Ambegaokar-Eckern-Schön theory for a collective spin: geometric Langevin noise. Supplemental Material

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A. CHOICE OF THE GAUGE.

Here we present a detailed justification of the gauge which is presented in MainTextEq. (6). Ideally we should have chosen a gauge that would lead to $Q_\parallel = 0$. Seemingly, this might have been achieved with the choice $\dot\chi(t) = \dot\phi(t)\,(1-\cos\theta(t))$ on both branches of the Keldysh contour. This choice, however, violates our desired boundary conditions as the integrals over $\dot\chi$ accumulated between $t=-\infty$ and $t=+\infty$ on the upper and on the lower Keldysh branches are different. Such a difference would show up as non-trivial boundary conditions on χ_q at either $t=-\infty$ or $t=+\infty$. In other words, had we selected $\dot\chi(t) = \dot\phi(t)\,(1-\cos\theta(t))$ we should have violated the requirement $\chi_q(t=\pm\infty)=0$. We note, though, that to linear order in the quantum components the condition $\dot\chi(t) = \dot\phi(t)\,(1-\cos\theta(t))$ yields $\dot\chi_q = \dot\phi_q(1-\cos\theta_c) + \theta_q \sin\theta_c\,\dot\phi_c$, leading to $\chi_q(t) = \int^t dt' \left[\dot\phi_q(t')(1-\cos\theta_c(t')) + \theta_q(t')\sin\theta_c(t')\,\dot\phi_c(t')\right] = \phi_q(t)(1-\cos\theta_c(t)) + \int^t dt' \sin\theta_c(t')\left[\theta_q(t')\,\dot\phi_c(t') - \dot\theta_c(t')\,\phi_q(t')\right]$. The first term vanishes at $t=\pm\infty$ but not the last term. We thus include only the first term in χ_q, leading to MainTextEq. (6), and consequently to a non-vanishing contribution to $\tilde Q_\parallel$ (MainTextEq. (7)).

B. SEMI-CLASSICAL EQUATIONS OF MOTION.

Here we present the derivation of the semiclassical equations of motion, MainTextEq. (10). Using the representation $R = A_0\sigma_0 + iA_x\sigma_x + iA_y\sigma_y + iA_z\sigma_z$, with $A_0 \equiv \cos\left[\frac{\theta}{2}\right]\cos\left[\frac{\chi}{2}\right]$, $A_x \equiv \sin\left[\frac{\theta}{2}\right]\sin\left[\phi-\frac{\chi}{2}\right]$, $A_y \equiv -\sin\left[\frac{\theta}{2}\right]\cos\left[\phi-\frac{\chi}{2}\right]$, $A_z \equiv -\cos\left[\frac{\theta}{2}\right]\sin\left[\frac{\chi}{2}\right]$ we rewrite the AES action (MainTextEq. (9)) as $\mathcal{S}_{AES} = \mathcal{S}^R_{AES} + \mathcal{S}^K_{AES}$, where

$$i\mathcal{S}^R_{AES} = -2ig\int dt_1 dt_2\, \alpha''_R(t_1-t_2)\sum_j A^q_j(t_1)A^c_j(t_2)\ , \tag{1}$$

and

$$i\mathcal{S}^K_{AES} = -\frac{g}{2}\int dt_1 dt_2\ \alpha_K(t_1-t_2)\sum_j A^q_j(t_1)A^q_j(t_2)\ . \tag{2}$$

Here $\alpha''_R(t) \equiv \mathrm{Im}\,\alpha_R(t)$ and $j=0,x,y,z$. The Keldysh part of the action (2) leads to random Langevin forces. This can be shown [1] using the Hubbard-Stratonovich transformation

$$e^{i\mathcal{S}^K_{AES}} = \int\left(\prod_{j=0,x,y,z} D\xi_j\right)\times \exp\left[\int dt\left\{i\sum_{j=0,x,y,z}\xi_j A^q_j\right\} + i\mathcal{S}_\xi\right]\ , \tag{3}$$

where the action $\mathcal{S}_\xi$ is given by

$$i\mathcal{S}_\xi = -\frac{1}{2g}\sum_j\int dt_1 dt_2\,[\alpha_K]^{-1}_{(t_1-t_2)}\,\xi_j(t_1)\xi_j(t_2)\ . \tag{4}$$

In other words, $\langle\xi_j(t_1)\xi_k(t_2)\rangle = \delta_{jk}\,g\,\alpha_K(t_1-t_2)$ and $\langle\xi_j\rangle = 0$. We obtain the Langevin equations MainTextEq. (10) from $\delta i\mathcal{S}_{total}/\delta\phi_q(t) = \delta i\mathcal{S}_{total}/\delta\theta_q(t) = 0$, where $i\mathcal{S}_{total} \equiv i\mathcal{S}_B + i\mathcal{S}_{WZNW} + i\mathcal{S}^R_{AES} + \int dt\,\sum_j i\xi_j A^q_j$. Here $i\mathcal{S}_B = -iS\gamma\,B\int dt\,\sin\theta_c\,\theta_q$ is the action related to the magnetic field (in z-direction). Prior to performing the variation of the action, the field χ is replaced according to the gauge fixing choice (MainTextEq. (6)). Finally, we use $\alpha''_R(t) = (\partial_t + C)\delta(t)$ (the constant C is important for causality but drops in our calculation) and obtain MainTextEqs. (10).

C. JUSTIFICATION OF THE SEMI-CLASSICAL EXPANSION

Here we justify why a semiclassical expansion of the action, leading to MainTextEq. (10), is applicable. It is instructive to rewrite the Berry phase action as $i\mathcal{S}_{WZNW} = i\oint_K dt\, p\,\dot{\phi}$, where $p \equiv S(1-\cos\theta)$. After the Keldysh rotation this gives

$$i\mathcal{S}_{WZNW} = i\int dt\,\left[p_c\dot{\phi}_q + p_q\dot{\phi}_c\right] = i\int dt\,\left[-\phi_q\dot{p}_c + p_q\dot{\phi}_c\right]\ , \qquad (5)$$

where we used the fact that the quantum component ϕ_q must vanish at $t=\pm\infty$. In contrast to MainTextEq. (8) we do not yet assume the quantum components to be small, thus, e.g., $p_q = p_u - p_d = -S(\cos\theta_u - \cos\theta_d)$. The rest of the action can in principle be also expressed using these variables.

The Berry phase part of the action (5) determines the canonical structure of our theory. Namely, we can define two pairs of canonically conjugate variables, i.e., $(-p_c,\phi_q)$ and (ϕ_c,p_q). Here $-p_c$ and ϕ_c play the role of canonical coordinates, whereas ϕ_q and p_q are their respective conjugate momenta. We are interested in diffusion, i.e., noise, of the coordinates ϕ_c and p_c. The well established way to estimate the latter is to obtain the generating function by introducing counting source fields (see, e.g., Ref [2]). The counting fields shift the conjugate momenta. For example, to calculate the generating function for cumulants of $-p_c$ the Keldysh partition function should be calculated with a shifted conjugate momentum $\phi_q \to \phi_q + \lambda$. The full action, including the dissipative terms, is periodic in ϕ_q. Thus the generating function is periodic in λ. This corresponds to the quantization of the conjugate coordinate $-p_c$ which is nothing but the classical component of $S_z - S$, where S_z is the z projection of the spin. Thus, in all processes described by our AES action S_z changes by $\Delta S_z = \pm 1$, as expected for a spin variable.

In this paper we assume $S \gg 1$. Thus, quantized jumps of S_z give rise to very small ($\sim 1/S$) changes of the angle θ. This allows us to consider the long time limit of continuous diffusion of θ. This limit is well described by a semi-classical approximation, in which the action is expanded up to the second order in the quantum components θ_q and ϕ_q. By performing this expansion we lose all cumulants higher than the second one. In the second cumulant (noise) the high frequency quantum noise is mixed (down-converted). This is due to the fact that the expanded Keldysh component of the action (2) still contains the classical components θ_c and ϕ_c. Thus, the resulting Langevin equation is "multiplicative", i.e., the noise terms (MainTextEq. (11)) contain the coordinates θ_c and ϕ_c. Similar mechanism led to the shot noise in the original AES case [3] (see also [4]).

The full action is not a periodic function of p_q (it is, of course, periodic as a function of $\theta_{u/d}$). Thus, no quantization corresponds to the second pair of conjugated variables (ϕ_c, p_q).

D. FEASIBILITY OF THE BANG-BANG EXPERIMENT

Below we argue that the proposed bang-bang experiment is, in fact, within the realm of the present day technology. We note that several works dealing with manipulations of qubits did encounter the problem of the resolution of the spin state. In particular the bang-bang technique has been successfully applied (see e.g., [5] (bang-bang in Fulerene qubits); [6] (bang-bang in Josephson qubits)). The spread of the initial spin state may be quantum-limited and could be less than 2% of a radian (cf. [7] on qubit tomography and supplemental material thereof). The state may broaden (on the Bloch sphere) through diffusion in the course of its evolution; even if this broadening is tiny, it may be resolved following repeated evolutions.

Let us discuss this in some detail, in the context of our large spin evolution. During the free evolution between two consecutive bang-bang π-pulses, the geometric diffusion constant is of order $D \sim gB/S^2$, whereas the relaxation rate (the inverse relaxation time) is of order $= \tau^{-1} \sim gB/S$. The time interval Δt between consecutive π-pulses of the bang-bang protocol is chosen so that the angle θ does not change much due to the deterministic relaxation. In other words, given that typically $\theta \sim \pi/4$, we request that $\Delta\theta^{\rm det} \sim 1/K \ll 1$ (here K is a large integer). We thus choose $\Delta t = \tau/K = 1/(\Gamma K)$. After N bang-bang pulses the spread due to the geometric diffusion is of the order of $\Delta\theta^{\rm diff} \sim \sqrt{DN\Delta t} = \sqrt{N/(KS)}$. For the latter quantity to be detectable, we require that it is of order A/S ($A \gg 1$), where $1/S$ is the minimal spread corresponding to quantum uncertainty. We assume here that any spread larger than the quantum uncertainty is detectable (this can be achieved by averaging over many repetitions of the same bang-bang procedure). This leads to a condition on the minimum number of bang-bang pulses, $N = KA^2/S$.

Let us assume for simplicity very strong π-pulses, i.e. $\Omega > B$, where Ω is the amplitude of a π pulse. Then the diffusion constant during the π-pulses is equal to $D^{\rm pulse} \sim g\Omega/S^2$, and the relaxation rate is given by $\Gamma^{\rm pulse} \sim g/S$. The pulse duration is of order $1/\Omega$ (remember we need half a rotation in a π-pulse). The deterministic change of θ due to and during N pulses (not counting the free evolution between the pulses) is given by

$$d\theta^{\rm N\ pulses}_{\rm det} \sim \Gamma^{\rm pulse} N/\Omega = Ng/S\ .$$

The diffusive spread due to and during N pulses (not counting the free evolution between pulses) is equal to

$$d\theta_{\rm diff}^{\rm N\ pulses} \sim \sqrt{ND^{\rm pulse}/\Omega} = \sqrt{Ng/S^2}\ .$$

Substituting $N = KA^2/S$ we obtain $d\theta_{\rm det}^{\rm N\ pulses} = F/S$ and $d\theta_{\rm diff}^{\rm N\ pulses} = \sqrt{F}/S$, where $F = gKA^2/S$.

It is clear that we need to estimate both $d\theta_{\rm det}^{\rm N\ pulses}$ and $d\theta_{\rm diff}^{\rm N\ pulses}$, as neither of them is cancelled by the bang-bang procedure. Both errors become of order $1/S$ for $F = 1$, that is for $g = S/(KA^2)$. Thus, if the tunneling conductance is smaller than this value the error due to the bang-bang pulses is smaller than the spread due to the geometric diffusion and therefore unimportant.

E. RELATION TO KONDO PROBLEM?

Below we expand the short argument given in the main text leading us to conclude that our model is unrelated to the Kondo problem. Our quantum dot Hamiltonian (MainTextEq. (1)) does not include a charging term, hence no Kondo physics. The best way to realize this model is to think of a large quantum dot with negligible charging energy, as was the case, e.g., in Refs. [8, 9]. As a result we are neither in a Coulomb valley, nor at a Coulomb peak. In this case three different types of fluctuations may take place: (i) Keeping the total S constant, the S_z component may fluctuate; (ii) S itself may fluctuate. We note that in the vicinity of the macroscopic Stoner instability (on either side), the distance in energy between an S and an $S+1$ configuration is much smaller than the level spacing δ (it is of order δ/S). Once the temperature (or the dot-lead tunneling strength, see below) is larger than this energy, such fluctuations in S are facilitated. (iii) Once the temperature is higher than the charging energy (or the tunneling strength becomes larger than the mean level spacing), the Coulomb energy is irrelevant, and fluctuations in the total number of electrons in the dot are allowed. Clearly, fluctuations of either type (ii) or (iii) (or both) take us beyond any Kondo model.

We note that the dissipative terms in our equations of motion are quadratic in the tunneling amplitude (linear in g, cf. for example MainTextEq. (10)). This has also been the case in Refs. [8, 9]. By contrast, cotunneling (facilitating fluctuations of S_z by 1, changing neither S nor the total charge), which would be the building blocks of high-order Kondo screening processes, is second order in g, hence Kondo physics is not present in our analysis.

In passing we note that one standard scenario where the charging energy, even if present, is not important refers to multi-channel leads (not to confuse with multi-channel Kondo). In this scenario each of the channels is weakly coupled to the dot (the tunneling coupling is $|V|^2$), but the sum of all those couplings renders the lead-dot conductance $g > 1$. Under these conditions the charging energy is suppressed, but perturbation in $|V|^2$ is allowed (note that the condition for an underdamped motion of the spin implies $g/S \ll 1$; this allows for $g \gg 1$).

F. RELEVANCE OF LONGITUDINAL FLUCTUATIONS OF M

In the main text we made an approximation $M(t) = M_0$, thus neglecting completely the longitudinal fluctuations of the magnetization. Here we discuss the effect of the latter and show that it is unimportant as far as our AES dynamics is concerned. As shown in our previous works [10–12], in the regime of mesoscopic Stoner instability the statistical fluctuations of M in an isolated dot are of the order $\Delta M \sim \sqrt{M_0 T}$. Close enough to Stoner instability $M_0 \gg T$ and, thus, $\Delta M \ll M_0$. For an isolated dot these are purely statistical fluctuations (fluctuations between different ensemble members) since the total spin is a constant of motion there. In an open dot considered here dynamical fluctuations of M become possible. One can show that these will be again limited by $\Delta M \sim \sqrt{M_0 T} \ll M_0$. In addition these fluctuation are slow (critical slowdown). Thus, the longitudinal fluctuations can be safely neglected in an analysis of the spin dynamics on the Bloch sphere. Clearly, in a ferromagnetic dot (ferromagnetic side of Stoner) the longitudinal fluctuations are even less noticeable.

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