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LAMP SERIES REPORT

(Laser, Atomic and Molecular Physics)

BRIGHT AND DARK SOLITARY WAVE PROPAGATION AND BISTABILITY IN THE ANOMALOUS DISPERSION REGION OF OPTICAL WAVEGUIDES WITH THIRD- AND FIFTH-ORDER NONLINEARITIES

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Preface

The ICTP-LAMP reports consist of manuscripts relevant to seminars and discussions held at ICTP in the field of Laser, Atomic and Molecular Physics (LAMP).

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ABSTRACT

A unified analysis of all existing solitary wave solutions of the cubic-quintic nonlinear Schrödinger equation is made concretely considering a monomode step-index optical fibre with third- and fifth-order nonlinearities in the anomalous dispersion frequency region. Different aspects of the saturation effect due to the fifth-order nonlinearity, such as solitary wave propagation and bistability, are considered. All results obtained are directly applicable to the cases of spatial solitons and optical beam self-action in highly nonlinear media. The possible applications in optical switching experiments and communication systems are discussed.

1. Introduction

During the last two decades the study of optical solitons attracted the attention of many research groups. Especially in the field of nonlinear waveguide optics the progress in the soliton subject seems to be the most significant in relation to its application in optical communication systems.

After the theoretical prediction of temporal bright and dark solitons in optical fibres [1,2] and their experimental observation [3,4] a great deal of effort was made to observe the corresponding spatial analogues - the spatial bright and dark solitons [5,6]. Spatial bright and dark solitons in planar waveguides like their counterparts in fibers are definitely stable [7] if they are solutions of the one-dimensional cubic nonlinear Schrödinger equation (CNLSE), i.e. if a Kerr-like nonlinearity is considered. However, as the incident fields become stronger or the field frequency approaches a resonant frequency of the medium, non-Kerr nonlinearities come into play changing essentially the physical features and the stability of optical soliton propagation. Generally, what is expected to happen at high optical field intensities and especially in materials with high nonlinear coefficients, e.g., semiconductor-doped glasses, organic polymers, thin liquid-filled capillaries and others, is the effect of saturation of third-order nonlinearity which can be accounted by modifying in an appropriate way [8,9,10] the nonlinear term in the

CNLSE.

In this paper we discuss solitary wave propagation when the saturation of the third-order nonlinearity is accounted by including a fifth-order term with an opposite sign, in respect to the third-order one, i.e., when the refractive index of the medium depend on the electric field amplitude E as $n = n_o + n_2|E|^2 + n_4|E|^4$ ($n_2 = 3\chi^{(3)}/(8n_o) > 0$, $n_4 = 5\chi^{(5)}/(32n_o)$, $\chi^{(3)} > 0$ and $\chi^{(5)} < 0$ being the components of the corresponding nonlinear dielectric tensors and n_o being the linear refractive index coefficient). More concretely we consider solitary wave pulse propagation in a monomode weakly guiding optical fibre and present in an appropriate analytical form the exact solitary wave solutions of the corresponding nonlinear wave equation in the anomalous dispersion fibre region discussing their stability and bistable character. Our attention is concentrated especially to the situations where the group velocity dispersion parameter $\beta_2 < 0$ (anomalous dispersion), the third-order nonlinear refractive index $n_2 > 0$ and the fifth-order one $n_4 < 0$ but in our final conclusions we also briefly discuss the situations where $n_2 < 0$, $n_4 > 0$ and $\beta_2 > 0$ (normal dispersion). In a double-doped optical fibre [11,12] the kind of nonlinearity and the two sign choices for the parameters of the waveguide structure, considered here, are physically reasonable depending on the doping materials, the operating frequency and optical pulse intensities. In addition, in

such fibres a large value of n_4 can be reached. For a seek of generality the study is not limited to the cases when n_4 is merely a small correction.

2. Solitary wave solutions of the CQNLSE

The dynamic equation satisfied by the slowly varying pulse envelope in a fibre with the refractive index given above is the so called cubic-quintic nonlinear Schrödinger equation (CQNLSE) and has the form:

$$i\frac{\partial A}{\partial z} + \frac{i}{v_g}\frac{\partial A}{\partial t} - \frac{1}{2}\beta_2\frac{\partial^2 A}{\partial t^2} + \varkappa|A|^2A - \gamma|A|^4A = 0, \qquad (1)$$

where A(z,t) is the pulse envelope, z is the longitudinal coordinate, t is the time, $1/v_g = \beta_1 = \partial\beta/\partial\omega|_{\omega = \omega_o}$ is the inversion of group velocity, $\beta_2 = \partial^2\beta/\partial\omega^2|_{\omega = \omega_o}$, $\varkappa = k_o n_2 \alpha_1$, $\gamma = -k_o n_4 \alpha_2$, $k_o = \omega_o/c$, ω_o being the carrier wave frequency and β - the wavenumber.

Eq. (1) was established for the first time by Pushkarov *et al.* [16,17]. The authors considered self-focussing and self-modulation of optical beams and pulses in highly nonlinear media and some of the exact solutions of equation (1) were presented. The CQNLSE does not belong to the class of integrable nonlinear evolution equations, which means that it can not be solved by the inverse-scattering method [18] but if materials with an appropriate choice of the values of the nonlinear refractive index coefficients n_2 and n_4 are considered then soliton-like pulses are expected to propagate. Following the approach of Pushkarov *et al.* [16,17,19] we will examine now the solitary wave solutions of equation (1) concerning nonlinear waveguide optics. In this case the slowly varying amplitude A(z,t) is related to the electric field E by the equality $E(r,z,t) = A(z,t)F(r)$, where r is the radial cylindrical coordinate and F(r) is the radial distribution of the optical fibre mode. The nonlinear dispersion coefficients α_1 and α_2, accounting for the radial mode distribution F(r), are connected with the third and fifth-order nonlinearity, respectively, and depend on the normalized frequency of the fibre $v = k_o R_o (n_{o1}^2 - n_{o2}^2)^{1/2}$ [13,14,15], where R_o is the core radius and n_{o1} and n_{o2} are the linear refractive indices of the core and the cladding.

Here it is worth noting that the equation describing self-trapping of light beams in nonlinear planar waveguides and bulk media is of the same type as (1) [20] and the results obtained in this work can be used for interpretation of both time-like and spatial solitons. More generally, equation (1) may be considered as a model equation describing self-phase modulation of pulses, self-focusing of beams and solitary wave evolution in any medium possessing the nonlinearity considered here.

2.1. Bright solitary wave solution

Let us consider first the bright solitary wave solution [16,17] of equation (1):

$$A(z,t) = A_o \exp(i\Phi)/[1 + B\cosh(\Psi)]^{1/2}, \qquad (2)$$

where $\Phi = qz - \Omega t + \varphi$, $\Psi = 2(t - t_o^c - z/V)/T_o$, q and Ω are the wavenumber and the frequency shift, respectively, T_o is related to the pulse width, t_o^c is the center of the pulse, φ is the initial phase, V is the transmission velocity, and A_o and B are real positive quantities. Using a normalization condition of the form

$$\int_{-\infty}^{+\infty} |A|^2 dt = P, \qquad (3)$$

typical of bright solitary wave solutions, we obtain the relations: $A_o^2 = P\tanh(b)/(T_o b)$, $B = \mathrm{sech}(b)$, $T_o = \tau\coth(b)$, $\tau =$, $(8\gamma|\beta_2|/3\varkappa^2)^{1/2}$, $q = \beta_1\Omega - |\beta_2|\Omega^2/2 + q_{NL}$, $q_{NL} = |\beta_2|\tanh^2(b)/(2\tau^2)$, $V = 1/(\beta_1 - |\beta_2|\Omega)$, and the following expression for the normalized pulse energy

$$b = P/P_o^b = \mathrm{arctanh}\,[Q^{1/2}], \qquad (4)$$

where $P_o^b = (3|\beta_2|/2\gamma)^{1/2}$ (the superscript b denotes "bright"), as we will see latter, is the characteristic pulse energy for the formation of the pulse (2), τ has a dimension of time and $Q = 2q_{NL}\tau^2/\beta_2$ is the normalized nonlinear wavenumber shift. It is worth noting that b is proportional to $\gamma^{1/2}$ and respectively to $(n_4)^{1/2}$. If for a fixed value of the pulse energy P we put $n_4 = 0$ (only cubic nonlinearity), then $B \to 1$ and the solution (2) turns into the well known bright soliton solution of the CNLSE - $\bar{A}(z,t) = \bar{A}_o \mathrm{sech}(\Psi/2)\exp(i\Phi)$, where $\bar{A}_o = A_o/2^{1/2}$. The width at 1/2 level of the pulse described by (2) can be expressed as a function of b (or γ) in the following way:

$$\tau_{1/2} = \tau_o^b \, b \coth(b) \, \mathrm{arccosh}[4 + 3\cosh(b)], \qquad (5)$$

where $\tau_o^b = 2|\beta_2|/(\varkappa P) = \tau/b$ has the value of T_o at $\gamma = 0$. If γ = b = 0 the expression (5) equals the width of $\bar{A}(z,t)$, i.e. $\bar{\tau}_{1/2} = \tau_o^b \, \mathrm{arccosh}(7) \cong 2.6 \, \tau_o^b$.

In Fig. 1 we show the dependence of the normalized pulse duration $t_o = \tau_{1/2}/\tau$ on the normalized pulse energy b. In contrast to the case of only cubic nonlinearity, the fifth-order one (FONL) makes the pulse duration a non monotonic function of the pulse energy. When the normalized pulse energy b varies slowly from zero to infinity the solitary wave peak amplitude $A_p = A_s \tanh(b)/[1 + 1/\cosh(b)]^{1/2}$ inccreases. The saturation effect begins to play a significant role at approximately b = 1 or when the pulse energy $P \cong P_o^b$ (in this sense the normalization quantity P_o^b is a characteristic pulse energy related to the saturation effect) and at very large values of b the peak amplitude A_p reaches its maximum value $A_s = (P_o/\tau)^{1/2} = [3\varkappa/(4\gamma)]^{1/2}$. At the same time (in the same variation range of b: 0 < b ∞) the normalized pulse duration t_o passes through a minimal value $t_o^{min} \cong 3.42$ at $b \cong 1.53$, being infinite at the both limits where the pulse evolves into plane waves. Thus, higher order

nonlinearities provide fundamental limitations to optical soliton shortening in fibres. The corresponding to t_o^{min} value of the pulse duration τ_{min} can be measured by slowly increasing pulse energy P operating with pulse intensities below the saturation intensity I_s and corresponding approximately to $P \cong 1.53P_o^b$. Then if a nonlinear material with known third-order nonlinear refractive index coefficient n_2 and group velocity dispersion parameter β_2 is considered, the value $t_o^{min} \cong 3.42$ allows us to determine the component (or the components) of the fifth-order nonlinear dielectric tensor $\chi^{(5)}_{ijklmn}$ on which the saturation effect is based. If, for example, the fundamental LP_{01}-mode is polarized along the $\hat{y}$ direction, then

$$\chi^{(5)}_{yyyyyy}(-\omega;\omega,-\omega,\omega,-\omega,\omega) = 1.28(n_o n_2^2 \tau_{min}^2)/(\lambda|\beta_2|\alpha),$$

where λ is the wavelength and the factor $\alpha = \alpha_2/\alpha_1^2$ accounts for the concrete radial mode distribution (see Appendix). For bulk materials $\alpha \cong 1$. If a special attention is paid to avoid the stimulated Raman light scattering in the fibre, losses are optimized and higher-order dispersion effects are neglected, the last formula can be used for measuring the fifth-order optical properties of transparent dielectric materials.

Thus, when $t_o > 3.42$ there exist two values of the pulse energy for one given value of the pulse width. This fact, pointed out in the work of Pushkarov *et al.* [16] and showing that two soliton-like pulses of the same width and different peak amplitudes can propagate in a highly nonlinear medium, was interpreted by some authors [11,12,21] as a kind of soliton bistability. On the other hand there is another criterion for soliton bistability [22,23] requiring that the normalized wavenumber shift Q has to be a multi-valued function of the normalized pulse energy b. If this criterion is applied, then it follows from (4) that no bistability takes places.

The stability of the solitary wave solution (2) is of prime importance for its physical feasibility. All studies till now show that it is stable for any value of the normalized pulse energy b. Indeed, the condition for the stability of the pulse (2) against small perturbations is $db/dQ > 0$ [24,25,22] which is true for any value of b and $A_p < A_s$. The numerical tests of the above stability condition [20,21] also showed the solitary wave behaviour of (2). Thus, in terms of the electric field peak amplitude the stability of the solitary wave pulse (2) is provided when $A_p < A_s = (3n_2\alpha_1/4n_4\alpha_2)^{1/2}$. The value of A_s is related to the saturation intensity $I_s = (\varepsilon_0 n_0 c/2)A_s^2$ and to the corresponding saturation peak power $P_s = I_s S_{eff}$, where S_{eff} is the effective area of the fundamental fibre mode. For SiO_2 [26,27] fibres (at $\lambda = 1.55\mu m$) $n_2 = 1.2\times10^{-22}(m/V)^2$, $n_4 = -4.4\times10^{-37}(m/V)^4$, $\alpha_2/\alpha_1 \cong 5$ (see Appendix) and therefore $P_S \cong 0.8W$. For semiconductor double doped glass fibres [11,12] (at $\lambda = 532nm$) $n_2 = 10^{-20}(m/V)^2$, $n_4 = -10^{-34}(m/V)^4$ and $P_s = 0.2W$. For comparison we give also the example of

$Al_{0.18}Ga_{0.82}As$-based waveguides [28] operating just below the one half band gap: $n_2 \cong 10^{-19}(m/V)^2$, $n_4 \cong -10^{-37}(m/V)^4$ and $P_s \cong 30kW$.

2.2 Kink (black) solitary wave solution

Let us consider now the second possible solution of the CQNLSE (1) for the same fibre parameters. It is of kink or black (the hole center has zero intensity) type and has the form

$$A(z,t) = A_0 \sinh(\Psi)\exp(i\Phi)/[1 + B\sinh^2(\Psi)]^{1/2}. \qquad (6)$$

Hence, due to the fifth-order nonlinearity (with $n_4 < 0$), kink pulses can propagate when the parameters of the fibre ($\beta_2 < 0$, $n_2 > 0$) are typical of cubic bright solitons. Although this fact was not pointed out till now and this explicit form of the solution (6) of equation (1) is reported here for the first time, it is worth noting that a more general solution of (1) has been given by Gagnon [29] and Gagnon and Winternitz [30] in another notation which can be reduced to the solution (6) as a special case. A solution of (1) of exactly the same analytical form as (6) exists in the theory of nonlinear excitations in deformable solids for parameter values corresponding to $\beta_2 > 0$ [19]. So, kink solitary waves can propagate in nonlinear systems described by the CQNLSE for both positive and negative values of the parameter corresponding to the group velocity dispersion parameter in optical fibres. A similar situation was reported for bright solitary wave solutions of (1) [29,30] and also of the so-called derivative nonlinear Schrödinger equation [32].

As $|A|^2 \to A_0^2/B \neq 0$ when $\Psi \to \pm\infty$, the energy corresponding to the hole propagating in a constant light background is equal to

$$\int_{-\infty}^{+\infty} (A_0^2/B - |A|^2)dt = P \tag{7}$$

with the following relations between the soliton parameters $A_0^2 = P\tanh(b_1)/[T_0 b_1 \cosh^2(b_1)]$, $B = \mathrm{sech}^2(b_1)$, $T_0 = 2\tau_0^d b_1[\tanh(b_1) - 1/\sinh(2b_1)]$, $Q = 4\sinh^2(b_1)[\sinh^2(b_1) - 2]/(2\sinh^2(b_1) - 1)]^2$. The normalized energy b_1 takes now the form

$$b_1 = P/P_0^d = \mathrm{arcsinh}\,(y^{1/2}), \tag{8}$$

where $y = [Q - 2 - (4 - 3Q)^{1/2}]/[2(Q - 1)]$ and the superscript d denotes "dark". The expressions for Φ, Ψ, τ, q and the normalization of q_{NL} are the same as for the bright solution. Now the normalization energy is $P_0^d = 2P_0^b$ and $\tau_0^d = 2\tau_0^b$ ensuring $\tau_0^d b_1 = \tau$. The width at 1/2 level of the pulse described by (6) has the form

$$\tau_{1/2} = T_0 \mathrm{arcsinh}\,[\cosh(b_1)/3^{1/2}] \tag{9}$$

with the corresponding expression for T_0 given above.

It must be pointed out that the solution (6) has no analogue in the $n_4 \to 0$ limit, because the requirements that T_0 and B have to be real and positive imply $b_1 > \mathrm{arcsinh}\,[1/2^{1/2}] \cong 0.66$. This means that for any given value of $n_4 < 0$ an "energy gap" exists for creating such a pulse. Similar energy gaps corresponding to nonexistense of bistable dark solitary waves in the normal

dispersion region of optical fibres were found to exist [31] for a kind of nonlinearity which is Kerr-like at small intensities, has a sufficiently steep jump at intermediate intensities, and becomes Kerr-like again or approach a constant value at large intensities.

In Fig. 2 the variations of the normalized pulse duration t_0 (curve 1) and wavenumber shift Q (curve 2) with the normalized pulse energy b_1 are shown. The pulse duration is a single valued function of the pulse energy and increases monotonically and almost linearly with increasing P. For $b_1 > 1.15$, $Q > 0$ and tends asymptotically to unity for larger values of b_1. This is a result of the saturation effect. When $b_1 < 1.15$, then $Q < 0$ and the fiber becomes an effectively self-defocussing medium. So, the more significant is the FONL term contribution in the CQNLSE, the smaller is the hole energy. This intuitively unexpected fact can be easily explained if the relation between the background field intensity of (6) and the hole energy is considered. That is why on Fig. 3 (curve 1) the dependence of the normalized background field intensity $I = a^2 = (2\gamma/\varkappa)A_0^2/B$ on the normalized hole energy b_1 is plotted. The hole energy decreases with invcreasing the background field intensity and vice versa. For $b_1 \cong 4$ the normalized background field intensity reaches its minimal possible value $I_{min} = 1.5$ below which the solitary wave solution (6) of equation (1) does not exist. So, a minimum background field intensity is needed for the existence of (6). As we mentioned above we do not restrict us only to nonresonant interactions, i.e. when the n_4 contribution to the refractive index is only a small correction. Nevertheless it is worth noting the following fact.

For nonresonant interactions the maximum of the ratio between the fifth and the third-order term in the power expansion of the refractive index with respect to $|E|^2$ is

$$\delta = (|n_4|/n_2)|E|^2 = (|n_4|/n_2)A_0^2/B < 1,$$

but the dynamics of the nonlinear pulse evolution in the fibre is governed by the ratio between the fifth and the third-order term in the CQNLSE: $\Delta = \delta\alpha_2/\alpha_1$. As shown in Appendix the ratio $\alpha_2/\alpha_1 > 1$ in the whole single mode frequency region and can reach the values 4 or 5. Hence, $\delta < 1$ is valid even when $1 \le \Delta < \alpha \cong 5$ and we can conclude that the light confinement in the fiber "helps" the role of the FONL effectively "increasing" the values of n_4. This fact has to be taken into account when high power optical pulse propagation experiments are interpreted. The value $\Delta = 1$ corresponds to $b_1 = 1.146$ and to a normalized background field intensity $I = 2$. So when non-resonant optical interactions of light with matter (without reaching the threshold damage level) are considered the relations $1.5 < I < 2$ have to be fulfilled. This threshold behaviour of the kink solitary wave (6) can find interesting applications in optical switching experiments. Also, if the optical intensity diminishes passing the critical value

$I_c = 1.5$ a solitary wave breakdown can take place. A critical background electric field amplitude $A_c = (3\varkappa/4\gamma)^{1/2}$ corresponds to I_c. As we pointed out, the maximum possible saturation intensity I_s of the bright solitary wave solution (2) is exactly equal to I_c. So, when the value of the peak electric field amplitude passes through the value $A_p = A_s = A_c$ the character of the solitary wave alters. The possibility for similar bright to dark soliton transitions (but with changing the carrier wave frequency through a particular value corresponding to a resonance) was pointed out in [33] where the role of material dispersion in dark soliton propagation in a nonlinear dielectric slab was considered. These transitions could be applicable in the all optical signal processing or the optical computing and numerical switching experiments in this direction could be of interest.

According to the two criteria for soliton bistability, cited above [11,12,21,22,23], the solitary wave (6) itself exhibits no bistability. Concerning the stability of (6) it can be mentioned that all possible kink solutions of the one-dimensional CQNLSE were found to be stable [34].

2.3. Gray solitary wave solution

There is also another dark solution of equation (1). It is of "gray" type (having a non zero value at the hole center) and has the form

$$A(z,t) = A_0 \cosh(\Psi)\exp(i\Phi)/[1 + B\cosh^2(\Psi)]^{1/2}. \tag{10}$$

When $\Psi \to \pm\infty$ then $|A|^2 \to A_0^2/B$ and $|A|^2/(1 + B)$ at the hole center ($\Psi = 0$).The condition (7) leads now to: $B = 1/\sinh^2(b_1)$, $A_0^2 = P\coth(b_1)/[T_0 b_1 \sinh^2(b_1)]$, $T_0 = \tau_0^d b_1[3\coth(b_1) - \tanh(b_1)]$, $Q = 4\cosh^2(b_1)[\cosh^2(b_1) + 2]/[2\cosh^2(b_1) + 1]$ and

$$P = P_0^d \operatorname{arccosh}[y^{1/2}], \tag{11}$$

with $y = [2 - Q + (4 - 3Q)^{1/2}]/[2(Q - 1)]$. All other parameters of (10) are given by the relations valid for the kink solution (6). The width of the hole can be expressed as

$$\tau_{1/2} = T_0 \operatorname{arccosh}\Big\{\sinh(b_1)[\cosh(b_1) + 1] \times [3\cosh(b_1) + 1]^{-1/2}[\cosh(b_1) - 1]^{-1/2}\Big\} \tag{12}$$

and is a two-valued function of b_1 (Fig. 4, curve 1) providing possibilities for successive propagation of gray solitary waves with the same widths but different hole energies. So, according to [11,12,21], the gray hole (10) is bistable. Now the minimal value of the normalized pulse duration $t_0^{min} = 3.36$ is reached at $b_1 = 1.35$. The only restriction for b_1 coming from the requirement $B > 0$ is $b_1 > 0$. Thus, the solution (10) exists only if $\gamma \neq 0$ and does not turn into any known soliton solution of the CNLSE for negligible values of n_4. In this sense the energy gap needed for creating (10) in a fiber is very small in comparison with that for creating (6). The normalized wavenumber shift Q begins from 4/3 for small values of b_1 and goes monotonically to unity with increasing pulse energy (Fig. 4, curve 2). Thus, according to the criterion suggested in [22,23], the propagation of the solitary

wave (10) in a fibre will not be bistable. In this case, in contrast to the previous one, the normalized background electric field intensity I increases with increasing normalized hole energy b_1 (Fig. 3, curve 2). Now, the requirements for $B > 0$ and $\tau_{1/2}$ to be positive imply $1 < I < 1.5$. So, the corresponding to $I_c = 1.5$ critical electric field peak amplitude A_c is the same as for the kink solitary wave (10) and is equal to the saturation peak amplitude A_s introduced for the bright solitary wave (6). So, a condition for the propagation of (10) is $A_p = A_0/B^{1/2} < A_c$. We remember that the corresponding condition for the bright pulse (6) is exactly the same. So, when $A_p < A_c$ then depending on the initial conditions at the fibre input and output bright and gray solitary waves can be supported.

For discussing the stability of (10) we have to define the parameter $C = (I - I_0)/I = 1/(1 + B)$, where I labels the normalized constant intensity background and I_0 corresponds to the minimal intensity at the hole center. This parameter characterizes the hole depth and often is called "contrast" of the hole. When expressed as a function of the normalized hole energy b_1, $C = \tanh^2(b_1)$ and therefore increases with increasing b_1. Enns and Mulder [31], studying the bistability of solitary wave holes in fibres with a Kerr-like nonlinearity at small intensities, a sufficiently steep jump at intermediate I, and Kerr-like again (or tending to a constant) at large I, introduced and checked numerically a criterion for the stability of the holes. This criterion requires that the derivative of b_1 in respect to the contrast C has to be positive: $db_1/dC > 0$. In our case $db_1/dC = 0.5(C)^{-1/2}/(1 - C) > 0$. So, the criterion of Enns and Mulder, generalized to the case of FONL, shows that the gray hole (10) is stable. Another theoretical work [34] also indicates the stability of (10).

3. Solitary wave bistability

A great deal of effort was done to search for different kind of appropriate nonlinearities providing soliton bistability. It is worth noting that the criterion for soliton bistability requiring that the soliton width has to be a multi-valued function of the pulse peak amplitude or energy is discussed only in the last few years and is not equivalent to the conventionally used, proposed by Kaplan [22,23]. Kaplan's criterion for soliton bistability requires that more than one solitary waves can exist carrying the same pulse energy but having different profiles and wavenumbers. In this sense the FONL, considered here, was not classified as perspective as long as other, more exotic ones [22,23], do. In this paper, after making a unified analysis of all possible solitary wave solutions of the CQNLSE (1), we would like to revise this common opinion. Indeed, when considering the dark quasi-solitons (6) and (10) we can see (Fig. 3) that they can propagate with equal hole energies and with slightly different background intensities and wavenumbers. In Fig. 5 we show the normalized

intensity distributions of the kink solitary wave (6) (curve 1) and the gray one (10) (curve 2) as functions of the normalized time coordinate Ψ (it is taken into account that the normalization quantities T_0 of the two solutions are different) for $b_1 = 1.5$. For this value of the normalized black and gray hole energies the corresponding normalized background field intensities are approximately 1.68 and 1.38, respectively, the normalized nonlinear wavenumber shifts being 0.7 (Fig. 2) and 1.18 (Fig. 4). Being of similar type (black and gray), the two holes can be considered as final evolution states of similar initial conditions. Thus, operating with energies a little over the energy gap needed for creating (6) and on the basis of appropriate initial conditions, switching between gray and kink quasi-solitons could be performed. In this sense the possibility for existence of two dark quasi-solitons of different shapes and equal hole energies in a highly nonlinear fiber can be interpreted as a new kind of soliton bistability. The fact that in the same time a bright solitary wave can propagate makes the possible switching experiments very attractive.

4. Conclusions

In this paper we examine all solitary wave solutions of the GNLSE (1) considering nonlinear pulse propagation in anomalous dispersion fibres with positive third-order nonlinear coefficient and negative fifth-order one discussing their stability and bistable behaviour. We show that the three kinds of solitary waves (bright, kink and gray one) can propagate in the fibre with comparable pulse (hole) energies. A method of measuring the fifth-order optical properties of transparent highly nonlinear dielectric materials is considered. Bright to kink and kink to bright solitary wave transitions are discussed and compared to similar ones in media with other nonlinearity laws. Considering the two possible dark solutions of (1) a new type of soliton bistability is predicted. All solitary wave parameters are presented as functions of the pulse (hole) energy and the role of the FONL for the nonlinear wave phenomena considered is discussed in detail. It must be pointed out that the exact solutions of the CQNLSE presented here, exist also for another choice of fibre parameters, namely: $\beta_2 > 0$, $n_2 < 0$ and $n_4 > 0$. In that case all expressions in this paper are the same, except those for the normalized wavenumber shifts - they have to be taken with the sign minus. Of course, the interpretation of the results will be similar but in this case it cannot be generalized for self-trapping of light beams in planar waveguides, because the diffraction is always positive and do not correspond to normal dispersion in fibres.

In fact, the choice of a fibre as a possible medium for the nonlinear wave phenomena, considered here, is not unique. That is why our aim was to emphasis on the physical consequences. Nevertheless, if fibres are considered, the problems in eventual experiments will be related to the limitation of the influence of

other nonlinear effects, such as stimulated Raman scattering, which could be triggered by the high field intensities. A possible solution of this problem can be the choice of a material or glass dopant with an appropriate (relatively large compared to the pulse width) phase relaxation time because the Raman gain depends essentially on it.

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Appendix: Derivation of the nonlinear dispersion coefficient α_2

The nonlinear dispersion coefficients α_1 and α_2, accounting for the radial electric field distribution, are connected with the third- and fifth-order nonlinearity, respectively. Conventionally they are assumed to be equal [26,27]. In addition, they are obtained by an approach based on the assumption that the transverse variation of the waveguide modes remains essentially unperturbed by the perturbing nonlinearity and the important effect of light confinement in the fiber is accounted by integration over the transverse cross-section of these linear fields instead of

applying the exact boundary conditions at the core-cladding interface including the nonlinear field distortions. The method we propose here is based on the derivation of the nonlinear dispersion relation [13,14] for the fundamental LP_{01}-mode of a weakly guiding step-index single mode optical fiber, but the procedure can be easily generalized for the case of graded multimode ones. The essential points in our consideration are the accounting for the transverse redistribution of the wave fields, caused by the nonlinearity, and the correct applying of the boundary conditions at the core cladding interface.

Let us consider the input electric field in the form:

$$\mathbf{E} = \frac{1}{2}\,\vec{y}\,E\exp[i(\beta_o z - \omega_o t)] + c.c., \tag{A1}$$

where $\vec{y}$ is a unit vector giving the polarization of the mode, $\beta_o = \omega_o n_{01}/c$ and the propagation is assumed to be in the z direction. If the nonlinear polarization is presented as

$$\mathbf{P}_{NL} = \frac{1}{2}\,\vec{y}\,P_{NL}\exp[i(\beta_o z - \omega_o t)] + c.c., \tag{A2}$$

the amplitude P_{NL} is

$$P_{NL} = \varepsilon_o\left(a|E|^2 + f|E|^4\right)E, \tag{A3}$$

where $a = (3/4)\chi^{(3)}_{yyyy}(-\omega;\omega,-\omega,\omega)$, $f = (5/8)\chi^{(5)}_{yyyyyy}(-\omega;\omega,-\omega,\omega,-\omega,\omega)$ and ε_o is the dielectric permittivity of the vacuum. Then the set of Maxwell equations reduces to

$$\Delta\mathbf{E} + k^2n^2\mathbf{E} = -(\varepsilon_o\varepsilon_L)^{-1}\nabla\nabla.\mathbf{P}_{NL} - (k^2/\varepsilon_o)\mathbf{P}_{NL}, \tag{A4}$$

$$\Delta\mathbf{H} + k^2n^2\mathbf{H} = i\omega\nabla\times\mathbf{P}_{NL}, \tag{A5}$$

where $k = \omega_o/c$ and $\varepsilon_L = n^2$ is the linear dielectric function. The z-component of the right-hand sides of equations (A4) and (A5) evaluate, respectively, to

$$P = -i\beta(\varepsilon_o\varepsilon_L)^{-1}\left(\frac{\partial}{\partial x}P_{NLx} + \frac{\partial}{\partial y}P_{NLy}\right), \tag{A6}$$

$$Q = i\omega\left(\frac{\partial}{\partial x}P_{NLy} - \frac{\partial}{\partial y}P_{NLx}\right) \tag{A7}$$

It should be observed that equations (A4) and (A5) must be evaluated within a weakly nonlinear assumption by using the zeroth order linear fields. Since the waveguide is cylindrical then coordinates (r, ϕ, z) will be used where r is a radius, ϕ is the azimuth and z is the propagation direction. Hence all field components will be separated into the form $F(r)e^{i\nu\phi}\,e^{i(\beta_o z - \omega_o t)}$ where $F(r)$ is a radial part, ν is an integer and the solutions are periodic in ϕ. The differential equations for the z-components of the radial part of the wave field are, therefore,

$r < R_o$:

$$\frac{\partial^2 E_z}{\partial \xi_1^2} + \frac{1}{\xi_1}\frac{\partial E_z}{\partial \xi_1} + \left[1 - \frac{\nu^2}{\xi_1^2}\right]E_z = \left(\frac{R_o}{u}\right)^2 P_1 e^{-i\nu\phi} \tag{A8}$$

$$\frac{\partial^2 H_z}{\partial \xi_1^2} + \frac{1}{\xi_1}\frac{\partial H_z}{\partial \xi_1} + \left[1 - \frac{\nu^2}{\xi_1^2}\right]H_z = \left(\frac{R_o}{u}\right)^2 Q_1 e^{-i\nu\phi} \tag{A9}$$

$r > R_o$:

$$\frac{\partial^2 E_z}{\partial \xi_2^2} + \frac{1}{\xi_2}\frac{\partial E_z}{\partial \xi_2} - \left[1 + \frac{\nu^2}{\xi_2^2}\right]E_z = \left(\frac{R_o}{w}\right)^2 P_2 e^{-i\nu\phi} \tag{A10}$$

$$\frac{\partial^2 H_z}{\partial \xi_2^2} + \frac{1}{\xi_2}\frac{\partial H_z}{\partial \xi_2} - \left[1 + \frac{\nu^2}{\xi_2^2}\right]H_z = \left(\frac{R_o}{w}\right)^2 Q_2 e^{-i\nu\phi} \tag{A11}$$

where $u^2 = (k^2n_{o1}^2 - \beta^2)R_o^2$, $w^2 = (\beta^2 - k^2n_{o2}^2)R_o^2$, $\xi_1 = ur/R_o$,

$\xi_2 = wr/R_o$. Equations (A8) to (A11), since $P_{1,2}$ and $Q_{1,2}$ use zeroth order fields, are now linear equations and can be integrated in a straightforward way. Hence, for the perturbed by the nonlinearity longitudinal fields of the HE_{11}-mode which corresponds to the LP_{01}-mode we obtain:

$r < R_o$:

$$E_{z1}(\xi_1) = A_1\frac{J_1(\xi_1)}{J_1(u)} + R_1(\xi_1) \tag{A12}$$

$$H_{z1}(\xi_1) = B_1\frac{J_1(\xi_1)}{J_1(u)} + S_1(\xi_1), \tag{A13}$$

where A_1 and B_1 are constants determined from the boundary conditions and

$$(R_1,\ S_1) = \frac{\pi}{2}\left\{Y_1(\xi_1)\int^{\xi_1} s\left(\frac{R_o}{u}\right)^2\left[P_1(s),\ Q_1(s)\right]J_1(s)ds - J_1(\xi_1)\int^{\xi_1} s\left(\frac{R_o}{u}\right)^2\left[P_1(s),\ Q_1(s)\right]Y_1(s)ds\right\}e^{-i\phi}. \tag{A14}$$

Similarly,

$r > R_o$:

$$E_{z2}(\xi_2) = A_2\frac{K_1(\xi_2)}{K_1(w)} + R_2(\xi_2) \tag{A15}$$

$$H_{z2}(\xi_2) = B_2\frac{K_1(\xi_2)}{K_1(w)} + S_2(\xi_2), \tag{A16}$$

where A_2 and B_2 are two further constants and

$$(R_2,\ S_2) = \left\{I_1(\xi_2)\int^{\xi_2} s\left(\frac{R_o}{w}\right)^2\left[P_2(s),\ Q_2(s)\right]K_1(s)ds\right.$$

$$- K_1(\xi_2)\int^{\xi_2} s\left(\frac{R_o}{w}\right)^2 \left[P_2(s),\ Q_2(s)\right] I_1(s)ds\Big\}\ e^{-i\phi}. \qquad (A17)$$

For an LP_{01} mode $E = (0, E_y, E_z)$ with $|E_z| << |E_y|$. The zeroth order (linear) field components, in terms of A, the complex amplitude of E_y at the core boundary, are [15]

$$E_y(\xi) = \begin{cases} A\dfrac{J_o(\xi_1)}{J_o(u)}, & r < R_o \\ A\dfrac{K_o(\xi_2)}{K_o(w)}, & r > R_o \end{cases} \qquad (A18)$$

where $n_1 \approx n_2 = n$, $A_1 = -A/2R_o kn$ and $B_1 = -in(\varepsilon_o/\mu_o)^{1/2}A_1$. So, the nonlinear terms P_1 and Q_1 take the forms

$$P_1 = -i\frac{k}{n}\, a_1\frac{|A|^2 A}{J_o^3(u)}\,\frac{\partial J_o^3(\xi_1)}{\partial y} - i\frac{k}{n}\, f_1\frac{|A|^4 A}{J_o^5(u)}\,\frac{\partial J_o^5(\xi_1)}{\partial y}, \qquad (A19)$$

$$Q_1 = i\omega\varepsilon_o a_1\frac{|A|^2 A}{J_o^3(u)}\,\frac{\partial J_o^3(\xi_1)}{\partial x} + i\omega\varepsilon_o f_1\frac{|A|^4 A}{J_o^5(u)}\,\frac{\partial J_o^5(\xi_1)}{\partial x}, \qquad (A20)$$

where the subscript "1" of the coefficients a_1 and f_1 label the core. The expressions for P_2 and Q_2 can be easily obtained by replacing J with K and the subscript "1" with "2" in (A19) and (A20). In practice, however, $a_1 \cong a_2$ and $f_1 \cong f_2$.

Now the electric field components (A12), (A13), (A15) and (A15) can obtain their concrete for the LP_{01}-mode form and can be used for obtaining the other needed components of the electric field. Then the applying of the boundary conditions at the core cladding interface leads to the following expression for the nonlinear dispersion relation of the LP_{01}-mode:

$$\frac{J_o(u)}{uJ_1(u)} - \frac{K_o(w)}{wK_1(w)} = k^2R_o^2|A|^2\left[a_1\frac{N_1(u)}{u^2} + a_2\frac{N_2(w)}{w^2}\right] + k^2R_o^2|A|^4\left[f_1\frac{M_1(u)}{u^2} + f_2\frac{M_2(w)}{w^2}\right], \qquad (A21)$$

where

$$N_1(u) = 1 + \frac{3}{uJ_o^3(u)J_1(u)}\int_0^u sJ_o^2(s)J_1^2(s)ds,$$

$$N_2(w) = 1 - \frac{3}{wK_o^3(w)K_1(w)}\int_w^\infty sK_o^2(s)K_1^2(s)ds,$$

$$M_1(u) = 1 + \frac{5}{uJ_o^5(u)J_1(u)}\int_0^u sJ_o^4(s)J_1^2(s)ds,$$

$$M_2(w) = 1 - \frac{5}{wK_o^5(w)K_1(w)}\int_w^\infty sK_o^4(s)K_1^2(s)ds,$$

The nonlinear terms in the dispersion equation

$$N = a_1\frac{N_1(u)}{u^2} + a_2\frac{N_2(w)}{w^2} \quad \text{and} \quad M = f_1\frac{M_1(u)}{u^2} + f_2\frac{M_2(w)}{w^2}$$

include not only the full nonlinear properties of the medium (the third- and fifth-order one, respectively), but also the transverse structure at nonlinear regime of the LP_{01}-mode. The differentiation of equation (A21), with respect to $|A|^2$ and $|A|^4$ will yield equations for $d\beta/d|E|^2$ and $d\beta/d|E|^4$ which are the nonlinear coefficients in the CQNLSE - $\varkappa$ and γ. Thus, the dispersion coefficients α_1 and α_2 take the form:

$$\alpha_1 = 2\left(\frac{uw}{v}\right)^2\left[\frac{N_1(u)}{u^2} + \frac{N_2(w)}{w^2}\right], \qquad (A22)$$

$$\alpha_2 = 2\left(\frac{uw}{v}\right)^2 \left[\frac{M_1(u)}{u^2} + \frac{M_2(w)}{w^2}\right]. \tag{A23}$$

In Fig. 6 are shown the variation of the nonlinear dispersion coefficients α_1 (curve 2) and α_2 (curve 1). We can see that in all the single mode frequency region ($v < 2.405$) the total fifth-order dispersion coefficient α_2 (core plus cladding nonlinear contributions) is significantly greater than the third-order one. In addition, what is not shown, when fifth-order nonlinearity is accounted in fibres the fifth-order cladding response is comparable to the third-order core one and always has to be taken into account.

Figure captions

Fig. 1. Dependence of the normalized pulse duration of the bright quasi-soliton (Eq. 2) on the normalized pulse energy b.

Fig. 2. Variation of the normalized pulse duration (curve 1) and the nonlinear wavenumber shift (curve 2) of the kink quasi-soliton (Eq. 6) with the normalized hole energy b_1.

Fig. 3. Variation of the normalized electric field background intensities I of the kink solitary wave (6) (curve 1) and of the gray solitary wave (10) with the normalized hole energy b_1.

Fig. 4 Variation of the normalized pulse duration (curve 1) and the nonlinear wavenumber shift (curve 2) of the gray quasi-soliton (Eq. 10) with the normalized pulse energy b_1.

Fig. 5 Normalized intensity distribution of the kink (curve 1) and gray (curve 2) solitary waves with the normalized time Ψ for $b_1 = 1.5$.

Fig. 6 Dependence of the nonlinear dispersion coefficients α_1 (curve 2) and α_2 (curve 1) with the normalized fibre frequency v in the single mode region of the fibre.

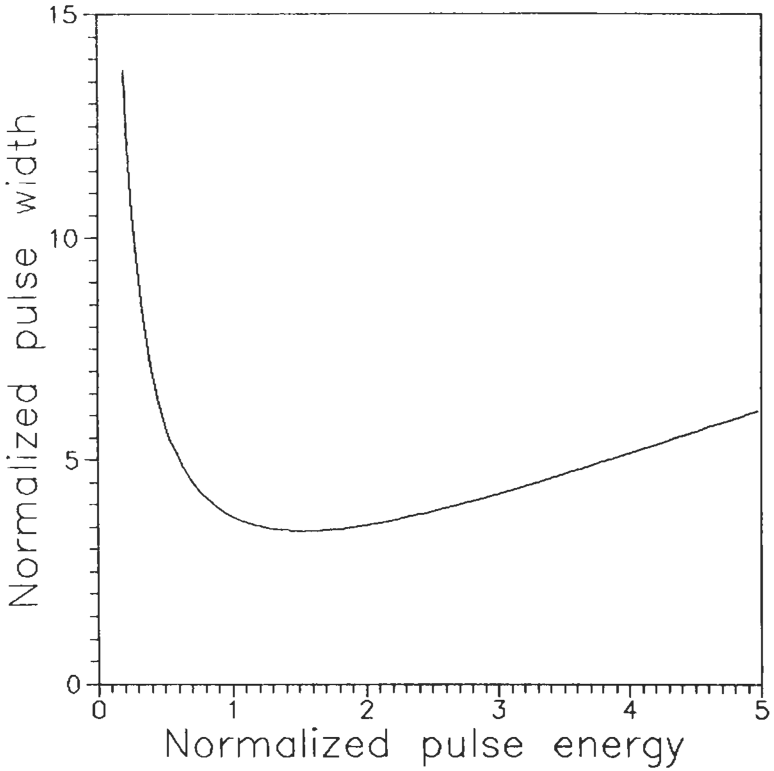

15
10
5
0
0
1
2
3
4
5
Normalized pulse width
Normalized pulse energy

Fig. 1

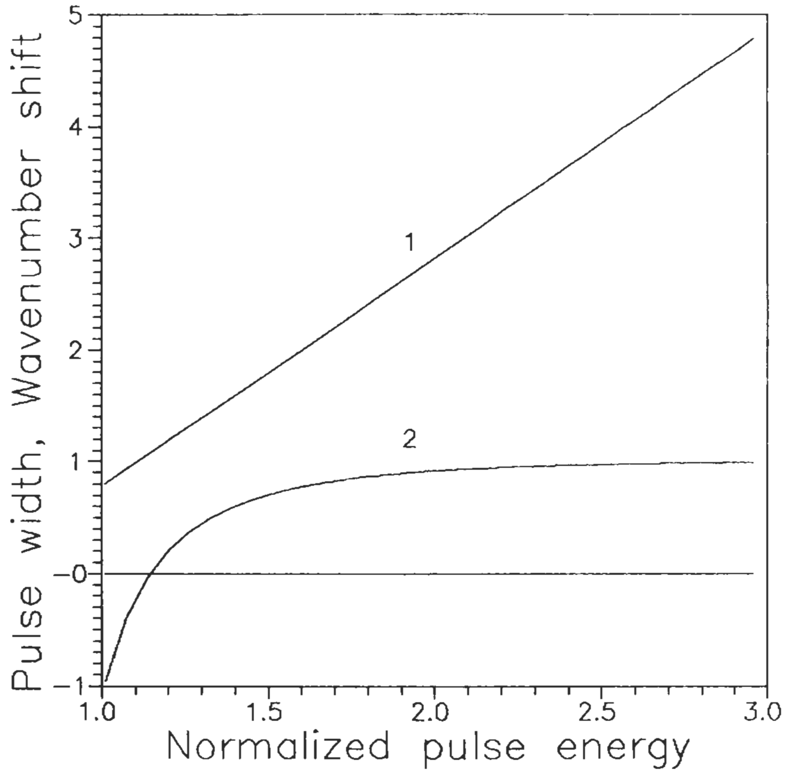
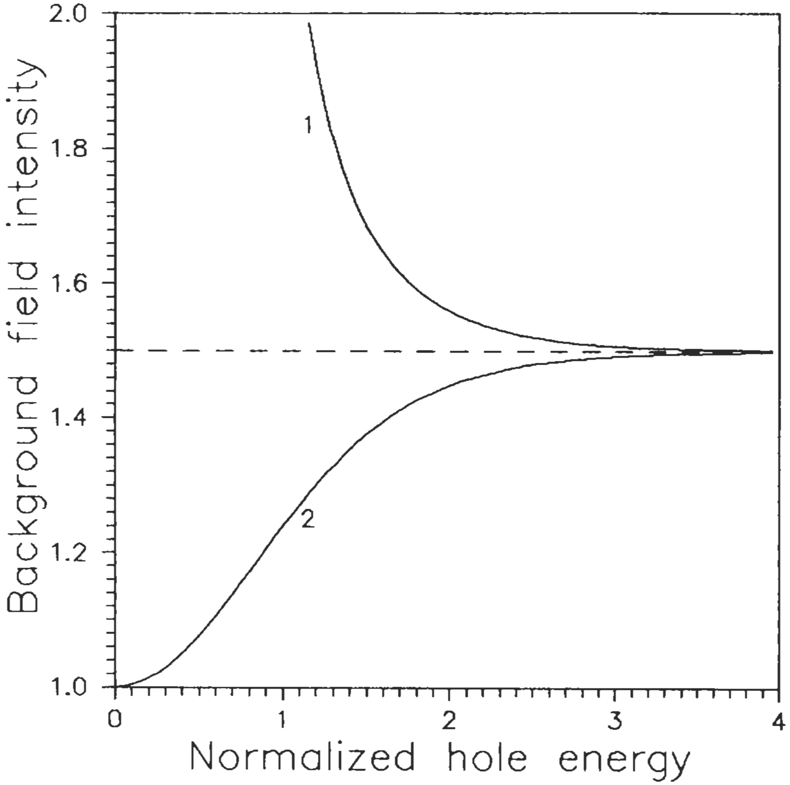
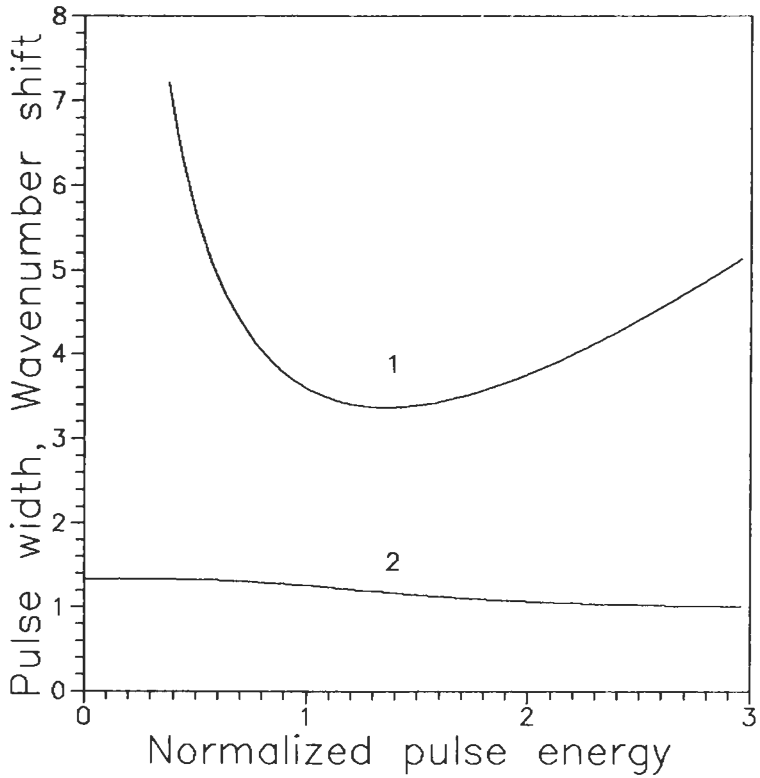

5
4
3
2
1
-0
-1
1.0
1.5
2.0
2.5
3.0
1
2
Pulse width, Wavenumber shift
Normalized pulse energy

Fig. 2



2.0
1.8
1.6
1.4
1.2
1.0
0
1
2
3
4
1
2
Background field intensity
Normalized hole energy

Fig. 3



8
7
6
5
4
3
2
1
0
0
1
2
3
1
2
Pulse width, Wavenumber shift
Normalized pulse energy

Fig. 4

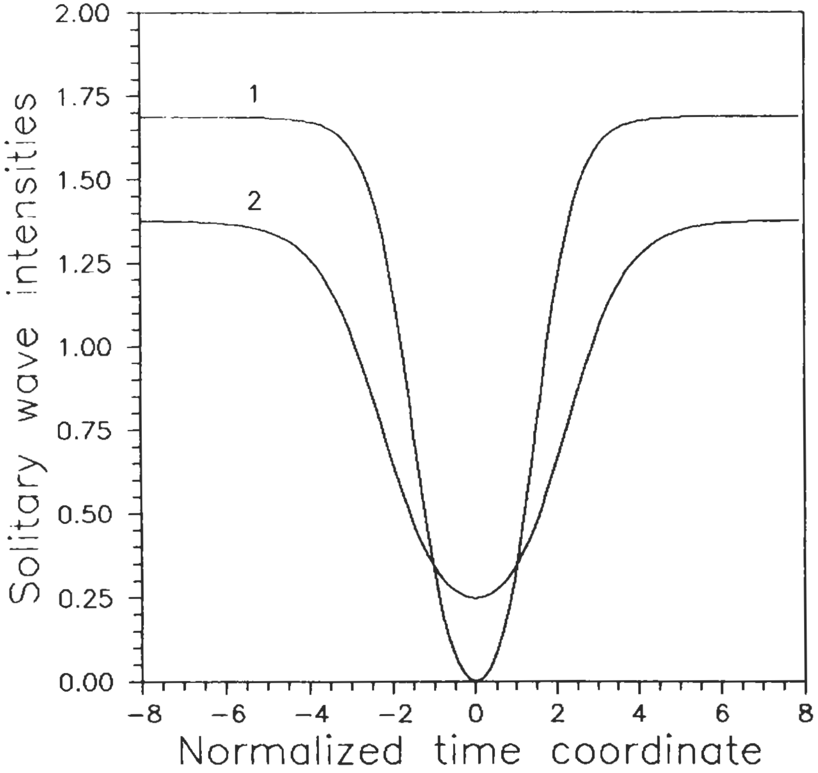

2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
Solitary wave intensities
1
2
−8
−6
−4
−2
0
2
4
6
8
Normalized time coordinate

Fig. 5

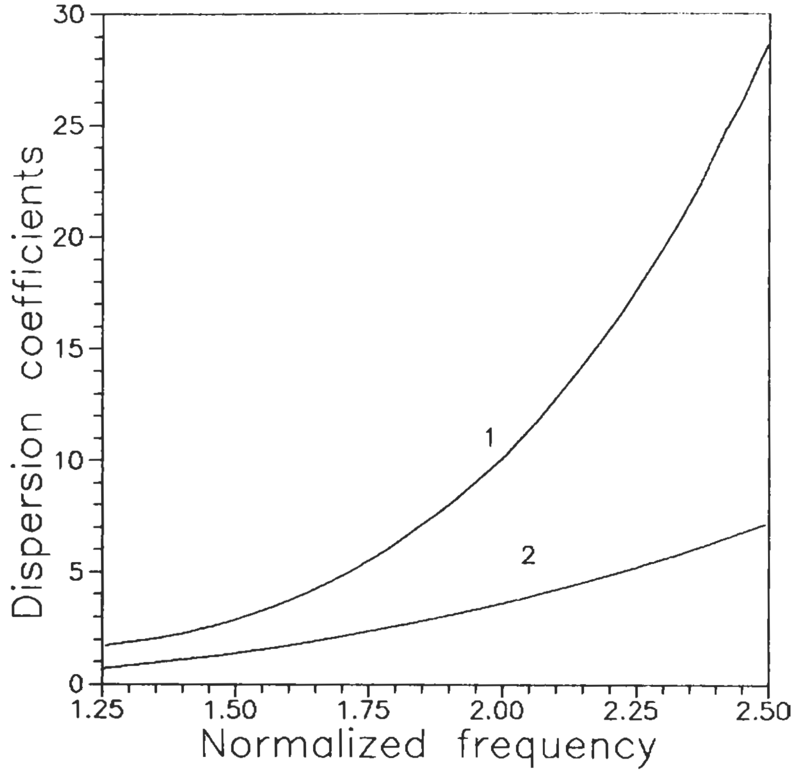

30
25
20
15
10
5
0
Dispersion coefficients
1
2
1.25
1.50
1.75
2.00
2.25
2.50
Normalized frequency

Fig. 6