This follows from

$$\sum_{p'} \tilde{T}_{p'}{}^0(P_0)R_{p'}(P_0)v_{k'}{}^\alpha$$

$$=\sum_{p'} \left(1-\frac{G_{k'}(\zeta_{l'}+z(l_0))-G_{k'}(\zeta_l)}{z(l_0)}\right)R_{p'}(P_0)v_{k'}{}^\alpha$$

$$=\frac{1}{z(l_0)}\sum_{k'} v_{k'}{}^\alpha \frac{1}{\beta}\sum_{l'} \{S_{k'l'}-S_{k',l'+l_0}\}$$

$$=0. \tag{6.8}$$

Now (6.7) is exactly the same as (4.32) for proper diagrams, and there is no difficulty passing to the infinite range limit. Thus (4.35) is valid for the proper vertex part.

Therefore, we conclude that the results (4.40)–(4.43) are valid for the proper vertex parts on the Fermi surface and that consequently all the identities of Sec. V still hold for proper scattering functions on the Fermi surface.

ACKNOWLEDGMENT

We would like to thank the Bell Telephone Laboratories, Murray Hill, New Jersey for their hospitality during the summer months of 1961, when a portion of this work was carried out.

PHYSICAL REVIEW VOLUME 127, NUMBER 5 SEPTEMBER 1, 1962

Derivation of the Landau Theory of Fermi Liquids. II. Equilibrium Properties and Transport Equation*

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(Received November 2, 1962)

Using the results of the preceding paper, it is shown that a large class of the conclusions of the Landau theory of Fermi liquids may be established within the framework of many-body perturbation theory. Both equilibrium and transport phenomena are discussed. The theory is also carried out for long-ranged Coulomb forces. Finally, it is shown that a rather simple general expression for the quasi-particle distribution function of Landau may be given.

I. INTRODUCTION

In this paper we shall be concerned with the verification of the results of the Landau theory of Fermi liquids for certain equilibrium and nonequilibrium phenomena. We shall do this first for short-ranged forces and then generalize to long-ranged Coulomb forces. The necessary formalism for this purpose has been given in the preceding paper.[1] In addition, we shall show that it is possible to give a very simple general definition of the "quasi-particle distribution function" which occurs in the Landau theory.

We begin our discussion with a brief summary of the Landau theory.[2] Consider first an ideal Fermi gas. A great many properties (heat capacity, compressibility, conductivity, etc.) of this system are determined (for temperatures much less than the degeneracy temperature) by the nature of the single particle states which lie in the immediate vicinity of the last occupied single-particle state. It is these same properties that the Landau theory tries to calculate for an interacting system of fermions. We can now state Landau's assumptions as follows:

(1) If the interaction is turned on, the single-particle states in the neighborhood of the last occupied one remain eigenstates of the system. We call these quasi-particle states, and say that a quasi-particle k is present if the state k is occupied. The low-lying excited states of the system are assumed to be in one-to-one correspondence with those of an ideal Fermi gas, the quasi-particle states just replacing the particle states. Therefore, since the number of real particles is conserved, in these low-lying excitations the number of quasi-particles must also be conserved. Adding a particle to the system must therefore also add a quasi-particle.

(2) The state of the Fermi liquid for weak excitation (equilibrium or nonequilibrium) is entirely described by the distribution function $n(k,x)$ of quasi-particles in

* Supported in part by the Office of Naval Research.

[1] P. Nozières and J. M. Luttinger, preceding paper [Phys. Rev. 126, 1423 (1962)]. We shall refer to this paper as I. The notation and assumptions used in the present paper will be the same as those of I.

[2] See A. A. Abrikosov and I. M. Khalatnikov, Soviet Phys.—Uspekhi 66, 68 (1958).

momentum and configuration space.[3] The normalization of $n(k,x)$ is such that $n(k,x)d^3kd^3x/(2\pi)^3$ is the number of quasi-particles in d^3kd^3x.[3] (This already implies the semiclassical limit, since both the momentum and position of the quasi-particles is given.) The distribution function for quasi-particles for the system in its ground state (n^0) must be

$$\begin{aligned} n^0 &= 1, \quad E_k^0 < \mu_0 \\ &= 0, \quad E_k^0 > \mu_0 \end{aligned} \tag{1.1}$$

when E_k^0 is the quasi-particle energy for the system in its ground state. μ_0 is the chemical potential at absolute zero, since the addition of one quasi-particle is the same as its addition of one real particle. If the system is excited, the quasi-particle energy will change. (In the Hartree-Fock approximation this would just be due to the self-consistent field of the neighboring particles.) Now by our general assumption the energy of the system must be some functional of n, i.e., $E=E\{n\}$. If the distribution function changes by an infinitesimal amount δn, and the total energy correspondingly by δE, then we define the quasi-particle energy $E(k,x)$ by

$$\delta E = \int E(k,x)\delta n(k,x)\frac{d^3kd^3x}{(2\pi)^3}. \tag{1.2}$$

That is,

$$E(k,x) = (2\pi)^3 \delta E/\delta n(k,x). \tag{1.3}$$

$E(k,x)$ is still a functional of n. If the distribution n does not differ very much from n^0, we may write

$$n = n^0 + n' \tag{1.4}$$

and expand in n'. That is,

$$E(k,x) = (2\pi)^3\left(\frac{\delta E}{\delta n(kx)}\right)_{n=n^0} + (2\pi)^3\int d^3k'd^3x' \times\left(\frac{\delta^2 E}{\delta n(kx)\delta n(k'x')}\right)_{n=n^0} n'(k',x') + \cdots. \tag{1.5}$$

If the forces are short-ranged and $n'(k',x')$ is a very slightly varying function of x' over the range of the forces (both assumed by Landau), then we may write

$$(2\pi)^6\left(\frac{\delta^2 E}{\delta n(x,k)\delta n(x',k')}\right)_{n=n^0} = f_{kk'}\delta(x-x') \tag{1.6}$$

and, therefore,

$$E(k,x) = E_k^0 + \frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'} n'(k',x) + \cdots. \tag{1.7}$$

The quantities E_k^0 and $f_{kk'}$ are the basic quantities necessary for the phenomenological theory of Landau.

[3] Units are such that $\hbar=1$ throughout this paper. We denote three-vectors k and x by italic symbols throughout.

Since $f_{kk'}$ is a second functional derivative, we have

$$f_{kk'} = f_{k'k}. \tag{1.8}$$

Further, it is assumed that if the original particles have a charge e then the quasi-particles also respond to external electromagnetic fields as if they had the same charge. That is, in an external field characterized by a vector potential A and a scalar potential φ, the Hamiltonian of the quasi-particles (h) will be given by

$$h(k,x) = E(k-(e/c)A,\, x) + e\varphi. \tag{1.9}$$

(3) The distribution function satisfies a Boltzmann-like transport equation

$$\frac{\partial n(k,x)}{\partial t} + \sum_{\alpha=1,2,3}\left(\frac{\partial n}{\partial x_\alpha}\frac{\partial h}{\partial k_\alpha} - \frac{\partial n}{\partial k_\alpha}\frac{\partial h}{\partial x_\alpha}\right) + \left(\frac{\partial n(k,x)}{\partial t}\right)_{\text{collisions}} = 0. \tag{1.10}$$

The term $(\partial n/\partial t)_{\text{collisions}}$ is absent in equilibrium or when there are no impurities or other scattering mechanisms present over and above the particle-particle interaction originally in the theory. We shall be concerned with a pure Fermi liquid in this paper, and shall not include this term.

It is easy to see from (1.10) that the total number of quasi-particles is conserved, and also that the current density defined by

$$\mathcal{J}_\alpha(x) = \frac{e}{(2\pi)^3}\int d^3k\, n(k,x)\frac{\partial h(k,x)}{\partial k_\alpha}, \tag{1.11}$$

satisfies the equation of continuity. Therefore, $\mathcal{J}_\alpha(x)$ is taken as the current density of the system. A similar expression may also be given for the energy current density $Q_\alpha(x)$:

$$Q_\alpha(x) = \frac{1}{(2\pi)^3}\int d^3k\, n(k,x)h(k,x)\frac{\partial h(k,x)}{\partial k_\alpha}. \tag{1.12}$$

We shall only be interested in the transport equation in the linear approximation, i.e., when the deviation of n from n^0 is very small. For completeness we shall also include an external magnetic field. If we write

$$n = n^0(k-(e/c)A) + n_1(k-(e/c)A,\, x) \tag{1.13}$$

and

$$\begin{aligned} n_1(k,x) &= -\frac{\partial n^0(k)}{\partial E_k^0}g(k,x) \\ &= \delta_k g(k,x), \end{aligned} \tag{1.14}$$

where

$$\delta_k \equiv \delta(\mu_0 - E_k^0), \tag{1.15}$$

then we see at once that (1.10) (without the collision term) becomes

$$\frac{\partial g}{\partial t}+\sum_{\alpha=1,2,3}\left\{V_k{}^\alpha\frac{\partial(g+E_1)}{\partial x_\alpha}+\frac{e}{c}(V_k\times H)_\alpha\frac{\partial(g+E_1)}{\partial k_\alpha}-eE_\alpha V_k{}^\alpha\right\}=0, \quad (1.16)$$

where

$$V_k{}^\alpha=\partial E_k{}^0/\partial k_\alpha, \quad (1.17)$$

$$E_1(k,x)=\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}\delta_{k'}g(k',x), \quad (1.18)$$

and E^α and H^α are the external electric and magnetic fields. Similarly, (1.11) becomes

$$\mathcal{J}_\alpha(x)=\frac{e}{(2\pi)^3}\int d^3x\, g(k,x)\delta_k \times\left\{V_k{}^\alpha+\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}\delta_{k'}V_{k'}{}^\alpha\right\}. \quad (1.19)$$

If the electric field has the form

$$E^\alpha=E_{q\omega}{}^\alpha e^{i(q\cdot x-\omega t)} \quad (1.20)$$

and we neglect the corresponding magnetic field, then so will g and the current density have this form. Putting

$$g(k,x)=g_{q\omega}{}^{(k)}e^{i(q\cdot x-\omega t)}, \quad (1.21)$$

$$\mathcal{J}_\alpha(x)=\mathcal{J}_{q\omega}{}^\alpha e^{i(q\cdot x-\omega t}, \quad (1.22)$$

we obtain finally

$$-i\omega g_{q\omega}(k)+\left[iq\cdot V_k+\frac{e}{c}(V_k\times H)\cdot\nabla_k\right]\left[g_{q\omega}(k)+\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}\delta_{k'}g_{q\omega}(k')\right]-eE_{q\omega}\cdot V_k=0, \quad (1.23)$$

and

$$\mathcal{J}_{q\omega}{}^\alpha=\frac{e}{(2\pi)^3}\int d^3k\, g_{q\omega}(k)\delta_k \times\left\{V_k{}^\alpha+\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}\delta_{k'}V_{k'}{}^\alpha\right\}. \quad (1.24)$$

In (1.24), H represents a constant external magnetic field. In this paper we shall not consider this case, and therefore (1.24) becomes

$$-i\omega g_{q\omega}(k)+iq\cdot V_k\left(g_{q\omega}(k)+\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}g_{q\omega}(k')\right)-eE_{q\omega}\cdot V_k=0. \quad (1.25)$$

To conclude this section, we list some other results of the Landau theory, for purposes of future reference. In equilibrium at temperature T ($\beta\equiv 1/kT$) the quasi-particle distribution is given by

$$n(k)=1/(e^{\beta(E_k-\mu)}+1), \quad (1.26)$$

where E_k is the quasi-particle energy at temperature T [itself a functional of $n(k)$]. It is easy to see, however, that the difference between E_k and $E_k{}^0$ is of the order of $(kT)^2/\mu$ and can be dropped. The heat capacity at constant volume is given by the usual formula for a Fermi gas, but with the unperturbed energy replaced by the quasi-particle energy $E_k{}^0$. If the system has translational invariance and if the unperturbed single particle energy is of the form $\epsilon_k=k^2/2m$, then Landau has shown that there is a relationship between the unperturbed velocity and the quasi-particle velocity on the Fermi surface. This is, for k on the Fermi surface,

$$v_k{}^\alpha=\frac{k_\alpha}{m}=V_k{}^\alpha+\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}\delta_{k'}V_{k'}{}^\alpha; \quad (1.27)$$

(1.27) becomes the Landau effective-mass equation if the interaction is isotropic so that the Fermi surface is spherical, and $f_{kk'}$ depends only on the angle between k and k' if k and k' are both on the Fermi surface.

Finally the compressibility (K), as is well known, is given by

$$K\equiv-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_V=\frac{V}{\bar{N}^2}\left(\frac{\partial\bar{N}}{\partial\mu}\right)_{V,\beta}. \quad (1.28)$$

In the Landau theory

$$\left(\frac{\partial\bar{N}}{\partial\mu}\right)_{V,\beta}=\sum_k\left(1-\frac{\partial E_k{}^0(\mu)}{\partial\mu}\right)\delta_k, \quad (1.29)$$

and

$$\frac{\partial E_k{}^0(\mu)}{\partial\mu}=\frac{1}{(2\pi)^3}\int d^3k'\, f_{kk'}\delta_{k'}\left(1-\frac{\partial E_{k'}{}^0(\mu)}{\partial\mu}\right). \quad (1.30)$$

In the isotropic case this enables us to solve at once for $(\partial E_k{}^0/\partial\mu)$ on the Fermi surface, and we at once get Landau's expression for the compressibility. In the more general case we have to solve the integral equation (1.30); in the next section we shall indicate how this may be done.

II. EQUILIBRIUM PROPERTIES

The equilibrium properties according to the many-body perturbation theory have been discussed by Luttinger.[4] It is clear that the result for the heat capacity is the same as Landau's, if we identify the "single-particle excitation energy at absolute zero" of Luttinger with Landau's quasi-particle energy $E_k{}^0$. This provides us therefore with a formula which could in

[4] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960). We shall refer to this paper as L.

principle be used to calculate E_k^0, though of course realistic calculations are difficult if not impossible.

Now we consider the compressibility. From (L. 70) we see that (1.29) is also a consequence of the microscopic theory. It only remains to verify that the Landau expression (1.30) for $\partial E_k^0/\partial\mu$ is the same as that obtained from the microscopic theory. From (I. 5.1) in the limit of $T=0$ we see that $\partial E_k^0/\partial\mu$ satisfies

$$\frac{\partial E_k^0}{\partial\mu}=\sum_{k'} f_{kk'}{}^{\omega}\delta_{k'}\left(1-\frac{\partial E_{k'}{}^0}{\partial\mu}\right)$$
$$=\frac{1}{(2\pi)^3}\int d^3k'\, V f_{kk'}{}^{\omega}\delta_{k'}\left(1-\frac{\partial E_{k'}{}^0}{\partial\mu}\right), \quad (2.1)$$

where $f_{kk'}{}^{\omega}$ is now the zero-temperature limit of $f_{kk'}{}^{\omega}$. Comparison with (1.30) shows that these are the same if we make the identification

$$f_{kk'}=Vf_{kk'}{}^{\omega}. \quad (2.2)$$

Therefore, we have expressions for both E_k^0 and $f_{kk'}$.

From (I. 5.2) we see that the general solution for $\partial E_k^0/\partial\mu$ is actually given by

$$\frac{\partial E_k^0}{\partial\mu}=\sum_{k'} f_{kk'}{}^{q}\delta_{k'}=\frac{V}{(2\pi)^3}\int d^3k'\, f_{kk'}{}^{q}\delta_{k'}. \quad (2.3)$$

Finally, we consider the "effective-mass" equation (1.27). From (I. 5.3) we have, under the same conditions that (1.27) is valid,

$$v_k{}^{\alpha}=V_k{}^{\alpha}+\sum_{k'} f_{kk'}{}^{\omega}\delta_{k'}V_{k'}{}^{\alpha}$$
$$=V_k{}^{\alpha}+\frac{1}{(2\pi)^3}\int d^3k'\, Vf_{kk'}{}^{\omega}\delta_{k'}V_{k'}{}^{\alpha},$$

which is the same as (1.27) with the previous identification of E_k^0 and $f_{kk'}$.

III. TRANSPORT EQUATION

We next wish to verify that the current density as calculated by the Landau prescription [(1.24) and (1.25)] agrees exactly with that calculated from the many-body perturbation theory point of view. To do this we first give an explicit expression for the Landau current density.

Let us consider (I. 3.13) and (I. 3.14). If we replace $2\pi i l_0/\beta$ by ω (where ω is imagined to have a very small negative imaginary part), these formulas provide us with an analytic function $f(P)$ of ω in the upper half ω plane, where now $P\equiv(q,\omega)$. Making use of this function (I. 3.13) and (2.2), we see by direct substitution that the solution of (1.25) is

$$g_{q\omega}(k)=\frac{eE_{q\omega}\cdot V_k}{i(q\cdot V_k-\omega)}-\frac{iq\cdot V_k}{i(q\cdot V_k-\omega)}\sum_{k'} f_{kk'}(P)\delta_{k'}\frac{eE_{q\omega}\cdot V_{k'}}{i(q\cdot V_{k'}-\omega)}. \quad (3.1)$$

Substituting this in (1.24) and using (I. 3.13) and (2.2), we obtain

$$\mathcal{J}_{q\omega}{}^{\alpha}=\frac{e^2}{V}\left(\sum_{k',\alpha'} V_k{}^{\alpha}V_k{}^{\alpha'}\frac{\delta_k E_{q\omega}{}^{\alpha'}}{i(q\cdot V_k-\omega)}-i\omega\sum_{k,k',\alpha'} V_k{}^{\alpha}\times\frac{\delta_k}{i(q\cdot V_k-\omega)}f_{kk'}(P)\frac{\delta_{k'}}{i(q\cdot V_{k'}-\omega)}V_{k'}{}^{\alpha'}E_{q\omega}{}^{\alpha'}\right)$$
$$=\frac{e^2}{Vi\omega}\Big(\sum_{k\alpha'} V_k{}^{\alpha}\tilde{\delta}_k(P)V_k{}^{\alpha'}E_{q\omega}{}^{\alpha'}-\sum_{kk'\alpha'} V_k{}^{\alpha}\tilde{\delta}_k(P)f_{kk'}(P)\tilde{\delta}_{k'}(P)V_{k'}{}^{\alpha'}E_{q\omega}{}^{\alpha'}\Big), \quad (3.2)$$

where we have used (I. 3.16).

Similarly, the charge density ($\rho_{q\omega}$) may be calculated directly or obtained from the equation of continuity:

$$iq\cdot\mathcal{J}_{q\omega}-i\omega\rho_{q\omega}=0, \quad (3.3)$$

yielding

$$\rho_{q\omega}=\frac{e^2}{Vi\omega}\Big\{\sum_{k\alpha'}\tilde{\delta}_k(P)V_k{}^{\alpha'}E_{q\omega}{}^{\alpha'}-\sum_{kk'\alpha'}\tilde{\delta}_k(P)f_{kk'}(P)\tilde{\delta}_{k'}(P)V_{k'}{}^{\alpha'}E_{q\omega}{}^{\alpha'}\Big\}. \quad (3.4)$$

We notice that $\mathcal{J}_{q\omega}{}^{\alpha}$ and $\rho_{q\omega}$ only depend on properties of the system on the Fermi surface because of the δ_k and $\tilde{\delta}_k$ functions.

Now, how can we verify (3.2)? In order to do this we must have an exact expression for $\mathcal{J}_{q\omega}{}^{\alpha}$ and then transform it into the form (3.2). The type of formula we need has already been given by many authors.[5] The particular form which we shall need is developed in Appendix A. From (A28)

$$\mathcal{J}_{q\omega}{}^{\alpha}=-\frac{e^2}{V}\sum_{\alpha'}\frac{E_{q\omega}{}^{\alpha'}}{i\omega}S^{\alpha'\alpha}(P)-\frac{Ne^2}{mV}\left(\frac{E_{q\omega}{}^{\alpha}}{i\omega}\right), \quad (3.5)$$

where $S^{\alpha'\alpha}(P)$ is defined by (I. 3.3). From (I. 5.22) we have

$$S^{\alpha'\alpha}(P)=S^{\alpha\alpha'}(P)=-\frac{N}{m}\delta_{\alpha\alpha'}-\sum_k V_k{}^{\alpha}\tilde{\delta}_k(P)V_k{}^{\alpha'}+\sum_{k,k'} V_k{}^{\alpha}\tilde{\delta}_k(P)f_{kk'}(P)\tilde{\delta}_{k'}(P)V_{k'}{}^{\alpha'}, \quad (3.6)$$

[5] R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957); P. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

Therefore,

$$\mathcal{J}_{q\omega}{}^{\alpha}=\frac{e^2}{V}\sum_{\alpha'}\frac{E_{q\omega}{}^{\alpha'}}{i\omega}\{\sum_k V_k{}^{\alpha}\tilde{\delta}_k(P)V_k{}^{\alpha'} -\sum_{k,k'}V_k{}^{\alpha}\tilde{\delta}_k(P)f_{kk'}\cdot\tilde{\delta}_{k'}(P)V_{k'}{}^{\alpha'}\}. \quad (3.7)$$

Comparison of (3.7) with (3.2) shows that they are identical. Therefore, the Landau theory does give the correct current density.

IV. TRANSPORT EQUATION WITH LONG-RANGE COULOMB FORCES

Let us consider the exact expressions (A23) and (A29) for the current and charge densities. Instead of expressing them in terms of the applied field $E_{q\omega}{}^{\alpha}$, let us express them in terms of the total field $\bar{E}_{q\omega}{}^{\alpha}$ in the sample. From Poisson's equation, this is

$$\bar{E}_{q\omega}=E_{q\omega}-iq_{\alpha}(Vu_q/e^2)\rho_{q\omega}, \quad (4.1)$$

where

$$u_q=4\pi e^2/Vq^2. \quad (4.2)$$

Therefore,

$$\rho_{q\omega}=\frac{1}{V}\sum_{\alpha}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle J_{-q}{}^{\alpha}(v)\rho_q\rangle\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega} +\frac{U_q}{e^2}\frac{\rho_{q\omega}}{i\omega}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle iq\cdot J_{-q}(v)\rho_q\rangle, \quad (4.3)$$

$$\mathcal{J}_{q\omega}{}^{\alpha}=\frac{1}{V}\sum_{\alpha'}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle J_{-q}{}^{\alpha'}(v)J_q{}^{\alpha}\rangle\frac{\bar{E}_{q\omega}{}^{\alpha'}}{i\omega} +\frac{U_q}{e^2}\frac{\rho_{q\omega}}{i\omega}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle iq\cdot J_{-q}{}^{\alpha'}(v)J_q{}^{\alpha}\rangle -\frac{N}{m}\frac{U_q\rho_{q\omega}}{i\omega}(iq_{\alpha}). \quad (4.4)$$

Now, by means of the operator identity,

$$iq\cdot J_q+i[\mathcal{H},\rho_q]=0 \quad (4.5)$$

(which is the operator form of the equation of continuity), we can simplify certain terms of (4.3) and (4.4). We have

$$\begin{aligned}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle iq\cdot J_{-q}(v)A\rangle &=i\int_0^{\beta}dv^{-vz(l_0)}\langle((\mathcal{H},\rho_{-q}(v))A\rangle\\ &=i\int_0^{\beta}dv\, e^{-vz(l_0)}\left\langle\frac{\partial\rho_{-q}(v)}{\partial v}A\right\rangle\\ &=i\left\{\langle\rho_{-q}(\beta)A\rangle-\langle\rho_{-q}A\rangle+z(l_0)\int_0^{\beta}e^{-vz(l_0)}\langle\rho_{-q}(v)A\rangle dv\right\}\\ &=i\left\{\langle(A,\rho_{-q})\rangle+\omega\int_0^{\beta}e^{-vz(l_0)}\langle\rho_{-q}(v)A\rangle dv\right\}. \quad (4.6)\end{aligned}$$

We also have

$$[\rho_q,\rho_{-q}]=0, \quad (4.7)$$

and

$$[J_q{}^{\alpha},\rho_{-q}]=(Ne^2/m)(q_{\alpha}). \quad (4.8)$$

Therefore, (4.3) and (4.4) become

$$\rho_{q\omega}=\frac{1}{V}\sum_{\alpha}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle J_{-q}{}^{\alpha}(v)\rho_q\rangle\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega} +\frac{u_q}{e^2}\rho_{q\omega}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle\rho_{-q}(v)\rho_q\rangle, \quad (4.9)$$

$$\mathcal{J}_{q\omega}{}^{\alpha}=\frac{1}{V}\int_0^{\beta}dv\, e^{-vz(l_0)} \times\sum_{\alpha'}\langle J_{-q}{}^{\alpha'}(v)J_q{}^{\alpha}\rangle\frac{\bar{E}_{q\omega}{}^{\alpha'}}{i\omega}-\frac{Ne^2}{mV}\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega} +\frac{u_q}{e^2}\rho_{q\omega}\int_0^{\beta}dv\, e^{-vz(l_0)}\langle\rho_{-q}(v)J_q{}^{\alpha}\rangle. \quad (4.10)$$

Using the same technique as in Appendix A, we have

$$\rho_{q\omega}=-\left(\frac{e^2}{m}\right)\frac{1}{V}\sum_{\alpha}\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega}\{\sum_p R_p(P)(k_{\alpha}+q_{\alpha}/2) +\sum_{pp'}(k_{\alpha}+q_{\alpha}/2)R_p(P)\Gamma_{pp'}(P)R_{p'}(P) -u_q\rho_{q\omega}\{\sum_p R_p(P)+\sum_{p,p'}R_p(P)\Gamma_{pp'}(P)R_{p'}(P)\}. \quad (4.11)$$

From (I. 3.3) and (I. 6.5), (4.11) becomes, with a little algebra,

$$\rho_{q\omega}=-\left(\frac{e^2}{m}\right)\frac{1}{V}\sum_{\alpha}\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega}\{\sum_p R_p(P)(k_{\alpha}+q_{\alpha}/2) +\sum_{p,p'}(k_{\alpha}+q_{\alpha}/2)R_p(P)\bar{\Gamma}_{pp'}(P)R_{p'}(P)\}, \quad (4.12)$$

where we have also used

$$S^{00}(P)=\frac{\bar{S}^{00}(P)}{1-u_q\bar{S}^{00}(P)}, \quad (4.13)$$

which also follows directly from the definitions and (I. 6.5).

Similarly, (4.10) becomes, using (4.12), (I. 3.3), and some manipulation,

$$\mathcal{J}_{q\omega}{}^{\alpha}=-\left(\frac{e}{m}\right)^2\frac{1}{V}\sum_{\alpha'}\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega}\{\sum_p R_p(P)(k_{\alpha'}+q_{\alpha'}/2) \times(k_{\alpha}+q_{\alpha}/2)+\sum_{p,p'}(k_{\alpha'}+q_{\alpha'}/2)R_p(P)\bar{\Gamma}_{pp'}(P) \times R_{p'}(P)(k_{\alpha}+q_{\alpha}/2)\}-\frac{Ne^2}{mV}\frac{\bar{E}_{q\omega}{}^{\alpha}}{i\omega}. \quad (4.14)$$

The expressions (4.12) and (4.14), which are still exact for all q, are of the same form as we used to derive the Landau theory in the case of a short-ranged interaction [(A27) and (A31)], with the scattering function Γ replaced by the proper scattering function $\tilde{\Gamma}$, and the external electric field $E_{q\omega}{}^{\alpha}$ replaced by the total field $\tilde{E}_{q\omega}{}^{\alpha}$. Therefore, since from I (Section VI), all the Ward identities we need hold for the proper vertex functions, it follows that the Landau transport equation will still be valid in the long-ranged Coulomb case if the electric field is replaced by the total field, and if we make the identification

$$f_{kk'}=V\tilde{f}_{kk'}{}^{\omega}=V(\tilde{\Gamma}_{pp'}{}^{\omega})_{p,p'\text{ on Fermi surface}}. \quad (4.15)$$

This result has been anticipated by Silin.[6] The explicit expression (4.15) for the interaction between quasi-particles in the Coulomb case has not been given previously.

V. INTERPRETATION OF THE QUASI-PARTICLE OCCUPATION NUMBER

We now show that it is possible to give a rather simple general expression for the "quasi-particle occupation number" which occurs in the Landau theory. Consider first the current density operator

$$\mathcal{J}^{\alpha}(x)=\frac{e}{2m}\{([p_{\alpha}-(e/c)A_{\alpha}]\psi)^{\dagger}\psi+\psi^{\dagger}[p_{\alpha}-(e/c)A_{\alpha}]\psi\}$$
$$=\frac{e}{mV}\sum_{k',k''}a_{k'}{}^{\dagger}a_{k''}\left[\frac{k_{\alpha}'+k_{\alpha}''}{2}-\left(\frac{e}{c}\right)A_{\alpha}(x)\right]e^{i(k''-k')\cdot x}. \quad (5.1)$$

If the system is in a state described by the density matrix ρ_T, then the average current density is given by

$$\langle\mathcal{J}^{\alpha}(x)\rangle_{\text{av}}=\text{Tr}(\rho_T\mathcal{J}^{\alpha}(x))=\sum_{k',k''}\frac{e}{mV}\left[\frac{k_{\alpha}'+k_{\alpha}''}{2}-\left(\frac{e}{c}\right)A_{\alpha}(x)\right]e^{i(k''-k')\cdot x}\mathfrak{D}_{k''k'}, \quad (5.2)$$

where

$$\mathfrak{D}_{k''k'}\equiv\text{Tr}(\rho_T a_{k'}{}^{\dagger}a_{k''}). \quad (5.3)$$

Now if we put

$$k'=k-q/2, \quad k''=k+q/2, \quad (5.4)$$

we have

$$\langle\mathcal{J}^{\alpha}(x)\rangle_{\text{av}}=\frac{e}{V}\sum_{k,q}(1/m)[k_{\alpha}-(e/c)A_{\alpha}(x)]e^{iq\cdot x}\mathfrak{D}_{k+q/2,k-q/2}$$
$$=\frac{e}{V}\sum_{k}[k_{\alpha}-(e/c)A_{\alpha}(x)]\mathfrak{D}(k,x)$$
$$=\frac{e}{(2\pi)^3}\int d^3k(1/m)[k_{\alpha}-(e/c)A_{\alpha}(x)]\mathfrak{D}(k,x), \quad (5.5)$$

[6] V. P. Silin, Soviet Phys.—JETP 6, 387 (1958).

where

$$\mathfrak{D}(k,x)\equiv\sum\nolimits_{q} e^{iq\cdot x}\mathfrak{D}_{k+q/2,k-q/2}. \quad (5.6)$$

From (5.5), we see that $\mathfrak{D}(k,x)$ plays exactly the role of the single particle distribution function at k, x, since $(1/m)[k_{\alpha}-(e/c)A_{\alpha}(x)]$ is just the velocity of a (classical) particle with momentum k at the point x. $\mathfrak{D}(k,x)$ is of course just the well-known Wigner semiclassical density matrix.

If the vector potential is finite, we can change the variables and write

$$\langle\mathcal{J}^{\alpha}(x)\rangle_{\text{av}}=\frac{e}{(2\pi)^3}\int d^3k\frac{k_{\alpha}}{m}\mathfrak{D}(k+(e/c)A(x),x). \quad (5.7)$$

We are interested in the response of the system to an external oscillating field described by A. Then to the first order in A we may write

$$\mathfrak{D}(k,x)=\mathfrak{D}^{(0)}(k,x)+\mathfrak{D}^{(1)}(k,x), \quad (5.8)$$

and

$$\mathfrak{D}(k+(e/c)A(x),x)=\mathfrak{D}^{0}(k,x)+(e/c)(A\cdot\nabla_k)\mathfrak{D}^{(0)}(k,x)+\mathfrak{D}^{(1)}(k,x). \quad (5.9)$$

Thus, the first-order single-particle distribution function $f_1(k,x)$ is given by

$$f_1(k,x)=(e/c)(A\cdot\nabla_k)\mathfrak{D}^{(0)}(k,x)+\mathfrak{D}^{(1)}(k,x), \quad (5.10)$$

and the average current density is

$$\langle\mathcal{J}^{\alpha}(x)\rangle_{\text{av}}=\frac{e}{(2\pi)^3}\int d^3k\frac{k_{\alpha}}{m}f_1(k,x). \quad (5.11)$$

We now ask, is it possible to give an expression analogous to (5.3) from which we can compute the quasi-particle distribution function? It is clear that in order to do this we need quasi-particle creation and destruction operators to take the place of the particle creation and destruction operators of (5.3). Now, in general, this is not possible since the quasi-particle states have a finite lifetime (i.e., they are not eigenstates of the system). This lifetime, however, goes to infinity as the momentum of the quasi-particle state approaches the Fermi surface, for the system in its ground state. Therefore, we would expect the quasi-particle states to be well defined on the Fermi surface, and any properties which just involve contributions from the Fermi surface to be rigorously calculated in terms of them.

Let us consider the following operator:

$$A_k=\frac{\eta_k}{\sqrt{z_k}}\int_0^{\infty}\exp(i\mathcal{H}t)a_k\exp(-i\mathcal{H}t)e^{iE_kt}e^{-\eta_kt}dt. \quad (5.12)$$

In (5.12), E_k is the energy of a quasi-particle of momentum k, z_k is defined by (I. 2.17), and η_k is an energy much less than the chemical potential but much greater than the reciprocal lifetime of the quasi-particle state k.

As k approaches the Fermi surface, we may let η_k approach zero. This quantity plays the role of the quasi-particle destruction operator, when k approaches the Fermi surface.[7] That is, as k approaches the Fermi surface, the state $A_k\psi_0$ (where ψ_0 is the ground state of the system) behaves for a longer and longer time like an exact normalized eigenstate of the system which has one less quasi-particle in the state k. Apart from normalization, the reason for this choice is fairly clear intuitively: $\exp(+i\mathcal{H}t)a_k\exp(-i\mathcal{H}t)$ acting on ψ_0 gives a distribution of single-particle states of momentum k. If the excitation energy of some such state is $\mathcal{E}_k$ then the integrand oscillates with frequency $\mathcal{E}_k-E_k$. Averaging over long times means we only get the contribution from the excitation of energy $\mathcal{E}_k=E_k$, if the time of averaging is not as long (much less than the lifetime) that the quasi-particle state with energy E_k decays away. That the normalization is correct will appear from what follows. We shall take (5.12) to be our *definition* of the quasi-particle destruction operator, and shall show it has all the properties which we need.

If we go into the representation in which $\mathcal{H}$ is diagonal [see Appendix A, (A15)] then we can do the integral in (5.12) at once, and we obtain

$$(A_k)_{nn'}=\frac{i\eta_k}{\sqrt{z_k}}\frac{(a_k)_{nn'}}{E_k+i\eta_k+E_n-E_{n'}}. \tag{5.13}$$

Similarly, the creation operator $A_k^\dagger$ becomes

$$(A_k^\dagger)_{nn'}=\frac{-i\eta_k}{\sqrt{z_k}}\frac{(a_k^\dagger)_{nn'}}{E_k-i\eta_k+E_{n'}-E_n}. \tag{5.14}$$

Let us first consider the quasi-particle distribution function for equilibrium. By definition this is $\langle A_k^\dagger A_k\rangle$:

$$\begin{aligned}\langle A_k^\dagger A_k\rangle&=\frac{\eta_k^2}{z_k}\sum_{n,n'}\rho_n\\&\quad\times\frac{(a_k^\dagger)_{nn'}(a_k)_{n'n}}{(E_k-i\eta_k+E_{n'}-E_n)(E_k+i\eta_k+E_{n'}-En)}\\&=\frac{\eta_k^2}{z_k}\int_{-\infty}^{\infty}d\xi\frac{1}{(E_k-\xi-i\eta_k)(E_k-\xi+i\eta_k)}\\&\quad\times\sum_{n,n'}\rho_n\delta(\xi+E_{n'}-E_n)(a_k^\dagger)_{nn'}(a_k)_{n'n}.\end{aligned} \tag{5.15}$$

Now

$$\sum_{n,n'}\rho_n\delta(x+E_{n'}-E_n)(a_k^\dagger)_{nn'}(a_k)_{n'n}=\rho_k(x)f^-(x),$$

$$f^-(x)\equiv\frac{1}{e^{\beta(x-\mu)}+1}, \tag{5.16}$$

[7] One of us (P.N.) has used this quantity extensively in a course given on the many-body problem during the past several years. Similar operators have also been introduced by N. M. Hugenholtz, Physica **23**, 481 (1957) [see Eq. (12.9) of that paper] from a somewhat different point of view.

where $\rho_k(x)$ is the spectral distribution function of the propagator (see I. 2.10). (This is easily seen by just writing down the definition of the propagator in terms of exact eigenfunctions.) Therefore, (5.15) takes the form

$$\langle A_k^\dagger A_k=\frac{1}{z_k}\int_{-\infty}^{\infty}d\xi\,\rho_k(\xi)f^-(\xi)\frac{\eta_k^2}{(E_k-\xi)^2+\eta_k^2}. \tag{5.17}$$

For k very near the Fermi surface, we have (I. 2.15)

$$\rho_k(\xi)=z_n\delta(\xi-E_k)+\tilde{\rho}_k(\xi), \tag{5.18}$$

where $\rho_k(\xi)$ is a perfectly smooth function. Therefore

$$\langle A_k^\dagger A_k\rangle=f^-(E_k)+\frac{1}{z_k}\int_{-\infty}^{\infty}d\xi\,\tilde{\rho}_k(\xi)f^-(\xi)\frac{\eta_k^2}{(E_k-\xi)^2+\eta_k^2}. \tag{5.19}$$

The last term of (5.19) vanishes for k near the Fermi surface since $\tilde{\rho}_k(\xi)$ is smooth, $f^-(\xi)$ bounded, and $\eta_k^2/(E_k-\xi)^2+\eta_k^2$ approaches $\eta_k\delta(\xi-E_k)$ as η_k approaches zero. Therefore for k near the Fermi surface

$$\langle A_k^\dagger A_k\rangle=f^-(E_k)=\frac{1}{e^{\beta(E_k-\mu)}+1}, \tag{5.20}$$

which just checks the quasi-particle distribution function (1.26) of Landau. From the result, we see the reason for the normalization factor $1/\sqrt{z_k}$ in the original definition of the quasi-particle destruction operator.

We next want to calculate the quasi-particle distribution function when the system is excited by an external oscillating electromagnetic field.

Analogously to (5.3), we define

$$D_{k''k'}=\mathrm{Tr}(\rho_T A_{k'}^\dagger A_{k''}). \tag{5.21}$$

Following Appendix A, we have

$$D_{k''k'}=D_{k''k'}^{(0)}+D_{k''k'}^{(1)}, \tag{5.22}$$

with

$$\begin{aligned}D_{k''k'}^{(0)}=\langle A_{k'}^\dagger A_{k''}\rangle&=\delta_{k'k''}\langle A_{k'}^\dagger A_{k'}\rangle\\&=\delta_{k'k''}f^-(E_{k'}),\end{aligned} \tag{5.23}$$

$$D_{k''k'}^{(1)}=\sum_\alpha\int_0^\beta dv\,e^{-vz(l_0)}\langle J_{-q}^\alpha(v)A_{k'}^\dagger A_{k''}\rangle\frac{E_{q\omega}^\alpha}{i\omega}, \tag{5.24}$$

where $z(l_0)$ is to be replaced by ω in the final results. Using (2.4) we get

$$\begin{aligned}D_{k'+q,k'}^{(1)}=&\frac{e}{m}\sum_{k,\alpha}(k_\alpha+q\alpha/2)\left(\frac{E_{q\omega}^\alpha}{i\omega}\right)\\&\times\int_0^\beta dv\,e^{-vz(l_0)}\langle a_{k+q}^\dagger(v)a_k(v)A_{k'}^\dagger A_{k'+q}\rangle.\end{aligned} \tag{5.25}$$

Now

$$\begin{aligned}&\langle a_{k+q}^\dagger(v)a_k(v)A_{k'}^\dagger A_{k'+q}\rangle\\&\quad=-\langle T(a_{k+q}^\dagger(v^+)A_{k'}^\dagger(0^+)a_k(v)A_{k'+q}(0))\rangle.\end{aligned} \tag{5.26}$$

The expression (5.26) would be exactly a conventional two-particle Green's function if the quasi-particle creation and destruction operators $A_{k'}^\dagger$, $A_{k'}$ were replaced by ordinary creation and destruction operators $a_{k'}^\dagger$, $a_{k'}$. Therefore, the expansion of (5.26) in diagrams has just the usual form, except that whenever the expressions $\langle a_{k'}^\dagger a_{k'}\rangle$, $\langle T(a_{k'}^\dagger a_{k'}(u))\rangle$, $\langle T(a_{k'+q}^\dagger(u)a_{k'+q})\rangle$ occur, they are to be replaced by $\langle A_{k'}^\dagger, A_{k'}\rangle$, $\langle T(A_{k'}^\dagger a_{k'}(u))\rangle$, $\langle T(a_{k'+q}^\dagger(u)A_{k'+q})\rangle$, respectively. Now define

$$Q_k(u_1,u_2)=\langle T(A_k^\dagger(u_1)a_k(u_2))\rangle,\qquad \bar{Q}_k(u_1,u_2)=\langle T(a_k^\dagger(u_1)A_k(u_2))\rangle. \tag{5.27}$$

Consider first $Q_k(u_1,u_2)$ for $u_1>u_2$. This is

$$\begin{aligned}Q_k(u_1,u_2)&=\sum_{n,n'}\rho_n e^{(u_1-u_2)(E_n-E_{n'})}(A_k^\dagger)_{nn'}(a_k)_{n'n}\\&=\sum_{n,n'}\rho_n e^{(u_1-u_2)(E_n-E_{n'})}\times\left(-\frac{i\eta_k}{\sqrt{z_k}}\right)\frac{(a_k^\dagger)_{nn'}(a_k)_{n'n}}{E_k-i\eta_k+E_{n'}-E_n}\\&=\int_{-\infty}^{\infty}dx\left(\frac{i\eta_k}{\sqrt{z_k}}\right)\frac{e^{(u_1-u_2)x}\rho_k(x)f^-(x)}{E_k-x-i\eta_k},\end{aligned} \tag{5.28}$$

by (5.16).

If k is near the Fermi surface, we make use of (5.18). Since η_k approaches zero, the regular part of ρ_k gives no contribution, and we get

$$Q_k(u_1,u_2)=(\sqrt{z_k})e^{(u_1-u_2)E_k}f^-(E_k),\quad (u_1>u_2). \tag{5.29}$$

Similarly, for $u_2>u_1$, we find

$$Q_k(u_1,u_2)=-(\sqrt{z_k})e^{(u_1-u_2)E_k}f^+(E_k),\quad (u_2>u_1)$$
$$f^+(E_k)=1-f^-(E_k). \tag{5.30}$$

Therefore, Q_k is a conventional unperturbed propagator with the unperturbed energy replaced by E_k, i.e.,

$$Q_k(u_1,u_2)=(\sqrt{z_k})\frac{1}{\beta}\sum_l\frac{e^{\zeta_l(u_1-u_2)}}{\zeta_l-E_k}. \tag{5.31}$$

Similarly,

$$\bar{Q}_k(u_1,u_2)=\sqrt{z_k}\frac{1}{\beta}\sum_l\frac{e^{\zeta_l(u_1-u_2)}}{\zeta_l-E_k}. \tag{5.32}$$

Therefore, just as in Appendix A (A26), we may write

$$\begin{aligned}\langle a_{k+q}^\dagger(v)a_k(v)A_{k'}^\dagger A_{k'+q}\rangle&=-\frac{(z_{k'+q}z_{k'})^{\frac12}}{\beta^3}\sum_{l_0}e^{z(l_0)v}\{\delta_{kk'}\beta\sum_l\tilde{S}_p\tilde{S}_{p+p}\\&\quad-\beta^3\delta_{p,0}S_{k'}'(0^+)f^-(E_{k'})\\&\quad+\sum_{l,l'}R_p(P)\Gamma_{pp'}(P)\tilde{S}_{p'}\tilde{S}_{p'+p}\}\end{aligned} \tag{5.33}$$

where

$$\tilde{S}_p\equiv1/(\zeta_l-E_k). \tag{5.34}$$

The sum over l in the first term of (5.33) is done at once (by the technique of I, Sec. 2, for example):

$$\frac{1}{\beta}\sum_l\tilde{S}_p\tilde{S}_{p+p}=-\delta_k(P), \tag{5.35}$$

using the notation of (I. 3.14).

The second term contributes nothing to (5.25). For the third term we have a sum of the form

$$\frac{1}{\beta}\sum_{l'}S_{p'}S_{p'+p}F(\zeta_{l'})\equiv I. \tag{5.36}$$

This can be treated by exactly the same method used for the sum in (I. 2.11), this giving for the contribution from the poles of $\tilde{S}_{p'}\tilde{S}_{p'+p'}-\delta_{k'}(P)F(\mu)$. The other term, arising from any singularities of $F(\zeta_l)$, can be dropped. This is not very easy to see directly from the sum (5.36), though it is not difficult to verify for some classes of lower order diagrams for $\Gamma_{pp'}(P)$. However, from the original definition of $D_{k'+q,k'}^{(1)}$ (5.25), the fact that the quasi-particle operators $A_{k'}^\dagger A_{k'+q}$ stand to the right means that we only get something when k' and $k'+q$ are on opposite sides of the Fermi surface (q assumed very small, but not zero). Therefore, the result is proportional to $f^-(E_{k'+q})-f^-(E_{k'})$. Now for the terms from the poles we get another factor of $1/[E_{k'+q}-E_{k'}-z(l_0)]$, so the pole term is of order unity. The term arising from the singularities of $F(\zeta_l)$ has however no singular behavior for small q or $z(l_0)$, and therefore this term is of lower order in P. Therefore, we may take

$$I=-\delta_{k'}(P)F(\mu). \tag{5.37}$$

We finally obtain, for small P, on putting this all together,

$$\begin{aligned}D_{k',k'+q}^{(1)}&=ez_{k'}\sum_\alpha\frac{E_{q\omega}^\alpha}{i\omega}[\delta_{k'}(P)v_{k'}^\alpha+\sum_p v_k^\alpha R_p(P)\Gamma_{pp'}(P)]_{\zeta_{l'}=\mu}\\&=ez_{k'}\sum_\alpha\frac{E_{q\omega}^\alpha}{i\omega}\delta_{k'}(P)[T_{p'}^\alpha(P)]_{\zeta_{l'}=\mu}\end{aligned} \tag{5.38}$$

using the notation of (I. 3.2). Since $\delta_{k'}(P)$ forces k' to be of the Fermi surface, we may write this as

$$D_{k',k'+q}^{(1)}=\sum_\alpha\frac{eE_{q\omega}^\alpha}{i\omega}\delta_{k'}(P)\tau_{k'}^\alpha(P), \tag{5.39}$$

from (I. 3.11).

From (I. 3.20) and (I. 4.31), we have

$$\tau_{k'}^\alpha(P)=V_{k'}^\alpha-\sum_{k'}f_{k'k''}(P)\tilde{\delta}_{k''}(P)V_{k''}^\alpha. \tag{5.40}$$

Comparison with (5.6) and (5.10) shows that we would expect the qth Fourier component of the first order quasi-particle distribution function $n_1(k,q)$ to be given by

$$n_1(k,q)=(e/c)(A_{q\omega}\cdot\nabla_k)D_{kk}{}^{(0)}+D_{k+q/2,k-q/2}{}^{(1)}. \quad (5.41)$$

Substituting (5.20), (5.39), and (5.40) into (5.41), we obtain

$$n_1(k,q)=\delta_k\left\{\frac{eE_{q\omega}\cdot V_k}{i(q\cdot V_k-\omega)}-\frac{iq\cdot V_k}{i(q\cdot V_k-\omega)}\times\sum_{k'}f_{kk'}(P)\frac{\delta_{k'}(eE_{q\omega}\cdot V_{k'}{}^{\alpha})}{i(q\cdot V_k-\omega)}\right. . \quad (5.42)$$

Direct comparison with (1.14) and (3.1) shows that this is identical with the quasi-particle distribution function obtained by solving the Landau transport equation. Therefore the quasi-particle distribution function is just given by exactly the same expression as the distribution function obtained from the usual single-particle density matrix, except that particle creation and destruction operators must be replaced by quasi-particle creation and destruction operators.

APPENDIX A. RESPONSE TO AN ARBITRARY FIELD

Consider a system in an arbitrary external electromagnetic field. Let this field be given by a vector potential A (we choose for simplicity a gauge with vanishing scalar potential). We shall be interested in the *linear* response of the system and shall therefore study only one Fourier component of A, i.e., put

$$A^{\alpha}=A_{q\omega}{}^{\alpha}e^{i(q\cdot x-\omega t)}, \quad (A1)$$

$$E^{\alpha}=-\frac{1}{c}\frac{\partial A}{\partial t}=E_{q\omega}{}^{\alpha}e^{i(q\cdot x-\omega t)}, \quad (A2)$$

$$E_{q\omega}{}^{\alpha}=(c/i\omega)A_{q\omega}{}^{\alpha}. \quad (A3)$$

In these expressions imagine that ω has a small positive imaginary part, corresponding to the field being turned on at $t=-\infty$.

If the Hamiltonian of the system in the absence of the field is

$$\mathcal{H}=\sum_i\epsilon(p_i)+U, \quad (A4)$$

where U is the potential energy of interaction, then when the field is present the total Hamiltonian is

$$\mathcal{H}_T=\sum_i\epsilon(p_i-(e/c)A_i)+U. \quad (A5)$$

For simplicity we shall take $\epsilon(p)=p^2/2m$. This is not at all necessary here, but it simplifies the writing considerably.

Then to the first order in the field,

$$\mathcal{H}_T=\mathcal{H}+\mathfrak{A}e^{-i\omega t}, \quad (A6)$$

$$\mathfrak{A}=-(1/i\omega)E_{q\omega}\cdot J_{-q}, \quad (A7)$$

where

$$J_{-q}{}^{\alpha}=\frac{e}{2m}\sum_i(p_i{}^{\alpha}e^{iq\cdot x_i}+e^{iq\cdot x_i}p_i{}^{\alpha}). \quad (A8)$$

The current density operator at the point x is given by

$$\mathcal{J}^{\alpha}(x)=\frac{e}{2}\sum_i\left\{\frac{[p_i{}^{\alpha}-(e/c)A_i{}^{\alpha}]}{m}\delta(x_i-x)+\delta(x_i-x)\frac{[p_i{}^{\alpha}-(e/c)A_i{}^{\alpha}]}{m}\right\}. \quad (A9)$$

If we Fourier-analyze this,

$$\mathcal{J}^{\alpha}(x)=\sum_q e^{iq\cdot x}\mathcal{J}_q{}^{\alpha}, \quad (A10)$$

we get

$$\mathcal{J}_q{}^{\alpha}=\frac{e}{2mV}\sum_i\{[p_i{}^{\alpha}-(e/c)A_i{}^{\alpha}]e^{-iq\cdot x_i}+e^{-iq\cdot x_i}[p_i{}^{\alpha}-(e/c)A_i{}^{\alpha}]\}$$
$$=\frac{1}{V}J_q{}^{\alpha}-\frac{Ne^2}{mV}\frac{E_{q\omega}{}^{\alpha}}{i\omega}. \quad (A11)$$

To find the average current density $\mathcal{J}_{q\omega}{}^{\alpha}$ we must have the density matrix ρ_T of the system. This satisfies

$$\partial\rho_T/\partial t=i[\rho_T,\mathcal{H}_T]. \quad (A12)$$

Writing

$$\rho_T=\rho+Fe^{-i\omega t}, \quad (A13)$$

where ρ is the equilibrium density matrix for a system with Hamiltonian $\mathcal{H}$, and evaluating (A12) to the first order in the field, we obtain

$$(\mathcal{H},F)-\omega F=(\rho,\mathfrak{A}). \quad (A14)$$

Let us label the representation which makes $\mathcal{H}$ diagonal as follows:

$$\mathcal{H}\psi_n=E_n\psi_n. \quad (A15)$$

Then the solution of (A14) is

$$F_{nm}=\frac{(\rho_n-\rho_m)\mathfrak{A}_{nm}}{E_n-E_m-\omega}. \quad (A16)$$

(A16) is regular because of the small positive imaginary part of ω.

Now suppose we want the expectation value of any quantity $\mathfrak{B}$. This will be given by

$$\langle\mathfrak{B}\rangle_T\equiv\mathrm{Tr}(\rho_T\mathfrak{B})=\mathrm{Tr}(\rho\mathfrak{B})+\mathrm{Tr}(F\mathfrak{B})e^{-i\omega t}\equiv\langle\mathfrak{B}\rangle+\bar{\mathfrak{B}}e^{-i\omega t}. \quad (A17)$$

Clearly

$$\bar{\mathfrak{B}}=\sum_{n,m}\frac{(\rho_n-\rho_m)\mathfrak{A}_{nm}\mathfrak{B}_{mn}}{E_n-E_m-\omega}. \quad (A18)$$

The question is, how can we calculate (A18) from the many-body perturbation theory of the Luttinger-Ward type, which involves only temperature and no time variables? This is particularly simple for the expression

(A18). Consider

$$Q(z(l_0))\equiv-\int_0^\beta e^{-vz(l_0)}\langle\mathfrak{A}(v)\mathfrak{B}\rangle dv, \quad \text{(A19)}$$

$$\mathfrak{A}(v)\equiv\exp(v\mathcal{H})\mathfrak{A}\exp(-v\mathcal{H}),\quad z(l_0)=2\pi i l_0/\beta. \quad \text{(A20)}$$

Expressing this in terms of the exact eigenfunctions of $\mathcal{H}$, we obtain

$$Q(z(l_0))=\sum_{n,m}\frac{\rho_n-\rho_m}{E_n-E_m-z(l_0)}\mathfrak{A}_{nm}\mathfrak{B}_{mn}. \quad \text{(A21)}$$

Therefore if we calculate (A19) (which just involves temperature variables) by the usual rules and in the *final result* replace $z(l_0)$ by ω, we get exactly (A18). We may express this symbolically as

$$\bar{\mathfrak{B}}=Q(\omega). \quad \text{(A22)}$$

Let us apply this to the calculation of the average current density $\mathcal{J}_{q\omega}{}^\alpha$. This gives

$$\mathcal{J}_{q\omega}{}^\alpha=\frac{1}{V}\sum_{\alpha'}\int_0^\beta dv\, e^{-vz(l_0)}\langle J_{-q}{}^{\alpha'}(v)J_q{}^\alpha\rangle\frac{E_{q\omega}{}^{\alpha'}}{i\omega}-\frac{Ne^2}{mV}\frac{E_{q\omega}{}^\alpha}{i\omega}, \quad \text{(A23)}$$

with $z(l_0)$ replaced by ω in the final result.

In second-quantized notation, we have

$$J_q{}^\alpha=\frac{e}{2m}\sum_k a_{k-q}{}^\dagger a_k(2k_\alpha-q_\alpha). \quad \text{(A24)}$$

Therefore, the first term of (A23) is just a two-particle Green's function,

$$\langle J_{-q}{}^{\alpha'}(v)J_q{}^\alpha\rangle=\left(\frac{e}{m}\right)^2\sum_{k,k'}(k_{\alpha'}+q_{\alpha'}/2)(k_\alpha'+q_\alpha/2)\times\langle a_{k+q}{}^\dagger(v)a_k(v)a_{k'}{}^\dagger a_{k'+q}\rangle. \quad \text{(A25)}$$

From (I. 2.1), (I. 2.2), and (I. 2.3), we have

$$\langle a_{k+q}{}^\dagger(v)a_k(v)a_{k'}{}^\dagger a_{k'+q}\rangle=-\frac{1}{\beta^3}\sum_{l_0}e^{z(l_0)v}\times\{\delta_{kk'}\beta\sum_l R_p(P)-\beta^3\delta_{p,0}S_k'(0^+)S_{k'}'(0^+)+\sum_{l,l'}R_p(P)\Gamma_{pp'}(P)R_{p'}(P)\}. \quad \text{(A26)}$$

Using (A26) and (A25) in (A23) and the fact that no current flows in equilibrium, we obtain

$$\mathcal{J}_{q\omega}{}^\alpha=-\left(\frac{e}{m}\right)^2\frac{1}{V}\sum_{\alpha'}\frac{E_{q\omega}{}^{\alpha'}}{i\omega}\{\sum_p R_p(P)(k_{\alpha'}+q_{\alpha'}/2)\times(k_\alpha+q_\alpha/2)+\sum_{p,p'}(k_{\alpha'}+q_{\alpha'}/2)R_p(P)\Gamma_{pp'}(P)\times R_{p'}(P)(k_\alpha+q_\alpha/2)\}-\frac{Ne^2}{mV}\frac{E_{q\omega}{}^\alpha}{i\omega}. \quad \text{(A27)}$$

For small q we can drop the terms coming from $q_{\alpha'}/2$, $q_\alpha/2$. Then (A27) becomes, in the notation of (I. 3.3),

$$\mathcal{J}_{q\omega}{}^\alpha=\frac{-e^2}{V}\sum_{\alpha'}\frac{E_{q\omega}{}^{\alpha'}}{i\omega}S^{\alpha'\alpha}(P)-\frac{Ne^2}{mV}\frac{E_{q\omega}{}^\alpha}{i\omega}. \quad \text{(A28)}$$

Similarly, the charge density is given by

$$\rho_q{}^n=\frac{1}{V}\sum_\alpha\int_0^\beta dv\, e^{-vz(l_0)}\langle J_{-q}{}^\alpha(v)\rho_q\rangle\frac{E_{q\omega}{}^\alpha}{i\omega}, \quad \text{(A29)}$$

where

$$\rho_q=e\sum_k a_k{}^\dagger a_{k+q} \quad \text{(A30)}$$

is the charge density operator times the volume. This gives

$$\rho_{q\omega}=-\frac{e^2}{m}\frac{1}{V}\sum_\alpha\frac{E_{q\omega}{}^\alpha}{i\omega}\{\sum_p R_p(P)(k_\alpha+q_\alpha/2)+\sum_{p,p'}(k_\alpha+q_\alpha/2)R_p(P)\Gamma_{pp'}(P)R_{p'}(P)\}, \quad \text{(A31)}$$

by the same reasoning that led to (A27).

ACKNOWLEDGMENTS

We would like to thank the Bell Telephone Laboratories, Murray Hill, New Jersey, for their hospitality during the summer months of 1961, when a portion of this work was carried out.

One of us (J. M. L.) would like to thank Dr. Abrikosov and Dr. Dzialoshinskii for some valuable discussion of the Landau theory.