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(Laser, Atomic and Molecular Physics)

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INTERNATIONAL ATOMIC ENERGY AGENCY

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DOWN–CONVERSION PROCESSES AND THE PARAMETRIC APPROXIMATION

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ABSTRACT

It is shown that the applicability of the parametric approximation to the processes of photon down–conversion depends on the value of frequency detuning between pump and signal field modes. It is found that for the multi–photon down–version processes the threshold value of detuning exists. If the detuning is smaller than the threshold value the behavior of the average number of photons in signal mode has a singular character in this approximation due to inapplicability of the parametric approximation in this case.

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Preface

The ICTP-LAMP reports consist of manuscripts relevant to seminars and discussions held at ICTP in the field of Laser, Atomic and Molecular Physics (LAMP).

These reports aim at informing LAMP researchers on the activity carried out at ICTP in their field of interest, with the specific purpose of stimulating scientific contacts and collaboration of physicists from Third World Countries.

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To describe an interaction of quantum systems with an intense coherent electromagnetic field in quantum optics the parametric approximation is commonly used [1]. In the parametric approximation the photon creation and annihilation operators of intense (pump) modes which are in coherent states are replaced by the coherent state amplitude: $a^+, a \rightarrow \alpha^* e^{i\omega_a t}, \alpha e^{-i\omega_a t}$. Here a and a^+ correspond to the photon annihilation and creation operators of the pump mode; α is the coherent state amplitude and ω_a is the pump mode frequency. In this approximation the effects of quantum system-pump field correlation and back action of quantum system on pump field are neglected.

The parametric approximation is successfully used to describe an atom-field interaction in presence of strong pump field. It seems obvious that the accuracy of the parametric approximation for this kind of problems improves when the pump field amplitude increases [1]. But recently it has been demonstrated that the parametric approximation can not be applied to arbitrary quantum system interacting with pump field. For example, for the multi-photon down-conversion processes it has been shown [2-4] that the parametric approximation under the condition of the full resonance for both pump and signal modes leads to divergencies of some quantum observables (photon number and et.c.). This divergencies are the result of neglecting the quantum nature of the pump field [3]. Thus the parametric approximation requires a detailed investigation of the applicability for each concrete case.

In this paper we discussed the applicability of the parametric approximation for the down-conversion processes. For illustration of the main concept of our discussion let us begin with the simplest case of the one-photon down-conversion. If both pump and signal modes are quantized the Hamiltonian in the rotating wave approximation is

$$H_{1com} = \hbar\omega_a a^+ a + \hbar\omega_b b^+ b + \hbar g(a^+ b + b^+ a), \tag{1}$$

where a^+, a and b^+, b correspond to the creation and annihilation operators of pump and signal modes. Let the initial state of signal mode be vacuum and the initial state of pump mode be coherent state $|\alpha>$. After using the parametric approximation the Hamiltonian (1) in the interaction picture takes the form

$$H_{1app} = \hbar\Delta b^+b + \hbar g(\alpha^* b + \alpha b^+), \tag{2}$$

where $\Delta = \omega_b - \omega_a$.

The Hamiltonian (1) leads to the following expression for the average number of photons in signal mode

$$< b^+(t)b(t) >= 2g^2|\alpha|^2\Big(1 - cos((\Delta^2 + 4g^2)^{1/2}t)\Big)/(\Delta^2 + 4g^2). \tag{3}$$

The approximate Hamiltonian (2) leads to

$$< b^+(t)b(t) >_{app}= 2g^2|\alpha|^2(1 - cos(\Delta t))/\Delta^2. \tag{4}$$

It can be seen that the expression (3) coincides with the expression (4) in the limit of the large detuning ($\Delta^2 \gg g^2$) for all amplitudes of the initial coherent state of pump field. Thus the possibility to apply the parametric approximation to the Hamiltonian (1) does not depend on the intensity of pump field, but it is determined by parameters of the Hamiltonian.

For zero detuning ($\Delta = 0$) the approximate Hamiltonian (2) leads to the divergent result
$< b^+(t)b(t) >_{app}= g^2|\alpha|^2t^2$,
that coincides with the exact solution
$< b^+(t)b(t) >_{com}= g^2|\alpha|^2(1 - cos(2gt))/2$
only for small time ($t \ll 1/g$). As it follows from this simple example, the applicability of the parametric approximation to the down-conversion processes depends on the frequency detuning between pump and signal modes.

Now let us consider the three-photon down-conversion process. The fully quantized Hamiltonian in the rotating wave approximation can be written in the form

$$H_{3com} = \hbar 3\omega_a a^+a + \hbar(\omega_a + \Delta)b^+b + \hbar g(a^+b^3 + b^{+3}a), \tag{5}$$

and, consequently, the parametrically approximated Hamiltonian can be written as

$$H_{3app} = \hbar\Delta b^+b + \hbar g(\alpha^* b^3 + \alpha b^{+3}). \tag{6}$$

The Hamiltonian (5) gives the following equation for the operator of photon number $n = b^+b$

$$d^2n/dt^2 = 9\Delta H_{3com} - 9\Delta^2 n + 6g^2\Big(I(9n^2+9n+6) - 4n^3 - 4n\Big), \tag{7}$$

where $I = a^+a + b^+b/3$ is the integral of motion of the Hamiltonian (5). The approximate Hamiltonian (6) gives the following equation for n

$$d^2n/dt^2 = 9\Delta H_{3app} - 9\Delta^2 n + 6g^2|\alpha|^2\Big(9n^2+9n+6\Big). \tag{8}$$

The equation (7) gives a finite value for average photon number $< n(t) >$ for all possible values of parameters of the Hamiltonian (5) due to existence of dominant nonlinear term $-24g^2n^3$ in the right-hand part of the equation (7). But the equation (8) for $\Delta = 0$ always leads to the divergencies of $< n(t) >$ [3,4]. However for each value of parameters g and α such region of values of detuning Δ can be found that the equation (8) will lead to finite result for $< n(t) >$. Let us estimate the threshold value of the detuning which separates the regions of convergence and divergence. After averaging over the initial vacuum state of signal mode b the equation (8) has the following form

$$d^2N/dt^2 = -9\Delta^2 N + 6g^2|\alpha|^2(9N^2+9N+6) + 54g^2|\alpha|^2\delta N^2, \tag{9}$$

where $< n(t) >$ is denoted by $N(t)$ and $\delta N^2 =< n^2(t) > - < n(t) >^2$. According to Schwartz inequality the relation

$< n^2 > \geq < n >^2$

is always hold and, consequently, $\delta N^2 > 0$. So the equation (9) without last term in the right-hand part has the solution that increases slower than the solution of the complete equation. The threshold value of the detuning that separates the regions of convergence and divergence for the truncated equation

$$d^2N/dt^2 = -9\Delta^2 N + 6g^2|\alpha|^2(9N^2+9N+6), \tag{10}$$

will be lower bound for the exact threshold value of the detuning for the complete equation (9). The truncated equation (10) has two stationary points given by the condition $d^2N/dt^2 = 0$, which are

$$N_{\pm} = \left(x - 1 \pm ((x-1)^2 - 8/3)^{1/2}\right)/2, \tag{11}$$

where $x = \Delta^2/6g^2|\alpha|^2$. Simple analyses shows that N_+ is unstable stationary point and N_- is stable stationary point. Thus the condition of existence of real N_- is actually the condition of existence of finite solution of the equation (10). According to the expression (11) this condition is

$x \geq 1 + 2(2/3)^{1/2}$.

In other words, when the inequality

$$\Delta^2 > 15.8g^2|\alpha|^2 \tag{12}$$

is violated the equation (9) always leads to the divergency of photon number $< n(t) >$.

If the detuning is sufficiently large so the inequality (12) is satisfied and the solution of the equation (9) is finite, the average number of photons $N(t)$ oscillates near small stationary value N_- ($N_- < (2/3)^{1/2}$) and, therefore, one can conclude that the exact $N(t)$ is small too. From this fact it follows that the equation (7) and the parametrically approximated equation (8) lead to the similar results for the average photon number, and the condition of convergence of the equation (9) solution is simultaneously the condition of the parametric approximation applicability to the three-photon down-conversion process.

In the case of small value of $N(t)$ the numerical calculations can be done directly from the Hamiltonian (6) by introducing of the time-dependent wave function

$|\psi(t)>= \sum_{k=0} C_k(t)|3k>$,

which gives the following set of equations

$$idC_k/dt = 3\Delta k C_k + g\alpha\Big(\big((3k+3)(3k+2)(3k+1)\big)^{1/2}C_{k+1} + \big((3k-2)(3k-1)3k\big)^{1/2}C_{k-1}\Big). \tag{13}$$

Cutting this system of equations off, i.e. supposing that $C_M(t) \equiv 0$ for sufficiently large M and numerically solving it the average number of photons can be found as

$$N(t) = 3\sum_{k=1}^{M} k|C_k(t)|^2. \tag{14}$$

An illustration of the function $N(t)$ given by the equation (14) is shown in Fig.1 . For the comparison the function $N(t)$ is plotted in Fig.2 when the condition (12) is violated. The above proposed method of cutting the system (13) off for sufficiently large M can be used in this case too but within time interval when the condition

$1 - \sum_{k=0}^{M} |C_k(t)|^2 \ll 1$

is satisfied and the mistake of cutting is negligible. This interval of time can be extended by increasing of M.

In conclusion, we have demonstrated that the parametric approximation for the three-photon down-conversion process gives a finite result and it is valid only for the very small (in comparison with the detuning) constants of interaction. Actually, the condition of the parametric approximation applicability is a small number photons in signal mode in comparison with the number of photons in pump mode. It is achieved by taking the sufficiently large detuning for both the single- and three-photon down-conversion. The above discussed analysis can be readily done for higher-order multi-photon down-conversion processes. So the conditions of the parametric approximation applicability for higher-order multi-photon down-conversion processes analogous to the inequality (12) can be obtained.

Acknowledgments

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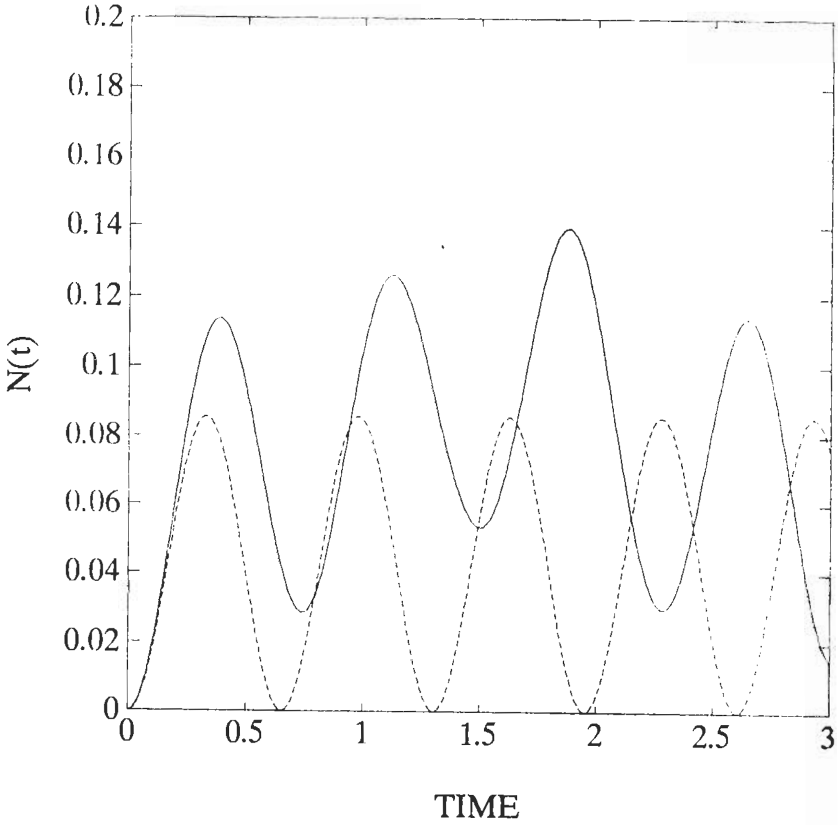
FIGURE CAPTIONS

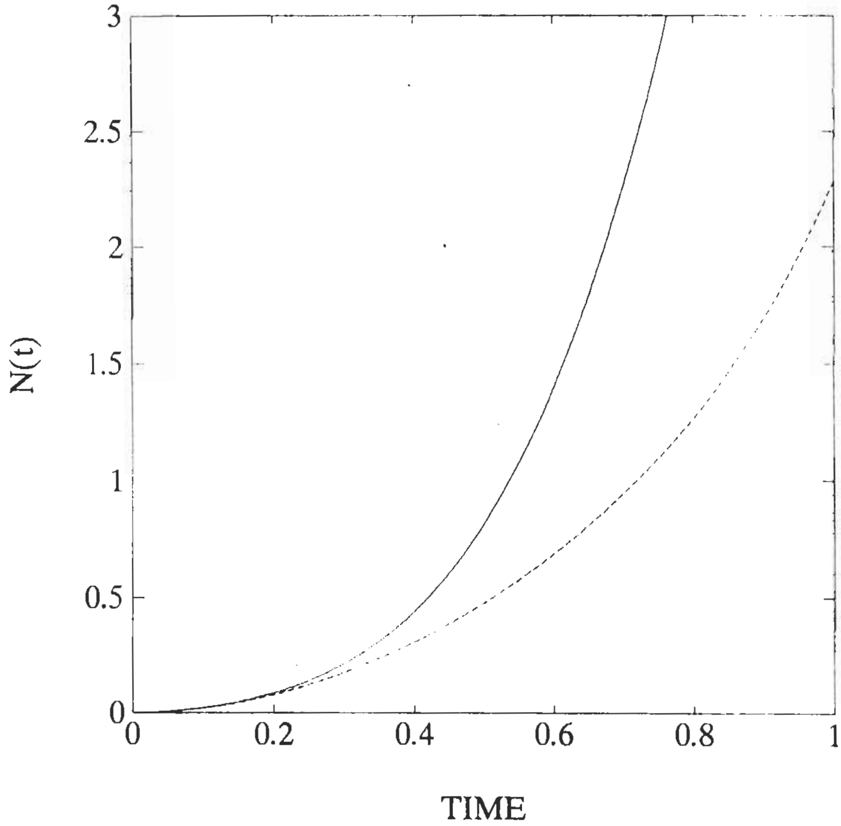
FIG. 1. A typical time-dependence of the average photon number for the case when the inequality (12) is satisfied. Solid line corresponds to the average number of photons given by the equation (14); dashed line corresponds to the average number of photons given by the truncated equation (10); parameter $x = 100$. The normalized timescale is used: time is plotted in the units $|g\alpha|^{-1}$.

FIG. 2. A typical time-dependence of the average photon number for the case when the inequality (12) is violated. Solid line corresponds to the average photon number given by the equation (14); dashed line corresponds to the average photon number given by the truncated equation (10); parameter $x = 9$.



0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.5
1
1.5
2
2.5
3
N(t)
TIME

Fig.1



3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
N(t)
TIME

Fig.2