The Imprint of the Equation of State on the Axial w-Modes of Oscillating Neutron Stars

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Abstract

We study the dependence of the pulsar spin frequencies of axial quasi-normal modes of a nonrotating neutron star upon the equation of state describing the star interior. The complex frequencies corresponding to a set of equations of state based on different physical assumptions have been computed. The numerical results, which appear to depend primarily on the stiffness of the equation of state, show that axial gravitational waves carry relevant information on both the structure of neutron star matter and the nature of the hadronic interactions.
1 Introduction

Among the astrophysical processes associated with emission of gravitational waves, the gravitational collapse and the coalescence of compact bodies are likely to be the most efficient sources. Both processes leave behind a compact object, either a neutron star or a black hole, which is expected to radiate the mechanical energy residual of its violent birth through gravitational waves, emitted at frequencies and with damping times characteristic of the quasi-normal modes of the source.

The complex frequencies of the quasi-normal modes carry information on the internal structure of the emitting source. For black holes, it has been shown that they only depend on the parameters identifying the spacetime geometry: mass, charge and angular momentum. For stars, the situation is far less simple, since the eigenfrequencies of the quasi-normal modes depend upon the equation of state (EOS) prevailing in the star interior, on which not much is known. It is therefore interesting to compute these frequencies for different EOS proposed to describe matter at supernuclear densities, and explore the possibility of extracting from them information on the internal structure of the star.

The quasi-normal mode frequencies can be obtained studying the source-free, adiabatic perturbations of an equilibrium configuration with an assigned EOS, and solving the linearized Einstein equations, coupled to the equations of hydrodynamics, with suitably posed boundary conditions.

If the unperturbed star is assumed to be static and non rotating, it is convenient to expand all perturbed tensors in tensorial spherical harmonics. As these harmonics have a different behaviour under the angular transformation $\theta \rightarrow \pi - \theta$, $\varphi \rightarrow \pi + \varphi$, the separated equations split into two decoupled sets: polar or even, with parity $(-1)^l$, and axial or odd, with parity $(-1)^{(l+1)}$. The polar equations are the relativistic generalization of the tidal perturbations of Newtonian theory, and couple the perturbations of the gravitational field to the perturbations of the fluid composing the star. In addition to the standard modes of newtonian theory, in general relativity there exists a new family of modes that are essentially spacetime modes, the corresponding motion of the fluid being negligible [1]. They are called w-modes, and are characterized by higher frequencies and much shorter damping times, i.e. these modes are strongly damped.

Unlike polar perturbations, axial perturbations are not coupled with fluid motion. They do not have a newtonian counterpart, and their radial evolution is described by a Schrödinger-like equation with a potential barrier that depends upon the distribution of energy and pressure in the interior of the star in the equilibrium configuration [2]. Thus, the EOS of the fluid has the sole role of determining the shape of the potential inside the star.

The axial quasi-normal modes can be further classified in two categories: the strongly damped w-modes, similar to the polar w-modes, and the slowly damped s-modes [3]. The latter appear if the star is extremely compact, so that the potential well in the interior becomes deep enough to allow for the existence of one or more quasi-stationary states, i.e. of quasi normal modes.
In this paper we calculate the frequencies of the axial quasi-normal modes for a set of EOS based on different physical assumptions and obtained using different theoretical approaches.

Although the slowly damped s-modes have been shown to exist for homogeneous stars with high enough compactness, it turns out that none of the EOS we have considered leads to a potential well deep enough to allow for the appearance of s-modes. Hence, we have focused on the imprint that different EOS leave on the strongly damped axial w-modes.

The equations governing the axial perturbations of a non rotating star are given in section 2, whereas section 3 is devoted to a short description of the EOS used in the calculations. Finally, the results are presented in section 4 and briefly discussed in section 5.

2 A Schrödinger-like equation for axial perturbations

After separating the variables, the equations for the axial perturbations of a non rotating star can be considerably simplified introducing a function \( Z_\ell(r) \), \( \ell \) being the harmonic index, constructed from the radial part of the axial metric components \[.\] \( Z_\ell(r) \) satisfies the equation

\[
\frac{d^2 Z_\ell}{dr_*^2} + \left[ \omega^2 - V_\ell(r) \right] Z_\ell = 0,
\]

where

\[
V_\ell(r) = \frac{\epsilon^{2\nu}}{r^{3\nu}} \left[ \frac{(l+1)r + r^3(\epsilon - p) - 6m(r)}{r^{3\nu}} \right]
\]

and

\[
r_* = \int_0^r e^{-\nu + \mu_2} dr .
\]

In eq. (2), \( \epsilon \) and \( p \) denote the energy-density and pressure of the perfect fluid composing the star, while the function \( \nu(r) \) can be obtained from the differential equation

\[
\nu_r = -\frac{p_r}{\epsilon + p}
\]

imposing the boundary condition that at the surface of the star the metric reduces to the Schwarzschild metric, i.e. that \( \epsilon^{2\nu}(R) = 1 - 2M/R \), where \( M = m(R) \) is the mass of the star. Outside the star both \( \epsilon \) and \( p \) vanish, and eq. (1) reduces to the well-known Regge-Wheeler equation:

\[
\frac{d^2 Z_\ell}{dr_*^2} + \left[ \omega^2 - V_\ell(r) \right] Z_\ell = 0,
\]

where

\[
V_\ell(r) = \frac{\epsilon^{2\nu}}{r^{3\nu}} \left[ \frac{(l+1)r - 6M}{r^{3\nu}} \right]
\]
and
\[ e^{2\nu} = 1 - \frac{2M}{r}. \] (7)

The above equations show that the axial perturbations of a star are fully described by a Schrödinger-like equation with a potential barrier that depends upon the distribution of energy-density and pressure inside the star in its equilibrium configuration.

The quasi-normal modes are defined to be solutions of eq.(1) satisfying the condition of being regular at \( r = 0 \) and behaving as purely outgoing waves at radial infinity, i.e. such that
\[ \lim_{r_\ast \to \infty} Z_{\ell \ell} \sim e^{-i\omega r_\ast}. \] (8)

3 Models of the neutron star matter equation of state

The calculations described in the present paper have been carried out using six different models of the neutron star matter EOS at supernuclear density (\( \rho > 2.8 \times 10^{14} \text{ g/cm}^3 \)). The low density regime has been described using the results of refs.[4, 5, 6] and [7]. However, it has to be stressed that the details of the EOS at subnuclear density have negligible influence on both neutron star properties and the calculated pulsation frequencies. The main features of the EOS at supernuclear density are described below. Whenever possible, we label the different EOS according to the classification scheme introduced in ref.[8].

- Pandharipande, model A [9]. The star is described in terms of pure neutron matter, the dynamics being dictated by a nonrelativistic hamiltonian containing a semi-phenomenological neutron-neutron potential fitted to deuteron properties and nucleon-nucleon scattering data. The calculation of the binding energy per particle as a function of density, needed to obtain the EOS, is carried out including only the contributions of two-neutron clusters.

- Pandharipande, model B [10]. A generalization of model A that allows both the occurrence of neutrino \( \beta \)-decay, leading to the appearance of protons, electrons and muons, and the appearance of heavier baryons (hyperons and nucleon resonances) at sufficiently high densities. The nucleon-nucleon interaction is described by the same potential employed in model A, whereas the hyperon-nucleon interaction is obtained rescaling the attractive nucleon-nucleon force. Due to the low hyperon density, the hyperon-hyperon interaction is disregarded.

- Wiringa, Fiks and Fabrocini, model WFF [11]. Neutron star matter is treated as a mixture of neutrons, protons, electrons and muons in \( \beta \)-equilibrium. In addition to the two-nucleon potential, the nonrelativistic nuclear hamiltonian includes a three-body potential, needed to account for the observed properties
of the three-nucleon bound states. Compared to models A and B, the ground state energy of neutron star matter is computed using a more sophisticated and accurate technique, that takes into account the contributions of a large number of many-body clusters.

- Akmal, Pandharipande and Ravenhall, model APR1 [12]. Similar to model WFF, but uses state of the art parametrizations of both the two- and three-body potentials. The two-body potential models developed over the past few years, as the Argonne $v_{18}$ potential [13] employed in model APR1 and APR2, represent a significant improvement upon the previously available ones. They are fitted to a larger and more complete database of nucleon-nucleon scattering data and explicitly include the effect of charge-symmetry-breaking terms in the scattering amplitude. The results of ref.[12] show that using a two-body potential of the last generation leads to qualitative changes in the density dependence of nuclear and neutron matter energy in the high density region, suggesting the possibility of a transition from the standard uniform phase to a spin-isospin ordered phase associated with condensation of neutral pions [14, 15].

- Akmal, Pandharipande and Ravenhall, model APR2 [12]. An improved version of model APR1, intended to gauge the relevance of relativistic corrections to the standard nuclear many-body approach. The nucleon-nucleon interaction is modified including boost corrections of order $(v/c)^2$, $v$ being the velocity of the center of mass of the interacting nucleon pair. The phenomenological three-nucleon interaction is also modified to preserve the agreement between theory and the three-nucleon data.

- Pandharipande & Smith 1975, model L [16]. Neutrons are assumed to interact through the exchange of two vector mesons, the $\omega$- and $\rho$-mesons, and one scalar meson, the $\sigma$-meson. While the exchange of heavy particles ($\omega$ and $\rho$) is described in terms of nonrelativistic potentials, the effect of the $\sigma$-meson is taken into account using relativistic field theory and the mean-field approximation. It has to be pointed out that the physical assumptions underlying this somewhat hybrid approach are different from those of the previous models.

It has to be mentioned that all models based on nonrelativistic many-body theory (like models A, B, WFF, APR1 and APR2) lead to a violation of causality, in that they predict a speed of sound exceeding the speed of light at very high density. However, this pathology only occurs at densities much greater than those relevant to the study of neutron star structure.

The supernuclear density behaviour of the EOS employed in this work is shown in fig. 1. It appears that the EOS can be classified according to their stiffness, providing a measure of the incompressibility of neutron star matter. For example, comparison between models A and B shows that the appearance of heavy baryons leads to a significant softening of the EOS at high density.
In the case of neutron matter, either pure or in $\beta$-equilibrium, the stiffness of the EOS is mainly determined by the nuclear hamiltonian. The EOS of models A, WFF, APR1 and APR2, in which the two-nucleon interaction reproduces the nucleon-nucleon scattering data, are rather close to each other. On the other hand, using a dynamics that is not constrained by nucleon-nucleon data, as in model L, may result in a significant difference in stiffness.

The EOS employed in this paper can be ordered according to increasing stiffness as follows: B, A, WFF, APR2, APR1, L.

In fig. 2 we plot the mass-radius relation for the above mentioned EOS in the range of stability against radial perturbations.

The EOS denoted by A, B, L and WFF have been widely used in the past to study equilibrium configurations of neutron stars [8, 11, 17] and, more recently, to compute their pulsation frequencies associated with emission of gravitational waves [18]. On the other hand, the models denoted APR1 and APR2 have been developed within the last few years and have never before been employed to study nonstatic neutron star properties.

## 4 Numerical results

As stated in section 1, the axial quasi-normal modes of a pulsating star are solutions of eq.(1) satisfying the boundary conditions imposed by physical requirements: $Z_\ell$ must be regular at the origin and behave as a purely outgoing wave at infinity. To compute the mode frequencies one has to find the (complex) values, $\omega = \omega_0 + i\omega_1$, for which these boundary conditions are satisfied.

The numerical determination of the quasi-normal mode frequencies is non-trivial,
especially for modes with large imaginary parts (strongly damped modes). Solutions of eq.(1) representing outgoing and ingoing waves at infinity have the $r_\star \to \infty$ asymptotic behaviour

$$Z_{\ell}^{\text{out}} \sim e^{r_\star/\tau}, \quad Z_{\ell}^{\text{in}} \sim e^{-r_\star/\tau},$$

where $\tau = 1/\omega_\ell$ is the damping time. Therefore, identifying by numerical integration the purely outgoing solutions (that is, those solutions for which $Z_{\ell}^{\text{in}}$ is zero) becomes increasingly difficult as the damping of the mode increases.

This problem, also occurring in the case of quasi-normal modes of black holes, was solved in the eighties by Leaver [19, 20], who was able to write a continued fraction relation that can be regarded as an implicit equation identifying the quasi-normal frequencies. The same method was subsequently reformulated and applied to the polar oscillations of a star in ref.[21].

We have employed the continued fraction method to find the frequency and damping time of the first axial w-mode, denoted by $\nu_{w0}$ and $\tau_{w0}$, respectively, for models of stars with the EOS discussed in section 3. The details of the calculatoins can be found in ref.[22].

In table 1 we summarize the characteristics of the considered stellar models. For each EOS, we choose the upper value of the central density (column 2) as that of the last radially stable configuration. The lower value is chosen as in [18] for EOS A, B, WFF and L, whereas for APR1 and APR2, that have never been used before to compute the oscillation modes of a star, it is chosen to give a value of $M/R$ comparable with that of the other models. In column 3 we give the ratio $M/R$ corresponding to the selected endpoints of $\rho_c$, the radius and mass of the star are given in columns 4 and 5, respectively, and the values of $\nu_{w0}$ and $\tau_{w0}$ are in columns 6 and 7.
### Table 1

<table>
<thead>
<tr>
<th>EOS</th>
<th>( \rho_c \times 10^{15} )</th>
<th>( M/R )</th>
<th>( R )</th>
<th>( M )</th>
<th>( \nu )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>1.995</td>
<td>0.165</td>
<td>8.7</td>
<td>0.97</td>
<td>11.2</td>
<td>20.2</td>
</tr>
<tr>
<td></td>
<td>5.91</td>
<td>0.296</td>
<td>7.1</td>
<td>1.42</td>
<td>10.6</td>
<td>71.7</td>
</tr>
<tr>
<td>A</td>
<td>1.259</td>
<td>0.157</td>
<td>9.9</td>
<td>1.05</td>
<td>9.76</td>
<td>21.6</td>
</tr>
<tr>
<td></td>
<td>4.11</td>
<td>0.291</td>
<td>8.4</td>
<td>1.65</td>
<td>9.11</td>
<td>72.4</td>
</tr>
<tr>
<td>WFF</td>
<td>0.8</td>
<td>0.118</td>
<td>11.1</td>
<td>0.89</td>
<td>9.20</td>
<td>18.6</td>
</tr>
<tr>
<td></td>
<td>2.6</td>
<td>0.274</td>
<td>9.8</td>
<td>1.83</td>
<td>8.11</td>
<td>62.3</td>
</tr>
<tr>
<td>APR2</td>
<td>0.75</td>
<td>0.116</td>
<td>11.8</td>
<td>0.93</td>
<td>8.82</td>
<td>19.4</td>
</tr>
<tr>
<td></td>
<td>2.75</td>
<td>0.325</td>
<td>10.0</td>
<td>2.20</td>
<td>6.69</td>
<td>165.3</td>
</tr>
<tr>
<td>APR1</td>
<td>0.65</td>
<td>0.116</td>
<td>12.4</td>
<td>0.97</td>
<td>8.35</td>
<td>20.4</td>
</tr>
<tr>
<td></td>
<td>2.36</td>
<td>0.326</td>
<td>10.8</td>
<td>2.38</td>
<td>6.19</td>
<td>177.0</td>
</tr>
<tr>
<td>L</td>
<td>0.398</td>
<td>0.120</td>
<td>14.9</td>
<td>1.21</td>
<td>6.70</td>
<td>25.5</td>
</tr>
<tr>
<td></td>
<td>1.42</td>
<td>0.288</td>
<td>13.6</td>
<td>2.66</td>
<td>5.62</td>
<td>76.7</td>
</tr>
</tbody>
</table>

For each EOS, the above table gives the minimum and maximum values of the central density \( [g/\text{cm}^3] \), chosen as explained in the text. In correspondence of these endpoints, columns 3-7 give the ratio \( M/R \) in geometrical units (column 3), the radius (km) and mass (\( M_{\odot} \)) of the star and the values of \( \nu_{\text{ao}} \) (kHz) and \( \tau_{\text{ao}} \) (ms) of the first axial w-mode.

Fig. 3 shows the frequency of the first axial w-mode, \( \nu_{\text{ao}} \), as a function of the star compactness, \( M/R \), for the EOS’ and central density ranges listed in Table 1. The corresponding damping times, \( \tau_{\text{ao}} \), are plotted in Fig. 4.

Our results clearly show that the frequencies of the axial gravitational waves emitted by neutron stars oscillating in their w-modes are mainly driven by the stiffness of the EOS, which is, in turn, dictated by the physical assumptions underlying the model describing the star interior. As a consequence, models WFF, APR1 and APR2, based on very similar assumptions (\( \beta \)-stable matter with hadronic interactions constrained by nucleon-nucleon scattering data) yield frequency ranges that significantly overlap with each other. On the other hand, from Table 1 and Fig. 3 it appears that the frequencies of the lower axial w-mode for the remaining EOS’, i.e. B, A and L range within intervals that are well separated from one another.

### 5 Conclusions

As axial perturbations of non rotating stars are described by a Schrödinger-like equation with a potential barrier shaped by the manner in which pressure and density are distributed inside the star, the frequencies of the axial quasi-normal modes are expected to depend on the equation of state. This was also shown to be the case for the polar w-modes [18] that are coupled to negligible fluid motion. In particular, both the axial and polar w-modes appear to depend essentially on the stiffness of the equation of state. However, while for each selected EOS the frequency of the polar w-modes is a rather steeply decreasing function of \( M/R \) [18], in the case
of the axial ones the dependence of $\nu_{\text{w}}$ on the star compactness turns out to be weak. As a consequence of this behaviour, illustrated in fig. 5, the axial w-modes provide more direct and explicit information, compared to the polar ones, on which equation of state prevails inside the star, regardless of its mass and radius. We emphasize that our results show that EOS based on different assumptions correspond to non overlapping frequency ranges, providing further constraints on the existing models, with regard to both the composition of neutron star matter and the description of the hadronic interactions.

The real problem is whether we will ever be able to detect an axial gravitational wave impinging on earth from a ringing star. Only detailed simulations of astrophysical processes, like gravitational collapse or the coalescence of compact bodies, will tell us to what extent the w-modes can be excited and be significant from the point of view of detection. If they would be significant, since these modes have frequencies higher than those detectable by current experiments, new high frequency detectors will need to be planned.

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Figure 4: Damping time of the first axial $w$-mode as a function of the compactness of the star.

Figure 5: Frequency of the first polar (dashed line) and axial (continuous line) $w$-modes, plotted as a function of the compactness of the star for EOS A, B, WFF, L.
References