Derivation of the Landau Theory of Fermi Liquids. I. Formal Preliminaries*

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The formal relationships necessary to derive the Landau theory of Fermi liquids are given. These include relationships between scattering functions for small energy and momentum transfers, vertex functions, and correlation functions. In addition certain identities (of the Ward type in quantum electrodynamics) are established which enable us to evaluate these quantities. Finally, the form of all these relationships when a long-ranged Coulomb force is present is given.

I. INTRODUCTION

ABOUT five years ago Landau[1] proposed a phenomenological theory of strongly interacting particles obeying Fermi-Dirac statistics. Such systems were called by him "Fermi liquids." The theory was intended for "normal" Fermi liquids, and excluded such possibilities as superfluidity or superconductivity. Landau was largely interested in He^3, and considered the case of short-ranged forces between the particles. For electrons in metals, however, the long-ranged Coulomb force is essential and the theory must be extended. Silin[2] has indicated how one must generalize Landau's results for this case.

In this, and in a subsequent paper, we shall be concerned with the establishment of these results of Landau and Silin under certain assumptions. The basic assumption is that we can use some form of many-body perturbation theory (to arbitrary order) to treat the interaction between the particles. In fact, essentially all that is necessary in order to obtain the Landau results is to extend a technique[3] used by Landau himself to justify his result for "zero-sound." This method, supplemented by certain identities (which are of the same type as those of Ward in quantum electrodynamics), enables one to show that the Landau prescription and the direct microscopic theory lead to identical results for the quantities one actually observes.

In this first paper we shall limit ourselves to the definitions of the basic quantities of the theory, and to the establishment of all the necessary relationships between them. This will be done both for short-ranged forces and for the long-ranged Coulomb interaction. Some of these results are already known (at least for the absolute zero of temperature), but we have felt it is of some interest to have them systematically developed and all together in a uniform notation. In the following paper they are applied to obtaining the Landau theory.

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[1] L. Landau, Soviet Phys.—JETP **3**, 920 (1956); **5**, 101 (1957).

[2] V. P. Silin, Soviet Phys.—JETP **6**, 387 (1958).

[3] L. Landau, Soviet Phys.—JETP **8**, 70 (1959). See also A. A. Abrikosov and I. M. Khalatnikov, Soviet Phys.—Uspekhi **66**, 68 (1958), Appendix 2.

II. SCATTERING FUNCTIONS

The form of the many-body perturbation theory which we shall use is that of Luttinger and Ward.[4] This enables us to carry out the entire theory at finite temperatures, and also avoid difficulties connected with the change in shape of the Fermi surface. In order to keep the notation as simple as possible, however, we shall consider spinless fermions and no external periodic potential. These are not essential simplifications, but make the entire discussion much easier to follow.

Following Landau,[3] we consider a quantity which we shall call the "scattering function" $\Gamma_{pp'}(P)$. This is the collection of all connected diagrams of the form indicated in Fig. 1. The index p is the "four-vector" (k,ζ_l), where k is the momentum carried by the line,[5] and $\zeta_l=\mu+\pi i(2l+1)/\beta$ $(\beta=1/kT)$, gives the (complex)

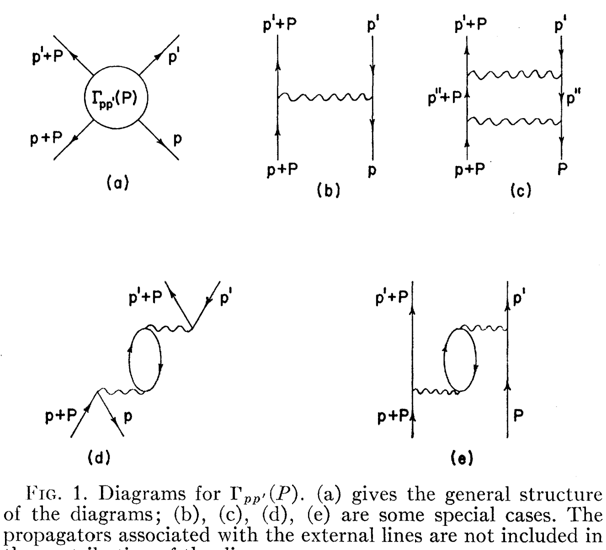

FIG. 1. Diagrams for $\Gamma_{pp'}(P)$. (a) gives the general structure of the diagrams; (b), (c), (d), (e) are some special cases. The propagators associated with the external lines are not included in the contribution of the diagram.

[4] J. M. Luttinger and J. Ward, Phys. Rev. **118**, 1417 (1960). We shall refer to the paper as LW and shall follow its notation whenever practical.

[5] For simplicity of notation we denote three-vectors **k** by italic symbols k. Their character will be evident from the context.

"energy" carried by the line. The quantity P is an arbitrary "four-vector" of the form $(q, 2\pi i l_0/\beta)$, giving the momentum and "energy" transferred in the scattering. It is often very convenient to have an expression for $\Gamma_{pp'}(P)$ in terms of the exact two-particle Green's function. Let us define

$$G_{k_1k_2k_3k_4}(v_1,v_2,v_3,v_4) = \langle T[a_{k_1}{}^\dagger(v_1)a_{k_2}{}^\dagger(v_2)a_{k_3}(v_3)a_{k_4}(v_4)]\rangle, \quad (2.1)$$

where T orders in increasing v from right to left and provides a plus or minus sign depending on whether or not this ordering requires an even or an odd permutation of the operators. The angular bracket is the usual average in the ground canonical distribution. It is easy to see[6] that (2.1) has an expansion of the form

$$G_{k_1k_2k_3k_4}(v_1v_2v_3v_4) = \frac{1}{\beta}\sum_{l_1l_2l_3l_4} \exp(\zeta_{l_1}v_1+\zeta_{l_2}v_2-\zeta_{l_3}v_3-\zeta_{l_4}v_4)G_{p_1p_2p_3p_4}. \quad (2.2)$$

Then one has, on putting

$$p_1=p+P,\quad p_2=p',\quad p_3=p,\quad p_4=p'+P: \quad (2.3)$$

$$G_{p+P,p',p,p'+P}=\beta^2\{\delta_{pp'}S_p'S_{p+P}'-\delta_{P,0}S_p'S_{p'}'\} +\beta S_p'S_{p+P}'\Gamma_{pp'}(P)S_{p'}'S_{p'+P}'. \quad (2.4)$$

In (2.5) S_p' is the exact single-particle propagator $S_k'(\zeta_l)$ defined in LW. It can easily be shown from this that $\Gamma_{pp'}(P)=\Gamma_{p'p}(P)$.

The diagrams for $\Gamma_{pp'}(P)$ may be decomposed into their so-called "irreducible parts." An irreducible scattering diagram is one which never has in its internal propagator lines a pair which differ by P due to momentum and ζ_l conservation. For example, in Fig. 1, (b) and (e) are irreducible, (c) and (d) are reducible in this sense. Call the contribution of all the irreducible diagrams $I_{pp'}(P)$. Then clearly from the diagrams (see Fig. 2)

$$\Gamma_{pp'}(P)=I_{pp'}(P)+\sum\nolimits_{p''} I_{pp''}{}^{(P)}S_{p''}'S_{p''+P}' \times I_{p''p'}(P)+\cdots, \quad (2.5)$$

where

$$\sum\nolimits_p\equiv(1/\beta)\sum\nolimits_{kl}. \quad (2.6)$$

Therefore, in a transparent matrix notation, we have

$$\begin{aligned}\Gamma(P)&=I(P)+I(P)R(P)\Gamma(P)\\ &=I(P)+\Gamma(P)R(P)I(P),\end{aligned} \quad (2.7)$$

where $R(P)$ is the diagonal matrix with elements

$$R_{pp'}(P)=\delta_{pp'}S_{p'}S_{p+P}'. \quad (2.8)$$

It will turn out that all the quantities of interest in the theory of Fermi liquids will be expressed in terms of $\Gamma(P)$ for P very small.[7] We next consider this case.

[6] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959), Sec. V.

[7] This means effectively $|q|\ll$Fermi momentum, $2\pi l_0/\beta|\ll\mu$.

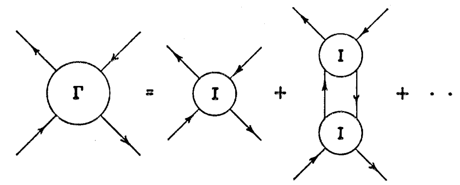

FIG. 2. Decomposition of Γ into irreducible diagrams.

The important thing to notice in this connection is that there is no unique limit as P approaches zero, but that the limit depends just how q and l_0 approach zero. This arises from the behavior of $R(P)$, the quantity $I(P)$ being, for short-ranged forces, perfectly regular as P approaches zero. Let us consider quantities of the type

$$\begin{aligned}Q_k(P)&=\frac{1}{\beta}\sum_l R_p(P)F(\zeta_l)\\ &=\frac{1}{\beta}\sum_l S_{k'}(\zeta_l)S_{k+q}'(\zeta_{l+l_0})F(\zeta_l).\end{aligned} \quad (2.9)$$

Now $S'(\zeta_l)$ has a spectral representation of the form[8]

$$S_k'(\zeta_l)=\int_{-\infty}^{\infty} d\xi\frac{\rho_k(\xi)}{\zeta_l-\xi}. \quad (2.10)$$

Therefore,

$$Q_k(P)=\int_{-\infty}^{\infty}\int d\xi d\xi'\rho_k(\xi)\rho_k(\xi')\frac{1}{\beta} \times\sum_l\frac{F(\zeta_l)}{(\zeta_l-\xi)(\zeta_{l+l_0}-\xi')}. \quad (2.11)$$

This may be converted into a contour integral (cf. LW Appendix A) by taking a contour C which is the sum of infinitesimal circles proceeding clockwise about each pole of $f(\zeta)\equiv1/(e^{\beta(\zeta-\mu)}+1)$. The analytic behavior of $F(\zeta)$ may be very complicated. We shall only assume that there is sufficient regularity near $\zeta_l=\mu$. Therefore,

$$\frac{1}{\beta}\sum_l\frac{F(\zeta_l)}{(\zeta_l-\xi)(\zeta_{l+l_0}-\xi')} =\frac{1}{2\pi i}\int_C f^-(\zeta)\frac{F(\zeta)}{(\zeta-\xi)(\zeta+2\pi il_0/\beta-\xi')}d\zeta. \quad (2.12)$$

Imagine the contour C deformed in such a manner that we surround the poles $\zeta=\xi$, $\xi-2\pi il_0/\beta$ and any cuts or poles of $F(\zeta)$. Call the contribution of the former poles

[8] J. M. Luttinger, Phys. Rev. **121**, 942 (1961). The proof is only carried out in detail for $T=0$ in that paper. However, the identical argument for $T\neq0$ leads to (2.10). $\rho_k(\xi)$ is, of course, a function of temperature now. For ξ near μ, Eq. (69) of that paper is no longer exactly valid, but the δ function is spread out over a region of the order of $(kT)^2/\mu$. All corrections coming from this source are extremely small and we shall drop them in what follows.

A. Then

$$A=\frac{f^-(\xi)F(\xi)-f^-(\xi'-2\pi il_0/\beta)F(\xi'-2\pi il_0/\beta)}{\xi-\xi'+2\pi il_0/\beta}. \quad (2.13)$$

Now in accordance with Luttinger,[8] we may write

$$\rho_k(\xi)=\delta(\xi-\epsilon_k-K_k(\xi))+\tilde{\rho}_k(\xi), \quad (2.14)$$

where $\tilde{\rho}_k(\xi)$ is a perfectly smooth function.

This may also be written

$$\rho_k(\xi)=z_k\delta(\xi-E_k)+\tilde{\rho}_k(\xi), \quad (2.15)$$

where E_k is the single-particle excitation energy given by

$$E_k-\epsilon_k-K_k(E_k)=0 \quad (2.16)$$

and

$$z_k^{-1}=(1-K_k(\xi)/\partial\xi)_{\xi=E_k}. \quad (2.17)$$

Therefore, the contribution to $Q_k(P)$ which comes from A and the singular part of (2.15) is

$$z_kz_{k+q}\frac{f^-(E_{k+q})F(E_{k+q}-2\pi il_0/\beta)-f^-(E_k)F(E_k)}{E_{k+q}-E_k-2\pi il_0/\beta}.$$

The leading term of this is, for $|q|$ and $|2\pi il_0/\beta|$ small,

$$z_kz_{k+q}\frac{f^-(E_{k+q})-f^-(E_k)}{E_{k+q}-E_k-2\pi il_0/\beta}F(E_k). \quad (2.18)$$

If we define the "quasi-particle" velocity by

$$V_k^\alpha=\partial E_k/\partial k_\alpha \quad (2.19)$$

and use

$$-\partial f^-(x)/\partial x=\delta(\mu-x)[1+O((kT)^2/\mu^2)], \quad (2.20)$$

(2.18) becomes

$$-z_k^2\frac{V_k\cdot q}{V_k\cdot q-(2\pi il_0/\beta)}\delta(\mu-E_k)F(\mu). \quad (2.21)$$

This is the part of the result for $Q_k(P)$ which depends on how P approaches zero, the rest being perfectly regular.[9] Let us put, for very small P,

$$R_p(P)=\Delta_p(P)+R_p, \quad (2.22)$$

where

$$\Delta_p(P)=-\beta z_k^2\frac{V_k\cdot q}{V_k\cdot q-(2\pi il_0/\beta)}\delta(\mu-E_k)\delta_{l,-1/2}. \quad (2.23)$$

In (2.23) $\delta_{l,-1/2}$ really means that $F(\zeta_l)$ is to be evaluated at $\zeta_l=\mu$. R_p is the $P=0$ limit of the part of $R_p(P)$ which doesn't depend on how P approaches zero.

There are two limits of particular interest. The first of these (which we shall call, following Landau, the ω limit) is if q approaches 0 and then l_0 approaches zero. Clearly, $\Delta_p(P)=0$ in the ω limit and we have

$$R_p{}^\omega=R_p. \quad (2.24)$$

The second limit is when $l_0=0$; then q approaches zero. We shall call this the q limit. For this case

$$[\Delta_p(P)]_{q\text{ limit}}=-\beta z_k^2\delta_p, \quad (2.25)$$

where

$$\delta_p\equiv\delta(\mu-E_k)\delta. \quad (2.26)$$

Therefore,

$$R_p{}^q=R_p-\beta z_k^2\delta_p. \quad (2.27)$$

One sees easily that the q limit is also what one obtains if one puts $q=0$, $l_0=0$ in the beginning for $R_p(P)$, i.e.,

$$R_p{}^q=S_p'^2. \quad (2.28)$$

Taking the ω and q limits of Eq. (2.7), we obtain at once

$$\Gamma^\omega=I+IR^\omega\Gamma^\omega, \quad (2.29)$$

$$\Gamma^q=I+IR^q\Gamma^q. \quad (2.30)$$

It is a straightforward matter to eliminate I in (2.7) in terms of Γ^ω or Γ^q. This gives[10]

$$\begin{aligned}\Gamma(P)&=\Gamma^\omega+\Gamma^\omega\Delta(P)\Gamma(P)\\&=\Gamma^\omega+\Gamma(P)\Delta(P)\Gamma^\omega\end{aligned} \quad (2.31)$$

and

$$\begin{aligned}\Gamma(P)&=\Gamma^q+\Gamma^q\tilde{\Delta}(P)\Gamma(P)\\&=\Gamma^q+\Gamma(P)\tilde{\Delta}(P)\Gamma^q,\end{aligned} \quad (2.32)$$

where

$$\begin{aligned}\tilde{\Delta}_p(P)&=\Delta_p(P)+\beta z_k^2\delta_p\\&=-\beta z_k^2\delta_p\frac{(2\pi il_0/\beta)}{V_k\cdot q-(2\pi il_0/\beta)}.\end{aligned} \quad (2.33)$$

The relationships (2.31) and (2.32) (which are due to Landau) are the basic equations for the scattering functions necessary to obtain a theory of Fermi liquids.

III. VERTEX AND CORRELATION FUNCTION IDENTITIES

In working out the Landau theory certain sums over scattering functions will be necessary. We now investigate these. The first set are the so-called "vertex" functions or vertex parts. Let the unperturbed single-particle energy be ϵ_k. Then (in order to have a compact notation) define the "four-velocity" v_k^α as

$$\begin{aligned}v_k^\alpha&=1 \quad \text{if} \quad \alpha=0\\&=\partial\epsilon_k/\partial k_\alpha \quad \text{if} \quad \alpha=1,2,3.\end{aligned} \quad (3.1)$$

The vertex part $T_p^\alpha(P)$ is then defined by

$$T_p^\alpha(P)=v_k^\alpha+\sum_{p'}\Gamma_{pp'}(P)R_{p'}(P)v_{k'}^\alpha. \quad (3.2)$$

[9] It should be mentioned that we have not established the results necessary for this conclusion with complete rigor. The analytic properties of the $F(\zeta_l)$ which came into our problem are not known with sufficient generality for this purpose. However, it is quite easy to check the results for a few diagrams of lower order.

[10] These relationships are also immediately obvious graphically if one expresses them in terms of diagrams.

Similarly the "correlation" function $S^{\alpha\alpha'}(P)$ is defined by

$$S^{\alpha\alpha'}(P)=\sum_p R_p(P)v_k^\alpha v_k^{\alpha'} +\sum_{p,p'} R_p(P)\Gamma_{pp'}(P)R_{p'}(P)v_k^\alpha v_{k'}^{\alpha'}. \quad (3.3)$$

We can express these functions for arbitrary but small P in terms of their values in the ω and q limits. Multiplying the p, p' matrix element of the first equation (2.31), for example, on the right by $R_{p'}(P)v_{k'}^\alpha$ and summing on p' we obtain, after a little algebra,

$$T_p^\alpha(P)=T_p^{\alpha\omega}+\sum_{p'}\Gamma_{pp'}{}^\omega\Delta_{p'}(P)T_{p'}^\alpha(P). \quad (3.4)$$

Similarly, taking the second equation of (2.31) and multiplying with $R_{p'}{}^\omega v_{k'}{}^\alpha$ yields

$$T_p^\alpha(P)=T_p^{\alpha\omega}+\sum_{p'}\Gamma_{pp'}(P)\Delta_{p'}(P)T_{p'}^{\alpha\omega}. \quad (3.5)$$

The same technique applied to (2.32) yields the equations

$$T_p^\alpha(P)=T_p^{\alpha q}+\sum_{p'}\Gamma_{pp'}{}^q\tilde{\Delta}_{p'}(P)T_{p'}^\alpha(P) \quad (3.6)$$

and

$$T_p^\alpha(P)=T_p^{\alpha q}+\sum_{p'}\Gamma_{pp'}{}^q(P)\tilde{\Delta}_{p'}(P)T_{p'}^{\alpha q}. \quad (3.7)$$

Next we write the correlation function as

$$S^{\alpha\alpha'}(P)=\sum_p R_p(P)v_p^\alpha T_p^{\alpha'}(P). \quad (3.8)$$

Making use of (3.5) and (2.22) and some algebra, we obtain at once

$$S^{\alpha\alpha'}(P)=S_\omega^{\alpha\alpha'}+\sum_p T_p^{\alpha\omega}\Delta_p(P)T_p^{\alpha'\omega} +\sum_{pp'} T_p^{\alpha\omega}\Delta_p(P)\Gamma_{pp'}(P)\Delta_{p'}(P)T_{p'}^{\alpha'\omega}. \quad (3.9)$$

Similarly,

$$S^{\alpha\alpha'}(P)=S_q^{\alpha\alpha'}+\sum_p T_p^{\alpha q}\tilde{\Delta}_p(P)T_p^{\alpha' q} +\sum_{p,p'} T_p^{\alpha q}\tilde{\Delta}_p\Gamma_{pp'}(P)\tilde{\Delta}_{p'}(P)T_{p'}^{\alpha' q}. \quad (3.10)$$

We shall mainly be interested in the different functions of p "on the Fermi surface," that is for k on the Fermi surface (F.S.) and $\zeta_l=\mu$. It is convenient to introduce the following notation:

$$\tau_k^\alpha(P)\equiv z_kT_p^\alpha(P)\,|_{\zeta_l=\mu, k \text{ on F.S}}, \quad (3.11)$$

$$f_{kk'}(P)=z_kz_{k'}\Gamma_{pp'}(P)\,|_{\zeta_l,\zeta_{l'}=\mu;\ k,k' \text{ on F.S.}} \quad (3.12)$$

On the Fermi surface, (2.31) takes on the following form in these quantities

$$f_{kk'}(P)=f_{kk'}{}^\omega-\sum_{k''}f_{kk'}{}^\omega\delta_{k''}(P)F_{k''k'}(P)$$

or, in matrix notation,

$$f(P)=f^\omega-f^\omega\delta(P)f(P) =f^\omega-f(P)\delta(P)f^\omega, \quad (3.13)$$

where

$$\delta_k(P)=\delta(\mu-E_k)\frac{(V_k\cdot q)}{V_k\cdot q-(2\pi il_0/\beta)}. \quad (3.14)$$

Similarly,

$$f(P)=f^q-f^q\tilde{\delta}(P)f(P) =f^q-f(P)\tilde{\delta}(P)f^q, \quad (3.15)$$

where

$$\tilde{\delta}_k(P)=\delta(\mu-E)\frac{(2\pi il_0/\beta)}{V_k\cdot q-(2\pi il_0/\beta)}. \quad (3.16)$$

We now list the form that the relationships of this section take on the Fermi surface:

$$\tau_k^\alpha(P)=\tau_k^{\alpha\omega}-\sum_{k'} f_{kk'}{}^\omega\delta_{k'}(P)\tau_{k'}{}^\alpha(P), \qquad \alpha=0, 1, 2, 3 \quad (3.17)$$

$$\tau_k^\alpha(P)=\tau_k^{\alpha\omega}-\sum_{k'} f_{kk'}(P)\delta_{k'}(P)\tau_{k'}{}^{\alpha\omega}, \quad (3.18)$$

$$\tau_k^\alpha(P)=\tau_k^{\alpha q}-\sum_{k'} f_{kk'}{}^q\tilde{\delta}_{k'}(P)\tau_{k'}{}^\alpha(P), \quad (3.19)$$

$$\tau_k^\alpha(P)=\tau_k^{\alpha q}-\sum_{k'} f_{kk'}(P)\tilde{\delta}(P)\tau_{k'}{}^{\alpha q}, \quad (3.20)$$

$$S^{\alpha\alpha'}(P)=S_\omega^{\alpha\alpha'}-\sum_k \tau_k^{\alpha\omega}\delta_k(P)\tau_k^{\alpha'\omega} +\sum_{k,k'} \tau_k^{\alpha\omega}\delta_k(P)f_{kk'}(P)\delta_{k'}(P)\tau_{k'}{}^{\alpha'\omega}, \quad (3.21)$$

$$S^{\alpha\alpha'}(P)=S_q^{\alpha\alpha'}-\sum_k \tau_k^{\alpha q}\tilde{\delta}_k(P)\tau_k^{\alpha' q} +\sum_{k,k'} \tau_k^{\alpha q}\tilde{\delta}_k(P)f_{kk'}(P)\tilde{\delta}_{k'}(P)\tau_{k'}{}^{\alpha' q}. \quad (3.22)$$

These relationships involve only properties of the vertex parts on the Fermi surface. These, as we shall see in the next section, may be obtained in some cases in the ω or q limits, and therefore the above relationships provide evaluations for general P.

IV. WARD IDENTITIES

We shall call certain identities which relate vertex parts with suitable derivatives of the proper self-energy part [the $G_k(\zeta_l)$ of LW] "Ward identities." There are four such identities which are of interest to us. They are all straightforward consequences of the definition of the scattering function $\Gamma_{pp'}(P)$. Let us consider the diagrams for $G_p\equiv G_k(\zeta_l)$. Imagine them expressed in terms of the true propagators. If we remove an internal line p' in all possible ways from G_p we obtain clearly (see Fig. 3) the irreducible scattering function $I_{pp'}$. If we differentiate G_p with respect to μ, we therefore obtain

$$\partial G_p/\partial\mu=\sum_{p'}I_{pp'}(\partial S_{p'}{}'/\partial\mu). \quad (4.1)$$

Since

$$\partial S_{p'}{}'/\partial\mu=S_{p'}{}^2[1-(\partial G_{p'}/\partial\mu)], \quad (4.2)$$

we obtain

$$\partial G_p/\partial\mu=-\sum_{p'}I_{pp'}S_{p'}{}^2[1-(\partial G_{p'}/\partial\mu)]. \quad (4.3)$$

Using (2.28), we may write this as

$$1=1-(\partial G_p/\partial\mu)-\sum_{p'} I_{pp'}R_{p'}{}^q[1-(\partial G_{p'}/\partial\mu)]. \quad (4.4)$$

On the other hand, from (2.30) we have the matrix identity

$$1-IR^q=(1+\Gamma^qR^q)^{-1}, \quad (4.5)$$

so that (4.4) becomes

$$1-(\partial G_p/\partial\mu)=1+\sum_{p'}\Gamma_{pp'}{}^qR_{p'}{}^q. \quad (4.6)$$

However, by (3.2) the right-hand side of (4.6) is just

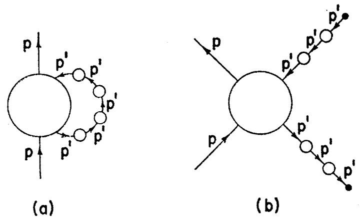

FIG. 3. Diagramatic representation of the Ward identity (4.1). (a) represents a diagram for G_p in which an internal line p' has been explicitly indicated. (b) represents the result of differentiating the middle unperturbed propagator of this line with respect to μ. The dot on the end of the p' lines means that those propagators have to be included and p' summed on in taking the contribution of the diagram.

$T_p{}^{0q}$, and therefore we obtain

$$T_p{}^{0q}=1-\partial G_p/\partial\mu, \tag{4.7}$$

which is the first "Ward" identity.

Similarly, differentiating G_p with respect to k_α gives another identity. Since G_p depends on k only through the matrix elements of the interaction potential, and these are invariant if all the momenta are increased by the same amount, differentiating with respect to k_α is the same as differentiating with respect to the k_α' of every internal line of the diagram. Therefore, the same reasoning as lead to (4.3) yields

$$\partial G_p/\partial k_\alpha=+\sum_{p'} I_{pp'}S_{p'}{}'^2[v_{k'}{}^\alpha+(\partial G_{p'}/\partial k_\alpha')]. \tag{4.8}$$

Or we may write

$$v_k{}^\alpha=v_k{}^\alpha+(\partial G_p/\partial k_\alpha) -\sum_{p'} I_{pp'}R_{p'}{}^q[v_{k'}{}^\alpha+(\partial G_{p'}/\partial k_\alpha')]. \tag{4.9}$$

Using (4.5) again, we obtain

$$v_k{}^\alpha+(\partial G_p/\partial k_\alpha)=v_k{}^\alpha+\sum_{p'} \Gamma_{pp'}{}^q R_{p'}{}^q v_{k'}{}^\alpha. \tag{4.10}$$

The right-hand side of (4.10) is by (3.2) just $T_p{}^{\alpha q}$, and therefore we have the second "Ward" identity[11]:

$$T_p{}^{\alpha q}=v_k{}^\alpha+(\partial G_p/\partial k_\alpha). \tag{4.11}$$

Next we consider the effect of changing ζ_l in G_p. Since $\zeta_{l+l_0}=\zeta_l+z(l_0)$, where

$$z(l_0)\equiv 2\pi i l_0/\beta, \tag{4.12}$$

we will consider the quantity

$$G_p(\zeta_l+z(l_0))-G_p(\zeta_l)/z(l_0)$$

in the limit where $z(l_0)$ is small, but not zero. To have a compact notation we write

$$\frac{\partial G(\zeta_l)}{\partial\zeta_l}\equiv\lim_{z(l_0)\to 0,\, l_0\neq 0}\frac{G_k(\zeta_l+z(l_0))-G_k(\zeta_l)}{z(l_0)} \tag{4.13}$$

[11] The identity (4.11) may also be proven directly from the gauge invariance of the theory. In fact, gauge invariance gives us a more general result relating to the vertex part for q arbitrary, $l_0=0$. We shall not need this, however, in what follows.

Now consider the left-hand side of (4.13). Since the only dependence of $G_k(\zeta_l)$ on l comes through the δ functions which "conserve" l, changing l by l_0 is equivalent to replacing the l' of any internal line by $l'+l_0$. This is the same as replacing $S_{k'}{}'(\zeta_{l'})$ by $S_{k'}{}'(\zeta_{l'}+z(l_0))$ in every internal line. Using the identity.

$$\frac{S_{k'}{}'(\zeta_{l'}+z(l_0))-S_{k'}{}'(\zeta_{l'})}{z(l_0)}=-S_{k'}{}'(\zeta_{l'}+z(l_0))S_{k'}{}'(\zeta_{l'}) \times\left[1-\frac{G_{k'}(\zeta_{l'}+z(l_0))-G_{k'}(\zeta_{l'})}{z(l_0)}\right], \tag{4.14}$$

$$\frac{\partial G_p}{\partial\zeta_l}=\lim_{z(l_0)\to 0,\, l_0\neq 0}\left\{-\Sigma_{p'}I_{pp'}S_{k'}{}'(\zeta_{l'}+z(l_0)) \times S_{k'}{}'(\zeta_{l'})\left(1-\frac{\partial G_{p'}}{\partial\zeta_{l'}}\right)\right\}$$
$$=-\Sigma_{p'}I_{pp'}R_{p'}{}^\omega\left(1-\frac{\partial G_{p'}}{\partial\zeta_{l'}}\right). \tag{4.15}$$

Writing (4.15) as

$$1=1-(\partial G_p/\partial\zeta_l) -\sum_{p'} I_{pp'}R_{p'}{}^\omega[1-(\partial G_{p'}/\partial\zeta_{l'})], \tag{4.16}$$

and making use of the matrix identity

$$1-IR^\omega=(1+\Gamma^\omega R^\omega)^{-1}, \tag{4.17}$$

which follows from (2.29), we obtain

$$1-(\partial G_p/\partial\zeta_l)=1+\sum_{p'} \Gamma_{pp'}{}^\omega R_{p'}{}^\omega. \tag{4.18}$$

The right-hand side of (4.18) is just the definition of $T_p{}^{0\omega}$ and therefore we have the third "Ward" identity:

$$T_p{}^{0\omega}=1-(\partial G_p/\partial\zeta_l). \tag{4.19}$$

To obtain the final Ward identity, consider the operator ($\mathcal{P}$) for the total momentum

$$\mathcal{P}^\alpha=\sum_k a_k{}^\dagger a_k k_\alpha,\quad a=1,2,3. \tag{4.20}$$

For the last identity we must assume that $\mathcal{P}^\alpha$ commutes with the Hamiltonian of the system. Consider the quantity

$$Q_k{}^\alpha(v,v')\equiv\langle T(a_k{}^\dagger(v)\mathcal{P}^\alpha(v')a_k)\rangle, \tag{4.21}$$

which is related to a certain class of vertex diagrams. Since $\mathcal{P}^\alpha$ commutes with the Hamiltonian,

$$\mathcal{P}^\alpha(v')=\mathcal{P}^\alpha. \tag{4.22}$$

Therefore, for $v>v'$,

$$Q_k{}^\alpha(v,v')=\langle T(a_k{}^\dagger(v)\mathcal{P}^\alpha a_k)\rangle =\langle T(a_k{}^\dagger(v)a_k\mathcal{P}^\alpha)\rangle-k_\alpha\langle T(a_k{}^\dagger(v)a_k)\rangle, \tag{4.23}$$

on using

$$(\mathcal{P}^\alpha,a_k)=-k_\alpha a_k. \tag{4.24}$$

For $v<v'$,

$$Q_k{}^\alpha(v,v')=\langle T(\mathcal{P}^\alpha a_k{}^\dagger(v)a_k)\rangle. \tag{4.25}$$

Assuming $v>0$, we therefore may write

$$Q_k^\alpha(v,v')=\langle(\mathcal{P}^\alpha a_k^\dagger(v)a_k)\rangle - k_\alpha\langle a_k^\dagger(v)a_k\rangle\theta(v-v') \quad (4.26)$$

where

$$\theta(v-v')=1,\quad v>v' \\ =0,\quad v<v'. \quad (4.27)$$

On the other hand, $Q_k^\alpha(v,v')$ is a two-particle Green's function. From (2.1), (2.2), and a little manipulation, we have

$$Q_k^\alpha(v,v')=\sum_{k'}k_\alpha'\langle T(a_k^\dagger(v)a_{k'}^\dagger(v'+0^+)a_{k'}(v')a_k)\rangle$$
$$=\sum_{k'}k_\alpha' G_{kk'k'k}(v,\, v'+0^+,\, v'0)$$
$$=\frac{1}{\beta^4}\sum_{k'}k_\alpha'\sum_{l,l',l_0}e^{\zeta_l v}e^{z(l_0)v'}\times G_{kk'k'k}(\zeta_l,\zeta_{l'+l_0},\zeta_{l'},\zeta_{l+l_0}). \quad (4.28)$$

Expanding the Green's function in (4.28) in a manner similar to that of (2.4), we obtain

$$Q_k^\alpha(v,v')=\frac{1}{\beta^2}\sum_{k'}k_\alpha'\sum_{l,l',l_0}e^{\zeta_l v}e^{z(l_0)v'}\bigg\{-\delta_{pp'}R_p(P_0)+\delta_{l_0,0}S_p'S_{p'}'-\frac{1}{\beta}R_p(P_0)\Gamma_{pp'}(P_0)R_{p'}(P_0)\bigg\}, \quad (4.29)$$

where

$$P_0=(0,z(l_0)). \quad (4.30)$$

On the other hand, if we use (4.26) to obtain a similar expansion for $Q_k^\alpha(v,v')$, we obtain (if $l^\dagger\neq 0$) for the expansion coefficients

$$\int_0^\beta\int_0^\beta e^{-\zeta_l v}e^{z(l_0)v'}Q_k^\alpha(v,v')dvdv'$$
$$=-k_\alpha\int_0^\beta dve^{-\zeta_l v}\int_0^v\langle a_k^\dagger(v)a_k\rangle e^{-z(l_0)v'}dv'$$
$$=k_\alpha\int_0^\beta dve^{-\zeta_l v}S_k'(v)\frac{e^{-z(l_0)v}-1}{z(l_0)}$$
$$=k_\alpha\frac{S_k'(\zeta_l+z(l_0))-S_k'(\zeta_l)}{z(l_0)}. \quad (4.31)$$

Comparing (4.31) with (4.29), we have, for $l_0\neq 0$,

$$k_\alpha\frac{S_k'(\zeta_l+z(l_0))-S_k'(\zeta_l)}{z(l_0)}=-\big[k_\alpha R_p(P_0)+\sum_{p'}R_p(P_0)\Gamma_{pp'}(P_0)R_{p'}(P_0)k_\alpha'\big], \quad (4.32)$$

or for $z(l_0)$ small,

$$+k_\alpha\bigg(1-\frac{\partial G_p}{\partial\zeta_l}\bigg)=k_\alpha+\sum_{p'}\Gamma_{pp'}{}^\omega R_{p'}{}^\omega k_\alpha'. \quad (4.33)$$

If the unperturbed single-particle energy is of the form

$$\epsilon_k=k^2/2m, \quad (4.34)$$

then (4.33) takes the form[12]

$$v_k^\alpha\bigg(1-\frac{\partial G_p}{\partial\zeta_l}\bigg)=v_k^\alpha+\sum_{p'}\Gamma_{pp'}{}^\omega R_{p'}{}^\omega v_{k'}{}^\alpha$$
$$=T_p{}^{\alpha\omega},\quad \alpha=1,2,3, \quad (4.35)$$

which has the same structure as the other Ward identities.

Finally, we evaluate these identities (4.7), (4.11), (4.19), (4.25) on the Fermi surface. Making use of (2.16) and (2.17), and differentiating (2.17) with respect to μ and k_α we obtain at once, on the Fermi surface,

$$\bigg(1-\frac{\partial G_p}{\partial\mu}\bigg)=\frac{1}{z_k}\bigg(1-\frac{\partial E_k}{\partial\mu}\bigg), \quad (4.36)$$

$$\bigg(v_k^\alpha+\frac{\partial G_p}{\partial k_\alpha}\bigg)=\frac{1}{z_k}V_k^\alpha, \quad (4.37)$$

$$\bigg(1-\frac{\partial G_p}{\partial\zeta_l}\bigg)=\frac{1}{z_k}, \quad (4.38)$$

$$v_k^\alpha\bigg(1-\frac{\partial G_p}{\partial\zeta_l}\bigg)=\frac{1}{z_k}v_k^\alpha. \quad (4.39)$$

Therefore, we have

$$\tau_k^{0q}=1-\partial E_k/\partial\mu, \quad (4.40)$$

$$\tau_k^{\alpha q}=V_k^\alpha,\quad \alpha=1,2,3 \quad (4.41)$$

$$\tau_k^{\alpha\omega}=1, \quad (4.42)$$

$$\tau_k^{\alpha\omega}=v_k^\alpha,\quad \alpha=1,2,3. \quad (4.43)$$

V. CONSEQUENCES OF WARD IDENTITIES FOR VERTEX AND CORRELATION FUNCTIONS

From the relationships (3.18) and (3.20) and (4.40)–(4.43), we obtain the following important results[13] [remembering that $\delta_k{}^q=\delta(\mu-E_k)$, $\delta_k{}^\omega=-\delta(\mu-E_k)$]:

$$1=1-(\partial E_k/\partial\mu)+\sum_{k'}f_{kk'}{}^\omega\delta(\mu-E_{\kappa'})\times[1-(\partial E_{k'}/\partial\mu)], \quad (5.1)$$

$$1-(\partial E_k/\partial\mu)=1-\sum_{k'}f_{kk'}{}^q\delta(\mu-E_{k'}), \quad (5.2)$$

$$v_k^\alpha=V_k^\alpha+\sum_{k'}f_{kk'}{}^\omega\delta(\mu-E_{k'})V_{k'}{}^\alpha, \quad (5.3)$$

$$V_k^\alpha=v_k^\alpha-\sum_{k'}f_{kk'}{}^q\delta(\mu-E_{k'})v_{k'}{}^\alpha. \quad (5.4)$$

[12] The identity (4.35) is not needed for the establishment of the Landau theory. It is, however, necessary if one wants to establish the Landau effective mass equation.

[13] For simplicity in this section, we shall always assume $\epsilon_k=k^2/2m$. Some of the relationships of this section are changed if we do not assume this, but none that we shall need for the Landau theory.

Actually (5.2) is equivalent to (5.1) and (5.4) is equivalent to (5.3). This follows from specializing (3.13) to the q limit. Then

$$f_{kk'}{}^{q}=f_{kk'}{}^{\omega}-\sum_{k''} f_{kk''}{}^{q}\delta(\mu-E_{k''})f_{k''k'}{}^{\omega}. \quad (5.5)$$

From this we see at once that if

$$B_k=A_k+\sum_{k'} f_{kk'}{}^{\omega}\delta(\mu-E_{k'})A_{k'}, \quad (5.6)$$

then

$$A_k=B_k-\sum_{k'} f_{kk'}{}^{q}\delta(\mu-E_{k'})B_{k'}. \quad (5.7)$$

[Multiply (5.5) by $\delta(\mu-E_{k'})A_{k'}$ and sum on k'.]

In order to obtain the correlation functions, we first need $S_q{}^{\alpha\alpha'}$ and $S_\omega{}^{\alpha\alpha'}$, where α, $\alpha'=0, 1, 2, 3$. Starting from (3.8) we have

$$S_q{}^{\alpha\alpha'}=\sum_p S_p{}'^2 v_k{}^{\alpha}T_p{}^{\alpha' q}, \quad (5.8)$$

$$S_\omega{}^{\alpha\alpha'}=\sum_p R_p v_k{}^{\alpha}T_p{}^{\alpha'\omega}. \quad (5.9)$$

From (4.7) and (4.11)

$$\begin{aligned} S_q{}^{\alpha\alpha'}&=\sum_p S_p{}'^2 v_k{}^{\alpha}(1-\partial G_p/\partial\mu), \quad \alpha'=0\\ &=\sum_p S_p{}'^2 v_k{}^{\alpha}(v_k{}^{\alpha'}+\partial G_p/\partial k_{\alpha'}), \quad \alpha'=1,2,3\\ &=\sum_p v_k{}^{\alpha}(-\partial S_p{}'/\partial\mu), \quad \alpha'=0\\ &=\sum_p v_k{}^{\alpha}(\partial S_p{}'/\partial k_{\alpha'}), \quad \alpha'=1,2,3\\ &=\sum_k v_k{}^{\alpha}(-\partial \bar{N}_k/\partial\mu), \quad \alpha'=0\\ &=\sum_k v_k{}^{\alpha}(\partial \bar{N}_k/\partial k_{\alpha'}), \quad \alpha'=1,2,3, \end{aligned} \quad (5.10)$$

where $\bar{N}_k$ is the mean occupation number for momentum k. Therefore

$$S_q{}^{00}=\partial\bar{N}(\mu)/\partial\mu, \quad (5.11)$$

$$S_q{}^{0\alpha'}=+S_q{}^{\alpha 0}=0, \quad \alpha,\alpha'=1,2,3 \quad (5.12)$$

$$S_q{}^{\alpha\alpha'}=-\sum_k (\partial^2\epsilon_k/\partial k_\alpha\partial k_{\alpha'})\bar{N}_k, \quad \alpha,\alpha'=1,2,3 \quad (5.13)$$

$$=-(N/m)\delta_{\alpha\alpha'}. \quad (5.14)$$

From (4.19) and (4.35), we have

$$\begin{aligned} S_\omega{}^{\alpha\alpha'}&=\Sigma_p R_p v_k{}^{\alpha}(1-\partial G_p/\partial\zeta_l), \quad \alpha'=0\\ &=\Sigma_p R_p v_k{}^{\alpha}v_k{}^{\alpha'}(1-\partial G_p/\partial\zeta_l), \quad \alpha'=1,2,3. \end{aligned} \quad (5.15)$$

However,

$$(1/\beta)\Sigma_l R_p(1-\partial G_p/\partial\zeta_l)=0, \quad (5.16)$$

so that

$$S_\omega{}^{\alpha\alpha'}=0, \quad \alpha,\alpha'=0,1,2,3. \quad (5.17)$$

To see (5.16), consider

$$(1/\beta)\Sigma_l[S_k{}'(\zeta_l)-S_k{}'(\zeta_{l+l_0})]e^{\zeta_l 0^+}=0.$$

This is the same as

$$\frac{1}{\beta}\sum_l S_k{}'(\zeta_l)S_k{}'(\zeta_{l+l_0})\left[1-\frac{G_k(\zeta_{l+l_0})-G_k(\zeta_l)}{z(l_0)}\right]=0, \quad (5.18)$$

if $l_0\neq 0$. Now let l_0 be small, but not zero. Then the first factor in (5.18) just goes over to R_p and the second to $(1-\partial G_p/\partial\zeta_l)$, so that (5.16) is proved.

Using these results, (3.21) and (3.22) become

$$\begin{aligned} S^{\alpha\alpha'}(P)=&-\sum_k v_k{}^{\alpha}\delta_k(P)v_k{}^{\alpha'}\\ &+\sum_{k,k'} v_k{}^{\alpha}\delta_k(P)f_{kk'}(P)\delta_{k'}(P)v_{k'}{}^{\alpha'}, \end{aligned} \quad (\alpha=0,1,2,3) \quad (5.19)$$

$$\begin{aligned} S^{00}(P)=&-\frac{\partial\bar{N}(\mu)}{\partial\mu}-\sum_k\left(1-\frac{\partial E_k}{\partial\mu}\right)\tilde{\delta}_k(P)\left(1-\frac{\partial E_k}{\partial\mu}\right)\\ &+\sum_{kk'}\left(1-\frac{\partial E_k}{\partial\mu}\right)\tilde{\delta}_k(P)f_{kk'}(P)\\ &\times\tilde{\delta}_{k'}(P)\left(1-\frac{\partial E_{k'}}{\partial\mu}\right), \end{aligned} \quad (5.20)$$

$$\begin{aligned} S^{0\alpha'}(P)=&-\sum_k\left(1-\frac{\partial E_k}{\partial\mu}\right)\tilde{\delta}_k(P)V_k{}^{\alpha'}\\ &+\sum_{k,k'}\left(1-\frac{\partial E_k}{\partial\mu}\right)\tilde{\delta}_k(P)f_{kk'}(P)\tilde{\delta}_{k'}(P)V_{k'}{}^{\alpha'}, \end{aligned} \quad (\alpha'=1,2,3) \quad (5.21)$$

$$\begin{aligned} S^{\alpha\alpha'}(P)=&-\sum_k\frac{\partial E_k}{\partial k_\alpha\partial k_{\alpha'}}\bar{N}_k-\sum_k V_k{}^{\alpha}\tilde{\delta}_k(P)V_k{}^{\alpha'}\\ &+\sum_{k,k'} V_k{}^{\alpha}\tilde{\delta}_k(P)f_{kk'}(P)\tilde{\delta}_{k'}(P)V_{k'}{}^{\alpha'}, \end{aligned} \quad (\alpha,\alpha'=1,2,3). \quad (5.22)$$

Of these only (5.22) will be of direct interest to us for the Landau theory. We also mention that by specializing these last relationships to the q or ω limits we get some interesting identities, but we shall not elaborate on these here.

VI. LONG-RANGE FORCES

Until now we have assumed that the forces between the particles are short-ranged. This has enabled us to go to the small-q limit without any difficulty. If, however, a Coulomb force is present and we have a diagram with a matrix element that has momentum transfer q, there will be a factor proportional to $1/q^2$ present. This diverges as q approaches zero. Such diagrams must clearly be given special treatment before the limit of small q is considered. Let us consider the diagrams for $\Gamma_{pp'}(P)$. We shall call a diagram *proper* if the momentum transfer q never appears in an internal interaction line,[14] and *improper* if it does. Fig. 4 gives a few examples.

Let us call the sum of all *proper* scattering diagrams, $\Gamma_{pp'}(P)$. This is perfectly regular as q approaches zero, since we are just not taking diagrams which lead to small q difficulties. We next express an arbitrary dia-

[14] Equivalently, a proper diagram is one which cannot be cut in two by cutting a single interaction line.

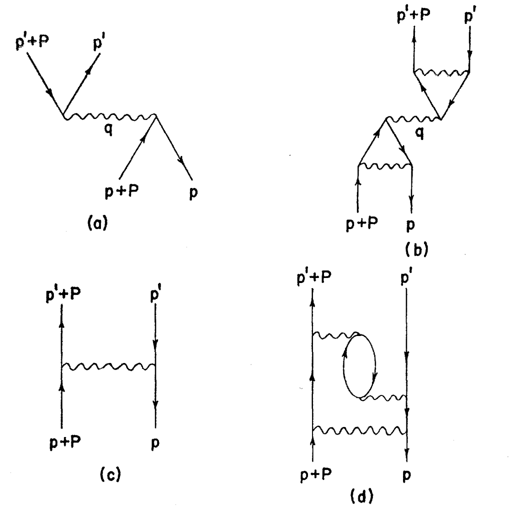

FIG. 4. Proper and improper scattering diagrams. (a) and (b) are improper; (c) and (d) are proper.

gram in terms of the proper ones. The general structure of $\Gamma_{pp'}(P)$ diagrams is indicated in Fig. 5. Clearly

$$\Gamma_{pp'}(P)=\tilde{\Gamma}_{pp'}(P)+\tilde{T}_p{}^0(P)V(P)\tilde{T}_p{}^0(P), \quad (6.1)$$

where

$$V(P)=\mu_q+u_q\tilde{S}^{00}(P)u_q+u_q\tilde{S}^{00}(P)u_q\tilde{S}^{00}\times(P)u_q\tilde{S}^{00}(P)+\cdots \quad (6.2)$$

Here, we have used the notation $\tilde{S},\tilde{T}$ to mean the proper correlation and vertex functions. In (6.2), u_q is just the matrix element of the Coulomb potential for momentum transfer q, i.e.,

$$u_q=4\pi e^2/\Omega q^2, \quad (6.3)$$

where Ω is the volume of the system.

The expression for $V(P)$ is just a geometric series, so that we have

$$V(P)=u_q/1-u_q\tilde{S}^{00}(P) \quad (6.4)$$

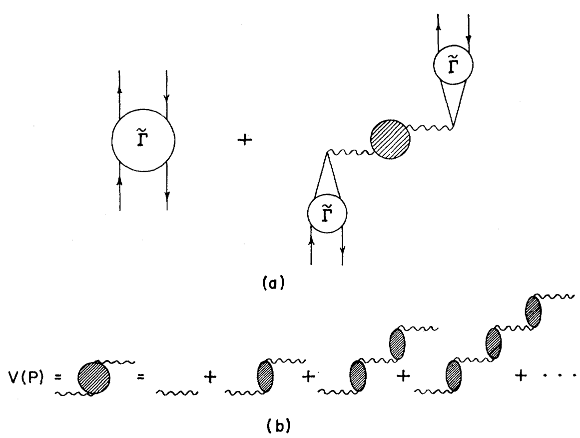

FIG. 5. The general structure of scattering diagrams: (a) gives the diagrams for $\Gamma_{pp'}(P)$. In (b) the oval-shaped parts are proper, i.e., they cannot be cut in two by cutting an interaction line.

and

$$\Gamma_{pp'}(P)=\tilde{\Gamma}_{pp'}(P)+\tilde{T}_p{}^0(P)\tilde{T}_{p'}{}^0(P)\times[u_q/1-u_q\tilde{S}^{00}(P)]. \quad (6.5)$$

The quantities $\tilde{\Gamma}$, $\tilde{T}$, $\tilde{S}$ now have well-defined q and ω limits, $\tilde{\Gamma}^q$, $\tilde{\Gamma}^\omega$, etc. The same analysis that led to (2.31) and (2.32) can now be repeated for proper diagrams. This tells us at once that these equations now hold with Γ replaced by $\tilde{\Gamma}$.

In addition, all the results of Sec. III, which are purely algebraic consequences of (2.31) and (2.32), now hold for the proper vertex and correlation functions. In particular (3.17)–(3.22) hold if a tilde is placed on every f, τ, and S.

Finally, we must consider the Ward identities of Sec. IV. Clearly (4.6) (because of the q limit) makes no sense for the long-range force case. This is also reflected in the fact that G_p itself diverges in this case because it contains the term (from the direct interaction) $Nu(0)$. Now in the Coulomb case, if we have a uniform background charge opposite to that of the interacting particles such terms are exactly canceled, and G_p is actually perfectly regular. That means terms like (*a*) and (*b*) of

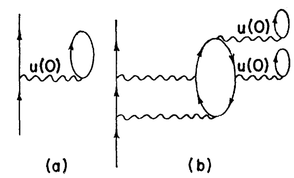

FIG. 6. Examples of proper self-energy diagrams which give trouble in long-range case.

Fig. 6 never occur. However, it is just such terms which when differentiated with respect to μ give rise to improper Γ diagrams. Therefore we have at once

$$\partial G_p/\partial\mu=-\Sigma_{p'}\tilde{\Gamma}_{pp'}{}^qS_{p'}{}'^2. \quad (6.6)$$

Similarly, (4.10) holds for $\tilde{\Gamma}_{pp'}{}^q$ and is replaced by $\tilde{\Gamma}_{pp'}{}^\omega$, since the type of insertion [Fig. 6(a)] which gives rise to difficulties has its l dependence completely independent of the incoming l. Therefore, changing l doesn't affect the contribution of such a line, and it does not contribute to $\partial G_p/\partial\zeta_l$.

Lastly, we come to the identity (4.35). This must be dealt with by another method (which incidently also works for other Ward identities). We consider first an interaction between the particles which is finite ranged, and at the end we let the range go to infinity. Then (4.32) still makes sense. Using (6.5), we obtain

$$\begin{aligned}v_k{}^\alpha(\partial S_p'/\partial\zeta_l)&=-R_p(P_0)T_p{}^\alpha(P_0)\\&=R_p(P_0)\tilde{T}_p{}^\alpha(P_0)-\sum_{p'}\tilde{T}_p{}^0(P_0)\tilde{T}_{p'}{}^0(P_0)\\&\qquad\times R_{p'}(P_0)v_{k'}{}^\alpha\frac{u_0}{1-u_0\tilde{S}^{00}(P_0)}\\&=R_p(P_0)\tilde{T}_p{}^\alpha(P_0).\end{aligned} \quad (6.7)$$

This follows from

$$\sum_{p'} \tilde{T}_{p'}{}^0(P_0) R_{p'}(P_0) v_{k'}{}^\alpha$$
$$= \sum_{p'} \left(1 - \frac{G_{k'}(\zeta_{l'}+z(l_0)) - G_{k'}(\zeta_l)}{z(l_0)}\right) R_{p'}(P_0) v_{k'}{}^\alpha$$
$$= \frac{1}{z(l_0)} \sum_{k'} v_{k'}{}^\alpha \frac{1}{\beta} \sum_{l'} \{S_{k' l'} - S_{k', l'+l_0}\}$$
$$= 0. \tag{6.8}$$

Now (6.7) is exactly the same as (4.32) for proper diagrams, and there is no difficulty passing to the infinite range limit. Thus (4.35) is valid for the proper vertex part.

Therefore, we conclude that the results (4.40)–(4.43) are valid for the proper vertex parts on the Fermi surface and that consequently all the identities of Sec. V still hold for proper scattering functions on the Fermi surface.

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Derivation of the Landau Theory of Fermi Liquids. II. Equilibrium Properties and Transport Equation*

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Using the results of the preceding paper, it is shown that a large class of the conclusions of the Landau theory of Fermi liquids may be established within the framework of many-body perturbation theory. Both equilibrium and transport phenomena are discussed. The theory is also carried out for long-ranged Coulomb forces. Finally, it is shown that a rather simple general expression for the quasi-particle distribution function of Landau may be given.

I. INTRODUCTION

In this paper we shall be concerned with the verification of the results of the Landau theory of Fermi liquids for certain equilibrium and nonequilibrium phenomena. We shall do this first for short-ranged forces and then generalize to long-ranged Coulomb forces. The necessary formalism for this purpose has been given in the preceding paper.[1] In addition, we shall show that it is possible to give a very simple general definition of the "quasi-particle distribution function" which occurs in the Landau theory.

We begin our discussion with a brief summary of the Landau theory.[2] Consider first an ideal Fermi gas. A great many properties (heat capacity, compressibility, conductivity, etc.) of this system are determined (for temperatures much less than the degeneracy temperature) by the nature of the single particle states which lie in the immediate vicinity of the last occupied single-particle state. It is these same properties that the Landau theory tries to calculate for an interacting system of fermions. We can now state Landau's assumptions as follows:

(1) If the interaction is turned on, the single-particle states in the neighborhood of the last occupied one remain eigenstates of the system. We call these quasi-particle states, and say that a quasi-particle k is present if the state k is occupied. The low-lying excited states of the system are assumed to be in one-to-one correspondence with those of an ideal Fermi gas, the quasi-particle states just replacing the particle states. Therefore, since the number of real particles is conserved, in these low-lying excitations the number of quasi-particles must also be conserved. Adding a particle to the system must therefore also add a quasi-particle.

(2) The state of the Fermi liquid for weak excitation (equilibrium or nonequilibrium) is entirely described by the distribution function $n(k,x)$ of quasi-particles in

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[1] P. Nozières and J. M. Luttinger, preceding paper [Phys. Rev. **126**, 1423 (1962)]. We shall refer to this paper as I. The notation and assumptions used in the present paper will be the same as those of I.

[2] See A. A. Abrikosov and I. M. Khalatnikov, Soviet Phys.—Uspekhi **66**, 68 (1958).