for the case of zero FI terms. For example, there is no obvious way to adapt it to the first problem as this theory admits no holomorphic gauge invariant observables.

There is a more general solution, which will turn out to have a fairly clear picture in terms of the physics of branes, but explaining this will require some additional formalism.

4 Pure quiver theories

Let us spend some time discussing the case with no superpotential first. There is quite a bit to say, and making careful definitions here will in fact carry us much of the way to the final results.

We can make the main points by considering the theory of two elementary branes B_1 and B_2 whose open string spectrum contains q bifundamental chiral multiplets X^i. This is described by the diagram

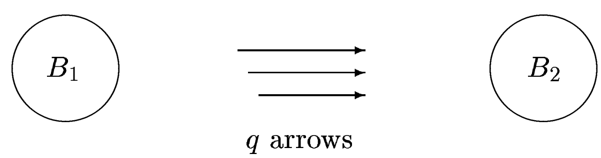

Fig.1. $U(N_1) \times U(N_2)$ quiver

Almost all of what we will say in this section generalizes to the general quiver theory with $W = 0$. In this case we will refer to matter fields between nodes i and j as $X_{i,j}$. Some of this generalizes directly to arbitrary superpotential, and we will say so when it does.

Let us consider a configuration with $N_1 B_1 + N_2 B_2$ elementary branes, and suppose that all such configurations are described by this $U(N_1) \times U(N_2)$ world-volume gauge theory. Each configuration of the q bifundamental chiral multiplets, modulo complex gauge equivalence, will provide a physically distinct bound state, if it solves the D-flatness conditions. Now even before we consider D-flatness, it is clear that each such configuration represents at most one physical state. Of course it might not be a single bound state; this will be the case only if the gauge symmetry is broken to $U(1)$.

Let us refer to a configuration as an "object" (or holomorphic object) in the category of quiver representations. One which breaks the complex gauge symmetry to $GL(1)$ is a simple object.

We now ask whether this quiver theory contains any simple objects, and if so what is the dimension of their moduli space. Let us refer to such a theory using the notation $(\vec{N})$ or $(N_1\ N_2)$; the moduli space of simple configurations will be $\mathcal{M}(\vec{N})$, while the vector $\vec{N}$ will be called the "charge" of the object. The two elementary branes are $(1\ 0)$ and $(0\ 1)$; let us denote these charge vectors as e_1 and e_2.

There is an obvious guess for the dimension of this moduli space, obtained by counting matter fields minus the number of broken gauge symmetries, and assuming that the resulting object is simple:

$$\dim \mathcal{M}(\vec{N}) = qN_1N_2 - N_1^2 - N_2^2 + 1.$$

One would expect that if this "expected dimension" $\dim \mathcal{M} \geq 0$, there will exist simple configurations and that their moduli space will have this dimension, while if $\dim \mathcal{M} < 0$ there will not exist simple configurations. This is true, although the mathematical proof of this fact is surprisingly complicated. A similar result holds for general quivers with $W = 0$, quoted in the appendix to [14].

We can also write the result in terms of a "Cartan matrix":

$$\begin{aligned} \dim \mathcal{M}(\vec{N}) &= 1 - \tfrac{1}{2} \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} 2 & -q \\ -q & 2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} & (7) \\ &\equiv 1 - \tfrac{1}{2}(N|N). & (8) \end{aligned}$$

In this language, we can distinguish three types of vector $\vec{N}$:

- Real roots with $(N|N) = 2$, which are "rigid" configurations with no moduli.

- Imaginary roots with $(N|N) \leq 0$, which are configurations with moduli.

- Vectors with $(N|N) > 2$ which do not correspond to simple configurations.

Using results which are probably familiar from Lie algebra theory, we can see that the general nature of the solutions depends on q in the following way:

- For $q = 0, 1$ there are a finite number of real roots and no imaginary roots, so this is the finite case.
- For $q = 2$, the real roots are $(n+1\ n)$ and $(n\ n+1)$, and the imaginary roots are $(n\ n)$. This is the affine case.
- For $q \geq 3$, the hyperbolic case, there are infinitely many roots of both types.

This last case is probably less familiar although as the reader may have guessed, all three cases indeed admit a relation to Kac-Moody algebras. Let us discuss it a bit more. The symmetries of a root system are generated by Weyl reflections, which act as follows:

$$r_i : \vec{N} \to \vec{N} - 2\frac{(N|e_i)}{(e_i|e_i)}e_i.$$

In the hyperbolic and affine cases, these reflections generate infinite discrete groups (which are cyclic in this simple case).

The condition $(N|N) \leq 0$ defines a region in the $(N_1\ N_2)$ plane in which all the points are imaginary roots. It contains infinitely many copies of a fundamental region, defined by the condition $r_i(\vec{N}) \geq \vec{N}$.

The real roots can all be obtained by Weyl reflection from the elementary roots, and thus form an infinite series which for $q = 3$ starts $(0\ 1), (1\ 3), (3\ 8), \ldots$.

Thus even for these very simple theories, the spectrum of branes has quite a bit of structure. One can even define explicit operations corresponding to the Weyl reflections which take one object into another. This is also discussed in the appendix to [14], and underlies the more complicated Seiberg dualities discussed in [4, 18, 9]. This structure is known to also be present in the quantum mechanical treatment of such branes for the finite and affine cases, and very likely is for the hyperbolic case.

We will say more about this structure and the nature of the Weyl reflections below, after introducing some more formalism.

4.1 Bound states and Ext

A question of primary interest for us is the following: given two objects A and A', when can they form a bound state?

That this question is nontrivial can be seen by considering the example of bound states of (0 1) and (1 n). Starting with (1 0), one can add successive (0 1)'s until one reaches (1 q). However (1 $q+1$) is clearly not a simple object as there are not enough matter fields to break $U(q+1)$ gauge symmetry. We would like a rule which tells us when this can happen, ideally depending only on the charges N and N'.

Clearly the answer to this question can be found by studying the $U(N_1+N_1') \times U(N_2+N_2')$ gauge theory which describes the combination of their constituents. Its matter will decompose in a block diagonal way:

$$X^a = \begin{pmatrix} X^a & \rho^a \\ \psi^a & (X')^a \end{pmatrix}. \tag{9}$$

If we can turn on matter fields ρ^a or ψ^a in a way which breaks the total gauge symmetry back to $U(1)$, these two objects will form a bound state. In other words, if after using all possible gauge symmetry to set components of ρ or ψ to zero, we are left with any nonzero components, we will find a bound state.

Let us consider only turning on ρ^a as we can then repeat the discussion, exchanging the two objects, to treat ψ^a. The gauge symmetries (5) can also be decomposed in block diagonal form, and the relevant parameters which act on X and can modify ρ are

$$g_1 = \begin{pmatrix} 1 & \epsilon_1 \\ 0 & 1 \end{pmatrix}; \qquad g_2^{-1} = \begin{pmatrix} 1 & -\epsilon_2 \\ 0 & 1 \end{pmatrix}.$$

The resulting gauge action is then

$$\delta\rho^a = X^a \epsilon_1 - \epsilon_2 (X')^a. \tag{10}$$

Note that the g_i are finite (not infinitesimal) complexified gauge transformations; in this sense this is not a linearized result but is exact. This also generalizes in the obvious way to any matter field $X_{i,j}$ and its off-diagonal part $\rho_{i,j}$ in any quiver.

The result can be seen much more quickly in terms of the following diagram:

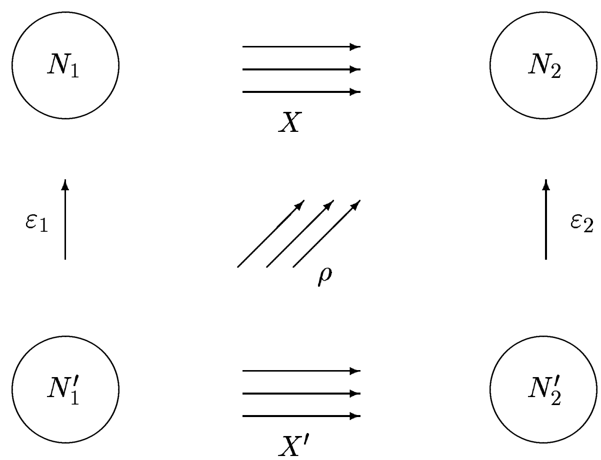

Fig.2. Combining two bound states

The vertical lines represent the off-diagonal degrees of freedom, and sensible (gauge invariant) products must respect the graphical structure.

Thus matter variations which cannot be gauged away are variations ρ^a which cannot be obtained from (10). These form a linear space which is denoted by

$$\mathrm{Ext}(A', A).$$

We are also interested in gauge transformations which are unbroken in the combined configuration, i.e. solutions to the equation $\delta\rho^a = 0$. These form another linear space which is denoted by

$$\mathrm{Hom}(A', A).$$

Note that some of the ranks N_i or N_i' might vanish in a particular example. The definitions still make sense if we just omit those gauge transformations or matter multiplets for which one of the ranks is zero. Often in these cases, the equation (10) will also degenerate; this is fine. One should check that this is clear for the simple examples

$$\dim \mathrm{Hom}(B_i, B_j) = \delta_{i,j} \tag{11}$$

$$\dim \mathrm{Ext}(B_i, B_j) = q\delta_{i,1}\delta_{j,2}. \tag{12}$$

All of these definitions generalize in a direct way to arbitrary quivers with $W = 0$. We will generalize them in the next sections to certain quiver theories with superpotentials as well, and they are quite important in all of the subsequent discussion.

The spaces Hom and Ext are examples of the spaces of morphisms associated with pairs of objects, which as discussed in the introduction define a category. The multiplication law is just the obvious composition of arrows and multiplication of the matrices associated with each arrow. This structure provides an obvious generalization of representations of groups, and it is in this spirit that quivers were first introduced in mathematics [20].

One can regard the equation (10) as defining a linear operator acting on the space of parameters ϵ_i and producing a configuration in the space of matter fields ρ^a, whose matrix elements depend on the configuration X and X'. Let us denote this operator as D, we then have

$$\mathrm{Hom}(A', A) = \ker D; \qquad \mathrm{Ext}(A', A) = \mathrm{coker}\ D.$$

and we are studying a cohomology problem. This formalizes the observation that, although the dimensions of these two spaces depend on the specific configuration, they can only change in a paired way, with an element disappearing or appearing on both sides, corresponding to the Higgsing or unHiggsing of an off-diagonal gauge boson.

This also shows that it is easy to compute the difference in the dimensions of these two spaces, since we can just do it for $X = 0$, by again counting multiplets. This number is called the relative Euler character; it is

$$\begin{aligned} \chi(A', A) &= \dim \mathrm{Hom}(A', A) - \dim \mathrm{Ext}(A', A) & (13) \\ &= \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}. & (14) \end{aligned}$$

It contains the information both of the intersection form and of the Cartan matrix,

$$\langle A|B\rangle = \chi(A, B) - \chi(B, A) \qquad (15)$$

$$(A|B) = \chi(A, B) + \chi(B, A). \qquad (16)$$

Although this does not directly determine either $\dim \mathrm{Hom}(A, B)$ or $\dim \mathrm{Ext}(A, B)$, it will give a lower bound for one of them, so given only the charges of the two objects we can often prove that one of $\dim \mathrm{Hom}$ or $\dim \mathrm{Ext}$ is nonzero. It is harder to prove that one of these is zero, although in typical examples where A and B are simple objects, one of these dimensions will in fact be zero.

This provides a procedure to decide whether two objects A and B can form a bound state, and a formal construct which is associated with each way of forming a bound state. There is a larger structure which this fits into: any $\mathrm{Ext}(A, B)$ and bound state E produced by turning it on will have an associated exact sequence

$$0 \longrightarrow B \xrightarrow{f} E \xrightarrow{g} A \longrightarrow 0. \tag{17}$$

By an exact sequence one means first of all that $g \cdot f = 0$ defined by composing these pairs of matrices. Furthermore, the terms $0 \longrightarrow B$ and $A \longrightarrow 0$ at the beginning and end of this sequence indicate that the map f must be injective (with no kernel), and g must be surjective (with no cokernel). In other words, E must incorporate all constituents of A and B, with nothing left over.

The object B is called a subobject of E, while A is a quotient object. Physically the Hom's represent the possibility of seeing that these two objects are contained in E by bringing up either one "next to it;" an enhanced complex gauge symmetry appears.

A simple example is provided by the bound states of the two elementary branes $B_1 = (1\ 0)$ and $B_2 = (0\ 1)$. Bound states with charge $(1\ 1)$ exist with a moduli space of dimension $q - 1$. Call one of these E; it will fit into the exact sequence

$$0 \longrightarrow (0\ 1) \xrightarrow{f} E \xrightarrow{g} (1\ 0) \longrightarrow 0$$

where $f \in \mathrm{Hom}(B_2, E)$ and $g \in \mathrm{Hom}(E, B_1)$ are easy to write down using our general definitions (exercise).

We can even write a "triangle" which completes the structure as follows: given $\phi \in \mathrm{Ext}(A, B)$, we have

$$B \xrightarrow{f} E \xrightarrow{g} A \xrightarrow{\phi} B \tag{18}$$

where $f \cdot \phi \cdot g$ is an $\mathrm{Ext}(E, E)$ we can vary in the resulting bound state. We will better define and use this structure later. It is present in general theories of D-branes on Calabi-Yau (and probably even more generally).

So far we have only talked about branes, and not their antibranes. The exact sequence (17) can also be interpreted as describing certain processes involving antibranes, namely the inverse to the bound state formation we just described. For example, given that A and B can form the bound state E, we might expect that E and $\bar{B}$ could partially annihilate to produce A, and also that $E + \bar{A} \to B$, and indeed these are all valid readings of (17). This does not get us too far on the problem of describing bound states of branes with antibranes, however, as all three of the objects involved are each made only from elementary branes or only from their antibranes. In general we will need to talk about bound states of elementary branes with antibranes, but this will require formalism we discuss later.

5 D-flatness and stability

In this section we will explain the general result promised earlier on D-flatness conditions; it applies to general quiver theories with or without superpotentials. The structure we described in the previous section will play an essential role.

As we saw, the question of whether an object really corresponds to a physical brane, i.e. solves the D-flatness conditions, can depend on the FI terms. This is how marginal stability will appear in the physical theories of branes on CY. Conversely, one might imagine that if by varying an FI term, an object becomes physically unstable, it will have to decay into constituents described by an exact sequence of the type we just discussed.

We can study the question of whether, given an $\mathrm{Ext}(A', A)$ and exact sequence (17), these two branes can actually form a physical bound state, by again considering the direct sum $U(N_1+N_1') \times U(N_2+N_2')$ gauge theory. Now we want to write down the D-term part of the potential $V = \frac{1}{2}\mathrm{tr}\, D^2$ for the mode ρ. Taking D from (4), we have

$$D_1 = -\sum_{a=1}^{q} X^a (X^a)^\dagger - \zeta_1 \tag{19}$$

$$D_2 = \sum_{a=1}^{q} (X^a)^\dagger X^a - \zeta_2. \tag{20}$$

We first note that there is a rather trivial but necessary condition for solving the D-flatness conditions, obtained by taking the trace of both and adding: one finds

$$0 = \sum_i (N_i + N_i')\zeta_i \equiv (\vec{N} + \vec{N}') \cdot \vec{\zeta}. \tag{21}$$

In this form it clearly holds for arbitrary quiver theories. We will further assume that $\vec{N} \cdot \vec{\zeta} = \vec{N}' \cdot \vec{\zeta} = 0$, so that both initial and final states solve the D-flatness conditions with the same FI terms. We will see below that this is the special case where all three branes preserve the same supersymmetry. The same analysis can be made without this assumption; see [26].

We now compute the quadratic term in V for the mode ρ, assuming that $D_i = 0$ before we turn it on. Substituting (9) into (19) we have

$$D_1 = \begin{pmatrix} \rho\rho^\dagger & \rho(X')^\dagger \\ X'\rho^\dagger & 0 \end{pmatrix}.$$

Using $X'X'^\dagger = \zeta_1$, and taking the trace, the quadratic term in $D_1^2/2$ becomes $N_1\zeta_1$. Adding similar contributions from each gauge group, the total mass squared for the ρ mode is

$$m_\rho^2 = \sum_i N_i \zeta_i.$$

If this is positive, the potential prevents us from turning on ρ to make a bound state, while if it is negative, ρ is tachyonic near the origin and this combination is in fact unstable to forming the bound state.

Since the vacuum energy is always bounded below in this supersymmetric theory, the process of tachyon condensation is guaranteed to stop, resulting in a bound state. Since one has the exact potential, one can be much more precise and prove that a solution of the D-flatness conditions (in the $G_{\mathbb{C}}$ orbit of the object E) can exist only if we have (21) and

$$\sum_i N_i' \zeta_i > 0. \tag{22}$$

If it exists, it will be unique.

The mathematical theorem [23] is even stronger than this and asserts the converse. Suppose we are interested in the particular object E and we want to know whether it is stable or not, i.e. whether it will decay into anything. The theorem states that the configuration E can solve D-flatness if and only if (21) is satisfied and if (22) is satisfied for every subobject of E. In other words, if the mode ρ A and A' becomes massive, not only is this process of bound state formation prevented, the product becomes unstable even if there are potentially other ways of forming it from brane pairs, and even if these other pairs would have led to tachyonic modes. Such a subobject is known as a destabilizing subobject.

This theorem is proven, and the general study of this type of problem (known as symplectic quotient), uses the methods of geometric invariant theory. In fact this type of necessary and sufficient condition was already known as stability, an amusing example in which mathematical and physical nomenclature actually coincide in meaning. The particular version defined here is known as θ-stability; other forms will appear below.

The theorem is not hard to prove, but we give only the general idea here. A general strategy for finding a solution of the D-flatness condition on a given orbit is to take the potential as a function of the group element g parameterizing the orbit,

$$V = \sum \mathrm{tr}\,(g^{-1}Xgg^{\dagger}X^{\dagger}g^{-1\dagger} - \zeta)^2$$

and minimize it by gradient descent. The simple form of the potential makes it possible to show that a minimum will be reached, but it is not guaranteed that the minimum will be on the orbit; it could be a limit of points on the orbit which is not on the orbit. This is illustrated by the second example above (the adjoint chiral field) and the nonnormal matrix. Its gauge orbit includes the matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x_1 & 1 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} x_1 & \lambda^2 \\ 0 & x_2 \end{pmatrix}$$

and one sees that the minimum of $\mathrm{tr}\,[X, X^{\dagger}]^2$ will be achieved as $\lambda \to 0$, a limit point not on the orbit but on a stable orbit with the same value of the invariants $\mathrm{tr}\,X^k$. One can show that whenever this happens, there is a similar one-parameter subgroup for which taking the limit decomposes the original unstable object into two objects, the subobject and quotient object (here 1-dimensional matrix configurations), and that conversely whenever the condition (22) is violated, a destabilizing one parameter subgroup can be constructed from it. Thus one obtains necessary and sufficient conditions for a solution to exist.

In our D-brane problems, the relation (21) typically will not be satisfied by any of the D-brane charges N, N' or $N + N'$. However it can be restored by taking advantage of the possibility mentioned earlier of making an overall shift ξ of the FI terms. This turns (21) into

$$0 = (\vec{N} + \vec{N}') \cdot (\vec{\zeta}_i + \xi\vec{e}) \tag{23}$$

where $\vec{e}$ is the vector with components $e_i = 1$. This can be solved for ξ.

One must then satisfy (22) with respect to the shifted FI terms. The resulting stability condition, with ξ eliminated using (23), is

$$(\vec{N}' \cdot \vec{\zeta})(\vec{e} \cdot \vec{N}) - (\vec{N}' \cdot \vec{e})(\vec{\zeta} \cdot \vec{N}) > 0. \tag{24}$$

This condition does not depend on $e \cdot \vec{\zeta}$ or on the overall scale of $\vec{\zeta}$. In the particular case of two nodes, only the ordering of the FI terms enters the final condition,

$$\operatorname{sgn}(\zeta_2 - \zeta_1)(N_1 N_2' - N_1' N_2) > 0.$$

Using methods we will not describe here, one can show that given two nodes, all simple bound states are stable on one side of the line $\zeta_1 = \zeta_2$, while on the other side only the two elementary branes and their antibranes are stable. This is very analogous to marginal stability in $\mathcal{N} = 2$ supersymmetric gauge theory, and in fact one can formulate that BPS spectrum in terms of representations of affine quivers [19].

5.1 D-brane stability near orbifold points

For quivers with more than two nodes, the condition (24) will have nontrivial dependence on the FI terms, leading to a complicated structure with an infinite number of lines of marginal stability. It is worth looking at this in detail, because this turns out to the the exact result for the spectrum in the neighborhood of orbifold points, and is a good illustration of the general structure.

A detailed analysis is rather involved (see [14]), but the basic idea can be illustrated by the following result: for any simple object, there is a line going into the point $\zeta = 0$ (which will be the orbifold point in our later examples) on which it is "most likely to be stable" (in many cases one can easily prove that it is). The idea is that we have a necessary condition for $\mathrm{Hom}(E', E) = 0$, namely $\chi(E', E) \leq 0$. Although there could still be $\mathrm{Hom}(E', E)$'s which cancel out of χ, this is not generic. Thus, if we could show that for every candidate destabilizing subobject E', i.e. one for which (24) fails, we had $\chi(E', E) \leq 0$, we would have good evidence for stability, while if this condition fails, we know that the object E is unstable.

This can be arranged by choosing $\vec{\zeta}$ so that

$$\vec{N}' \cdot \vec{\zeta} = \chi(\vec{N}', \vec{N}),$$

i.e. $\zeta_i = \chi_{ij} N_j$ in an obvious notation. Given this choice, the opposite of (24) becomes

$$\chi(\vec{N}', \vec{N}) \leq \chi(\vec{N}, \vec{N}) \frac{\vec{e} \cdot \vec{N}'}{\vec{e} \cdot \vec{N}}.$$

Now by considerations in the previous sections, we know that $\chi(\vec{N}, \vec{N}) \leq 1$ for simple objects, and $\vec{e}\cdot\vec{N}' < \vec{e}\cdot\vec{N}$ for subobjects, so on this line $\chi(E', E) \leq 0$ follows.

On the other hand, every object has some elementary brane as subobject which will destabilize it by taking its FI term negative, so for every object there will also be a line on which it is unstable. These two lines must be separated by lines of marginal stability, which may be associated with the elementary brane we just discussed, or may be associated with larger subobjects.

Furthermore, one can easily check that the line for decay into a given subobject (say one of the elementary branes) is different for objects with different charge. This will also be clear from more conventional (BPS charge) marginal stability considerations, so there will be infinitely many lines of marginal stability in these problems, and a rather intricate structure.

6 Orbifold quiver theories

The simplest source of physical theories of D-branes on non-flat spaces is the orbifold construction. This is described in many places including [11].

We start with D3-branes extending in 3 + 1 Minkowski space and at points in an internal space $\mathbb{C}^n$, and choose an orbifold group Γ, an action of Γ on $\mathbb{C}^n$ denoted $r(g)$, and another N-dimensional representation of Γ acting on the "Chan-Paton factors" and denoted $\gamma(g)$. One then takes the world-volume $U(N)$ gauge theory and applies the projection

$$\gamma(g)^{-1}\phi\gamma(g) = r(g)\phi \tag{25}$$

to all the fields, where $r(g)$ acts on ϕ in the appropriate representation (scalar for the gauge fields, vector for the coordinates, spinor for the spinors). For $r(g)$ preserving supersymmetry one can also take the action directly on the superfields, which is what we will do.

The representation γ can be written as a direct sum over irreducibles,

$$\gamma = \oplus_i N_i r_i,$$

and one finds that the gauge group of the resulting theory is

$$G = \otimes_i U(N_i)$$

and the chiral matter spectrum is that of a quiver theory with

$$n_{i,j} = [r \otimes r_i^* \otimes r_j]$$

where $[R]$ denotes the number of times the trivial representation r_0 appears in R, e.g. $[r_i] = \delta_{i,0}$.

There will also be a superpotential and FI terms. If the theory arises from a $\mathbb{C}^3$ orbifold, the superpotential is the projection of that of $\mathcal{N} = 4$ $U(N)$ super Yang-Mills,

$$W_{N=4} = \operatorname{tr} X^1 [X^2, X^3].$$

We will confine ourselves to the example of $\Gamma = \mathbb{Z}_k$ with action

$$X^m \to \exp \frac{2\pi i a_m}{k} X^m,$$

in which case there are k irreps and the result of the projection is

$$\begin{aligned} W &= \textstyle\sum_{m=1}^{3} \sum_{i=0}^{k-1} \\ &\quad \operatorname{tr} X^1_{i,i+a_1} X^2_{i+a_1,i+a_1+a_2} X^3_{i+a_1+a_2,i} \\ &- \operatorname{tr} X^1_{i,i+a_1} X^3_{i+a_1,i+a_1+a_3} X^2_{i+a_1+a_3,i}. \end{aligned} \tag{26}$$

There are k FI terms which are fairly general, satisfying the relation

$$\sum_i \zeta_i = 0$$

and in certain examples additional relations. They can be derived from world-sheet considerations of couplings to twist sector moduli and their physical interpretation is as blowup modes for the orbifold singularity; they can be considered as real coordinates for the complexified Kähler moduli space in the neighborhood of the orbifold point.

These quiver theories can be used to describe a large number of D-branes anywhere near the orbifold point, along the lines we just described, as bound states of arbitrary numbers of elementary branes, which are usually called fractional branes in this context. We might even be able to describe all branes, if these branes span the charge lattice.

We will focus particularly on the simplest case of $\mathbb{C}^3/\mathbb{Z}_3$, which has the quiver diagram

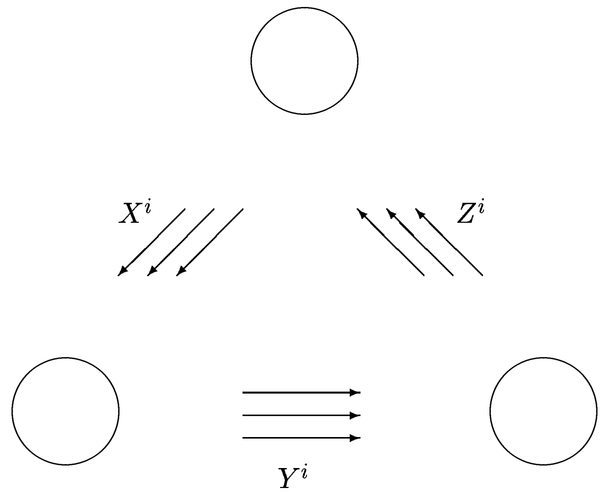

Fig.3. The $\mathbb{C}^3/\mathbb{Z}_3$ quiver

and two independent FI terms related in some way to the single complexified Kähler modulus. For simplicity, denote the three groups of chiral multiplets as X^i, Y^i and Z^i, then the superpotential is

$$W = \epsilon_{ijk} \mathrm{tr}\, Z^i Y^j X^k.$$

The formal orbifold construction can be generalized to higher dimensional complex space. Later we will see that in fact D-branes in Gepner models can also be understood using this construction, essentially because they can be defined as Landau-Ginzburg orbifolds. The case we will consider is $\mathbb{C}^5/\mathbb{Z}_k$, and we will argue that these theories can be derived as the orbifold projection of a $U(N)$ gauge theory with chiral multiplets X^i in the 5 of a global $SU(5)$, chiral multiplets $Y_{[ij]}$ in the $\bar{1}0$, and superpotential

$$W = \mathrm{tr}\, X^i X^j Y_{[ij]} + \ldots.$$

where $\ldots$ indicates higher order terms which we will pretty much ignore in these lectures. Applying the projection leads to a superpotential much like (26).

Both of these superpotentials are cubic, and impose commutativity conditions on the matrices X^i,

$$X^m_{i,i+a_m} X^n_{i+a_m,i+a_m+a_n} = X^n_{i,i+a_n} X^m_{i+a_n,i+a_n+a_m}. \tag{27}$$

In fact they have a much more important common property: they express the condition that a certain D operator squares to zero.

6.1 $D^2 = 0$ superpotentials and higher Ext groups

We now generalize our previous discussion of bound states and the moduli space to these quiver theories with superpotential.

The formula (10) for a gauge transformation, and the definition of $\mathrm{Hom}(A, B)$ as the kernel of this operator D, go over unchanged.

However, the massless matter variations are different, because some of these can be lifted by the superpotential. We need to supplement the condition that a matter variation be in the cokernel of D with an additional condition that it satisfy the superpotential constraint. For the quadratic constraints (27), we can write this as

$$\begin{aligned} 0 = &X^m_{i,i+a_m} \rho^n_{i+a_m,i+a_m+a_n} \\ &+\rho^m_{i,i+a_m} (X')^n_{i+a_m,i+a_m+a_n} \\ &-X^n_{i,i+a_n} \rho^m_{i+a_n,i+a_n+a_m} \\ &-\rho^n_{i,i+a_n} (X')^m_{i+a_n,i+a_n+a_m}. \end{aligned} \tag{28}$$

Note that this is again linear in ρ and in the configuration variables X and X'. Thus we can define a *new* operator D which takes a variation ρ and maps it to the right hand side of (28). Gauge invariance requires $D^2 = 0$ and in terms of this D, the massless off-diagonal matter variations are now the ρ in $\ker D / \mathrm{Im}\, D$.

Thus we regain the interpretation of $\mathrm{Ext}(A', A)$ as the cohomology of the operator D. One sees now however that to write a formula like (13), we will have to keep track of the cokernel of (28), because now matter variations can pair up with these and become massless. We thus define this space as

$$\mathrm{Ext}^2(A', A)$$

and rename the previous space $\mathrm{Ext}^1(A', A)$ (often it is still called Ext without the superscript).

One naturally wants to know what is the physical meaning of Ext^2. From their construction, these are superpotential constraints which are not actually used in the matter configuration, and do not lift matter fields. This is related to one of the major difficulties in working with general $\mathcal{N}=1$ theories, namely the fact that the dimension of moduli space need not be constant; it can consist of several branches with various dimensions, because of the relatively unconstrained form of the superpotential and the fact that generic cubic terms will produce such a structure. Indeed the naive estimate for the dimension of moduli space in an $\mathcal{N}=1$ theory is that it is always zero, because there are as many equations in $W'=0$ as there are unknowns. This naive estimate can fail because of unused and redundant superpotential constraints, and this is reflected in the fact that the generalization of (13) is not going to involve just $\dim \mathrm{Hom}$ and $\dim \mathrm{Ext}$ but additional terms which can be different on different branches of moduli space.

This story could clearly be repeated again by defining a similar operator acting on new fields (not present in the original Lagrangian) $\rho^{[ab]}$ with two indices, and so forth up to n index fields (one will need to continue to antisymmetrize indices to keep $D^2=0$). Should we do this?

The answer appears to be yes, for specific reasons which will appear shortly, and because in general it is useful to keep track of redundancies between the superpotential relations. Suppose we found that a two index element of $\mathrm{coker}\, D$ was not in fact in $\ker D$, so that it drops out of Ext^2 (being paired with a three-index field). This means that some of the unused superpotential constraints become redundant when multiplied by additional powers of the fields X. The presence of such redundancies in solving equations was one of the main motivations for introducing the homological algebra techniques we are now discussing.

The whole setup can be defined at once by introducing a Grassmann algebra with n generators,

$$e_m e_n + e_n e_m = 0,$$

and defining

$$D_X = X^n e_n$$

where X^n is the direct sum of the various matrices $X^n_{i,i+a_n}$ acting on the direct sum of Hilbert spaces associated to all the nodes. It is easy to see that $D^2=0$ then precisely reproduces (28).

The operator D of our previous discussion is then $D_X - D_{X'}$ (the relative sign is not important) where D_X acts on the sum

$$\epsilon + \rho^a e_a + \rho^{[ab]} e_a e_b + \dots$$

on the left, and D'_X acts on it on the right.

The same argument as before suffices to compute the relative Euler character, now defined as

$$\chi(A,B) \quad = \textstyle\sum_i (-1)^i h^1 \quad \equiv \displaystyle\sum_i (-1)^i \dim \mathrm{Ext}^i(A,B) \tag{29}$$

where we let $\mathrm{Ext}^0(A,B) \equiv \mathrm{Hom}(A,B)$. For the configuration $X = X' = 0$, every field ϵ and ρ contributes with the sign $(-1)^i$, leading to a formula bilinear in the charges N_i of the two representations.

In practice, it is often useful to assume that certain of the X or Y links are zero, and also drop the fields ρ which cannot appear under this assumption, to get a definition of χ with stronger consequences. We already followed this policy in the $\mathbb{C}^5$ case in not considering the constraints $\partial W/\partial X = 0$; these can be automatically solved by setting $Y = 0$, leading to the equations we kept. Example of such simplifications are to assume that some nodes are not present, or that all links which cross a certain "line" on the quiver (say all links from i to $j < i$) are zero, in which case all fields ρ which cross this line can also be dropped.

A simplified formula which will be of relevance below is to consider the $\mathbb{C}^3/\mathbb{Z}_3$ quiver with the $X^a_{3,1} = 0$. In this case one obtains

$$\chi(A,B) = \vec{N}_A \cdot \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \cdot N_B. \tag{30}$$

This corresponds to the "naive" dimension one obtains by taking the number of matter fields minus unbroken gauge symmetries and superpotential constraints, and is indeed the correct dimension (it turns out that $h^2 = 0$) for all cases except the D0-brane, (1 1 1).

This case is instructive. The result $\chi(D0, D0) = 0$ corresponds to a moduli space dimension for the D0 of 1. Of course we left out the $X_{3,1}$ links and this could increase the dimension; however these satisfy superpotential constraints of their own and one finds that they provide one extra modulus, leading to the incorrect answer 2.

The resolution of this is that in fact $h^2 = 1$ in this case. The general solution of the constraints $0 = X^i_{1,2}X^j_{2,3}\epsilon_{ijk}$ is to take $X^i_{1,2} = \lambda X^i_{2,3}$, for a general complex number λ. This is two conditions, and comparing to the three superpotential constraints, we see there is one redundancy. Thus, the result $\chi = 1 = h_0 - h_1 + h_2$ predicts $h_1 = 2$. Similarly $X^i_{3,1} = \lambda' X^i_{2,3}$ solves the remaining constraints, leading to the correct moduli space dimension 3 for an object moving on $\mathbb{C}^3/\mathbb{Z}_3$.

This discussion was of course overkill for such a simple example, but the point is that it illustrates the general (non-trivial) relation between χ and the naive dimension formula, and the true dimension of moduli space. In general, moduli spaces in $\mathcal{N} = 1$ theories can have branches with different dimensions; this will happen because the h^n with $n \geq 2$ can differ on different branches.

We could also extend the definition in the $\mathbb{C}^5$ case to include the $\partial W/\partial X = 0$ superpotential constraints as well, by defining

$$D = e_a X^a + \epsilon^{abcde} e_a e_b e_c Y_{[de]}.$$

Both the forms $\rho^{[a_1 \ldots a_k]} e_{a_1} \ldots e_{a_k}$, and the forms representing Ext^k, are in many ways like differential forms. In particular they have a wedge product which obeys the same formal rules. These rules can be axiomatized in the structure of an "abelian category" as described in Gelfand and Manin [22] and many other textbooks.

The field ρ with n indices (the "top form") is special as it is a sum of adjoints and thus one can take its trace to get the analog of an integral. This trace provides a bilinear form on $\mathrm{Ext}^k(A, B) \times \mathrm{Ext}^{n-k}(B, A)$. If we work in the original orbifold quiver category (i.e. not setting any links to zero), one can see that this form is non-degenerate and thus the dimensions of these two spaces will be the same. This is a quiver analog of "Serre duality" (on which more later).

The D0 problem again provides an example. There, the number of extra moduli from the link $X_{3,1}$ was equal to h^2. This is an example of the duality $\dim \mathrm{Ext}^1(D0, D0) = \dim \mathrm{Ext}^2(D0, D0)$ valid for the orbifold quiver category; the extra Ext^1 is the dual of the h^2 we derived earlier.

Having defined the higher Ext groups, we remark that the relation between Ext^1 and bound states, its further relations to exact sequences and triangles, and the role of these in solving D-flatness conditions, all go through in exactly the same way as before.

7 Some large volume considerations

We more or less have all the tools we need to find the spectrum of BPS branes in simple orbifolds, anywhere near the orbifold point. Indeed it is quite easy to analyze the states which can be formed from arbitrary numbers of two fractional branes, and one might do some exercises of this sort to get familiar with the ideas.

The description does not look much like the way one is used to describing D-branes at large volume, but on a deeper level the two limits can be made part of the same formalism. The key to this is the "decoupling statement" which asserts that all of the holomorphic structure of the branes must be independent of Kähler moduli. This will imply that quiver objects and all the information in the F-flatness conditions must in some sense be the same as the set of holomorphic bundles or suitable generalizations of these.

To see this, we will need to review some of the theory of D-branes in "large volume," i.e. when stringy corrections are negligible. We assume some familiarity with the material in Polchinski's TASI lectures [27].

The problem of finding supersymmetric embeddings of branes governed by the supersymmetrized Nambu-Born-Infeld Lagrangian has been much studied and the general equations governing these are known. In the $\alpha' \to 0$ limit (we will be a bit more general below), and if no other background fields are turned on, the general answer is that a supersymmetric brane must embed into a calibrated submanifold, and the gauge fields must preserve supersymmetry in the usual Yang-Mills sense, except that one can use an inhomogeneous supersymmetry as well: an unbroken supersymmetry is given by two spinor parameters (ϵ, ϵ') which must satisfy

$$0 = \delta\chi = \Gamma^{IJ} F_{IJ} \epsilon + \epsilon'. \tag{31}$$

The inhomogeneous supersymmetry simply corresponds to shifts of the non-interacting gaugino (or its diagonal component in $U(N)$ theory). Such a supersymmetry is guaranteed to be present because the D-brane spontaneously broke half of the supersymmetry of the bulk theory. However, we see that which half is preserved can depend on the world-volume background fields, and in general will be some linear combination of the ϵ and ϵ' bulk supersymmetries.

7.1 Calibrated geometry

A calibration is a p-form λ which is closed, $d\lambda = 0$, and which provides a lower bound for the volume: for any p-dimensional linear subspace V of the tangent space at any point, one has

$$\lambda|_V \leq (\text{volume})_V \tag{32}$$

considered as an equation between oriented p-forms.

Given a calibration, one has an easy way to find minimal volume manifolds: a calibrated submanifold, defined as a submanifold Σ whose tangent bundle saturates (32) at each point, is necessarily minimal volume. This is because Stokes' theorem tells us that $\int_\Sigma \lambda$ will be preserved under any continuous variation of the submanifold Σ, while the bound (32) then tells us that volume can only increase under these variations.

Examples in flat space $\mathbb{C}^n$ are not hard to find and check. They include the Kähler form

$$\omega = \frac{1}{i}\sum_{I=1}^{n} dz^I \wedge d\bar{z}^I,$$

and the various real parts of the holomorphic n-form,

$$\Omega_\theta \equiv \Re e^{i\theta}\Omega$$

where

$$\Omega = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n.$$

In general, calibrations arise from covariantly constant spinors and thus are closely associated with supersymmetry. Suppose we have a covariantly constant spinor ϵ; then we claim that

$$\lambda = \epsilon^+ \Gamma^{(p)} \epsilon$$

is a calibration, where

$$\Gamma^{(p)} = i^{(p(p-1)/2} \prod_I \Gamma_I dx^I$$

and we choose p to make the product non-zero.

To see this, consider a linear p-dimensional subspace V of the cotangent space to a point, with orthonormal basis e^I, and consider the operator

$$\Gamma_V = i^{(p(p-1)/2} \prod_I \Gamma_I e^I.$$

This operator satisfies the equation

$$(1 - \Gamma_V)^2 = 2(1 - \Gamma_V)$$

and thus

$$\epsilon^+(1 - \Gamma_V)\epsilon \geq 0$$

which implies the bound.

This argument also shows that branes wrapped on calibrated submanifolds preserve supersymmetry: taking V to be a tangent space to the brane, we have $\Gamma_V = \Gamma_D$, so saturating the bound implies $\epsilon = \Gamma_D \epsilon$.

7.2 B type calibrated submanifolds

Each of the types of calibrated manifold has its own distinctive geometry. The most intensively studied case is the Calabi-Yau manifolds, with $SU(n)$ holonomy, for which there are two types of calibration.

The first (called "B-type" for reasons given later) is with respect to the Kähler form, or powers of the Kähler form. It is not hard to see that these are holomorphic submanifolds, defined by an holomorphic map from a p-dimensional complex manifold into space, or else as the zero set of $n - p$ complex equations in space. This also includes branes which embed in the entire CY manifold, and the brane which embeds in a point. All of the methods of algebraic geometry which were so useful in analyzing the geometry of Calabi-Yau manifolds will be just as useful in analyzing these branes.

Gauge field backgrounds which preserve supersymmetry (admit solutions to (31)) can be found by slightly generalizing a traditional argument used in heterotic string compactification (as given in GSW); they are the hermitian Yang-Mills connections. The argument is simply to use the Kähler structure of the manifold to rewrite the Dirac algebra $\{\Gamma^I, \Gamma^J\} = 2g^{IJ}$ as an algebra of fermionic creation and annihilation operators, which naturally act on the space of (antiholomorphic) $(0,p)$-forms. Explicitly, using complex coordinates z^I and $\bar{z}^{\bar{I}}$,

$$\Gamma^{\bar{I}} \to d\bar{z}^{\bar{I}} \tag{33}$$

$$g_{I\bar{J}}\Gamma^I \to i_{\bar{J}} \qquad s.t.\{i_{\bar{J}}, d\bar{z}^{\bar{I}} = \delta^{\bar{I}}_{\bar{J}}. \tag{34}$$

The two spinor representations of $SO(2n)$ then reduce to the direct sum of even or odd differential forms. Taking for ϵ and ϵ' the zero form, (31) becomes

$$F^{(2,0)} = F_{IJ} dz^I \wedge dz^J = 0 \tag{35}$$
$$F^{(0,2)} = F_{\bar{I}\bar{J}} d\bar{z}^{\bar{I}} \wedge d\bar{z}^{\bar{J}} = 0 \tag{36}$$

and finally, for $F^{(1,1)} = F_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}$, we have

$$F^{(1,1)} \wedge \omega^{n-1} = c\omega^n$$

where ω is the Kähler form and c is an arbitrary constant.

7.2.1 BPS central charge

The constant c determines which $\mathcal{N} = 1$ subalgebra of the bulk $\mathcal{N} = 2$ supersymmetry is unbroken. The $\mathcal{N} = 2$ supersymmetry algebra admits a moduli space of $\mathcal{N} = 1$ subalgebras parameterized by a phase θ, defined by

$$0 = \operatorname{Im} e^{i\theta} Q^\alpha.$$

In terms of the parameters ϵ and ϵ' we can write

$$0 = \operatorname{Im} e^{i\theta}(i\epsilon + \epsilon')$$

which if we insert the solution gives

$$0 = \operatorname{Im} e^{i\theta}(\omega + i l_s^4 F) \wedge \omega^2 \tag{37}$$

This is only an $l_s \to 0$ estimate and one can easily do better. Another way to determine the unbroken $\mathcal{N} = 1$ is computing the BPS central charge of the brane; its phase will be $e^{i\theta}$. The BPS central charge of a D-brane is determined by its Ramond-Ramond charges, which are those of a source

$$\int_\Sigma C \wedge \operatorname{Tr} e^{B-F}. \tag{38}$$

Assuming that in the large volume limit a pure D$2p$-brane has central charge $\int (-iV)^p$ leads to the expression

$$Z = \int_\Sigma e^{-F+B+i\omega} = \sum_p \frac{1}{(d-p)!} ch_p (B + iJ)^{d-p} \tag{39}$$

where the Chern character ch_p is the $2p$-form in the expansion of $\operatorname{Tr} e^F$.

These considerations lead to a formula whose leading $l_s \to 0$ limit is (37). They can be confirmed microscopically from an analysis using the supersymmetrized Nambu-Born-Infeld action, leading to an equation derived by Mariño, Minasian, Moore and Strominger (the MMMS equation) [25]. They take into account all powerlike corrections in l_s, but not effects due to world-sheet instantons.

7.2.2 The Donaldson-Uhlenbeck-Yau theorem

There is a standard mathematical approach to solving such equations, based on what is called the Hitchin-Kobayashi correspondence. For hermitian Yang-Mills this is encapsulated in the theorems of Donaldson and Uhlenbeck-Yau, but the approach is in fact more general and applies to the MMMS equations and indeed in a sense we will describe to the general case of string scale Calabi-Yaus.

We start by assuming we can find a solution to the equation $F^{0,2} = 0$. This is the integrability condition for the antiholomorphic part of the connection,

$$[\bar{\partial} + \bar{A}, \bar{\partial} + \bar{A}] = 0,$$

and thus one can locally trivialize the bundle by some holomorphic transformation which acts on sections as $\psi \to g(z)\psi$. Globally, the bundle need not be trivial, but all of the transition functions will be holomorphic. Thus each solution to this equation (up to complex gauge equivalence) oorresponds to a holomorphic bundle. This is a notion which can be defined purely in terms of the complex structure of the manifold, so this part of the problem does not depend on the Kähler moduli of the CY. A hermitian connection will then automatically solve $F^{2,0} = 0$.

This leaves the equation on $F^{(1,1)}$, and the DUY theorems then state necessary and sufficient conditions that there exist a particular connection in this orbit of the complexified gauge group which solves this equation.

First of all, one knows that the first Chern class $c_1 = \int \mathrm{Tr} F \wedge J \wedge J$ is a topological invariant, so it is computable just knowing the holomorphic bundle. Integrating the equation shows that the constant c from above is determined in terms of c_1.

We then define the slope of a bundle E, $\mu(E)$, as the ratio

$$\mu(E) = \frac{c_1(E)}{rk(E)} = \frac{1}{rk(E)} \int \mathrm{Tr} F \wedge J^{n-1}.$$

A holomorphic bundle E is then μ-stable if, for all subbundles E', we have

$$\mu(E') < \mu(E). \tag{40}$$

The DUY theorems then state that an irreducible (simple) solution to $\omega F^{1,1} = c$ will exist if and only if E is μ-stable. Both the equation and this condition depend on the Kähler form and it can be seen in examples that if $b^{(1,1)} > 1$

(so there is a choice of Kähler form) this dependence is nontrivial; there are "walls" in Kähler moduli space on which the list of stable bundles changes.

Note the very close parallel between this condition and the θ-stability condition we discussed that controls the solvability of D-flatness conditions. Both can be understood using the ideas of geometric invariant theory and in fact the DUY theorem is proven by the same idea we discussed earlier of flowing within a complexified gauge orbit to a minimum of a potential, here the Yang-Mills action. Again the stability condition is what guarantees that the minimum is still on the original orbit.

7.3 A type calibrated submanifolds

The other possibility is to calibrate with respect to one of the n-forms Ω_θ. These are A-type or special Lagrangian (sL) submanifolds. One usually wants to keep track of θ and distinguish A_θ or sL_θ submanifolds, because θ determines the unbroken $\mathcal{N}=1$ supersymmetry just as in our previous discussion.

The reason for the name special Lagrangian is that one can show (by an easy local argument) that the calibration condition is equivalent to the pair of conditions

$$\omega|_\Sigma \quad = 0 \tag{41}$$

$$\mathrm{Im}\, e^{i\theta}\Omega_\Sigma \quad = 0. \tag{42}$$

For A branes in other than two dimensions, a supersymmetric gauge connection must satisfy $F=0$, i.e. it is a Wilson line. The case of two dimensions is special and is better understood by bringing in the formalism of hyperkähler geometry, which we will not do here.

One of the basic results about these branes is that the moduli space of a smooth special Lagrangian Σ has real dimension $b_1(\Sigma)$. Combining this with the moduli of the flat gauge connection, the D-brane moduli space has complex dimension $b_1(\Sigma)$.

The BPS central charge of an A brane Σ is simply

$$Z = \int_\Sigma \Omega. \tag{43}$$

7.4 Mirror symmetry and comparison of the two pictures

Mirror symmetry will exchange the A and B branes. The basic physics behind this is a realization of mirror symmetry proposed by Strominger, Yau and Zaslow. The idea is to describe the CY_3 as a T^3 fibration, and then interpret mirror symmetry as T-duality along the fibers. This exchanges the D0-brane with a distinguished D3-brane with topology T^3; since $b_1(T^3) = 3$ one can rederive the original Calabi-Yau as the moduli space of this D3-brane on the mirror. Similarly the T-duality will exchange every B brane on M, with its moduli space and all other physics, with a corresponding A brane on W.

The special Lagrangian picture has some advantages and disadvantages over the holomorphic brane picture. Its main disadvantage at present is that it is much less well understood mathematically, but this situation may improve.

Sometimes in string theory, one finds that duality can exchange a description in which some observable receives quantum corrections, with a dual description in which the corresponding observable is always equal to its classical value. In this case one is usually much better off using the description which is always classical for detailed computation, while the other description retains its primary importance only in limits in which it becomes classical.

Indeed mirror symmetry is the example *par excellence*, as the observables relating to the special geometry of the $\mathcal{N} = 2$ compactified theory, namely the prepotential and BPS central charges, are classically exact in the IIb compactification, in which the BPS central charges are just (43), the periods of the holomorphic three-form, which depend only on complex structure and are independent of Kähler moduli. This is in contrast to the IIa compactification in which the BPS central charges are independent of complex structure and depend only on Kähler moduli; although they take simple values in the large volume limit, in general they receive world-sheet instanton corrections. Physically, mirror symmetry is usually used as a tool for summing these instanton corrections to derive exact prepotentials and solve $\mathcal{N} = 2$ string compactifications.

7.5 The decoupling statement

We should ask the same question, whether there is any dual picture in which the world-volume theory can be determined purely classically, in our D-brane context. It turns out to have a pretty answer [8]: of the two semi-independent parts of our $\mathcal{N} = 1$ world-volume Lagrangian, the holomorphic part and the D-flatness conditions, one of them will receive quantum corrections while the other will not. In the B brane picture, one can see that the holomorphic structure is exact at large volume, while the D-flatness conditions receive instanton corrections. In the A brane picture, it is the other way around.

There are various arguments for this. On one side, we can consider the topologically twisted theory of the open strings ending on the D-branes. This model only depends on a subset of the CY moduli, the complex structure in the B model and the Kähler structure in the A model, and since B model observables do not depend on the volume they must be exact at large volume. One can show that this topological theory can be used to compute the holomorphic structure and superpotential of the world-volume theories.

This leaves the questions of how the Kähler moduli affect B branes, and how complex structure moduli affect A branes. These moduli directly control the BPS central charges of the branes and the most striking physics resulting from this is the variation of the BPS spectrum expressed in lines of marginal stability: as we vary the moduli, a brane can decay into constituents, or new bound states can form. As we discussed, this type of behavior is controlled by the D-flatness conditions, leading to the idea that these moduli couple only through the FI terms. Since A brane central charges are exact at large volume (they are the D-particles of IIb theory), this more or less requires that the A brane D flatness conditions are exact at large volume.

This is explained from a world-sheet point of view in [15]. A simple space-time argument for this can be made by using the decomposition of the moduli of the $\mathcal{N} = 2$ bulk theory under the $\mathcal{N} = 1$ supersymmetry of the D-branes. This argument is dimension dependent and it is most convenient to make it for the $3 + 1$ world volume theories we have been discussing so far. In this case, B branes naturally live in IIb theory, while A branes naturally live in IIa theory. Let us consider the case of IIb theory; then the $\mathcal{N} = 2$ vector multiplets contain complex structure moduli, while the hypermultiplets contain Kähler moduli and partner RR scalars. Under $\mathcal{N} = 1$ supersymmetry, the vector must decompose into a vector and a chiral multiplet, so this chiral multiplet is purely NS-NS and is just a complex structure modulus. The hypermultiplet decomposes into two chiral multiplets and one can check that each of these contains one real NS-NS Kähler modulus and one real partner RR scalar. (The simplest case to check is the related type I theory which keeps only one of these chiral multiplets).

Now, since the $\mathcal{N} = 1$ superpotential is holomorphic, if it depends on a chiral multiplet, it must depend on both of its real components. This is fine for the complex structure multiplet, but if it depends on a Kähler modulus, this means it will depend on an RR scalar, call it C_i. Now perturbative string amplitudes very generally do not produce nonderivative couplings to the RR fields; for each RR scalar there is an exact symmetry $\delta C_i = \epsilon_i$. This contradiction implies that the superpotential is independent of Kähler moduli (reproducing our previous claim).

Fayet-Iliopoulos terms are real however and in fact naturally depend only on one real component of a chiral multiplet. The generic coupling of this type to a world-volume vector superfield V is

$$\int d^4\theta \; (\phi + \phi^+ - V)^2.$$

This is gauge invariant under $\delta V = \Sigma + \Sigma^+$ and $\delta\phi = \Sigma$. This shifts the real part of ϕ, but a world-volume gauge transformation cannot act on the bulk fields: thus this must be an exact symmetry of the bulk theory. This will be true only if the real part of ϕ is a RR scalar, and this allows FI terms to be controlled by Kähler moduli but not complex moduli.

The strongest test of the decoupling statement would of course be to simply derive the F or D-flatness conditions in the appropriate brane world-volume theories and check that they are the same in the cases they are supposed to be. Let us consider this problem in an example. Since the F flatness conditions are primary, one should start with these, and thus with the B brane picture.

8 Introduction to the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

We start by reviewing the basic picture of strings compactified on this space. Defining closed strings on this orbifold leads to an $\mathcal{N} = 2$ theory with marginal operators in the twisted sectors corresponding to a single complexified Kähler modulus. The geometric interpretation of turning these on should correspond to some operation which fixes the complex structure but introduces an element of $H^{1,1}$.

Such an operation is known mathematically and is called a blow-up. The idea is that one can take any point in an n-dimensional complex manifold and replace it by a $\mathbb{CP}^{n-1}$ (henceforth just called $\mathbb{P}^{n-1}$) parameterizing the various tangent vectors to the original point. This can be made precise by the following equations in local coordinates z^i: to blow up $z = 0$, introduce a $\mathbb{P}^{n-1}$ with homogeneous coordinates $w^i \cong \lambda w^i$, $1 \leq i \leq n$, and impose the equations

$$z^i w^j = z^j w^i \qquad \forall i, j.$$

Away from $z = 0$ one can solve for w, while at $z = 0$ they are unconstrained.

Since we define our $\mathbb{Z}^3$ orbifold as $z^i \cong \omega z^i$ with $\omega = e^{2\pi i/3}$, this operation is well defined on the orbifold and replaces the singularity with a $\mathbb{P}^2$. One can also check that the holomorphic three-form has no zeroes, so the result is a Calabi-Yau. In fact it is (the total space of) the line bundle $M = \mathcal{O}_{\mathbb{P}^2}(-3)$.

Thus finite energy D-branes must wrap cycles in the $\mathbb{P}^2$. Now $H^p(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ for $p = 0, 2, 4$ and integrating RR potentials over these three cycles leads to three conserved RR charges (agreeing with the orbifold) which we can call D0, D2 and D4 charge.

8.1 Line bundles on $\mathbb{P}^2$

Let us discuss holomorphic branes on large volume $\mathbb{P}^2$ a bit (much more detail can be found in [14], and in fact the mathematical classification of stable sheaves is completely known in this case). The simplest ones are the line bundles, which we will denote $\mathcal{O}(n)$. These can be defined for $n \geq 0$ as the bundles which admit sections which are degree n polynomials in the homogeneous coordinates w^i. For $n < 0$, one can either talk about sections with poles, or define $\mathcal{O}(-n)$ as the dual object to $\mathcal{O}(n)$ such that multiplication of sections from the two produces a function. All of these are stable and correspond to a D4-brane with n units of D2 charge turned on.

From (38), the RR charge of $\mathcal{O}(n)$ is given by the Chern character $\mathrm{ch}(\mathcal{O}(n)) = e^{nJ}$, where $J = c_1(\mathcal{O}(1))$ is the generator of $H^2(\mathbb{P}^n, \mathbb{Z})$ (the unit of magnetic flux), and we use conventions where the $\sqrt{\hat{A}}$ term has been factored out. The successive terms in the expansion in J (up to $o(J^2)$) are the D4, D2 and D0 charges. Thus for $n \neq 0$ these objects also carry D0 charge.

It turns out that these objects already provide a basis for the K theory of $\mathbb{P}^2$ (and the K theory with compact support of M). Since M is not compact, we cannot directly check this from the intersection form, but as suggested in section 2 we will instead look at the index of the Dirac operator on $\mathbb{P}^2$. In the Kähler context this becomes the index of the $\bar{\partial}$ operator, which is also known as the relative Euler character:

$$\chi(E, F) \equiv \sum_p (-1)^p \dim H^{0,p}(M, E^* \otimes F).$$

The index theorem in this case reduces to the Grothendieck-Riemann-Roch formula, which is (2) with $\hat{A}(M) = Td(M)$.

We now proceed to compute $\chi(\mathcal{O}(m), \mathcal{O}(n)) = \chi(\mathcal{O}, \mathcal{O}(n-m)) \equiv \chi(\mathcal{O}(n-m))$ on $\mathbb{P}^N$. For those who want to do this directly from the index theorem (it is not too hard), we quote $Td(\mathbb{P}^N) = (J/(1-e^{-J}))^{N+1}$.

A shortcut to this is to compute $\chi(\mathcal{O}(n))$ for large n, by explicitly computing $\dim H^0(\mathcal{O}(n))$ and appealing to the following vanishing theorem ([21], p. 159),

Theorem B. Let M be a compact complex manifold and $L \longrightarrow M$ a positive line bundle (i.e. there exists a connection such that $F_{i\bar{j}}$ is everywhere positive; for $\mathbb{P}^n$ we just need $c_1 > 0$). Then for any holomorphic vector bundle E, there exists μ_0 such that

$$H^{0,q}(M, L^\mu \otimes E) = 0 \qquad \forall q > 0, \mu > \mu_0.$$

to conclude that $\chi(\mathcal{O}(n)) = \dim H^0(\mathcal{O}(n))$. But we also know from (2) that χ is polynomial in n, so computing it at large n determines it for all n.

Since the sections of $\mathcal{O}_{\mathbb{P}^N}(n)$ are degree n polynomials in $N+1$ homogeneous variables, we can conclude that

$$\chi(\mathcal{O}_{\mathbb{P}^N}(n)) = \frac{1}{N!} \prod_{i=1}^{N} (n+i).$$

In particular, the three line bundles $\mathcal{O}, \mathcal{O}(1)$ and $\mathcal{O}(2)$ have

$$\chi(\mathcal{O}(m), \mathcal{O}(n)) = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \tag{44}$$

which is a unimodular matrix, and thus this set can be used as a basis. For example, the K theory class of the D0 is determined by solving $N_0 + N_1 e^J + N_2 e^{2J} = J^2 + o(J^3)$ to be

$$[\mathcal{O}_z] = [\mathcal{O}(-1)] - 2[\mathcal{O}] + [\mathcal{O}(1)].$$

We have given the D0 at the point z its mathematical name $\mathcal{O}_z$, the structure sheaf of the point z.

Readers with any familiarity with D-branes on orbifolds will immediately recognize that the three fractional branes B_i of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold cannot be these, as these satisfy

$$[\mathcal{O}_z] = [B_1] + [B_2] + [B_3]. \tag{45}$$

So what are the B_i ?

8.2 Identifying the fractional branes

This problem was first solved by mirror symmetry techniques [10]. The idea [8] to take expressions for the BPS central charges computed using mirror symmetry and explicitly compare them between the large volume Dp-brane basis and a basis at the orbifold point.

We need to discuss the structure of the stringy Kähler moduli space of M, i.e. the complex structure moduli space of its mirror. It is a Riemann sphere with three singularities. One is the large volume limit near which the BPS charges are (39). The second is the orbifold point, described by our quiver theory. Finally there is a third singularity called "conifold point" at which one of the BPS central charges vanishes. Directly continuing this to large volume, one finds that the corresponding brane is $\mathcal{O}$, the "pure" D4.

As usual in $\mathcal{N}=2$ theories, the simplest attribute of a singularity is the monodromy it induces on the charges. In large volume this is $B \to B+1$ which takes $\mathcal{O}(n) \to \mathcal{O}(n+1)$. Around the conifold point it is determined by the usual considerations involving a massless particle, while the orbifold point has an associated $\mathbb{Z}_3$ monodromy which permutes the fractional branes and the FI terms of the quiver theory.

The results which suffice to determine the identification in this case are that $\mathcal{O}$ is one of the fractional branes (tested by continuing its period to the conifold point), and the $\mathbb{Z}_3$ monodromy expressed in the large volume basis using mirror symmetry.

We did not quote the final identification of the fractional branes as it turns out that there is a simpler way to do this.

9 The McKay correspondence

An independent way to identify the fractional branes follows what is called the "generalized McKay correspondence" in mathematics [28, 5], which we summarize. It can be physically motivated [17] and agrees with the mirror symmetry prediction wherever this has been tested.

The idea is that it is relatively easy to geometrically identify a dual set of "fractional branes" which fill the noncompact space M, and to find their intersection form with the original fractional branes. This data then turns out to determine the original fractional branes.

An extended fractional brane can be taken as a D9-brane filling $\mathbb{C}^3/\Gamma$. Now the orbifold projection (25) acts on the spatial coordinates as well; for the Yang-Mills connection it is

$$\gamma^{-1}(g)A_i(z)\gamma(g) = r_i^j A_j(g(z)). \tag{46}$$

This is a twisted boundary condition and its interpretation is rather clear, at least far from the singularity. It means that scalar matter in the fundamental, i.e. a section of the associated bundle, must transform as

$$\gamma\phi(z) = \phi(g(z)). \tag{47}$$

A particularly simple case is to take γ to be the regular representation, in which case we can consider $\phi(z)$ as a vector-valued field indexed by an element of Γ, so (47) becomes

$$\phi_{gh}(z) = \phi_h(g(z)). \tag{48}$$

This bundle is referred to as the "tautological bundle" over the quotient space. It can be decomposed as a direct sum over bundles R_i associated to irreps γ which if Γ is abelian are line bundles; these are the tautological line bundles.

Both types of fractional brane are labelled by a choice of group representation, and we can write a quiver theory summarizing the massless fermion content of any combination of these, again associating each brane to a quiver node. Let R_i be the D9 node corresponding to r_i and S^j be the D3 node corresponding to r_j, i.e. the original fractional branes.

The spectrum of $(3, 9)$-strings between a pair (R_i, S^j) is also determined by the orbifold projection. In fact, massless fermions with such boundary conditions (Dirichlet-Neumann boundary conditions in all the transverse dimensions) transform like scalars in $\mathbb{C}^3$, so this projection acts as

$$\gamma_3^{-1}(g)\chi\gamma_9(g) = \chi \tag{49}$$

so we have $n_{ij} = \delta_i^j$ such fermions in each sector. As in section 2, this implies that the intersection form between the two types of branes should be

$$\langle R_i, S^j \rangle = \delta_i^j. \tag{50}$$

This is the natural Poincaré duality on our noncompact space M, between $K(M)$ and the K theory of bundles with compact support $K_c(M)$ (meaning bundles over compact submanifolds) and we see that it indeed gives a perfect pairing.

This relation can then be used to determine the K theory classes of the S_j (and, given more formalism, even identify them as specific holomorphic objects). We need to know the intersection form for the R_i in an explicit basis to make this definition concrete. For example, if we have

$$\langle R_i, R_j \rangle \equiv (I^{-1})_{ij}, \tag{51}$$

then we can write

$$S^j = I^{ij} R_i \tag{52}$$

for which

$$\langle S^j, S^k \rangle = I^{jk}. \tag{53}$$

In terms of the K theory classes, (52) becomes

$$[S^j] = I^{ij}[R_i], \tag{54}$$

a simple explicit formula for the K theory classes of the fractional branes given those of the tautological line bundles.

In practice, we will restrict the bundles R_i from the total space M to the exceptional divisor, and then use $\chi(R_i, R_j)$ on this space as our intersection form in these formulas.

9.1 The $\mathbb{C}^3/\mathbb{Z}_3$ example

We have done almost all the work already if we can convince ourselves. that the R_i for $i = 1, 2, 3$ are in fact the bundles $O(i-1)$ of our previous discussion. This can also be seen fairly directly from the quiver diagram using the fact that the D0 is the object (1 1 1).

The moduli space of this theory is then the space M itself (as seen by the D0). If we are just interested in the $\mathbb{P}^2$, we can solve a simpler problem obtained by setting one set of the links to zero, say $Z^i = 0$. One already sees a $\mathbb{P}^2$ for the moduli space of (say) X^i (with appropriate signs of the FI terms; this went into the choice of which link to set to zero) and it is easy to check that the constraints $Y^{[i}X^{j]} = 0$ then determine $Y^i = X^i$ up to complex gauge equivalence.

The three branes R_i are then distinguished by which node S_i their associated link ψ_i is charged under. The line bundle interpretation of R_i is then determined by the transformation properties or the allowed values of ψ_i, which must be a section of the associated bundle. We can identify these by comparing the gauge invariant observables constructed from each, as we know that both X^i and Y^i correspond to the homogeneous coordinates. Only the relative transformation properties are defined; one can define one of the bundles to be $\mathcal{O}$.

Looking at the figure,

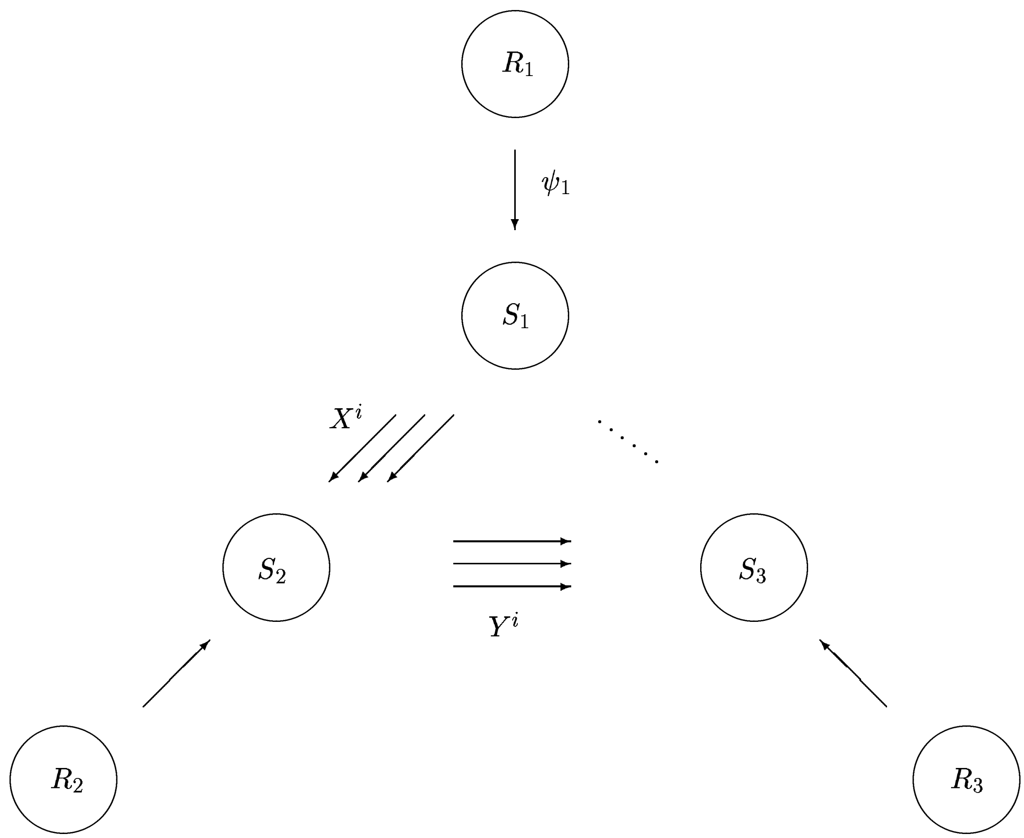

Fig.4. Dual bases

we see that gauge invariant combinations involving these variables are $\psi_1 X^i Y^i \sim \psi_2 X^i \sim \psi_3$. This implies that gauge invariant sections must look like $\psi_1 \sim 1$, $\psi_2 \sim x$, and $\psi_3 \sim x^2$, which establishes the claim.

Thus $\chi(R_i, R_j)$ is given by (44). We need to invert it; this is easy if we realize that it can also be thought of as multiplication of functions $a+bz+cz^2$ by a formal power series $(1-z)^{-3}$, so its inverse is $(1-z)^3$ or

$$\chi(S_i, S_j) = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{55}$$

We can now implement (54) to find that

$$[S_3] = [R_1] = [\mathcal{O}] \tag{56}$$
$$[S_2] = [R_2] - 3[R_1] = e^J - 3 \tag{57}$$
$$[S_1] = [R_3] - 3[R_2] + 3[R_1] \tag{58}$$
$$= e^{2J} - 3e^J + 3 = e^{-J} + o(J^3). \tag{59}$$

We already knew that one of the fractional branes was $\mathcal{O}$ (this seems to be a very general result). The third relation is compatible with

$$S_1 = \mathcal{O}(-1)$$

and indeed a large volume monodromy avoiding the conifold point could clearly turn this into $\mathcal{O}$, so this is consistent with expectations.

The identity of the second brane may not be as obvious but there is a natural exact sequence which this formula suggests, and a much more developed framework in which this can be seen to be necessary. It is

$$0 \longrightarrow \bar{S}_2 \longrightarrow \mathcal{O}^3 \xrightarrow{f} \mathcal{O}(1) \longrightarrow 0. \tag{60}$$

The obvious map f takes a vector of three functions ψ_i (a section of $\mathcal{O}^3$) and produces $z^i\psi_i$. Thus a section of S_2 is a set of functions satisfying $z^i\psi_i = 0$. In fact one can see that this is the cotangent bundle twisted by (tensored with) a line bundle,

$$S_2 = \bar{\Omega}_{\mathbb{P}^2}(1) \equiv \bar{\Omega}_{\mathbb{P}^2} \otimes \mathcal{O}(1).$$

This is seen perhaps more easily in its dual form

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^3 \longrightarrow T_{\mathbb{P}^2} \longrightarrow 0$$

which says that a tangent vector to $\mathbb{P}^2$ can be written as

$$v^i_j z^j \frac{\partial}{\partial z^i} + \lambda z^i \frac{\partial}{\partial z^i}$$

where the choice of λ (a section of $\mathcal{O}$) drops out.

Note that S_2 is not a coherent sheaf but an "antibrane" to a coherent sheaf, with D4 charge -2. That such a thing would be necessary was already clear from (45) and does not seem so remarkable at first, but it is a sign that we are going to have to start thinking harder about bound states of branes and antibranes than we have done so far.

10 Moduli spaces of coherent sheaves on $\mathbb{P}^2$

Having identified the fractional branes, we are at a point where we can make nontrivial comparisons between large volume and the orbifold point. Recall that we have an extremely strong prediction: the holomorphic objects and their moduli spaces should be literally the same in these two limits.

The simplest thing to compare is the intersection form, or $\chi(E,F) - \chi(F,E)$. This is supposed to count massless fermions between pairs of branes, and the claim is that as we vary the Kähler moduli, the massless fermions must carry over unchanged between the two limits.

There is a subtlety in this interpretation of (30), however, because we added fields ρ^{ab} which were not present in the original quiver theory in making our definition. This interpretation is still correct however because we did not count the fermions in the links Z^c which we set to zero to simplify the analysis. These contribute exactly as the term we added, and in fact these two fields are "dual" in a clear sense.

Now $\chi(E,F)$ is (55) at large volume and (30) at the orbifold point, so indeed the intersection forms agree; in fact we have an even stronger statement that $\chi(E,F)$ itself agrees. The simplest argument that this had to happen is that the dimensions of moduli spaces must agree in the two limits, and this and the intersection form is enough information to reconstruct χ. In fact this is a very small part of the equivalence between the two descriptions: all objects, all moduli spaces, and all morphisms must agree as well.

In fact the equivalence between representations of the $\mathbb{C}^3/\mathbb{Z}_3$ quiver (with the Z link set to zero) and a large subset of sheaves on $\mathbb{P}^2$ had been observed in [14] and follows from a mathematical theorem due to Beilinson.

Without describing this in all detail, let us show how a quiver representation E can correspond to a sheaf $S(E)$. If all of the matrices in the quiver representation were zero, we know the correspondence from the above, we have

$$S_0(E) = \oplus N_i S_i = \sum_i V_i \otimes S_i$$

using notations established in section 3.

To incorporate the quiver configuration, we first note that there are natural maps $\hat{e}^a : S_i \longrightarrow S_{i+1}$ for $1 \leq a \leq 3$. For $i = 2$ this is clear from (60), and these are just the maps

$$\hat{e}^a(\psi_i) = \psi_a.$$

For $i = 1$, one needs to check that $\mathcal{O}(-1) \cong \Lambda^2\Omega(1)$; in other words a section of $\mathcal{O}(-1)$ can be written as a vector of three functions $\psi_{[ij]}$ satisfying $z^i\psi_{[ij]} = 0$. Then one has $\hat{e}^a(\psi_{[ij]})_k = \psi_{ak}$.

Because of the antisymmetrization in $\psi_{[ij]}$, these maps satisfy the relations

$$\hat{e}^a\hat{e}^b + \hat{e}^b\hat{e}^a = 0. \tag{61}$$

Using these, the quiver configuration $X^a_{i,j}$ can be used to construct a natural operator on $S_0(E)$,

$$\hat{D} = \sum_a X^a \hat{e}^a.$$

From (61) and (27), one sees that $\hat{D}^2 = 0$, so the operator D has a cohomology, which is a sheaf. This is the sheaf which corresponds to the original quiver configuration. One can show that this relation is one to one; furthermore the construction can be reversed and used to show equivalences between all morphisms as well.

This provides a very detailed equivalence for many objects in the large volume limit, and at the orbifold point, and is the sense in which the quiver theory really does know about all the geometry of coherent sheaves at large volume. However, on reflection one can see that the set of quiver representations and the set of coherent sheaves on $\mathbb{P}^2$ cannot be literally identical. The simplest counterexample is the D2-brane. This also has a simple representation as an exact sequence,

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{f} \mathcal{O} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0.$$

Here Σ is a $\mathbb{P}^1$ contained in $\mathbb{P}^2$ and $\mathcal{O}_\Sigma$ is its structure sheaf, the D2. The map f is just linear, $f = a_i z^i$, and Σ is the curve $f = 0$ in $\mathbb{P}^2$, leaving $\mathcal{O}_\Sigma$ as its cokernel.

The main point we want to make about this is that the D2 is a bound state of a fractional brane with a fractional antibrane, i.e. it is $(-1\ 0\ 1)$. Since we have a complete basis, there is no other way to make it. Physically, in fact, such a bound state cannot exist at the orbifold point, as its BPS central charge would have vanished. Thus there is no contradiction, but we see that the quiver representation framework as we have defined it so far cannot describe all holomorphic objects at all points in moduli space.

If one's goal is just to describe each brane separately, there are easy ways around this problem. For example, the D2 with a flux turned on, $\mathcal{O}_\Sigma(1)$, can be described with fractional branes – it is $(0\ 1\ 2)$ – and from large volume considerations we know this has the same moduli space as $\mathcal{O}_\Sigma$. In this sense, the quiver does give an adequate description of all sheaves, and is used for this purpose in their classification.

However, one would prefer to have a uniform description of all the branes. Worse, it looks like we have found a contradiction to the decoupling statement. However the agreement between the objects that do exist in both limits is so precise, one feels that there must be some way to extend it to these cases as well. We will eventually find that this can be done, by using the derived category.

11 Flow of gradings

As one explores the relationship between the quiver description of holomorphic branes and the coherent sheaf description, more subtle differences start to appear.

Let us consider the line bundles. Although $\chi(\mathcal{O}(m), \mathcal{O}(n))$ agrees between the two descriptions, the groups $\mathrm{Ext}^{p*}(\mathcal{O}(m), \mathcal{O}(n))$ as defined by the quiver theory do not always agree with the groups $H^p(M, \mathcal{O}(n-m))$ defined by sheaf theory. (We distinguish the quiver groups with a "*" because there is also a definition of $\mathrm{Ext}^p(A, B)$ for sheaves, which is what one would use in a more general discussion. It is equal to $H^p(M, A * \otimes B)$ if A and B are bundles.)

This is fairly clear without detailed analysis as the groups $\mathrm{Ext}^{p*}(\mathcal{O}(m), \mathcal{O}(n))$ do not in fact depend only on $n-m$. Let us look at some specific examples. One can check that $\mathcal{O}(-3) = (6\ 3\ 1)$, $\mathcal{O}(-2) = (3\ 1\ 0)$, $\mathcal{O}(-1) = (1\ 0\ 0)$, $\mathcal{O} = (0\ 0\ 1)$, $\mathcal{O}(1) = (0\ 1\ 3)$, and $\mathcal{O}(2) = (1\ 3\ 6)$. Then one can compute

$A \to B$	Ext^p	Ext^{p*}
$\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)$	3 0 0	3 0 0
$\mathcal{O}(0) \longrightarrow \mathcal{O}(1)$	3 0 0	3 0 0
$\mathcal{O}(0) \longrightarrow \mathcal{O}(-1)$	0 0 0	0 0 0
$\mathcal{O}(0) \longrightarrow \mathcal{O}(-2)$	0 0 0	0 0 0
$\mathcal{O}(-2) \longrightarrow \mathcal{O}$	6 0 0	0 0 6
$\mathcal{O}(-1) \longrightarrow \mathcal{O}(1)$	6 0 0	0 0 6
$\mathcal{O}(-1) \longrightarrow \mathcal{O}$	3 0 0	0 0 3
$\mathcal{O}(0) \longrightarrow \mathcal{O}(-3)$	0 0 1	1 0 0 .

The last of these is computed using Serre duality, which states that $H^p(\mathbb{P}^2, E) \cong H^{2-p}(\mathbb{P}^2, E^* \otimes O(-3))$.

These results show a simple pattern and can be summarized by the following rule. Let us introduce a notation where we put the gradings of the morphisms into the objects, as so:

$$\mathrm{Ext}^n(A, B) \equiv \mathrm{Hom}(A, B[n]) \equiv \mathrm{Hom}(A[m], B[m+n])$$

Then, to go from Ext to Ext*, all the branes $\mathcal{O}(n)$ with $n \geq 0$ become $\mathcal{O}(n)[1]$, while all the branes $\mathcal{O}(n)$ with $n < 0$ become $\mathcal{O}(n)[-1]$.

The explicit shifts $+1$ and -1 are determined by the following rule. Let the BPS central charges of a brane E at large volume be $Z(E;1)$ and at the orbifold point be $Z(E;2)$ (we could compare any two points in stringy Kähler moduli space). We define the "grade" of the brane E at a point x as

$$\varphi(E;x) = \frac{1}{\pi}\mathrm{Im}\log Z(E;x). \tag{62}$$

The "flow of grading" $E \to E[\Delta\varphi]$ from 1 to 2 is then determined as

$$\Delta\varphi = \varphi(E;2) - \varphi(E;1).$$

In this formula, the branches of the logarithm are determined by analytic continuation; defining this in (62) requires some additional discussion.

In the example, all of these branes have Z_2 real and positive, while $Z_1 = (-n+iV)^2$ is approximately real and negative, but with positive imaginary part for $n < 0$ and negative imaginary part for $n \geq 0$ (the case $n = 0$ is actually $n > 0$ as we are taking a path with decreasing V and $B > 0$ to avoid the conifold point). Furthermore the central charges for the two types of branes stay on their respective sides of the origin, positive or negative imaginary part, which leads to the rule we cited.

Although this rule may seem as if it was pulled out of a hat, it not only describes these explicit results but can be justified by a combination of physical and mathematical arguments.

The physical interpretation of the grading q of a morphism $\mathrm{Ext}^q(A, B)$ is that it is in fact the world-sheet $U(1)$ charge of the corresponding (bosonic) open string. Its most direct physical consequence is to determine the mass squared of the boson, which by conformal field theory arguments must be

$$m^2 = \frac{1}{2}(q-1)$$

in string units.

This leads to a very direct conformal field theory generalization of the type of stability argument we gave in our discussion of D-flatness conditions, and the necessary and sufficient conditions found there [15]. When we bring two branes (or a brane-antibrane pair) A and B together, we need to check the gradings of all of the morphisms $\mathrm{Ext}^q(A, B)$. If any satisfy $q < 1$, then bound state formation may be possible. Conversely, one can show that an object E goes unstable if the grading q of an $\mathrm{Ext}^q(A, B)$ between any of its quotient and subobjects goes above 1. This is because one of the Hom's in the triangle (18) would have its degree become negative, but negative $U(1)$ charges for chiral operators in unitary SCFT are not allowed, a contradiction which can only be resolved by the decay of E.

12 Antibranes and the derived category

The considerations of variation of central charge we made above have an even more striking consequence: namely, the distinction between "branes" and "antibranes" is not universal but in fact depends on where one is in Kähler moduli space.

This is not to say that there is any ambiguity in claiming that a particular $\bar{B}$ is "the" antibrane which annihilates B. However this is the only case in which there is no ambiguity. In other cases, the only clear distinction between brane and antibrane is the relative phase of the BPS central charge: for branes these are aligned or roughly aligned, while for brane and antibrane they are antialigned or roughly antialigned.

However, the central charges vary drastically as we move around the moduli space. Let us illustrate this with our example of $\mathbb{C}_3/\mathbb{Z}_3$. We saw that Z changes sign for $\mathcal{O}(n)$ as we go from large volume to orbifold point, so in some sense these branes become antibranes. The other fractional brane S_2 of course started out as an antibrane (meaning negative D4 charge) at large volume and indeed the phase of its central charge does not change during the flow, $\Delta\varphi = 0$, so it stays an antibrane. Thus we have a consistent picture in which all three fractional branes are simultaneously "branes" at the orbifold point. However, they are brane-antibrane pairs at large volume, and indeed using flow of gradings the bosons in $S_1 \longrightarrow S_2 \longrightarrow S_3$ can all be seen to be standard brane-antibrane tachyons which one expects at large volume.

Having realized that this distinction is so fluid, we now see that any description of all of the holomorphic objects which could make sense everywhere in moduli space had better treat branes and antibranes on a very equal footing, and indeed allow continuous evolution between them. It would seem rather hard to imagine such a thing, but as it turns out such a formalism already exists in mathematics, the formalism of the derived category. Indeed the observation that this should be relevant in describing D-branes on Calabi-Yau goes back to Kontsevich's homological mirror symmetry proposal of 1993 [24], so in some sense this part of the story predates D-branes!

It is possible to motivate (and in some sense "derive") the derived category as a systematic extension of the framework of topological open string theory to allow the BRST operator to have a general matrix (Chan-Paton) structure, as is done in [15, 3]. We will not get into this here but instead just describe the resulting formalism as it will appear in our primary application, that of finding the spectrum of BPS branes.

One starts with an abelian category of the sort we have been implicitly describing, of coherent sheaves, quiver representations, or whatever. One can think of the exact sequences (17) and the associated triangles (18) as the primary structure of interest. Since the last arrow in (18) was an Ext^1, however, we write it as

$$B \longrightarrow E \longrightarrow A \longrightarrow B[1].$$

Furthermore, since $\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(X[n],Y[n])$, this sequence can be continued in both directions indefinitely:

$$\longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow B[1] \longrightarrow E[1] \longrightarrow \dots. \tag{63}$$

This is called a "distinguished triangle" and it plays the role of the exact sequence in the derived category. Note that it contains less information, however: the exact sequence picked out one object as special (the one in the middle), while the distinguished triangle does not. This is an advantage and a disadvantage for our purposes. It is an advantage because it unifies the various processes we discussed before of brane-brane and brane-antibrane bound state formation. It is also a disadvantage because one does not know which of the objects is the bound state and which are the constituents. This shows up mathematically in the statement that one cannot define a notion of "subobject;" indeed every morphism $B \longrightarrow E$ can be completed to a distinguished triangle (63) for some A (called the "cone" of the morphism).

However, this is the universal structure which remains invariant under variations of Kähler moduli. The precise statement of the theorem of Beilinson we referred to, and many similar results on Calabi-Yau monodromies and Fourier-Mukai transforms, is that the natural equivalences and monodromy actions in general do not take sheaves to sheaves, or any other known subclass of holomorphic objects into itself, but instead act on the derived category.

Variation of the Kähler moduli has only two effects on this structure. First, it induces the flow of gradings we discussed. This preserves the only essential constraint on the gradings of the morphisms in (63), namely that they sum to 1, and the two ideas fit very naturally together. Second, it changes the stability of objects, in some way generalizing the orbifold point and large volume phenomena we discussed. We now turn to this.

13 Π-stability

The construction of the derived category now gives a precise meaning to the decoupling statement, at least on the holomorphic side. On the other hand, the flow of gradings we discussed only depended on BPS central charges, and since these are geometric in the A picture, if we could base our discussion of stability only on these, we would have effectively implemented it on the other side.

Let us say a bit more about this. We have a good (though still somewhat abstract) description of the set of all possible F-flat configurations for orbifolds and Calabi-Yaus with a Gepner model realization, as the derived category of representations of a McKay quiver. We could go on to try to formulate and solve analogous stringy D-flatness conditions. However, the discussion we gave of how to find $\mathcal{N} = 1$ supersymmetric vacua suggests a simpler strategy. The procedure we ended up with was to find F flat configurations or objects, but then instead of solving the D flatness conditions, we instead found a necessary and sufficient criterion for such a solution to exist, the θ-stability condition. Furthermore, the DUY theorem shows that the problem of describing BPS branes at large volume can be stated in precisely the same paradigm, we first find holomorphic bundles or objects and then check their stability. Finally, we have now discussed the sense which the holomorphic objects in the B picture are the same in these two limits and indeed everywhere in Kähler moduli space.

All this suggests that we rephrase the problem. Instead, we will try to find a stringy version of the stability condition, which reduces to the conditions we already saw in the large volume and orbifold limits.

Such a condition can be found and is called "Π-stability." We will just state it without the detailed definitions and arguments, which can mostly be found in [13, 15, 1].

We start with a simplified version of Π-stability which was proposed in [13] and is adequate for problems not involving both branes and antibranes. It is essentially to replace the stability conditions (24) at the orbifold point and (40) with the single condition that E is stable if for every subobject E' of E,

$$\varphi(E') < \varphi(E). \tag{64}$$

All of the dependence on Kähler moduli is contained in (62).

This is good when one can define subobject, but there is no concept of subobject in the derived category. Furthermore, comparison of the definitions of subobject in large volume and at the orbifold point shows that they are different (for example, $\mathcal{O}$ is a subobject of $\mathcal{O}(-3)$ at the orbifold point). Thus we must get by without it.

A refined version of Π-stability which can treat this problem was proposed in [15, 1]. One has to start with a list of stable objects, which might be found at large volume or at the orbifold point using the previous definitions. The stability condition is then the following: two stable objects A and B cannot participate in morphisms of negative degree. Taking into account the definition of flow of gradings, this is essentially equivalent to (64), but the difference comes when we cross a line on which this condition is violated.

One can check that the definition of φ in terms of the phases of central charges means that if one of the morphisms in (63) has grade 0, the others must be integral. If the three objects involved were stable, one will be grade 0 and the other will be grade 1. The rule is then that the object between the 0's decays. This is always the heaviest of the three objects, so physically there is no doubt that this is the correct rule.

Conversely, if a morphism between two stable objects drops in degree below 1, the third element of the corresponding distinguished triangle (the "cone") becomes stable. This corresponds to a massless boson becoming tachyonic.

13.1 Examples

A number of examples of these rules are worked out in [15, 16, 1]. Another simple example can already be understood at large volume, namely the decay of a high degree D$n-2$-brane. Consider a compact CY_3 and the following exact sequence:

$$0 \longrightarrow \mathcal{O}(-\frac{N}{2}) \xrightarrow{f} \mathcal{O}(\frac{N}{2}) \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0. \tag{65}$$

The map f is a polynomial of degree N and generically vanishes on a non-singular hypersurface of degree N, i.e. a brane with D4 charge N. The total charge is $e^{NJ/2} - e^{-NJ/2}$ and one sees that this brane has zero D2 charge but D0 charge of order N^3.

According to the central charge formula (39), such a brane has central charge

$$Z = -3NV^2 + \frac{N^3}{4}.$$

Although it is large at large volume, $Z = 0$ at $V = N/\sqrt{12}$ in string units. If N is large, this is clearly a nonsingular point in moduli space, so the brane must decay before reaching it. Furthermore, world-sheet instantons are clearly unimportant at this scale, so there is no loophole in this.

The natural exact sequence which might govern the decay of this brane is just (65) as we don't know of any others in general. The map f, as a Hom, has degree 0 in large volume (this is the brane-antibrane tachyon of $m^2 = -1/2$). However, as we decrease V, the central charges $(-N/2 + iV)^3$ and $(N/2 + iV)^3$ will vary in precisely the same way we described above for the line bundles $\mathcal{O}(n)$ in $\mathbb{C}^3/\mathbb{Z}_3$, and with the same effect: the morphism f will increase in degree until it reaches 1, at which point the D4 will decay. This will happen when these two central charges antialign (so the brane and antibrane charges align), i.e.

$$\begin{aligned} 0 &= (-N/2 + iV)^3 + (N/2 + iV)^3 && (66)\\ &= (iV)^3 + 3iV\left(\tfrac{N}{2}\right)^2, && (67) \end{aligned}$$

which is at $V = \sqrt{3}N/2$, long before the problematic point.

The same thing happens in the known examples with instanton corrections; the "mysterious brane" in the quintic Gepner model, and even the D2 on $\mathbb{C}^3/\mathbb{Z}^3$, are very similar examples.

14 Parting words

In these lectures, we have given an introduction to a framework for studying and classifying BPS branes in string theory compactified on Calabi-Yau manifolds which, although not complete, has achieved a definite form in which concrete problems can be solved. This work has also shed new light on the structure of $\mathcal{N} = 1$ supersymmetry and provides new methods for studying $\mathcal{N} = 1$ theories, inspired both by the physics of branes and by modern mathematics.

Developments continue in the various directions we discussed; let us mention a few. First, one should be able to get a much better picture of the geometry of string-scale Calabi-Yaus from the behavior of the spectrum of stable BPS branes, and particularly from the D0-brane, in the spirit of "D-geometry" [11, 12]. A good example of this is that the connections between topologically distinct Calabi-Yaus which were visible from the linear sigma model and toric geometry [29, 2] could be rederived by seeing how D0-brane moduli spaces change under variation of stability. Second, one should be able to completely understand Seiberg duality, at least as an equivalence between classical moduli spaces of dual $\mathcal{N} = 1$ effective field theories, along the lines of [4, 18, 9], as examples of Fourier-Mukai transforms, which are the general symmetries of the category of branes on a Calabi-Yau, and thus fit them into a larger, stringy framework.

We feel that a very promising longer term direction for this work is to extend it to describe $\mathcal{N} = 1$ compactifications with branes which make sense as quantum theories (i.e. cancel anomalies), either in type I theory, in type II orientifold theories, or eventually in more general contexts and with quantum corrections, perhaps by using dualities with these constructions. Given our experience with duality, it does not seem unreasonable to hope that a classical moduli space of D-brane configurations could be equivalent to the exact quantum moduli space of some (perhaps very different looking) dual theory. We might look at these constructions as providing "solvable" $\mathcal{N} = 1$ models, somewhat analogous to the role Calabi-Yau compactification has played in the study of $\mathcal{N} = 2$ models.

It seems to us that systematic exploration of $\mathcal{N} = 1$ vacua is the central problem for string/M theory in the coming years, and it seems to us that this is the first framework which gives any usable description at all of complete nontrivial moduli spaces of $\mathcal{N} = 1$ supersymmetric theories arising from string theory, so it will be exciting to see if these ideas can be extended in these ways.

Acknowledgments. These ideas were developed in discussions with many people, and particularly in collaborations with P. Aspinwall, D.-E. Diaconescu, B. Fiol and C. Römelsberger.

This research was supported in part by DOE grant DE-FG05-90ER40559.

References

[1] P. S. Aspinwall and M. R. Douglas, *D-Brane Stability and Monodromy,* hep-th/0110071.

[2] P. Aspinwall, B. R. Greene and D. R. Morrison, Nucl. Phys. B420 (1994) 184-242; hep-th/9311042.

[3] P. S. Aspinwall and A. Lawrence, *Derived Categories and Zero-Brane Stability*, JHEP 0108 (2001) 004, hep-th/0104147.

[4] C. E. Beasley, M. R. Plesser, *Toric Duality Is Seiberg Duality*, JHEP 0112 (2001) 001, hep-th/0109053.

[5] T. Bridgeland, A. King, and M. Reid, *Mukai implies McKay,* math.AG/9908027.

[6] I. Brunner and J. Distler, *Torsion D-Branes in Nongeometrical Phases*, hep-th/0102018.

[7] I. Brunner, J. Distler, and R. Mahajan *Return of the Torsion D-Branes*, hep-th/0106262.

[8] I. Brunner, M. R. Douglas, A. Lawrence, and C. Römelsberger, *D-branes on the quintic,* JHEP 08(2000)015, hep-th/9906200.

[9] F. Cachazo, B. Fiol, K. Intriligator, S. Katz and C. Vafa, *A Geometric Unification of Dualities*, hep-th/0110028.

[10] D.-E. Diaconescu and J. Gomis, *Fractional Branes and Boundary States in Orbifold Theories*, JHEP 0010 (2000) 001, hep-th/9906242.

[11] M. R. Douglas, *Two Lectures on D-Geometry and Noncommutative Geometry,* in *Nonperturbative Aspects of Strings, Branes and Supersymmetry,* World Scientific (1999), hep-th/9901146.

[12] M. R. Douglas, *Topics in D-geometry,* Class. Quant. Grav. **17** (2000) 1057, hep-th/9910170.

[13] M. R. Douglas, B. Fiol, and C. Römelsberger, *Stability and BPS branes,* hep-th/0002037.

[14] M. R. Douglas, B. Fiol, and C. Römelsberger, *The spectrum of BPS branes on a noncompact Calabi–Yau,* hep-th/0003263.

[15] M. R. Douglas, *D-branes, Categories and $N = 1$ Supersymmetry,* J. Math. Phys. 42 (2001) 7, 2818-2843, hep-th/0011017.

[16] M. R. Douglas, *D-Branes and N=1 Supersymmetry,* to appear in the proceedings of Strings 2001, Mumbai, India, hep-th/0105014.

[17] M. R. Douglas and D.-E. Diaconescu, *D-branes on Stringy Calabi-Yau Manifolds,* hep-th/0006224.

[18] B. Feng, A. Hanany, Y.-H. He and A. M. Uranga, *Toric Duality as Seiberg Duality and Brane Diamonds*, hep-th/0109063.

[19] B. Fiol, *The BPS Spectrum of N=2 SU(N) SYM and Parton Branes*, hep-th/0012079.

[20] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscr. Math. 6 (1972) 71-103.

[21] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).

[22] S. I. Gelfand and Yu. I. Manin, *Homological Algebra*, Springer 1999.

[23] A. D. King, *Moduli of Representations of Finite Dimensional Algebras*, Quart. J. Math. Oxford (2), 45 (1994), 515-530.

[24] M. Kontsevich, *Homological Algebra of Mirror Symmetry,* Proceedings of the 1994 ICM, alg-geom/9411018.

[25] M. Marino, R. Minasian, G. Moore and A. Strominger, *Nonlinear Instantons from Supersymmetric p-Branes,* JHEP 0001 (2000) 005, hep-th/9911206.

[26] R. Minasian and A. Tomasiello, *Variations on stability,* hep-th/0104041.

[27] J. Polchinski, *TASI Lectures on D-branes,* hep-th/9611050.

[28] M. Reid, *La correspondance de McKay,* Séminaire Bourbaki (novembre 1999), no. 867, math.AG/9911165.

[29] E. Witten, *Phases of N=2 theories in two dimensions,* Nucl. Phys. **B403** (1993) 159, hep-th/9301042.