Institut für Theoretische Physik und Astrophysik

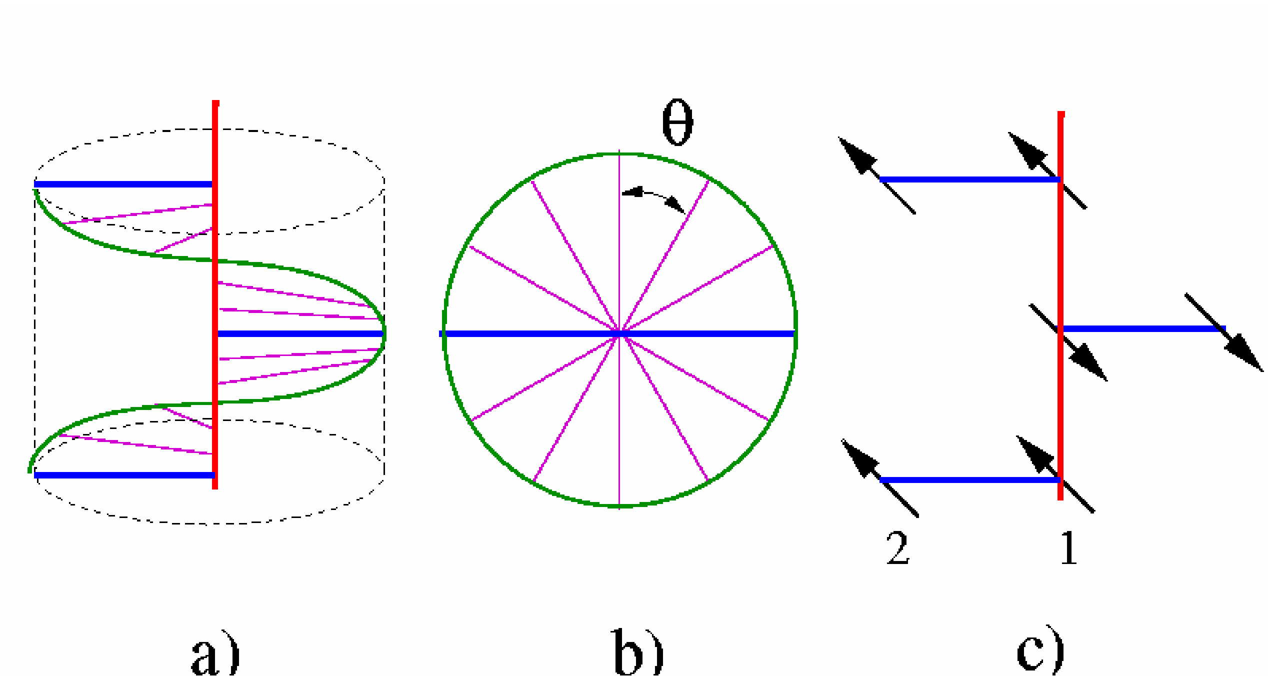
Universität Würzburg

M.N.Kiselev, D.N.Aristov[1] and K.Kikoin[2]

Spin Gap in a Spiral Staircase Model

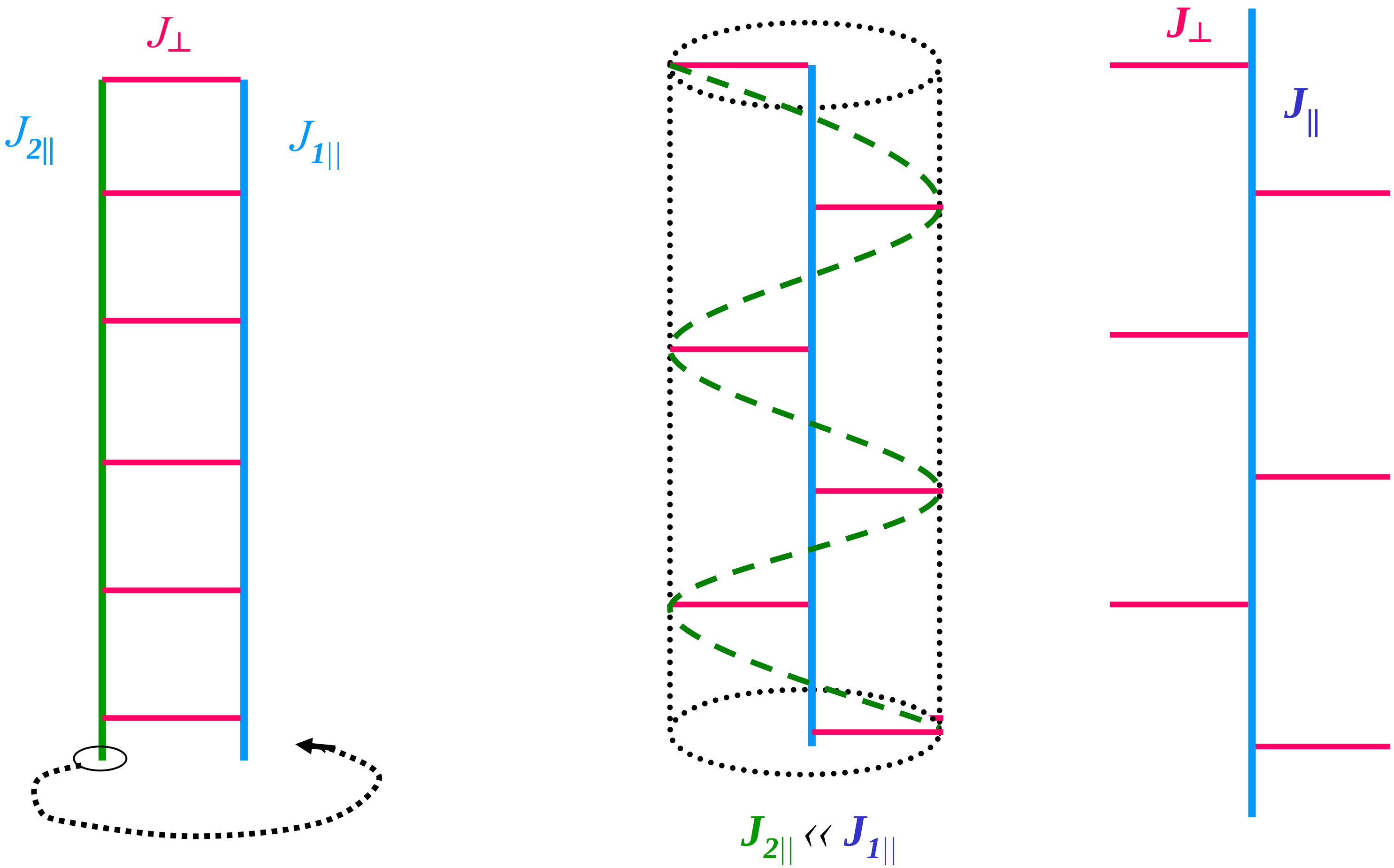
1. Max-Planck Institut für Festkörperforschung, Stuttgart, Germany
2. Ben-Gurion University of the Negev, Beer-Sheva, Israel

Spin Staircase Model



$$H = J_{\parallel} \sum_i \left(\vec{s}_{1,i}\vec{s}_{1,i+1} + \cos^2\left(\frac{\theta}{2}\right)\vec{s}_{2,i}\vec{s}_{2,i+1}\right) - J_{\perp}\sum_i \vec{s}_{1i}\vec{s}_{2i}$$

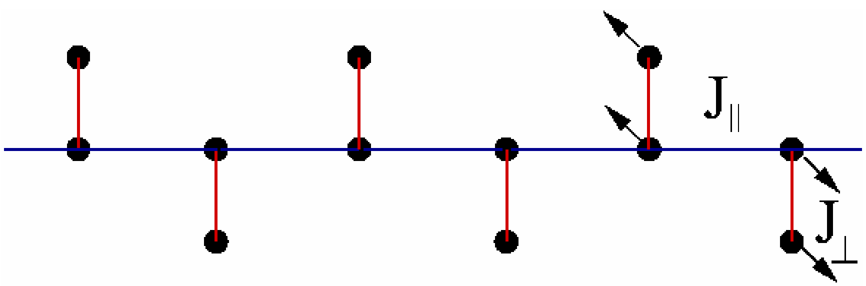
FROM LADDER TO BARBED WIRE



$\theta = 0$ ← Two flat realizations → $\theta = \pi$

Spin Rotator Chain (SRC) Model

$$\theta = \pi$$



$$\vec{S}_i = \vec{s}_{1,i} + \vec{s}_{2,i}$$

$$\vec{R}_i = \vec{s}_{1,i} - \vec{s}_{2,i}$$

$$H = J_{\|} \sum_i \vec{s}_{1,i} \vec{s}_{1,i+1} - J_{\perp} \sum_i \vec{s}_{1i} \vec{s}_{2i}$$

$$H = \frac{J_{\|}}{4} \sum_i \left[\vec{S}_i \vec{S}_{i+1} + \vec{S}_i \vec{R}_{i+1} + (\vec{S} \leftrightarrow \vec{R}) \right] - \frac{J_{\perp}}{4} \sum_i \left(\vec{S}_i^2 - \vec{R}_i^2 \right)$$

Isotropic and anisotropic Spin Staircases

Isotropic (Heisenberg) model

$$H = \sum_i \left[\frac{J_{\|}}{4} \left\{ \left(\vec{S}_i \vec{S}_{i+1} + \vec{R}_i \vec{R}_{i+1} \right) \left(1 + \cos^2 \left(\frac{\theta}{2} \right) \right) \right. \right.$$
$$\left. \left. + \sin^2 \left(\frac{\theta}{2} \right) \left(\vec{S}_i \vec{R}_{i+1} + \vec{R}_i \vec{S}_{i+1} \right) \right\} - \frac{J_{\perp}}{4} \left(\vec{S}_i^2 - \vec{R}_i^2 \right) \right]$$

XXZ Spin Rotator Chain Model

$$\begin{aligned} H_{\|} &= \frac{J_{\|}^x}{8} \sum_i \left[S_i^+ S_{i+1}^- + S_i^+ R_{i+1}^- + (S \leftrightarrow R) + h.c. \right] \\ &+ \frac{J_{\|}^z}{4} \sum_i \left[S_i^z S_{i+1}^z + S_i^z R_{i+1}^z + (S^z \leftrightarrow R^z) \right], \\ H_{\perp}^i &= \frac{J_{\perp}^x}{8} \left((R_i^x)^2 + (R_i^y)^2 \right) + \frac{J_{\perp}^z}{4} (R_i^z)^2 - (\vec{R}_i \leftrightarrow \vec{S}_i) \end{aligned}$$

SO(4) group

$$[D_{\alpha\beta} D_{\mu\nu}] = (\delta_{\alpha\mu} D_{\beta\nu} - \delta_{\alpha\nu} D_{\beta\mu} - \delta_{\beta\mu} D_{\alpha\nu} + \delta_{\beta\nu} D_{\alpha\mu})$$

$$D = -i \begin{pmatrix} 0 & L_3 & -L_2 & M_1 \\ & 0 & L_1 & M_2 \\ & & 0 & M_3 \\ & & & 0 \end{pmatrix}$$

$$[L_j, L_k] = ie_{jkl} L_l, \;\; [M_j, M_k] = ie_{jkl} L_l, \;\; [M_j, L_k] = ie_{jkl} M_l.$$

o(4) algebra of Spin-Rotator

Permutations

$$\left[S_j, S_k\right] = i\varepsilon_{jkl} S_l \qquad \left[R_j, R_k\right] = i\varepsilon_{jkl} S_l \qquad \left[S_j, R_k\right] = i\varepsilon_{jkl} R_l$$

Casimir Operators

$$\vec{S} \cdot \vec{R} = 0 \qquad S^2 + R^2 = 3$$

Jordan-Wigner Transformation

(isolated rung)

Two-component fermionic field

$$S^+ = a^\dagger + e^{i\pi a^\dagger a} b^\dagger, \; S^- = a + b e^{-i\pi a^\dagger a}, \; S^z = a^\dagger a + b^\dagger b - 1$$

$$R^+ = a^\dagger - e^{i\pi a^\dagger a} b^\dagger, \; R^- = a - b e^{-i\pi a^\dagger a}, \; R^z = a^\dagger a - b^\dagger b$$

Commutation relations

$$[S^+, S^-] = 2S^z, \;\; \{S^+, S^-\} = 2 + 2(a^\dagger b + b^\dagger a), \;\; (S^z)^2 = 1 - (n_a - n_b)^2$$

$$[R^+, R^-] = 2S^z, \;\; [R^+, S^-] = 2R^z, \;\; (R^z)^2 = (n_a - n_b)^2$$

$$\{R^+, R^-\} = 2 - 2(a^\dagger b + b^\dagger a), \;\; \{R^+, S^-\} = -\{R^-, S^+\} = 2(a^\dagger b - b^\dagger a)$$

Jordan-Wigner Transformation

(rotated basis)

$$f_\uparrow = (a-b)/\sqrt{2}, \qquad f_\downarrow^\dagger = (a+b)/\sqrt{2}$$

SO(4) generators

$$\begin{aligned}
S_j^+ &= \sqrt{2}\left(f_{\uparrow j}^\dagger(1-n_{\downarrow j})K_j + K_j^\dagger f_{\downarrow j}(1-n_{\uparrow j})\right), \\
S_j^- &= \left(S_j^+\right)^\dagger, \qquad S_j^z = n_{\uparrow j} - n_{\downarrow j}, \\
R_j^+ &= \sqrt{2}\left(f_{\uparrow j}^\dagger n_{\downarrow j} K_j + K_j^\dagger f_{\downarrow j} n_{\uparrow j}\right), \\
R_j^- &= \left(R_j^+\right)^\dagger, \qquad R_j^z = f_{\uparrow j}^\dagger f_{\downarrow j}^\dagger + f_{\downarrow j} f_{\uparrow j}
\end{aligned}$$

String operator

$$K_j = \exp[i\pi \textstyle\sum_{k<j,\sigma} n_{\sigma k}] = \prod_{k<j}(1-2n_{\uparrow k})(1-2n_{\downarrow k})$$

Jordan-Wigner Transformation

(Hubbard Operators)

Transverse components

$$S_j^+ = \sqrt{2}\left(X_j^{10}K_j + K_j^\dagger X_j^{0\bar{1}}\right), \quad S_j^- = \sqrt{2}\left(X_j^{\bar{1}0}K_j + K_j^\dagger X_j^{01}\right)$$

$$R_j^+ = \sqrt{2}\left(X_j^{2\bar{1}}K_j - K_j^\dagger X_j^{12}\right), \quad R_j^- = -\sqrt{2}\left(X_j^{21}K_j - K_j^\dagger X_j^{\bar{1}2}\right)$$

Longitudinal components

$$S_j^z = X_j^{11} - X_j^{\bar{1}\bar{1}}, \quad R_j^z = X_j^{20} + X_j^{02}$$

String Operator

$$K_j = K_j^{c,d} = \exp\left(i\pi \textstyle\sum_{k<j}(S_k^z)^2\right) = \exp\left(i\pi \textstyle\sum_{k<j}\left(X_k^{11} + X_k^{\bar{1}\bar{1}}\right)\right) = \mathcal{K}_j \exp\left(i\pi \textstyle\sum_{k<j} S_k^z\right)$$

Counting degrees of freedom

$SU(2)$ 3 generators, 1 Casimir operator

2S+1 components

$S = 1/2$ Spinless fermions (two component field)

$SU(2) \otimes SU(2) = SO(4)$ 6 generators, 2 Casimir operators

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two $S = 1/2$ 4-component field $(n_\uparrow, n_\downarrow)$ triplet $(1,0)$ $(0,0)$ $(0,1)$

singlet $(1,1)$

Projecting out singlet state

$$\vec{S}_j^2 = 2[1 - n_{\uparrow j} n_{\downarrow j}]$$

Hidden constraint – additional Hubbard interaction

Hidden symmetries

Particle-hole transformation $f_\uparrow \to f_\uparrow^\dagger, f_\downarrow \to f_\downarrow^\dagger$

means $$S^+ \to R^+,\ S^- \to R^-,\ S^z \to S^z,\ R^z \to R^z$$

Flavor transformation $f_\uparrow \to f_\downarrow$

means $$S^+ \to R^-,\ S^- \to R^+,\ S^z \to -S^z,\ R^z \to -R^z$$

Particle-hole flavor transformation $f_\uparrow \to f_\downarrow^\dagger,\ f_\downarrow \to f_\uparrow^\dagger$

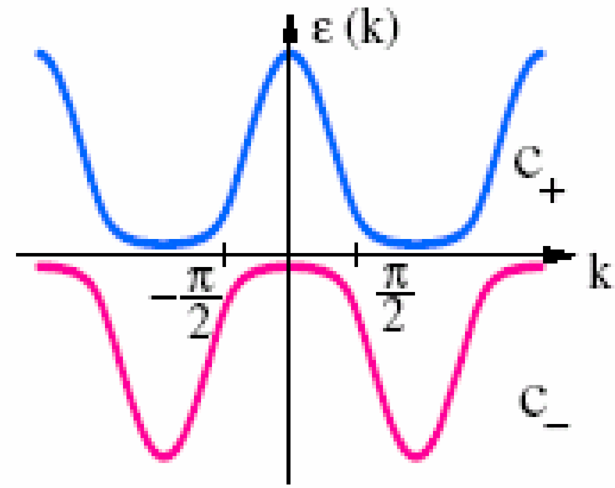
means $$S^+ \to S^-,\ R^+ \to R^-,\ S^z \to -S^z,\ R^z \to -R^z$$

and other discrete rotations…

Effective model

XXZ SRC Model

$$H_{\parallel} = J_{\parallel}^{x} \sum_i \left(a_i^{\dagger} a_{i+1} + a_{i+1}^{\dagger} a_i\right) \cos(\pi n_i^b) + J_{\parallel}^{z} \sum_i \left(n_i^a - \frac{1}{2}\right)\left(n_{i+1}^a - \frac{1}{2}\right)$$



$$H_{\perp,i} = -\frac{J_{\perp}^{x}}{2}(a_i^{\dagger} b_i + b_i^{\dagger} a_i) - J_{\perp}^{z}(n_i^a - \frac{1}{2})(n_i^b - \frac{1}{2})$$

u - v transformation

$$u_{\pm}^2(p) = \pm\varepsilon_{\pm}(p)/(\varepsilon_{+}(p) - \varepsilon_{-}(p)),$$
$$\varepsilon_{\pm}(p) = J_{\parallel}^{x} \cos p \pm [(J_{\parallel}^{x} \cos p)^2 + (J_{\perp}^{x})^2]^{1/2}$$

Locality vs. Nonlocalality

How to deal with a kinematic factor $\cos(\pi n_b)$?

- **Unitary transformation** $\tilde{H} = U^\dagger H U$

$$U = \exp(i\pi \sum_{l,j>l} n_j^a n_l^b)$$

$$\tilde{H}^x_{\perp,i} = -\frac{1}{2} J^x_\perp \left(a_i^\dagger b_i e^{-i\pi \sum_{j<i} [a_j^\dagger a_j + b_j^\dagger b_j]} + h.c. \right)$$

- **g-ology**

Kinematic term may be treated as a part of effective interaction.

g-ology

$$J_\perp \ll J_\parallel$$

$$H_0 = \sum_{p,\lambda=\pm} \varepsilon_\lambda(p) c^\dagger_{\lambda,p} c_{\lambda,p}$$

$$H^{XY}_{int} = \frac{1}{2} \sum_{\{\mu,\nu,\alpha\}=\pm 1,q} g^{\nu\nu'}_{\mu\mu'}(q) \rho_{\mu\mu',\alpha}(q) \Lambda_{\nu\nu',\alpha'}(-q)$$

$$\rho_{\mu\mu',\alpha}(q) = \sum_k c^\dagger_{\alpha,\mu,k-q/2} c_{\alpha,\mu',k+q/2}$$

$$\Lambda_{\nu\nu',\alpha}(q) = -\alpha \sum_k k \;\; c^\dagger_{\alpha,\nu,k-q/2} c_{\alpha,\nu',k+q/2},$$

Two gaps

$$J_\perp \ll J_\parallel$$

Direct gap $\Delta \sim J_\perp^x$

Gap in two-point particle-hole correlation function

Gap separating excited S=0 state on a rung from the triplet S=1 state

This gap is renormalized by backward scattering processes

Indirect gap $\Delta_{min} \sim (J_\perp^x)^2 / J_\parallel$

Gap in the density of spin-fermion states

What is Haldane Gap?

Bosonization

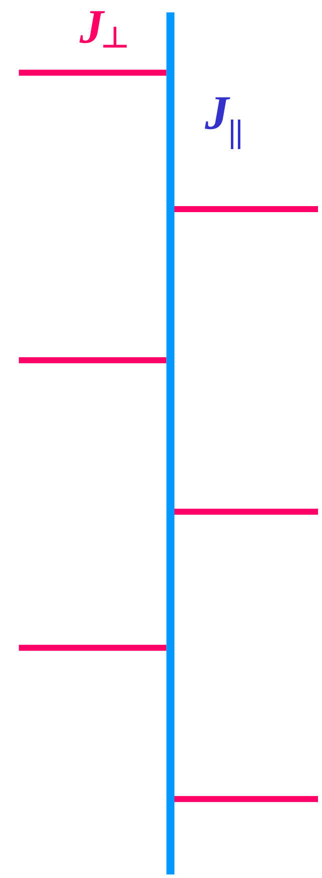
Continuum representation

$$
\begin{aligned}
s^{\pm}(x) &\sim e^{\pm i\theta}(\cos(\pi x) + \cos(2\phi)), \\
s^{z}(x) &\sim \pi^{-1}\partial_x\phi + \cos(\pi x + 2\phi)
\end{aligned}
$$

Hamiltonian

$$
\begin{aligned}
H &= \int dx \left(\frac{\pi u K}{2} \Pi_a^2 + \frac{u}{2\pi K} (\partial_x \phi_a)^2 \right) \\
&+ J_\perp^x \cos(\theta_a - \theta_b) + J_\perp^z \cos(2\phi_a)\cos(2\phi_b)
\end{aligned}
$$

Spin Gap scaling $J_\perp \ll J_\parallel$



$J_\perp^x \neq 0, \quad J_\perp^z = 0$ **Easy plane model**

$$\Delta \sim J_\parallel (J_\perp^x / J_\parallel)^{2/3}$$

$J_\perp^x = 0, \quad J_\perp^z \neq 0$ **Easy axis model**

$$\Delta \sim J_\parallel (J_\perp^z / J_\parallel)^{2/3}$$

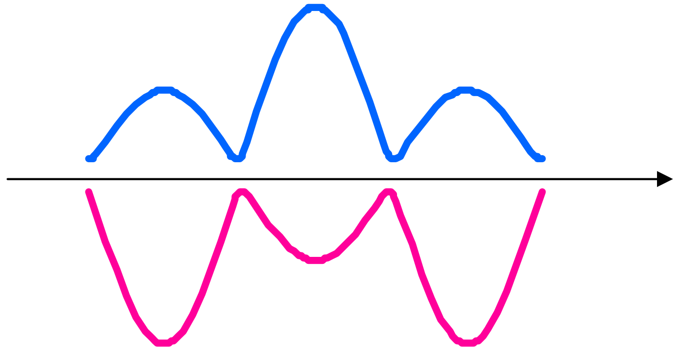
$J_\perp^x = J_\perp^z = J_\perp$ **Isotropic model**

$$\Delta \sim J_\parallel (J_\perp / J_\parallel)^{2/3}$$

Spin Staircase Model with arbitrary twist

XXZ Spin Staircase

$$
\begin{aligned}
H_{\parallel} &= J_{\parallel}^{x} \sum_i \left\{ \left(a_i^{\dagger} a_{i+1} + a_{i+1}^{\dagger} a_i \right) \cos(\pi n_i^{b}) \right. \\
&- \left. \cos^2\left(\frac{\theta}{2}\right) \left(b_i^{\dagger} b_{i+1} + b_{i+1}^{\dagger} b_i \right) \cos(\pi n_{i+1}^{a}) \right\} \\
&+ J_{\parallel}^{z} \sum_i \left(n_i^{a} - \frac{1}{2} \right) \left(n_{i+1}^{a} - \frac{1}{2} \right)
\end{aligned}
$$



$$\theta = \pi / 2$$

$$H_{\perp} = -\frac{J_{\perp}^{x}}{2} \sum_i (a_i^{\dagger} b_i + b_i^{\dagger} a_i) - J_{\perp}^{z} \sum_i (n_i^{a} - \tfrac{1}{2})(n_i^{b} - \tfrac{1}{2})$$

Two different Fermi velocities in uncoupled railings

Two energy scales (gaps) in staircase model

Two stage renormalization in continuum limit

Conclusions

- Spin Staircase Model describes spin systems intermediate between S=1 Spin chains and two-leg ladders
- Spin Gap is intrinsic property of the Spin Staircase Model
- Spin Staircase Model allows a mapping on two-component fermion interacting model, while four-fermion scattering can be treated both by renormalization group and bosonization approaches
- Spin Rotator Chain Model unlike conventional two-leg ladder model shows fractional scaling dependence of the spin gap as a function of the coupling along rung