Thermoelectric Transport in a Three-Channel Charge Kondo Circuit Supplemental Material

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S1. PERTURBATIVE CALCULATIONS OF THERMOELECTRIC COEFFICIENT G_T

We present here the details of perturbative calculations accounting for the backscattering processes. We use notations and definitions of the *Letter*. All calculations are performed for 3CK model in the spirits of Matveev-Andreev theory [A. V. Andreev and K. A. Matveev, Phys. Rev. Lett. **86**, 280 (2001); K. A. Matveev and A. V. Andreev, Phys. Rev. B **66**, 045301 (2002)], which concern the saddle-point method. We first evaluate Gaussian integral $Z(\tau) = \int \exp[-S_0 - S_C(\tau) - S'] \prod_\alpha \mathcal{D}\phi_\alpha(x,t)$ under the assumption $S' = 0$. At zero order, the saddle point, based on the principle of the action minimum, is found as

$$\phi_{c,\tau}(x,t) = \frac{\pi N}{\sqrt{3}} - \sqrt{3} T E_C \sum_{\omega_n} \frac{\exp\left[-{|\omega_n x|}/{v_F}\right]}{|\omega_n| + {3E_C}/{\pi}} n_\tau(\omega_n)\, e^{-i\omega_n t}, \tag{S1.1}$$

with $\omega_n = 2\pi n T$ are bosonic Matsubara frequencies and the Fourier transform of $n_\tau(t)$ is $n_\tau(\omega_n) = {(e^{i\omega_n\tau}-1)}/{i\omega_n}$.

In the calculation $K_0(\tau) = Z(\tau)/Z(0)$, the integrals over the fluctuations of the field $\phi_c(x,t)$ about the saddle points in the numerator and the denominator cancel each other. Therefore, the value of $K_0(\tau)$ is evaluated by the integrals at the saddle point values. In the condition $\tau \gg E_C^{-1}$ and $T \ll E_C$, we find

$$[S_0 + S_C(\tau)]_{\phi=\phi_{c,\tau}(x,t)} = \frac{3E_C}{2\pi^2 T} \sum_{n=1}^{\infty} \frac{[1-\cos(2\pi T n\tau)]}{n\left[n + {3E_C}/{2\pi^2 T}\right]}, \tag{S1.2}$$

and $[S_0 + S_C(\tau)]_{\phi=\phi_{c,0}(x,t)} = 0$. The correlator $K_0(\tau)$

$$K_0(\tau) = \left[\frac{\pi^2 T}{3\gamma E_C}\frac{1}{|\sin(\pi T\tau)|}\right]^{2/3}. \tag{S1.3}$$

Plugging in the formula (S1.3) into equation (6) we find the electric conductance as a function of the temperature:

$$G = \frac{G_L \pi^{11/6}\Gamma({4}/{3})}{2(3\gamma)^{2/3}\Gamma({11}/{6})}\left[\frac{T}{E_C}\right]^{2/3}. \tag{S1.4}$$

Substituting formula (S1.3) into equation (7) we find that the thermoelectric coefficient G_T vanishes. At zero order of perturbation theory, the PH symmetry is conserved (as explained in the main text). One needs to proceed the perturbation calculation at the first non-vanishing order. To proceed we express S' in terms of the products of charge mode, pseudo-spin mode, and flavor mode. However, only the fluctuations of the charge mode is suppressed at low frequencies by the charging energy term. The fluctuations of the pseudo-spin mode and flavor mode are not suppressed by the charging energy. Therefore, the average $\langle S' \rangle = 0$.

At second order of perturbation theory,

$$K(\tau) = K_C(\tau)\left[1 + \frac{1}{2}\left(\langle S'^2\rangle_\tau - \langle S'^2\rangle_0\right)\right] \tag{S1.5}$$

and therefore it is necessary to calculate $\langle S'^2 \rangle_\tau$. For the purposes of illustration we use parametrization of charge, pseudospin and flavor modes given by Eq.(1).

$$\langle S'^2\rangle_\tau = \frac{D^2}{\pi^2}|r|^2 \int_0^\beta dt \int_0^\beta dt' \left[\left\langle \cos\left[\frac{2}{\sqrt{3}}\phi_c(t) - \frac{2\sqrt{2}}{\sqrt{3}}\phi_f(t)\right] \cos\left[\frac{2}{\sqrt{3}}\phi_c(t') - \frac{2\sqrt{2}}{\sqrt{3}}\phi_f(t')\right]\right\rangle \right.$$
$$\left. +4\left\langle \cos\left[\frac{2}{\sqrt{3}}\phi_c(t) + \frac{\sqrt{2}}{\sqrt{3}}\phi_f(t)\right] \cos\left[\frac{2}{\sqrt{3}}\phi_c(t') + \frac{\sqrt{2}}{\sqrt{3}}\phi_f(t')\right]\right\rangle \left\langle \cos\left[\sqrt{2}\phi_s(t)\right]\cos\left[\sqrt{2}\phi_s(t')\right]\right\rangle\right], \tag{S1.6}$$

where we use shorthand notations $\phi_\alpha(t) \equiv \phi_\alpha(0,t)$. Since charge, pseudo-spin, flavor modes are independent, we decouple them as

$$\langle S'^2 \rangle_\tau = \frac{D^2}{2\pi^2}|r|^2 \int_0^\beta dt \int_0^\beta dt' \Big\{ \mathrm{Re}\left[\kappa_c^+(t,t',\tau)\,\widetilde{\kappa}_f^+(t,t',\tau) + \kappa_c^-(t,t',\tau)\,\widetilde{\kappa}_f^-(t,t',\tau)\right] \\ +2\mathrm{Re}\left[\kappa_c^+(t,t',\tau)\,\kappa_f^+(t,t',\tau) + \kappa_c^-(t,t',\tau)\,\kappa_f^-(t,t',\tau)\right]\mathrm{Re}\left[\kappa_s^+(t,t',\tau) + \kappa_s^-(t,t',\tau)\right]\Big\}, \tag{S1.7}$$

with $\kappa_j^\pm(t,t',\tau) \equiv \langle \exp[ia_j[\phi_j(t) \pm \phi_j(t')]]\rangle$. We apply the saddle-point method one more time in order to perform the integrals in the formula (S1.7)

$$\kappa_j^\pm(t,t',\tau) = \exp[ia_j(\phi_{j\tau}(t) \pm \phi_{j\tau}(t'))] \exp\left[-a_j^2\left(\langle\varphi_j^2(t)\rangle \pm \langle\varphi_j(t)\,\varphi_j(t')\rangle\right)\right], \tag{S1.8}$$

with $\phi_j(t) = \phi_{j\tau}(t) + \varphi_j(t)$. Calculation of the correlator $\langle\varphi_j(t)\,\varphi_j(t')\rangle$ is achieved by using generating functional method as follows.

To evaluate the correlator (S1.8) we introduce the generating functional

$$W[\{J_j(\omega_n)\}] = \left\langle \exp\left[-T\sum_{\omega_n} J_j(\omega_n)\,\varphi_j(-\omega_n)\right]\right\rangle. \tag{S1.9}$$

The value of the Gaussian integral is completely defined by the saddle point

$$W[\{J_j(\omega_n)\}] = \exp\left[-\frac{T}{2}\sum_{\omega_n} J_j(\omega_n)\,\varphi_j^J(-\omega_n)\right], \tag{S1.10}$$

in which $\varphi_j^J(t)$ is the saddle-point value of the field φ_j.

Now, we consider the charge mode: the fluctuations of $\varphi_j(t)$ coincide with those of $\phi_c(0,t)$ at $N=0$, $n_\tau = 0$ and $n_\tau(t)$ plays the role of a source term similar to $J_c(t)$ as $J_c(t) = 2\sqrt{3}E_C n_\tau(t)/\pi$, or $n_\tau(\omega_n) = \pi J_c(\omega_n)/2\sqrt{3}E_C$. We re-write the generating functional as follows:

$$W[\{J_c(\omega_n)\}] = \exp\left[\frac{\pi T}{4}\sum_{\omega_n}\frac{J_c(\omega_n)\,J_c(-\omega_n)}{|\omega_n| + \frac{3E_C}{\pi}}\right]. \tag{S1.11}$$

The correlator $\langle\varphi_c(-\omega_n)\,\varphi_c(\omega_m)\rangle$ can be obtained by differentiating the functional W in formula (S1.9) with respect to $J_c(\omega_n)$ and $J_c(-\omega_m)$. In time representation, this correlator is written as

$$\langle\varphi_c(t)\,\varphi_c(t')\rangle = \frac{\pi T}{2}\sum_{\omega_n}\frac{e^{i\omega_n(t-t')}e^{-|\omega_n|/D}}{|\omega_n| + \frac{3E_C}{\pi}}.$$

At the limits we are interested in, this correlator behaves as

$$\langle\varphi_c(t)\,\varphi_c(t')\rangle = \begin{cases} \frac{1}{2}\ln\frac{\pi D}{3\gamma E_C\sqrt{1+[D(t-t')]^2}}, & |t-t'| \ll E_C^{-1} \\ \frac{\pi^4 T^2}{2(3E_C)^2\sin^2[\pi T(t-t')]}, & |t-t'| \gg E_C^{-1}. \end{cases} \tag{S1.12}$$

Therefore, at $T \ll E_C$, we obtain

$$\kappa_c^+(t,t',\tau) \approx \left[\frac{3\gamma E_C}{\pi D}\right]^{\frac{2}{3}} e^{i\frac{4\pi N}{3}} e^{-i\frac{2}{3}\chi_\tau(t)} e^{-i\frac{2}{3}\chi_\tau(t')}, \tag{S1.13}$$

and

$$\kappa_c^-(t,t',\tau) \approx \left[\frac{3\gamma E_C}{\pi D}\right]^{\frac{2}{3}} e^{-i\frac{2}{3}\chi_\tau(t)} e^{+i\frac{2}{3}\chi_\tau(t')}, \tag{S1.14}$$

with

$$\chi_\tau(t) = \frac{3E_C}{4\pi^2 T}\sum_{n=-\infty}^{\infty}\frac{e^{-i2\pi Tn(t-\tau)} - e^{-i2\pi Tnt}}{in\left(|n| + \frac{3E_C}{2\pi^2 T}\right)} = [\pi n_\tau(t) + \delta\chi_\tau(t)], \tag{S1.15}$$

$$\delta\chi_\tau(t) = \sum_{n=1}^{\infty} \frac{\sin[2\pi T n(t-\tau)] - \sin[2\pi T n t]}{n + \frac{3E_C}{2\pi^2 T}} \approx \frac{\pi^2 T}{3E_C}\left\{\cot[\pi T(t-\tau)] - \cot[\pi T t]\right\}. \tag{S1.16}$$

Similarly, we consider pseudospin and flavor modes. We find that if there exists a "flavoring energy term" E_f and a "spinning energy term" E_s (in the same meaning as charging energy term E_C), the relation between $n_\tau(t)$ and $J_{f/s}(t)$ should be $J_f(t) = 2\sqrt{6}E_f n_\tau(t)/\pi$ and $J_s(t) = 2\sqrt{2}E_s n_\tau(t)/\pi$. However, after obtaining the correlator $\left\langle \varphi_{s/f}(t)\varphi_{s/f}(t')\right\rangle$, we need to take the limits $E_{s/f} = 0$. At the end, we have

$$\langle\varphi_s(t)\varphi_s(t')\rangle = \langle\varphi_f(t)\varphi_f(t')\rangle = \frac{\pi T}{2}\sum_{\omega_n}\frac{e^{i\omega_n(t-t')}e^{-|\omega_n|/D}}{|\omega_n|} = -\frac{1}{4}\ln\left[1 + e^{-\frac{4\pi T}{D}} - 2e^{-\frac{2\pi T}{D}}\cos[2\pi T(t-t')]\right]. \tag{S1.17}$$

We obtain the result as shown in the second line in the limit $T \ll D$.

$$\kappa_s^-(t,t',\tau) \approx \left[\frac{\pi T}{D}\right]\frac{1}{|\sin(\pi T(t-t'))|}, \tag{S1.18}$$

$$\kappa_f^-(t,t',\tau) \approx \left[\frac{\pi T}{D}\right]^{\frac{1}{3}}\frac{1}{|\sin(\pi T(t-t'))|^{\frac{1}{3}}}, \tag{S1.19}$$

$$\widetilde{\kappa}_f^-(t,t',\tau) \approx \left[\frac{\pi T}{D}\right]^{\frac{4}{3}}\frac{1}{|\sin(\pi T(t-t'))|^{\frac{4}{3}}}. \tag{S1.20}$$

If an electron comes from QPC2, we only take into account the first term in Eq. (S1.7) and $\phi_{s\tau}(x,t) = 0$, $\phi_{f\tau}(x,t) = {}^{-2\pi N}/_{\sqrt{6}}$ then

$$\widetilde{\kappa}_f^+(t,t',\tau) \approx e^{i\frac{8\pi N}{3}}\left[\frac{4\pi T}{D}\right]^{\frac{4}{3}}|\sin(\pi T(t-t'))|^{\frac{4}{3}}. \tag{S1.21}$$

The N-dependent term is

$$\langle S'^2\rangle_{\tau,\, odd\, N} \approx 4\left[\frac{6\gamma E_C}{\pi^2}\right]^{\frac{2}{3}} T^{\frac{4}{3}}|r|^2\sin[4\pi N]\int_0^\beta dt\sin\left[\frac{2}{3}\chi_\tau(t)\right]\int_0^\beta dt'\cos\left[\frac{2}{3}\chi_\tau(t')\right]|\sin(\pi T(t-t'))|^{\frac{4}{3}}. \tag{S1.22}$$

We find that

$$\cos\left[\frac{2}{3}\chi_\tau(t')\right] = \cos\left[\frac{2}{3}\pi n_\tau(t') + \frac{2}{3}\delta\chi_\tau(t')\right] \approx \begin{cases} -\frac{1}{2} - \frac{\sqrt{3}}{3}\delta\chi_\tau(t'), & \text{if } 0 \le t' \le \tau, \\ 1, & \text{if } \tau < t' \le \beta, \end{cases} \tag{S1.23}$$

$$\sin\left[\frac{2}{3}\chi_\tau(t)\right] = \sin\left[\frac{2}{3}\pi n_\tau(t) + \frac{2}{3}\delta\chi_\tau(t)\right] \approx \begin{cases} \frac{\sqrt{3}}{2} - \frac{1}{3}\delta\chi_\tau(t), & \text{if } 0 \le t \le \tau, \\ \frac{2}{3}\delta\chi_\tau(t), & \text{if } \tau < t \le \beta. \end{cases} \tag{S1.24}$$

The function $\delta\chi_\tau(t)$ as shown in formula (S1.16) makes the integrals to diverge logarithmically . At $|t-t'| \gg E_C^{-1}$ and $T \ll E_C$, we take into account only the terms which contain the first order in $\delta\chi_\tau(t)$, we obtain

$$\langle S_2'^2\rangle_{\tau,\, odd\, N} \approx -\frac{8\left(2-\sqrt{3}\right)}{3}\left[\frac{6\gamma}{\pi^2}\right]^{\frac{2}{3}}|r|^2\left[\frac{T}{E_C}\right]^{\frac{1}{3}}\ln\left[\frac{E_C}{T}\right]\sin[4\pi N]\,F_2(\tau), \tag{S1.25}$$

with

$$F_2(t) = \cos(\pi T t)\left[F_2^1\left[\frac{1}{2},\frac{5}{6},\frac{3}{2},\cos^2(\pi T t)\right] + 3|\sin(\pi T t)|^{\frac{1}{3}}\right]. \tag{S1.26}$$

The second order contribution to the correlator $K(\tau)$ corresponding to this process is

$$K_{odd\,N}(\tau) \approx -\frac{2^{\frac{2}{3}}4\left(2-\sqrt{3}\right)}{3}|r|^2\frac{T}{E_C}\ln\left[\frac{E_C}{T}\right]\sin\left[4\pi N\right]\frac{F_2(\tau)}{|\sin(\pi T\tau)|^{\frac{2}{3}}}. \tag{S1.27}$$

The equation for the thermoelectric coefficient G_T accounting for an electron coming from QPC2 is given by

$$G_T = C_2\frac{G_L}{e}|r|^2\frac{T}{E_C}\ln\left[\frac{E_C}{T}\right]\sin\left[4\pi N\right], \tag{S1.28}$$

with

$$C_2 = \frac{2^{\frac{2}{3}}2\left(2-\sqrt{3}\right)}{3}\pi\int_{-\infty}^{\infty}\frac{\sinh^2(x)\left[F_2^1\left[\frac{1}{2},\frac{5}{6},\frac{3}{2},-\sinh^2(x)\right]+3\cosh^{\frac{1}{3}}(x)\right]}{\cosh^{\frac{11}{3}}(x)}dx = 0.936794. \tag{S1.29}$$

If an electron comes from QPC1 or QPC3, we only take into account the second term in Eq. (S1.7) and $\phi_{s\tau}(x,t) = {}^{\pm\pi N}/\sqrt{2}$, $\phi_{f\tau}(x,t) = {}^{\pi N}/\sqrt{6}$ then

$$\kappa_s^+(t,t',\tau) \approx e^{i2\pi N}\left[\frac{4\pi T}{D}\right]|\sin(\pi T(t-t'))|, \tag{S1.30}$$

$$\kappa_f^+(t,t',\tau) \approx e^{i\frac{2\pi N}{3}}\left[\frac{4\pi T}{D}\right]^{\frac{1}{3}}|\sin(\pi T(t-t'))|^{\frac{1}{3}}. \tag{S1.31}$$

As we have discussed in the main text, G_T must be an odd function of the gate voltage N as well as odd function of τ. Plugging in all $\kappa_j^{\pm}(t,t',\tau)$ given by equations Eq.(S1.18-S1.31) into formula (S1.7), we obtain the temperature scaling of $\langle S'^2\rangle_{\tau,\,odd\,N}$.We find that only two terms $2\mathrm{Re}\left[\kappa_c^+(t,t',\tau)\,\kappa_f^+(t,t',\tau)\right]\mathrm{Re}\left[\kappa_s^+(t,t',\tau)\right]$ and $2\mathrm{Re}\left[\kappa_c^+(t,t',\tau)\,\kappa_f^+(t,t',\tau)\right]\mathrm{Re}\left[\kappa_s^-(t,t',\tau)\right]$ contribute to G_T. The first one contributes to G_T the same result as shown in Eq. (S1.28). Let us illustrate the calculations for the latter term:

$$\langle S'^2\rangle_{\tau,\,odd\,N} \approx 2\left[\frac{6\gamma E_C}{\pi^2}\right]^{\frac{2}{3}}T^{\frac{4}{3}}|r|^2\sin\left[2\pi N\right]\int_0^{\beta}dt\sin\left[\frac{2}{3}\chi_\tau(t)\right]\int_0^{\beta}dt'\cos\left[\frac{2}{3}\chi_\tau(t')\right]|\sin(\pi T(t-t'))|^{-\frac{2}{3}}. \tag{S1.32}$$

The function $|\sin(\pi T(t-t'))|^{-\frac{2}{3}}$ exhibits *integrable* power-law divergence at $t'\to t$,. We calculate the integrals in Eq. (S1.32) in the same way as we did for Eq. (S1.22). At the end, we obtain

$$\langle S'^2\rangle_{\tau,\,odd\,N} \approx -\frac{10}{3}\left[\frac{6\gamma}{\pi^2}\right]^{\frac{2}{3}}\left[\frac{T}{E_C}\right]^{\frac{1}{3}}|r|^2\sin\left[2\pi N\right]\ln\left[\frac{E_C}{T}\right]F_1(\tau), \tag{S1.33}$$

with

$$F_1(t) = \cos(\pi Tt)\,F_2^1\left[\frac{1}{2},\frac{5}{6},\frac{3}{2},\cos^2(\pi Tt)\right]. \tag{S1.34}$$

We plug formula (S1.33) into formula (S1.5) and obtain the correlator $K(\tau)$ at second order as

$$K_{odd\,N}(\tau) \approx -\frac{2^{\frac{2}{3}}5|r|^2}{3}\frac{T}{E_C}\ln\left[\frac{E_C}{T}\right]\sin\left[2\pi N\right]\frac{F_1(\tau)}{|\sin(\pi T\tau)|^{\frac{2}{3}}}. \tag{S1.35}$$

From (7) and (S1.35), we obtain the thermoelectric coefficient G_T which is contributed by $2\mathrm{Re}\left[\kappa_c^+(t,t',\tau)\,\kappa_f^+(t,t',\tau)\right]\mathrm{Re}\left[\kappa_s^-(t,t',\tau)\right]$. We collect the contributions to the thermoelectric coefficient G_T accounting for electrons coming from either QPC1 or QPC3 and add corresponding contribution originating from QPC2. Finally performing symmetrization over all QPCs index permutations corresponding to all possible ways to re-numerate QPCs as explained in the main text we get

$$G_T = C_1\left(1 + a\cos[2\pi N]\right)\frac{G_L}{e}|r|^2\frac{T}{E_C}\ln\left[\frac{E_C}{T}\right]\sin\left[2\pi N\right], \tag{S1.36}$$

with $a = 2C_2/C_1 \approx 2.87$,

$$C_1 = \frac{5}{9}2^{2/3}\pi\int_{-\infty}^{\infty}\frac{\sinh^2(x)F_2^1\left[\frac{1}{2},\frac{5}{6},\frac{3}{2},-\sinh^2(x)\right]}{\cosh^{\frac{11}{3}}(x)}dx = 0.653. \tag{S1.37}$$

We demonstrated that the temperature scaling of G_T and TP is given by $T\log T$ and $T^{1/3}\log T$ correspondingly.